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TETRAD-GAUGE THEORY OF GRAVITY

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We present a tetrad–gauge theory of gravity based on the local Lorentz group in a four-dimensional Riemann–Cartan space–time. Using the tetrad formalism allows avoiding problems connected with the noncompactness of the group and includes the possibility of choosing the local inertial reference frame arbitrarily at any point in the space–time. The initial quantities of the theory are the tetrad and gauge fields in terms of which we express the metric, connection, torsion, and curvature tensor. The gauge fields of the theory are coupled only to the gravitational field described by the tetrad fields. The equations in the theory can be solved both for a many-body system like the Solar System and in the general case of a static centrally symmetric field. The metric thus found coincides with the metric obtained in general relativity using the same approximations, but the interpretation of gravity is quite different. Here, the space–time torsion is responsible for gravity, and there is no curvature because the curvature tensor is a linear combination of the gauge field tensors, which are absent in the case of pure gravity. The gauge fields of the theory, which (together with the tetrad fields) define the structure of space–time, are not directly coupled to ordinary matter and can be interpreted as the fields describing dark energy and dark matter.

Keywords: tetrad formalism, torsion, gauge field, gravity, dark matter, dark energy

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1. Introduction

We construct a tetrad–gauge theory of gravity (TGTG) as a gauge theory of gravitational and gauge fields in a four-dimensional Riemann–Cartan space–time U_4 [1]. This approach differs radically from general relativity (GR) in both the mathematical tools used and the physical interpretation. We define the metric as in [2]: $d\tau^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu$ with the signature $(+,-,-,-)$. The indices $\mu,\nu,\lambda,\dots$ from the middle of the Greek alphabet are used as coordinate indices in an arbitrary reference frame where gravity is present (general coordinate indices) and take the values 0, 1, 2, and 3. The interval $d\tau$ becomes invariant under parallel translation and is therefore a useful concept in the usual space–time physics if the metric $g_{\mu\nu}(x)$ is covariantly constant (this means that the nonmetricity tensor is zero [1]). The tetrad formalism can include a nonzero nonmetricity tensor in constructing the theory of gravity in a general affine-metric space (see [3], where there are many references that allow viewing the history of using the tetrad formalism and its applications in different approaches to developing the modern theory of gravity). The TGTG is based on the local Lorentz group, whose noncompactness (as shown below) creates no obstacles to constructing the gauge theory [4]. The nonsymmetric connection in the space U_4 is

$$\Gamma^\lambda_{\mu\nu} = G^\lambda_{\mu\nu} + S_{\mu\nu}{}^\lambda + S^\lambda{}_{\mu\nu} + S^\lambda{}_{\nu\mu},$$

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where $G^{\lambda}_{\mu\nu} = g^{\lambda\rho}(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu})/2$ is the purely metric term (Christoffel symbols) [2]; the torsion tensor $S_{\mu\nu}{}^{\lambda}$, which is antisymmetric in the first two indices, is the metric-independent part of the connection [1]; and the third and fourth terms are obtained from the torsion tensor by raising and lowering the indices using the metric tensor $g_{\mu\nu}$, which is covariantly constant in this connection. The last three terms of the connection form the contorsion tensor, which is antisymmetric in the last two indices: $K_{\mu\nu}{}^{\lambda} \equiv -(S_{\mu\nu}{}^{\lambda} + S^{\lambda}{}_{\mu\nu} + S^{\lambda}{}_{\nu\mu})$ [1].

This paper is structured as follows. In Sec. 2, we introduce the basic conventions and notation related to the Lorentz group [5], [6], the tetrad formalism in the theory of gravity [7], and gauge fields in the tetrad formalism. In Sec. 3, we construct the Lagrangian for the system of tetrad and gauge fields, which provides the lower bound for the energy of small perturbations, and we derive a system of equations for the tetrad and gauge fields in presence of point particles from the least action principle. In Sec. 4, we consider the post-Newtonian expansion procedure applied to the TGTG. In Sec. 5, we solve the TGTG equations in the Newtonian and first post-Newtonian approximation. In Sec. 6, we find the metric of a static isotropic gravitational field as a result of solving the TGTG equations. In Sec. 7, we give brief conclusions and present some prospects for the TGTG.

2. Basic conventions and notation

2.1. Basic quantities and relations associated with the Lorentz group. The Lorentz group transformations in the Minkowski space are defined by the relation $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$, where $\Lambda = (\Lambda^{\alpha}{}_{\beta})$ is the Lorentz matrix and $\eta = (\eta_{\alpha\beta})$ is the Minkowski metric with the signature $(+,-,-,-)$.

We introduce the single notation ${}_4t_a$ for all six group generators, where the index $a = (a', a'')$ ranges six values (the indices a' and a'' take the values 1, 2, and 3): $({}_4t_{a'})^{\alpha}{}_{\beta} = -i\delta^{\alpha}_i\,\delta^{\beta}_j\varepsilon_{a'ij}$ are the matrix elements of the generator of the space rotation around the axis labeled a', $\varepsilon_{a'ij}$ is a completely antisymmetric unit rank-3 pseudotensor ($\varepsilon_{123} = 1$), and $({}_4t_{a''})^{\alpha}{}_{\beta} = i(\delta^{\alpha}_0\,\delta^{\beta}_{a''} + \delta^{\alpha}_{a''}\,\delta^{\beta}_0)$ are the matrix elements of the boost generator in the direction of the space axis labeled a''. The indices $\alpha, \beta, \gamma, \delta, \ldots$ from the beginning of the Greek alphabet are used for the locally inertial coordinates, where there is no gravity (Lorentz indices), and take the values 0, 1, 2, and 3. In this notation, the commutation relations for ${}_4t_a$ are $[{}_4t_a, {}_4t_b] = iC^{c}{}_{ab}\,{}_4t_c$, where the only nonzero group constants are $C^{c'}{}_{a'b'} = \varepsilon_{c'a'b'}$, $C^{c'}{}_{a''b''} = -\varepsilon_{c'a''b''}$, and $C^{c''}{}_{a'b''} = \varepsilon_{c''a'b''}$. We consider real spaces and, instead of complex matrices ${}_4t_a$, therefore use real matrices $\tau_a = -i_4t_a$ satisfying the commutation relations $[\tau_a, \tau_b] = C^{c}{}_{ab}\,\tau_c$. Eliminating the upper index in $({}_4t_a)^{\alpha}{}_{\beta}$ via the metric $\eta_{\alpha\beta}$, we find that $(\tau_a)_{\alpha\beta} = -(\tau_a)_{\beta\alpha}$. The Lorentz group is related to its Lie algebra by the exponential map $\Lambda(\theta) = \exp(\tau_a\theta^a)$.

The generators ${}_6t_a$ of the adjoint representation [5] for the Lorentz group are purely imaginary, and their matrix elements are $({}_6t_a)^{b}{}_{c} = iC^{b}{}_{ac}$. Below, we use the real quantities $\chi_a = -i_6t_a$ with the matrix elements $(\chi_a)^{b}{}_{c} = C^{b}{}_{ac}$ satisfying the same commutation relations $[\chi_a, \chi_b] = C^{c}{}_{ab}\chi_c$ as τ_a. The metric ${}_6\eta = ({}_6\eta_{ab})$ in the space of the adjoint representation for the Lorentz group is expressed in terms of the group constants:

$$
{}_6\eta_{ab} = {}_6\eta^{ab} = -\frac{1}{4}\,\mathrm{Tr}(\chi_a\chi_b) = \frac{1}{4}C^{c}{}_{ad}C^{d}{}_{bc},
$$

$$
{}_6\eta_{a'b'} = \delta_{a'b'}, \qquad {}_6\eta_{a''b''} = -\delta_{a''b''}, \qquad {}_6\eta_{a'b''} = {}_6\eta_{a''b'} = 0.
$$

The generators χ_a satisfy the relation $\chi^{\mathrm{T}}_a\,{}_6\eta = -{}_6\eta\chi_a$, which implies the equality ${}_6\Lambda^{\mathrm{T}}\,{}_6\eta\,{}_6\Lambda = {}_6\eta$ satisfied by the matrix ${}_6\Lambda$ of the adjoint representation. In the adjoint representation, there is a standard exponential map between the Lorentz group and its Lie algebra: ${}_6\Lambda(\theta) = \exp(\chi_a\theta^a)$. The metric in the space of the adjoint representation can also be expressed in terms of the Lorentz group generators τ_a as ${}_6\eta_{ab} = -\,\mathrm{Tr}(\tau_a\tau_b)/2 = (\tau_a)^{\alpha}{}_{\beta}(\tau_b)^{\beta}{}_{\alpha}/2$.

2.2. Tetrad formalism in the theory of gravity. Using the tetrad formalism in the theory of gravity is unavoidable because of the requirement to describe the spinor fields in this theory. According to the principle of the local equivalence of gravity and inertial force at each point of the space–time, we can introduce a locally inertial reference frame with the basis vectors $e^\alpha(x)$ having the components $e^\alpha{}_\mu(x)$ in the reference frame with the coordinates x^μ, and we can hence represent the metric as $g_{\mu\nu}(x) = e^\alpha{}_\mu(x) e^\beta{}_\nu(x)\eta_{\alpha\beta}$. Equivalently, we can regard $e^\alpha{}_\mu(x)$ as components of four basis coordinate vectors $e_\mu(x)$ with respect to the basis $e^\alpha(x)$. The indices of $e^\alpha{}_\mu(x)$ can be raised and lowered using the respective matrices $\eta_{\alpha\beta}$ and $g_{\mu\nu}(x)$, and each of the two indices can therefore have either the upper or the lower position. It is important that the left index is the Lorentz index of the locally inertial reference frame while the right index is a general coordinate index.

The choice of the dynamical variables describing the field is important in constructing a field theory. Therefore, if the metric is chosen as a dynamical variable describing gravity, then the least action principle leads to a GR theory that is not a gauge theory. The possibility promised by GR to choose a locally inertial reference frame arbitrarily at each space–time point essentially means the presence of a symmetry described by the local Lorentz group, but this symmetry remains beyond the scope of the mathematical setting of GR, which admits the presence of such a symmetry but cannot describe it, because of the unsuitable choice of the dynamical variable, which cannot take this symmetry into account. In fact, the metric is invariant under local Lorentz transformations.

Choosing the tetrad as a dynamical variable describing the gravitational field, we must use the derivatives of the tetrad components to construct the Lagrangian, and this breaks the covariance of the theory. To restore this covariance, we must introduce the gauge fields. We note that the Lorentz group is noncompact, but this does not prevent constructing a physically suitable gauge theory, as we show below.

2.3. Tetrad formalism and gauge fields. Together with the fields $e^\alpha{}_\nu(x)$, the Lagrangians must contain their derivatives $\partial_\mu e^\alpha{}_\nu(x)$ with respect to the coordinates. Under infinitesimal local Lorentz transformations, the tetrad $e^\alpha{}_\nu(x)$ transforms according to the rule $e^\alpha{}_\nu(x) \to e'^\alpha{}_\nu(x) = e^\alpha{}_\nu(x) + (\tau_a)^\alpha{}_\beta\, \theta^a(x) e^\beta{}_\nu(x)$. In this case, the derivative transformation rule $\partial_\mu e^\alpha{}_\nu(x)$ contains terms with the derivatives $\partial_\mu \theta^a(x)$, which prevent constructing invariant Lagrangians. To eliminate these derivatives, we use the well-known method of introducing gauge fields [6] by replacing $\partial_\mu e^\alpha{}_\nu(x)$ with the *gauge-covariant derivative of the tetrad*

$$[D_\mu(B) e_\nu(x)]^\alpha \equiv [D_\mu(B)]^\alpha{}_\beta\, e^\beta{}_\nu(x) \equiv [\delta^\alpha_\beta\, \partial_\mu + (\tau_a)^\alpha{}_\beta\, B^a{}_\mu(x)] e^\beta{}_\nu(x), \tag{1}$$

where $B^a{}_\mu(x)$ are six coordinate covariant vector fields (gauge fields) that transform under local Lorentz transformations according to the rule

$$B^a{}_\mu(x) \to B'^a{}_\mu(x) = B^a{}_\mu(x) + C^a{}_{cb}\, B^b{}_\mu(x)\, \theta^c(x) - \partial_\mu \theta^a(x). \tag{2}$$

The transformation rule for the gauge-covariant derivative of the tetrad under infinitesimal local Lorentz transformations,

$$[D_\mu(B) e_\nu(x)]^\alpha \to [D_\mu(B') e'_\nu(x)]^\alpha = [D_\mu(B) e_\nu(x)]^\alpha + (\tau_a)^\alpha{}_\beta\, \theta^a(x) [D_\mu(B) e_\nu(x)]^\beta, \tag{3}$$

has the same form as for the tetrad itself, as it should for a Lorentz vector.

We introduce the contraction of the derivative $[D_\mu(B) e_\nu(x)]^\alpha$ and $e_\alpha{}^\lambda(x)$ over the Lorentz index α:

$$\Gamma^\lambda_{\mu\nu}(x) \equiv e_\alpha{}^\lambda(x) [D_\mu(B) e_\nu(x)]^\alpha = e_\alpha{}^\lambda(x)\, \partial_\mu e^\alpha{}_\nu(x) + (\tau_a)^\alpha{}_\beta\, e_\alpha{}^\lambda(x)\, e^\beta{}_\nu(x)\, B^a{}_\mu(x). \tag{4}$$

Because of the derivative, the first term in the right-hand side is not a tensor, $\Gamma^\lambda_{\mu\nu}(x)$ therefore behaves as a space–time connection under a coordinate change, and we identify it with the connection. The covariant derivative of the tetrad in this connection (we omit the explicit indication of the coordinate dependence for quantities below) is linear in the gauge fields:

$$e^{\alpha}{}_{\mu;\lambda} = \partial_\lambda e^{\alpha}{}_{\mu} - \Gamma^{\nu}_{\lambda\mu}\, e^{\alpha}{}_{\nu} = -(\tau_a)^{\alpha}{}_{\beta}\, e^{\beta}{}_{\mu}\, B^{a}{}_{\lambda}. \tag{5}$$

It hence follows that $g_{\mu\nu;\lambda} = 0$, and the metric therefore agrees with connection (4). The four-dimensional Riemann–Cartan space–time U_4, which is completely defined by the connection $\Gamma^{\lambda}_{\mu\nu}$ and the metric $g_{\mu\nu}$, can hence be described in terms of the tetrad field $e^{\alpha}{}_{\mu}$ and the gauge fields $B^{a}{}_{\mu}$. In this case, we must impose six independent conditions in the form of tensor equalities on the field components to eliminate the six-parameter redundancy of the number of independent field components $e^{\alpha}{}_{\mu}$ and $B^{a}{}_{\mu}$ compared with the number of independent components of the metric $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}_{\mu\nu}$ (there are 74 components of the metric $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}_{\mu\nu}$ that satisfy the 40 conditions $g_{\mu\nu;\lambda}$=0, and there are 40 components of $e^{\alpha}{}_{\mu}$ and $B^{a}{}_{\mu}$).

Connection (4) can be represented as a sum $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)} + S_{\mu\nu}{}^{\lambda}$, where the term $\Gamma^{\lambda}_{(\mu\nu)} \equiv (\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\mu})/2$ symmetric in the lower indices is not a tensor and the antisymmetric term

$$S_{\mu\nu}{}^{\lambda} \equiv \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}) = \frac{1}{2} e_{\alpha}{}^{\lambda}[(\partial_\mu e^{\alpha}{}_{\nu} - \partial_\nu e^{\alpha}{}_{\mu}) + (\tau_a)^{\alpha}{}_{\beta}(e^{\beta}{}_{\nu}\, B^{a}{}_{\mu} - e^{\beta}{}_{\mu}\, B^{a}{}_{\nu})] \tag{6}$$

is a torsion tensor that is nonzero even at $B^{a}{}_{\mu} = 0$.

The curvature tensor, defined as usual [1], [7] via connection (4), in terms of the variables $e^{\alpha}{}_{\mu}$ and $B^{a}{}_{\mu}$ is

$$R^{\mu}{}_{\nu\lambda\rho} \equiv \partial_\lambda \Gamma^{\mu}_{\rho\nu} - \partial_\rho \Gamma^{\mu}_{\lambda\nu} + \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\lambda\nu} = (\tau_a)_{\alpha\beta}\, e^{\alpha\mu} e^{\beta}{}_{\nu}\, F^{a}_{\lambda\rho}, \tag{7}$$

where we use the notation for the six gauge field tensors

$$F^{a}_{\lambda\rho} \equiv \partial_\lambda B^{a}{}_{\rho} - \partial_\rho B^{a}{}_{\lambda} + C^{a}{}_{bc}\, B^{b}{}_{\lambda}\, B^{c}{}_{\rho}. \tag{8}$$

Unlike the gauge fields $B^{a}{}_{\mu}$, the coordinate tensor $F^{a}_{\lambda\rho}$ is a vector (with respect to the index a) in the adjoint representation space of the Lorentz group, which we can easily verify in the case of infinitesimal transformations. The tensor $R_{\mu\nu\lambda\rho}$ is asymmetric with respect to the indices of the first and the second pairs, but these two pairs of indices are completely different: the first is related only to the tetrad fields, and the second is related only to the gauge field tensor. Therefore, the space U_4 can have a curvature only in the presence of gauge fields, while the torsion exists even without them. As a consequence of the different nature of the first and second pairs of the torsion tensor indices, the Ricci tensor $R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$ is antisymmetric.

3. System of equations for tetrad and gauge fields in the presence of classical particles

3.1. Invariants. We construct the Lagrangian for the system of tetrad and gauge fields using all the invariants that can be formed from the torsion, curvature, and metric tensors. Using the components $S_{\mu\nu}{}^{\lambda}$ and $g^{\mu\nu}$, we can construct three linearly independent scalars

$$S_1 = g^{\mu\nu} S_{\mu\rho}{}^{\lambda} S_{\nu}{}^{\rho}{}_{\lambda}, \qquad S_2 = g^{\mu\nu} S_{\mu\rho}{}^{\lambda} S_{\nu\lambda}{}^{\rho}, \qquad S_3 = S^{\nu}{}_{\lambda}{}^{\lambda} S_{\nu\rho}{}^{\rho}$$

and two linearly independent antisymmetric rank-2 tensors

$${}_1S_{\sigma\pi} = S_{\sigma\pi}{}^{\lambda} S_{\lambda\rho}{}^{\rho}, \qquad {}_2S_{\sigma\pi} = (S_{\sigma}{}^{\lambda}{}_{\pi} - S_{\pi}{}^{\lambda}{}_{\sigma})\, S_{\lambda\rho}{}^{\rho}.$$

Using the components $R_{\mu\nu\lambda\rho}$ and $g^{\mu\nu}$, we can construct three linearly independent scalars expressed by virtue of (7) in terms of $F^a_{\lambda\rho}$ and $e^\alpha{}_\mu$:

$$R_1 = \frac{1}{2} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = {}_6\eta_{ab}\, g^{\lambda\sigma} g^{\rho\pi} F^a_{\lambda\rho} F^b_{\sigma\pi},$$

$$R_2 = R_{\mu\nu\lambda\rho} R^{\lambda\rho\mu\nu} = (\tau_a)_{\alpha\beta}\, e^{\alpha\sigma} e^{\beta\pi} (\tau_b)_{\gamma\delta}\, e^{\gamma\lambda} e^{\delta\rho} F^a_{\lambda\rho} F^b_{\sigma\pi},$$

$$R_3 = R_{\mu\nu\lambda\rho} R^{\mu\lambda\nu\rho} = -(\tau_a)_{\beta\alpha} (\tau_b)^\alpha{}_\delta\, e^{\delta\lambda} g^{\rho\pi} e^{\beta\sigma} F^a_{\lambda\rho} F^b_{\sigma\pi}.$$

Using the Ricci tensor $R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$, where $R_{(\mu\nu)}$ is the symmetric part and $R_{[\mu\nu]}$ is the antisymmetric part of the tensor $R_{\mu\nu}$, we construct three more scalars:

$$R_4 = R_{(\mu\nu)} R^{(\mu\nu)} = \frac{1}{2} (\tau_a)^\gamma{}_\delta (\tau_b)^\alpha{}_\beta\, e_\alpha{}^\sigma\, e_\gamma{}^\lambda (\eta^{\beta\delta} g^{\pi\rho} + e^{\delta\pi} e^{\beta\rho}) F^a_{\lambda\rho} F^b_{\sigma\pi},$$

$$R_5 = R_{[\mu\nu]} R^{[\mu\nu]} = \frac{1}{2} (\tau_a)^\gamma{}_\delta (\tau_b)^\alpha{}_\beta\, e_\alpha{}^\sigma\, e_\gamma{}^\lambda (\eta^{\beta\delta} g^{\pi\rho} - e^{\delta\pi} e^{\beta\rho}) F^a_{\lambda\rho} F^b_{\sigma\pi},$$

$$R_6 = R^\mu{}_\mu = R_{(\mu\nu)} g^{\mu\nu} = (\tau_a)_{\alpha\beta}\, e^{\alpha\lambda} e^{\beta\nu} F^a_{\lambda\nu}.$$

Calculating the square of the invariant R_6, we obtain the last invariant, $R_7 = R_6^2$. These seven invariants R_k exhaust all the invariants not exceeding the fourth order in the gauge fields.

In the case of a compact symmetry group, there is a single invariant analogous to R_1 [4] for constructing the Lagrangian of the gauge fields, but the metric of the adjoint representation space is positive-definite in this case, which does not lead to any problems. In the case of the noncompact symmetry group considered here, the metric ${}_6\eta_{ab}$ included in R_1 of the adjoint group-representation space is not a sign-definite metric, which eventually results in the energy of small perturbations of the gauge fields being unbounded from below. Essentially, this means that the Lagrangian constructed using just one invariant R_1 is inconsistent with the least action principle.

Therefore, we must construct a linear combination of the reduced invariants and find coefficients of these invariants such that the least action principle is satisfied, i.e., such that small perturbations of the energy of the gauge fields are bounded from below.

3.2. Preliminary form of the Lagrangian. We first note that the tetrad and gauge fields are massless. It is impossible to form a mass term (i.e., a term including the field components without derivatives) from $e^\alpha{}_\mu$ because the only contraction that can be constructed using only the components $e^\alpha{}_\mu$ is the number $e^\alpha{}_\mu\, e_\alpha{}^\mu = 4$. The gauge fields $B^a{}_\mu$ are not vectors of the adjoint representation space with respect to the group index a. Therefore, a quadratic term playing the role of a mass term cannot be constructed using only the components $B^a{}_\mu$.

As the TGTG Lagrangian, we choose a linear combination of the above invariants including constraints in it:

$$L_{\mathrm{GC}\lambda}(e, \partial e, B, \partial B, \lambda) \equiv L_{\mathrm{G}}(e, \partial e, B) + L_{\mathrm{C}}(e, B, \partial B) + \frac{c^3}{2\pi G} \lambda^{\sigma\pi} \Phi_{\sigma\pi}(e, \partial e, B, \partial B), \tag{9}$$

where

$$L_{\mathrm{G}}(e, \partial e, B) \equiv \frac{c^3}{2\pi G} \sum_{k=1}^{3} B_k S_k,$$

$$L_{\mathrm{C}}(e, B, \partial B) \equiv -\frac{\hbar}{4} \left(\sum_{k=1}^{5} A_k R_k + A_7 R_7 \right) + A_6 \frac{c^3}{2\pi G} R_6.$$

In the first of these definitions, B_k are unknown numerical coefficients, and the dimensional factor is formed by the speed of light c and gravitational constant G. In the second expression, A_k are unknown numerical coefficients, the Plank constant $\hbar$ is taken as the dimensional factor, and the dimensional factor before R_6 is the same as in L_{G} because R_6 and S_k have the same dimensionality (the tetrad fields are dimensionless, while the gauge fields have the dimension of inverse length). The third term in Lagrangian (9) is a linear combination of the left-hand sides of the constraint equations $\Phi_{\sigma\pi} = 0$, where we take $\Phi_{\sigma\pi} \equiv \alpha_1\, {}_1S_{\sigma\pi} + \alpha_2\, {}_2S_{\sigma\pi} + \alpha_3 R_{[\sigma\pi]}$, i.e., a linear combination of the available antisymmetric tensors ${}_kS_{\sigma\pi}$ and $R_{[\sigma\pi]}$. Here, α_k are the numerical factors. The Lagrange factors $\lambda^{\sigma\pi}$ in the third term of (9) are the components of the antisymmetric tensor. The factor 2π is included in the denominators for consistency with the Newton theory.

It can be seen that if there are no gauge fields, then only the first and third terms remain in (9), which include both tetrad fields and their derivatives with respect to the space–time coordinates. As a result, the Lagrangian describes only the tetrad fields (by tetrad fields, we mean the differences $e^{\alpha}{}_{\mu} - \delta^{\alpha}_{\mu}$), which must be identified with pure gravity. Hence, the torsion, not the curvature, which is absent if there are no gauge fields (see relations (7) and (8)), is responsible for the gravitational interaction in the TGTG. The gauge fields of the TGTG are coupled only to the tetrad fields, i.e., only to gravity. Only gravity is directly coupled to ordinary matter (classical particles and electromagnetic and spinor-type fields). This follows because in the presence of gravity, all the coordinate fields must transform into scalars via the contraction with the tetrad components and the derivatives of the coordinate scalar fields are expressed in terms of the tetrad fields and their derivatives [7].

The gravitational source is the energy–momentum tensor for a material body [7]. As the simplest such material body, we take a system of N free classical point particles with the masses m_n $(n = \overline{1,N})$ moving along trajectories $x^{\mu}_n(x^0)$. The energy–momentum tensor for this system is symmetric and has the form

$$T^{\mu\nu}(x) = g^{-1/2}(x) \sum_{n=1}^{N} m_n c^2 \frac{dx^{\mu}_n(x^0)}{dx^0} \frac{dx^{\nu}_n(x^0)}{dx^0} \left(\frac{d\tau_n}{dx^0}\right)^{-1} \delta^4(\mathbf{x} - \mathbf{x}_n(x^0)). \tag{10}$$

3.3. Constructing the numerical coefficients in the Lagrangian. To avoid an unacceptably large size of this paper, we explain the method for finding the numerical coefficients of Lagrangian extremely briefly.

By the least action principle, we obtain a system of TGTG equations consisting of the tetrad field equations, gauge field equations, constraint equations, and equations of motion for particles. To find the numerical coefficients in the expression for L_{G}, we consider the case of free weak gravitational field generated by a point mass m at rest. It is natural to assume that the tetrad $e^{\alpha}{}_{\mu}$ in this case is $e^{\alpha}{}_{\mu} = \delta^{\alpha}_{\mu} + f^{\alpha}{}_{\mu}$, where $|f^{\alpha}{}_{\mu}| \ll 1$. Writing the system of equations, we restrict ourself to first-order terms in the small quantities $|f^{\alpha}{}_{\mu}|$. The constraint equations become identities $0 \equiv 0$, which means that there are no Lagrange multipliers in this approximation. The requirement for obtaining the Newtonian result leads to the constraints $f^{i}{}_{j} = -\delta^{i}_{j} f^{0}{}_{0}$, $2B_1 + B_2 + B_3 = 0$, $B_3 = -1$, $|f^{0}{}_{i}| \ll |f^{0}{}_{0}|$, and $|f^{i}{}_{0}| \ll |f^{0}{}_{0}|$ in accordance with the tetrad field equations. The absence of gauge fields does not mean that the other gauge field equations automatically become identities. These equations disappear if the conditions $A_6 = -1/4$ and $\alpha_3 = 0$ are satisfied. As a result, we have the only nonidentity equation, according to which $f^{0}{}_{0}(r) = -r_0/r$, where $r_0 = Gm/c^2$, and $g_{00}(r) = 1 + 2f^{0}{}_{0}(r)$ is hence the sought Newtonian approximation of the metric.

Of course, this result can be obtained using the post-Newtonian approximation [7], which allows finding the corrections in the next order of smallness to the Newtonian result. It turns out that $r_0/r \sim \bar{v}^2/c^2$, where $\bar{v}$ is the mean particle velocity in the gravitational field generated by the point source at rest. Hence, the Newtonian contribution to the metric is a quantity of the second order of smallness in the parameter $\bar{v}/c$. The next corrections to the metric are quantities of the third and fourth orders of smallness in $\bar{v}/c$. In particular, the equations for the quantities of the third order of smallness yield $|f^i{}_0| \sim \bar{v}^3/c^3$ and $|f^0_i| \sim \bar{v}^3/c^3$, which agree with the results of solving the pure gravity equations in the Newtonian approximation given above. In addition, the equation of the third order of smallness implies that $B_1 = 1/4$ and $B_2 = 1/2$.

Considering the problem of finding the static isotropic metric [7] in the framework of the tetrad formalism in the pure gravity case allows finding the numerical coefficients α_1 and α_2. In framework of this problem, the constraint equations become identities $0 \equiv 0$ only if $\alpha_1 = \alpha_2$, which means that there are no Lagrange multipliers. We choose these coefficients to be unity, which is allowed because they, being equal to each other, play the role of a scale factor of the Lagrange multipliers. In addition, the remaining pure gravity equations lead to a solution corresponding to the Schwarzschild solution for the metric in GR. Using the well-known experimental status of pure gravity in these problems thus allows finding not only all the numerical coefficients directly related to the tetrad fields but also the coefficients A_6 and α_3 directly related to the gauge fields.

The experimental status of the gauge fields of the TGTG is unclear, which does not allow finding all the numerical coefficients related to the gauge fields, but the fundamental requirement that the "kinetic energy" of the gauge fields be positive, i.e., the condition that squares of linear combinations of the time derivatives of all gauge field components are included with positive signs, imposes five conditions on the six coefficients, and these conditions allow finding some of those numerical coefficients. To simplify seeking the coefficients $A_1, \ldots, A_5$, and A_7, we consider the case of free gauge fields where $e^\alpha{}_\mu = \delta^\alpha_\mu$ and the metric tensor coincides with the Minkowski metric.

The only quantity including the derivatives of gauge fields is gauge field tensor (8). Therefore, in the first term $L_{\mathrm{C}}(e, B, \partial B)$, it suffices to restrict ourself to only the terms of gauge field tensors that include derivatives, i.e., we consider only $F^a_{\lambda\rho} \approx \partial_\lambda B^a{}_\rho - \partial_\rho B^a{}_\lambda$ (see (8)) in $L_{\mathrm{C}}(e, B, \partial B)$. In $L_{\mathrm{C}}(e, B, \partial B)$, we keep only the set $L_{\mathrm{C}00}$ of terms considerable in this case that are quadratic in time derivatives of the gauge fields. We obtain $L_{\mathrm{C}00} \geq 0$ if the conditions $A_4 + A_5 = 0$, $2A_1 + A_3 = 0$, $4A_2 + A_3 + A_4 = 0$, $A_3 \leq 0$, and $A_4 + 4A_7 \leq 0$ are satisfied.

The unclear experimental status of the gauge fields thus does not allow uniquely defining all the numerical coefficients. We choose $A_1 = 1$. Then $A_3 = -2$, which ensures that the numerical coefficient of the highest-order time derivative $\partial_0\partial_0 B'^k{}_k$ is unity in the gauge field equations. Here, we assume summation over repeating indices and also take into account that the indices a' and a'' take the same three values 1, 2, and 3, as the indices k and n. It is therefore convenient to replace them with k and n and transfer the signs $'$ and $''$ to the field components. For instance, for $a' = 1$, we change $B^{a'}{}_n \equiv B'^1{}_n$. This convention cannot lead to misunderstandings.

Considering the structure of the gauge field Lagrangian, we see that the choice $A_4 = -2$ and $A_7 = -1$ provides an equal status for all those terms in $L_{\mathrm{C}}(e, B, \partial B)$ that are quadratic in the gauge field tensor (see below). In this case, $A_5 = 2$, $A_2 = 1$, and $L_{\mathrm{C}00}$ becomes

$$L_{\mathrm{C}00} = \frac{\hbar}{2}(\partial_0 B'^k{}_k\, \partial_0 B'^n{}_n + 3\partial_0 B''^k{}_k\, \partial_0 B''^n{}_n).$$

The additional arguments in favor of this choice for A_4 and A_7 are that the gauge fields of the TGTG are coupled only to gravity and we can therefore relate them to the only currently known physical objects of this kind, which are dark matter and dark energy, whose densities are related in the ratio one to three [8].

3.4. Final form of the Lagrangian. Using the definitions of invariants and taking the indicated values for the numerical coefficients into account, we obtain

$$L_{\mathrm{G}}(e, \partial e, B) = \frac{c^3}{2\pi G} g^{\mu\nu}\left(\frac{1}{4}S_{\mu\rho}{}^\lambda S_\nu{}^\rho{}_\lambda + \frac{1}{2}S_{\mu\rho}{}^\lambda S_{\nu\lambda}{}^\rho - S_{\mu\lambda}{}^\lambda S_{\nu\rho}{}^\rho\right), \tag{11}$$

$$L_{\text{C}}(e,B,\partial B) = -\frac{\hbar}{4} M_{ab}{}^{\lambda\rho\sigma\pi}(e) F^a_{\lambda\rho} F^b_{\sigma\pi} - \frac{1}{4}\frac{c^3}{2\pi G}(\tau_a)^{\alpha\beta} e_\alpha{}^\lambda\, e_\beta{}^\rho\, F^a_{\lambda\rho}. \tag{12}$$

Here, the quantities depending only on the tetrad,

$$M_{ab}{}^{\lambda\rho\sigma\pi}(e) \equiv \frac{1}{2}\big\{{}_6\eta_{ab}(g^{\rho\pi}g^{\lambda\sigma} - g^{\lambda\pi}g^{\rho\sigma}) + 2(\tau_a)_{\alpha\beta}\, e^{\alpha\pi}e^{\beta\sigma}(\tau_b)_{\gamma\delta}\, e^{\gamma\rho}e^{\delta\lambda} +$$
$$+ (\tau_a)^\beta{}_\gamma(\tau_b)^\gamma{}_\delta[e^{\delta\rho}(e_\beta{}^\pi g^{\lambda\sigma} - e_\beta{}^\sigma g^{\lambda\pi}) - e^{\delta\lambda}(e_\beta{}^\pi g^{\rho\sigma} - e_\beta{}^\sigma g^{\rho\pi})] -$$
$$- (\tau_a)^\alpha{}_\beta(\tau_b)^\gamma{}_\delta(e_\gamma{}^\pi e^{\beta\sigma} - e_\gamma{}^\sigma e^{\beta\pi})(e_\alpha{}^\rho e^{\delta\lambda} - e_\alpha{}^\lambda e^{\delta\rho}) - 2(\tau_a)_{\alpha\beta}\, e^{\alpha\rho}e^{\beta\lambda}(\tau_b)_{\gamma\delta}\, e^{\gamma\pi}e^{\delta\sigma}\big\}, \tag{13}$$

have the symmetry $M_{ab}{}^{\lambda\rho\sigma\pi} = M_{ba}{}^{\sigma\pi\lambda\rho}$, which provides the symmetry of the considered Lagrangian term with respect to the two multipliers $F^a_{\lambda\rho}$ and $F^b_{\sigma\pi}$, and also provides the antisymmetry $M_{ab}{}^{\lambda\rho\sigma\pi} = -M_{ab}{}^{\rho\lambda\sigma\pi} = -M_{ab}{}^{\lambda\rho\pi\sigma} = M_{ab}{}^{\rho\lambda\pi\sigma}$ under independent permutations of the coordinate indices of the first pair and of the coordinate indices of the second pair, which is agrees with the antisymmetry of each multiplier $F^a_{\lambda\rho}$ and $F^b_{\sigma\pi}$. The constraint equations have the form

$$\Phi_{\sigma\pi} \equiv (S_{\sigma\pi}{}^\lambda + S_\sigma{}^\lambda{}_\pi - S_\pi{}^\lambda{}_\sigma)S_{\lambda\rho}{}^\rho = 0. \tag{14}$$

3.5. The TGTG equations for the point-particle system. According to (9)–(14), the variation of the action of the considered system is

$$\delta I_{mGC\lambda}[x_n,e,B,\lambda] = \sum_{n=1}^N m_n c\int_{P_n}^{Q_n} d\tau_n\, g_{\lambda\sigma}(x_n)\left[\frac{d^2x_n^\lambda}{d\tau_n^2} + G^\lambda_{\mu\nu}(x_n)\frac{dx_n^\mu}{d\tau_n}\frac{dx_n^\nu}{d\tau_n}\right]\delta x_n^\sigma -$$
$$- \frac{1}{c}\int_\Omega d^4x\,\sqrt{g(x)}\,2T^{\mu\nu}(x)e_{\alpha\nu}(x)\,\delta e^\alpha{}_\mu(x) +$$
$$+ \delta\int_\Omega d^4x\,\sqrt{g(x)}L_{\text{GC}\lambda}(e,\partial e,B,\partial B,\lambda). \tag{15}$$

Equating action variation (15) to zero and taking the independence of the variations δx_n^ν, $\delta e^\alpha{}_\mu(x)$, $\delta B^a{}_\mu(x)$, and $\delta\lambda^{\sigma\pi}(x)$ into account, in accordance with the least action principle, we obtain the equations of motion for particles in the gravitational field

$$\frac{d^2x_n^\lambda}{d\tau_n^2} + G^\lambda_{\mu\nu}(x_n)\frac{dx_n^\mu}{d\tau_n}\frac{dx_n^\nu}{d\tau_n} = 0, \tag{16}$$

constraint equations (14), and also the equations

$$\partial_\nu Q_\alpha{}^{\mu\nu} + G^\rho_{\rho\nu}Q_\alpha{}^{\mu\nu} - Q_\gamma{}^{\mu\nu}(\tau_a)^\gamma{}_\alpha\, B^a{}_\nu + \frac{1}{2}E_a{}^{\mu\rho}e_\alpha{}^\lambda F^a_{\lambda\rho} - Z^\mu{}_\rho\, e_\alpha{}^\rho -$$
$$- \partial_\nu V_\alpha{}^{\mu\nu} - G^\rho_{\rho\nu}V_\alpha{}^{\mu\nu} + V_\gamma{}^{\mu\nu}(\tau_a)^\gamma{}_\alpha\, B^a{}_\nu + W^\mu{}_\rho\, e_\alpha{}^\rho +$$
$$+ \frac{2\pi G}{c^3}\hbar M_{ab}{}^{\rho\lambda\pi\mu}e_\alpha{}^\sigma\, F^a_{\lambda\rho}F^b_{\sigma\pi} + \frac{2\pi G}{c^3}(L_{\text{G}} + L_{\text{C}})e_\alpha{}^\mu - \frac{4\pi G}{c^4}T^{\mu\nu}e_{\alpha\nu} = 0 \tag{17}$$

for the tetrad fields and

$$\hbar[\partial_\nu(M_{ab}{}^{\mu\nu\sigma\pi}F^b_{\sigma\pi}) + G^\lambda_{\lambda\nu}\, M_{ab}{}^{\mu\nu\sigma\pi}F^b_{\sigma\pi} + C^d_{ac}\, B^c{}_\nu\, M_{db}{}^{\mu\nu\sigma\pi}F^b_{\sigma\pi}] -$$
$$- \frac{c^3}{2\pi G}\left\{\frac{1}{2}(\partial_\nu E_a{}^{\nu\mu} + G^\lambda_{\lambda\nu}E_a{}^{\nu\mu} + C^d_{ac}\, B^c{}_\nu\, E_d{}^{\nu\mu}) + \left(E_a{}^{\mu\rho}S_{\rho\lambda}{}^\lambda + \frac{1}{2}E_a{}^{\lambda\rho}S_{\lambda\rho}{}^\mu\right) +\right.$$
$$\left. + \lambda^{\sigma\pi}\left(E_{a\sigma\pi}S^\mu{}_\rho{}^\rho - \frac{1}{2}E_a{}^\mu{}_\nu(S_{\sigma\pi}{}^\nu + 2S_\sigma{}^\nu{}_\pi)\right)\right\} = 0 \tag{18}$$

for the gauge fields. Here, to understand the equations better, we introduce notation for the antisymmetric rank-2 coordinate tensors dependent on local transformations of the tetrad

$$Q_\alpha{}^{\mu\nu}(e,\partial e,B) \equiv \frac{1}{2}(S^{\mu\nu}{}_\rho + S^\mu{}_\rho{}^\nu - S^\nu{}_\rho{}^\mu)e_\alpha{}^\rho + (S^\nu{}_\rho{}^\rho\, e_\alpha{}^\mu - S^\mu{}_\rho{}^\rho\, e_\alpha{}^\nu), \tag{19}$$

$$V_\alpha{}^{\mu\nu}(e,\partial e,B,\lambda) \equiv (\lambda^{\nu\mu} S^\rho{}_\sigma{}^\sigma + \lambda^{\nu\rho} S^\mu{}_\sigma{}^\sigma - \lambda^{\mu\rho} S^\nu{}_\sigma{}^\sigma)e_{\alpha\rho} + \\ + \frac{1}{2}\lambda^{\sigma\rho}[e_\alpha{}^\mu(S_{\sigma\rho}{}^\nu + 2S_\sigma{}^\nu{}_\rho) - e_\alpha{}^\nu(S_{\sigma\rho}{}^\mu + 2S_\sigma{}^\mu{}_\rho)], \tag{20}$$

$$E_a{}^{\mu\nu} \equiv (\tau_a)^{\alpha\beta} e_\alpha{}^\mu e_\beta{}^\nu \tag{21}$$

and also for the coordinate tensors independent of local transformations of the tetrad

$$Z^\mu{}_\rho(e,\partial e,B) \equiv (S^{\mu\sigma}{}_\pi + S^\mu{}_\pi{}^\sigma - S^\sigma{}_\pi{}^\mu)S_{\rho\sigma}{}^\pi - 2S^\mu{}_\sigma{}^\sigma S_{\rho\pi}{}^\pi - 2S^\sigma{}_\pi{}^\pi S_{\sigma\rho}{}^\mu, \tag{22}$$

$$W^\mu{}_\rho(e,\partial e,B,\lambda) \equiv 2\lambda^{\sigma\mu} S_{\pi\nu}{}^\nu S_\sigma{}^\pi{}_\rho + \\ + \lambda^{\sigma\pi}[2S^\mu{}_\nu{}^\nu S_{\rho\sigma\pi} + (S_{\sigma\pi}{}^\nu + 2S_\sigma{}^\nu{}_\pi)S_{\rho\nu}{}^\mu - (S_{\sigma\pi}{}^\mu + 2S_\sigma{}^\mu{}_\pi)S_{\rho\nu}{}^\nu]. \tag{23}$$

From (19) and (20), we can immediately see that $Q_\alpha{}^{\mu\nu}(e,\partial e,B)$ and $V_\alpha{}^{\mu\nu}(e,\partial e,B,\lambda)$ are Lorentz vectors with respect to the index α. Using the matrix ${}_6\Lambda(\theta)$ and the commutation relations for the generators χ_a of the adjoint representation of the Lorentz group, we verify that $E_a{}^{\mu\nu}$ is a vector in the adjoint representation space with respect to the index a. Taking the definition of the covariant derivative and connection (4) into account, we find that the left-hand sides of (17) and (18) are both coordinate and Lorentz vectors and the equations themselves are tensor equalities, i.e., they preserve their form under both arbitrary coordinate transformations and local Lorentz transformations (double covariance).

4. Post-Newtonian expansion of the TGTG equations

4.1. Problem setting. We consider the solution of Eqs. (14) and (16)–(23) for a system of particles that are coupled like the Sun and planets by mutual gravitational attraction. For solving such a problem in the framework of GR, the method of post-Newtonian expansion was developed; it is described in detail in [7]. We use this method to solve the equations after adapting it to the TGTG.

Let $\bar{m}$ and $\bar{v}$ be the mean particle mass and velocity and $\bar{r}$ be the mean distance between the particles. According to the virial theorem in Newtonian mechanics, the mean kinetic energy, which is of the order of $\bar{m}\bar{v}^2/2$, equals the characteristic potential energy $G\bar{m}^2/\bar{r}$, i.e., $\bar{v}^2 \sim G\bar{m}/\bar{r}$ up to an order of magnitude. Therefore, for instance, a particle moving in a circular orbit of radius r around a central body of mass m has a velocity v defined by the exact expression $v^2 = Gm/r$ in the Newtonian mechanics. The post-Newtonian approximation can be regarded as a method describing the motion of a system of gravitationally interacting bodies that is one order of the small parameters $G\bar{m}/\bar{r}c^2$ and $\bar{v}^2/c^2$ more precise than the Newtonian mechanics.

We first formulate the problem confronting us. Using equations of motion (16) for particles in the theory of gravity, we find the components of the particle acceleration in the gravitational field, whose detailed form is [7]

$$\frac{d^2x^k}{dt^2} = -c^2 G^k_{00} - 2c\, G^k_{0i}\frac{dx^i}{dt} - G^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \\ + \left(c^2 G^0_{00} + 2c\, G^0_{0i}\frac{dx^i}{dt} + G^0_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)\frac{1}{c}\frac{dx^k}{dt}. \tag{24}$$

All the velocities are assumed to vanish in the Newtonian approximation, and in constructing the component $g_{00}(\mathbf{x})$ of the metric tensor, we keep only terms of the first order in the parameter $v^2/c^2 \sim Gm/rc^2 \equiv r_0/r$, where $r_0 \equiv Gm/c^2$ [7]. In this case, according to (24),

$$\frac{d^2x^k}{dt^2} = -c^2 G^k_{00}(\mathbf{x}) = -\frac{c^2}{2}\partial_k g_{00}(\mathbf{x}). \tag{25}$$

It follows from relation (25) that to find the acceleration in the Newtonian approximation, i.e., acceleration of the order of v^2/r, we must know the Christoffel symbols up to the order of Gm/rc^2 or, in other words, up to the order of v^2/c^2. Therefore, the goal of the first post-Newtonian approximation is to calculate the derivative d^2x^k/dt^2 up to $\bar{v}^4/c^2\bar{r}$. For this, according to (24), we must know the Christoffel symbols with the accuracy $G^0_{ij} \sim \bar{v}/c\bar{r}$, $G^0_{0i} \sim \bar{v}^2/c^2\bar{r}$, $G^0_{00} \sim \bar{v}^3/c^3\bar{r}$, $G^k_{ij} \sim \bar{v}^2/c^2\bar{r}$, $G^k_{0i} \sim \bar{v}^3/c^3\bar{r}$, and $G^k_{00} \sim \bar{v}^4/c^4\bar{r}$.

4.2. Tetrad and metric expansion in powers of the small parameter. We assume that there is a reference frame where the tetrad components can be written as

$$e^\alpha{}_\mu \equiv \delta^\alpha_\mu + f^\alpha{}_\mu, \tag{26}$$

where the corrections $f^\alpha{}_\mu$ can be expanded in powers of $(Gm/rc^2)^{1/2} \sim \bar{v}/c$. In particular, we assume that

$$\begin{aligned} f^0_{\ 0} &= \overset{2}{f}{}^0{}_0 + \overset{4}{f}{}^0{}_0 + \ldots, \qquad & f^i{}_j &= \overset{2}{f}{}^i{}_j + \overset{4}{f}{}^i{}_j + \ldots, \\ f^0_{\ i} &= \overset{3}{f}{}^0{}_i + \overset{5}{f}{}^0{}_i + \ldots, \qquad & f^i{}_0 &= \overset{3}{f}{}^i{}_0 + \overset{5}{f}{}^i{}_0 + \ldots, \end{aligned} \tag{27}$$

where $\overset{N}{f}{}^\alpha{}_\mu$ means the term of the order of $(\bar{v}/c)^N$ in $f^\alpha{}_\mu$. In this case, the metric tensor

$$g_{\mu\nu} = \eta_{\alpha\beta}\, e^\alpha{}_\mu\, e^\beta{}_\nu = \eta_{\mu\nu} + \eta_{\mu\mu} f^\mu{}_\nu + \eta_{\nu\nu} f^\nu{}_\mu + \eta_{\alpha\beta} f^\alpha{}_\mu\, f^\beta{}_\nu \tag{28}$$

admits the expansions

$$g_{00} = 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \ldots, \qquad g_{ij} = -\delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \ldots, \qquad g_{0i} = \overset{3}{g}_{0i} + \overset{5}{g}_{0i} + \ldots, \tag{29}$$

where

$$\overset{2}{g}_{00} = 2\overset{2}{f}{}^0{}_0, \qquad \overset{2}{g}_{ij} = -\overset{2}{f}{}^i{}_j - \overset{2}{f}{}^j{}_i, \qquad \overset{3}{g}_{0i} = \overset{3}{g}_{i0} = \overset{3}{f}{}^0{}_i - \overset{3}{f}{}^i{}_0, \tag{30a}$$

$$\overset{4}{g}_{00} = (\overset{2}{f}{}^0{}_0)^2 + 2\overset{4}{f}{}^0{}_0, \qquad \overset{4}{g}_{ij} = -\overset{2}{f}{}^k{}_i\,\overset{2}{f}{}^k{}_j - \overset{4}{f}{}^i{}_j - \overset{4}{f}{}^j{}_i. \tag{30b}$$

Just as in expansions (27), $\overset{N}{g}_{\mu\nu}$ here means the term of the order of $(\bar{v}/c)^N$ in $g_{\mu\nu}$. The dependence on odd powers of $\bar{v}$ in $f^0_{\ i}$ and $f^i_{\ 0}$ and therefore in g_{0i} arises because the components g_{0i} must change sign under time reversal, i.e., under replacing t with $-t$. The verification of this expansion is given below. We show that it leads to a self-consistent solution of the gravitational field equations.

The inverse tetrad $e_\alpha{}^\mu$ is defined by either of the two equations $e^\alpha{}_\mu\, e_\alpha{}^\nu = \delta^\nu_\mu$ or $e^\alpha{}_\mu\, e_\beta{}^\mu = \delta^\alpha_\beta$. Similarly, we assume that $e_\alpha{}^\mu = \delta^\mu_\alpha + f_\alpha{}^\mu$ by analogy with (26). Then, taking (26) into account, from the first equation defining the inverse tetrad, we obtain the relations

$$e^\alpha{}_\mu\, e_\alpha{}^\nu - \delta^\nu_\mu = (\delta^\alpha_\mu + f^\alpha{}_\mu)(\delta^\nu_\alpha + f_\alpha{}^\nu) - \delta^\nu_\mu = f^\nu{}_\mu + f_\mu{}^\nu + f^\alpha{}_\mu\, f_\alpha{}^\nu = 0 \tag{31}$$

and represent them in a form convenient for the further transformations,

$$f^0_{\ 0} + f_0^{\ 0} + f^\alpha_{\ 0}\, f_\alpha^{\ 0} = 0, \qquad f^0_{\ i} + f_i^{\ 0} + f^\alpha_{\ i}\, f_\alpha^{\ 0} = 0,$$
$$f^i_{\ 0} + f_0^{\ i} + f^\alpha_{\ 0}\, f_\alpha^{\ i} = 0, \qquad f^i_{\ j} + f_j^{\ i} + f^\alpha_{\ j}\, f_\alpha^{\ i} = 0. \tag{32}$$

We assume that the inverse tetrads admit expansions similar to (27),

$$f_0^{\ 0} = \overset{2}{f}{}_0^{\ 0} + \overset{4}{f}{}_0^{\ 0} + \ldots, \qquad f_i^{\ j} = \overset{2}{f}{}_i^{\ j} + \overset{4}{f}{}_i^{\ j} + \ldots,$$
$$f_i^{\ 0} = \overset{3}{f}{}_i^{\ 0} + \overset{5}{f}{}_i^{\ 0} + \ldots, \qquad f_0^{\ i} = \overset{3}{f}{}_0^{\ i} + \overset{5}{f}{}_0^{\ i} + \ldots. \tag{33}$$

Substituting expansions (27) and (33) in (32) and restricting ourself to terms of the order of smallness not exceeding $(\bar{v}/c)^4$, we obtain

$$\overset{2}{f}{}^0_{\ 0} + \overset{2}{f}{}_0^{\ 0} + \overset{2}{f}{}^0_{\ 0}\,\overset{2}{f}{}_0^{\ 0} + \overset{4}{f}{}^0_{\ 0} + \overset{4}{f}{}_0^{\ 0} = 0, \qquad \overset{3}{f}{}^0_{\ i} + \overset{3}{f}{}_i^{\ 0} = 0,$$
$$\overset{3}{f}{}^i_{\ 0} + \overset{3}{f}{}_0^{\ i} = 0, \qquad \overset{2}{f}{}^i_{\ j} + \overset{2}{f}{}_j^{\ i} + \overset{2}{f}{}^k_{\ j}\,\overset{2}{f}{}_k^{\ i} + \overset{4}{f}{}^i_{\ j} + \overset{4}{f}{}_j^{\ i} = 0. \tag{34}$$

Equations (34) allow expressing the inverse tetrad through the original tetrad:

$$\overset{2}{f}{}_0^{\ 0} = -\overset{2}{f}{}^0_{\ 0}, \qquad \overset{2}{f}{}_j^{\ i} = -\overset{2}{f}{}^i_{\ j}, \qquad \overset{3}{f}{}_i^{\ 0} = -\overset{3}{f}{}^0_{\ i}, \qquad \overset{3}{f}{}_0^{\ i} = -\overset{3}{f}{}^i_{\ 0},$$
$$\overset{4}{f}{}_0^{\ 0} = -\overset{4}{f}{}^0_{\ 0} + (\overset{2}{f}{}^0_{\ 0})^2, \qquad \overset{4}{f}{}_j^{\ i} = -\overset{4}{f}{}^i_{\ j} + \overset{2}{f}{}^i_{\ k}\,\overset{2}{f}{}^k_{\ j}. \tag{35}$$

We find the expansion of the inverse metric tensor using the equation $g^{\mu\nu} = \eta^{\alpha\beta}\, e_\alpha^{\ \mu}\, e_\beta^{\ \nu} = \eta^{\mu\nu} + \eta^{\mu\mu} f_\mu^{\ \nu} + \eta^{\nu\nu} f_\nu^{\ \mu} + \eta^{\alpha\beta} f_\alpha^{\ \mu} f_\beta^{\ \nu}$ analogous to (28), where we must use expansions (35) for the inverse tetrad. As a result, we obtain

$$g^{00} = 1 + \overset{2}{g}{}^{00} + \overset{4}{g}{}^{00} + \ldots, \qquad g^{ij} = -\delta^{ij} + \overset{2}{g}{}^{ij} + \overset{4}{g}{}^{ij} + \ldots, \qquad g^{0i} = \overset{3}{g}{}^{0i} + \overset{5}{g}{}^{0i} + \ldots, \tag{36}$$

where

$$\overset{2}{g}{}^{00} = -2\overset{2}{f}{}^0_{\ 0}, \qquad \overset{2}{g}{}^{ij} = \overset{2}{f}{}^i_{\ j} + \overset{2}{f}{}^j_{\ i}, \qquad \overset{3}{g}{}^{0i} = \overset{3}{g}{}^{i0} = \overset{3}{f}{}^0_{\ i} - \overset{3}{f}{}^i_{\ 0}, \tag{37a}$$

$$\overset{4}{g}{}^{00} = -2\overset{4}{f}{}^0_{\ 0} + 3(\overset{2}{f}{}^0_{\ 0})^2, \qquad \overset{4}{g}{}^{ij} = \overset{4}{f}{}^i_{\ j} + \overset{4}{f}{}^j_{\ i} - \overset{2}{f}{}^i_{\ k}\,\overset{2}{f}{}^j_{\ k} - \overset{2}{f}{}^i_{\ k}\,\overset{2}{f}{}^k_{\ j} - \overset{2}{f}{}^j_{\ k}\,\overset{2}{f}{}^k_{\ i}. \tag{37b}$$

4.3. Expansion of the Christoffel symbols. In calculating the Christoffel symbols, we must take into account that $\bar{r}$ and $\bar{r}/\bar{v}$ are selected as the respective distance and time scales in the considered system. We therefore assume that the space and time derivatives have the orders $\partial_i \sim 1/\bar{r}$, $\partial_t \sim \bar{v}/\bar{r}$, and $\partial_0 \sim \bar{v}/c\bar{r}$. Using expansions (29) and (36), we find the Christoffel symbols G^k_{00}, G^k_{ij}, and G^0_{0i} in Eq. (24) with the required accuracy. At even powers of the velocity components, we have

$$G^k_{00} = \overset{2}{G}{}^k_{00} + \overset{4}{G}{}^k_{00}, \qquad \overset{2}{G}{}^k_{00} = \frac{1}{2}\,\partial_k\,\overset{2}{g}_{00}, \qquad \overset{4}{G}{}^k_{00} = \frac{1}{2}\,\partial_k\,\overset{4}{g}_{00} - \partial_0\overset{3}{g}_{0k} + \frac{1}{2}\overset{2}{g}_{kn}\,\partial_n\overset{2}{g}_{00}, \tag{38a}$$

$$G^k_{ij} = \overset{2}{G}{}^k_{ij} = -\frac{1}{2}(\partial_j\overset{2}{g}_{ik} + \partial_i\overset{2}{g}_{jk} - \partial_k\overset{2}{g}_{ij}); \qquad G^0_{0i} = \overset{2}{G}{}^0_{0i} = \frac{1}{2}\,\partial_i\overset{2}{g}_{00}. \tag{38b}$$

At odd powers of the velocity components, we have

$$G^0_{00} = \overset{3}{G}{}^0_{00} = \frac{1}{2}\,\partial_0\,\overset{2}{g}_{00}, \qquad G^k_{0i} = \overset{3}{G}{}^k_{0i} = -\frac{1}{2}(\partial_i\overset{3}{g}_{0k} + \partial_0\overset{2}{g}_{ik} - \partial_k\overset{3}{g}_{0i}), \qquad \overset{1}{G}{}^0_{ij} = 0. \tag{39}$$

It follows from these formulas that to calculate the required Christoffel symbols, we must know g_{ij} up to terms of the order of $(\bar{v}/c)^2$, g_{0i} up to terms of the order of $(\bar{v}/c)^3$, and g_{00} up to terms of the order of $(\bar{v}/c)^4$. According to (29) and (30), this means that we must write the expansions for the metric tensor

$$g_{00} = 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} = 1 + 2\overset{2}{f}{}^0{}_0 + (\overset{2}{f}{}^0{}_0)^2 + 2\overset{4}{f}{}^0{}_0, \tag{40a}$$

$$g_{ij} = -\delta_{ij} + \overset{2}{g}_{ij} = -\delta_{ij} - \overset{2}{f}{}^i{}_j - \overset{2}{f}{}^j{}_i, \qquad g_{0i} = \overset{3}{g}_{0i} = \overset{3}{f}{}^0{}_i - \overset{3}{f}{}^i{}_0. \tag{40b}$$

For comparison, we note that regarding the Newtonian approximation, we need g_{00} up to terms of the order of $(\bar{v}/c)^2$ (see (25)) and only the zeroth approximation for g_{0i} and g_{ij}. This means that we remove (i.e., equate to zero) the quantities $(\overset{2}{f}{}^0{}_0)^2$, $\overset{4}{f}{}^0{}_0$, $\overset{2}{f}{}^i{}_j$, $\overset{3}{f}{}^0{}_i$, and$\overset{3}{f}{}^i{}_0$ in formulas (40). Everything outlined above clarifies the meaning of the Newtonian approximation and verifies it consistently.

4.4. Expansion of gauge fields and Lagrange factors. According to the above, the gauge fields do not affect the motion of the considered system of particles. Nevertheless, these fields are included in not only the gauge field equations but also the tetrad field equations, which requires also extending the post-Newtonian expansion method to these fields. For the internal consistency of the expansion, we require that the gauge fields and Lagrange factors $\lambda^{\mu\nu}$ can be expanded in the series

$$\begin{aligned} B'^m{}_0 &= \overset{3}{B}{}'^m{}_0 + \overset{5}{B}{}'^m{}_0 + \ldots, \qquad & B'^m{}_j &= \overset{2}{B}{}'^m{}_j + \overset{4}{B}{}'^m{}_j + \ldots, \\ B''^m{}_0 &= \overset{2}{B}{}''^m{}_0 + \overset{4}{B}{}''^m{}_0 + \ldots, \qquad & B''^m{}_j &= \overset{3}{B}{}''^m{}_j + \overset{5}{B}{}''^m{}_j + \ldots, \end{aligned} \tag{41}$$

$$\lambda^{0i} = \overset{1}{\lambda}{}^{0i} + \overset{3}{\lambda}{}^{0i} + \ldots, \qquad \lambda^{ij} = \overset{2}{\lambda}{}^{ij} + \overset{4}{\lambda}{}^{ij} + \ldots. \tag{42}$$

4.5. Expansions of gauge field tensors and the torsion tensor. Using gauge field expansions (41) and the explicit forms for the group constants, we find the expansions of the gauge field tensors (8) keeping terms not exceeding the fourth order of smallness:

$$F'^i_{0j} = \overset{3}{F}{}'^i_{0j} = -\overset{3}{F}{}'^i_{j0} = \partial_0 \overset{2}{B}{}'^i{}_j - \partial_j \overset{3}{B}{}'^i{}_0, \tag{43a}$$

$$F'^i_{mj} = \overset{2}{F}{}'^i_{mj} + \overset{4}{F}{}'^i_{mj}, \tag{43b}$$

$$\overset{2}{F}{}'^i_{mj} = \partial_m \overset{2}{B}{}'^i{}_j - \partial_j \overset{2}{B}{}'^i{}_m, \qquad \overset{4}{F}{}'^i_{mj} = \partial_m \overset{4}{B}{}'^i{}_j - \partial_j \overset{4}{B}{}'^i{}_m + \varepsilon_{ikn} \overset{2}{B}{}'^k{}_m \overset{2}{B}{}'^n{}_j, \tag{43c}$$

$$F''^i_{0j} = \overset{2}{F}{}''^i_{0j} + \overset{4}{F}{}''^i_{0j}, \tag{43d}$$

$$\overset{2}{F}{}''^i_{0j} = -\overset{2}{F}{}''^i_{j0} = -\partial_j \overset{2}{B}{}''^i{}_0, \qquad \overset{4}{F}{}''^i_{0j} = -\overset{4}{F}{}''^i_{j0} = \partial_0 \overset{3}{B}{}''^i{}_j - \partial_j \overset{4}{B}{}''^i{}_0 - \varepsilon_{ikn} \overset{2}{B}{}'^k{}_j \overset{2}{B}{}''^n{}_0, \tag{43e}$$

$$F''^i_{mj} = \overset{3}{F}{}''^i_{mj} = \partial_m \overset{3}{B}{}''^i{}_j - \partial_j \overset{3}{B}{}''^i{}_m. \tag{43f}$$

Using tetrad component expansions (27), gauge field expansions (41), and the explicit forms of the Lorentz group generators, we find the expansions of torsion tensors (6), keeping terms not exceeding the fourth order of smallness:

$$S_{0i}{}^0 = \overset{2}{S}{}_{0i}{}^0 + \overset{4}{S}{}_{0i}{}^0, \tag{44a}$$

where

$$\overset{2}{S}{}_{0i}{}^0 = -\overset{2}{S}{}_{i0}{}^0 = \frac{1}{2}(-\partial_i \overset{2}{f}{}^0{}_0 + \overset{2}{B}{}''^i{}_0), \tag{44b}$$

$$\overset{4}{S}{}_{0i}{}^0 = -\overset{4}{S}{}_{i0}{}^0 = \frac{1}{2}(\overset{2}{f}{}^0{}_0\, \partial_i \overset{2}{f}{}^0{}_0 + \partial_0 \overset{3}{f}{}^0{}_i - \partial_i \overset{4}{f}{}^0{}_0 - \overset{2}{f}{}^0{}_0\, \overset{2}{B}{}''^i{}_0 + \overset{2}{f}{}^j{}_i\, \overset{2}{B}{}''^j{}_0 + \overset{4}{B}{}''^i{}_0), \tag{44c}$$

and

$$S_{0i}{}^{j} = -S_{i0}{}^{j} = \overset{3}{S}_{0i}{}^{j} = \frac{1}{2}(\partial_0 \overset{2}{f}{}^{j}{}_{i} - \partial_i \overset{3}{f}{}^{j}{}_{0} - \overset{3}{B}{}''^{j}{}_{i} + \varepsilon_{ijk} \overset{3}{B}{}'^{k}{}_{0}), \tag{44d}$$

$$S_{ij}{}^{0} = \overset{3}{S}_{ij}{}^{0} = \frac{1}{2}(\partial_i \overset{3}{f}{}^{0}{}_{j} - \partial_j \overset{3}{f}{}^{0}{}_{i} + \overset{3}{B}{}''^{j}{}_{i} - \overset{3}{B}{}''^{i}{}_{j}), \tag{44e}$$

$$S_{ij}{}^{k} = \overset{2}{S}_{ij}{}^{k} + \overset{4}{S}_{ij}{}^{k}, \tag{44f}$$

where

$$\overset{2}{S}_{ij}{}^{k} = \frac{1}{2}(\partial_i \overset{2}{f}{}^{k}{}_{j} - \partial_j \overset{2}{f}{}^{k}{}_{i} + \varepsilon_{njk} \overset{2}{B}{}'^{n}{}_{i} - \varepsilon_{nik} \overset{2}{B}{}'^{n}{}_{j}), \tag{44g}$$

$$\overset{4}{S}_{ij}{}^{k} = \frac{1}{2}\big[- \overset{2}{f}{}^{k}{}_{n}(\partial_i \overset{2}{f}{}^{n}{}_{j} - \partial_j \overset{2}{f}{}^{n}{}_{i}) + (\partial_i \overset{4}{f}{}^{k}{}_{j} - \partial_j \overset{4}{f}{}^{k}{}_{i}) + \overset{2}{f}{}^{k}{}_{n}(\varepsilon_{min} \overset{2}{B}{}'^{m}{}_{j} - \varepsilon_{mjn} \overset{2}{B}{}'^{m}{}_{i}) +$$

$$+ (\varepsilon_{nmk} \overset{2}{f}{}^{m}{}_{j} \overset{2}{B}{}'^{n}{}_{i} - \varepsilon_{nmk} \overset{2}{f}{}^{m}{}_{i} \overset{2}{B}{}'^{n}{}_{j} + \varepsilon_{njk} \overset{4}{B}{}'^{n}{}_{i} - \varepsilon_{nik} \overset{4}{B}{}'^{n}{}_{j})\big]. \tag{44h}$$

According to the presented expansions, the torsion tensor components with an even number of zero indices have even orders of smallness $(2, 4, \dots)$, and the torsion tensor components with an odd number of zero indices have odd orders of smallness $(3, 5, \dots)$. It follows from formulas (44) that the chosen gauge field expansions (41) are consistent with the tetrad field expansions. According to (43) and (44), the higher terms in the expansions of the gauge field tensors and torsion tensors have the second order of smallness.

4.6. Expansion of the energy–momentum tensor in the presence of gravity. Energy–momentum tensor (10) for particles in a gravitational field (this tensor is the source of the gravitational field) must also be represented as an expansion in the small parameter. As a result of simple but cumbersome calculations, we find the expansions

$$T^{00}(x) = \overset{0}{T}{}^{00}(x) + \overset{2}{T}{}^{00}(x), \tag{45a}$$

where

$$\overset{0}{T}{}^{00}(x) = \sum_{n=1}^{N} m_n c^2\, \delta^3(\mathbf{x} - \mathbf{x}_n(x^0)), \tag{45b}$$

$$\overset{2}{T}{}^{00}(x) = \sum_{n=1}^{N} m_n c^2 \left[-2\overset{2}{f}{}^{0}{}_{0}(x) - \overset{2}{f}{}^{i}{}_{i}(x) + \frac{1}{2}\frac{\mathbf{v}_n^2}{c^2}\right] \delta^3(\mathbf{x} - \mathbf{x}_n(x^0)), \tag{45c}$$

and

$$T^{0k}(x) = \overset{1}{T}{}^{0k}(x) + \overset{3}{T}{}^{0k}(x), \tag{45d}$$

where

$$\overset{1}{T}{}^{0k}(x) = \sum_{n=1}^{N} m_n c^2 \frac{dx_n^k(x^0)}{dx^0}\, \delta^3(\mathbf{x} - \mathbf{x}_n(x^0)), \tag{45e}$$

$$\overset{3}{T}{}^{0k}(x) = \sum_{n=1}^{N} m_n c^2 \frac{dx_n^k(x^0)}{dx^0} \left[-2\overset{2}{f}{}^{0}{}_{0}(x) - \overset{2}{f}{}^{i}{}_{i}(x) + \frac{1}{2}\frac{\mathbf{v}_n^2}{c^2}\right] \delta^3(\mathbf{x} - \mathbf{x}_n(x^0)), \tag{45f}$$

and

$$T^{kj}(x) = \overset{2}{T}{}^{kj}(x) + \overset{4}{T}{}^{kj}(x), \tag{45g}$$

where

$$\overset{2}{T}{}^{kj}(x) = \sum_{n=1}^{N} m_n c^2 \frac{dx_n^k(x^0)}{dx^0} \frac{dx_n^j(x^0)}{dx^0} \delta^3(\mathbf{x} - \mathbf{x}_n(x^0)), \tag{45h}$$

$$\overset{4}{T}{}^{kj}(x) = \sum_{n=1}^{N} m_n c^2 \frac{dx_n^k(x^0)}{dx^0} \frac{dx_n^j(x^0)}{dx^0} \left[-2\overset{2}{f}{}^0{}_0(x) - \overset{2}{f}{}^i{}_i(x) + \frac{1}{2}\frac{\mathbf{v}_n^2}{c^2} \right] \delta^3(\mathbf{x} - \mathbf{x}_n(x^0)). \tag{45i}$$

It follows from the TGTG equations and the energy–momentum tensor expansions that in the first post-Newtonian approximation, we need the energy–momentum tensor only up to the quantities of the second order of smallness, in other words, two order of smallness more precise than in the Newtonian approximation, i.e., in addition to $\overset{0}{T}{}^{00}(x)$, we need only $\overset{1}{T}{}^{0k}(x)$, $\overset{2}{T}{}^{00}(x)$, and $\overset{2}{T}{}^{kj}(x)$. Hence, to find the gravitational field source (the energy–momentum tensor) in this approximation, we need the tetrad field components only in the approximations preceding the considered approximation. Therefore, the post-Newtonian expansion is self-consistent.

5. Solving the TGTG equations in the Newtonian and first post-Newtonian approximations

5.1. Newtonian approximation. If we restrict ourself to the Newtonian and first post-Newtonian approximations in solving the TGTG equations, then we can simplify the equations significantly by omitting terms with obvious orders of smallness exceeding the fourth order. Such terms in Eqs. (17) are $W^{\mu}{}_{\rho}\, e_{\alpha}{}^{\rho}$, $G^{\rho}_{\rho\nu} V_{\alpha}{}^{\mu\nu}$, $V_{\gamma}{}^{\mu\nu}(\tau_a)^{\gamma}{}_{\alpha}\, B^a{}_{\nu}$, and the term with $\hbar$, which is small even disregarding the smallness of $\hbar$. The term with $\hbar$ in Eqs. (18) is small compared with the second term and can be omitted. A further analysis of the equations is possible only after choosing values of the indices α and μ corresponding to the time and space coordinates, while the group indices a must be assigned values corresponding to rotations or boosts.

Using the above expansions, we find a system of equations in the Newtonian approximation consisting of just the tetrad field equations of the second order of smallness. These equations for $\mu = 0$ and $\alpha = 0$ are

$$\frac{1}{2}(-\partial_i\partial_i \overset{2}{f}{}^j{}_j + \partial_i\partial_j \overset{2}{f}{}^j{}_i) = \frac{4\pi G}{c^4} \overset{0}{T}{}^{00}. \tag{46}$$

For $\mu = k$ and $\alpha = n$, they are

$$\partial_i\partial_i(\overset{2}{f}{}^k{}_n + \overset{2}{f}{}^n{}_k) - \partial_n\partial_i(\overset{2}{f}{}^k{}_i + \overset{2}{f}{}^i{}_k) - \partial_k\partial_i(\overset{2}{f}{}^i{}_n + \overset{2}{f}{}^n{}_i) + \\ + 2\partial_n\partial_k(\overset{2}{f}{}^0{}_0 + \overset{2}{f}{}^i{}_i) + 2\delta^k_n(-\partial_i\partial_i \overset{2}{f}{}^0{}_0 - \partial_i\partial_i \overset{2}{f}{}^j{}_j + \partial_i\partial_j \overset{2}{f}{}^j{}_i) = 0. \tag{47}$$

There are no gauge fields and Lagrange factors in these equations, which follows directly from the used expansions (41) and (42). For

$$\overset{2}{f}{}^i{}_j = -\delta^i_j \overset{2}{f}{}^0{}_0, \tag{48}$$

Eq. (47) becomes an identity $0 \equiv 0$, and Eq. (46) becomes

$$\partial_i\partial_i \overset{2}{f}{}^0{}_0 = \frac{4\pi G}{c^4} \overset{0}{T}{}^{00}. \tag{49}$$

The solution of this equation for the tetrad field $\overset{2}{f}{}^0{}_0(x^0,\mathbf{x})$ can be written as an integral over the whole three-dimensional space:

$$\overset{2}{f}{}^0{}_0(x^0,\mathbf{x}) = -\frac{G}{c^4}\iiint d^3x'\,\frac{\overset{0}{T}{}^{00}(x^0,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \equiv \frac{\phi(x^0,\mathbf{x})}{c^2}, \tag{50}$$

where $\phi(x^0,\mathbf{x})$ is the Newtonian potential. We note that $\phi(x^0,\mathbf{x})$ is defined by formula (50) for any source of the gravitational field. Using (45b) in the case of a point-mass system, we obtain

$$\overset{2}{f}{}^0{}_0(x^0,\mathbf{x}) = -\sum_{n=1}^{N}\frac{Gm_n}{c^2}\frac{1}{|\mathbf{x}-\mathbf{x}_n(x^0)|}. \tag{51}$$

In the particular case where the source of the gravitational field is a point particle of rest mass m, Eq. (51) implies that

$$\overset{2}{f}{}^0{}_0(r) = -\frac{Gm}{c^2}\frac{1}{r} = -\frac{r_0}{r}, \tag{52}$$

where $r\equiv|\mathbf{x}|$ is the absolute value of the vector $\mathbf{x}$ from the source of the gravitational field (a point particle of mass m) to the observation point. According to (29) and (30a) in the Newtonian approximation, we have

$$g_{00}(x^0,\mathbf{x}) = 1 + 2\overset{2}{f}{}^0{}_0(x^0,\mathbf{x}) \tag{53}$$

(the sign of $g_{00}(x^0,\mathbf{x})$ differs from the sign in [7] because we use a different signature).

Therefore, solving the equations of the second order of smallness obtained in the standard post-Newtonian expansion of the system of TGTG equations allows not only establishing the Newtonian result, i.e., finding $\overset{2}{f}{}^0{}_0(x^0,\mathbf{x})$ given by (50) but also obtaining $\overset{2}{f}{}^i{}_j(x^0,\mathbf{x})$ given by (48), which necessary in the first post-Newtonian approximation. As a result, we have (see (29) and (30a))

$$g_{ij}(x^0,\mathbf{x}) = -\delta_{ij} + 2\overset{2}{f}{}^0{}_0(x^0,\mathbf{x})\delta_{ij}. \tag{54}$$

5.2. Solving equations of the third order of smallness. Similarly, we find a system of equations of the third order of smallness related to the first post-Newtonian approximation. This system consists of the tetrad field equations of the third order of smallness with the form at $\mu=0$ and $\alpha=k$

$$\begin{aligned}&\frac{1}{4}\,\partial_i\partial_i(\overset{3}{f}{}^0{}_k - \overset{3}{f}{}^k{}_0) - \frac{1}{4}\,\partial_k\partial_i(\overset{3}{f}{}^0{}_i - \overset{3}{f}{}^i{}_0) + \partial_k\partial_0\overset{2}{f}{}^0{}_0 - \\ &\quad - \frac{1}{2}\partial_i\big[\overset{1}{\lambda}{}^{0i}(\partial_k\overset{2}{f}{}^0{}_0 + \overset{2}{B}{}''^k{}_0 - \varepsilon_{kjn}\overset{2}{B}{}'^j{}_n) + \overset{1}{\lambda}{}^{0k}(\partial_i\overset{2}{f}{}^0{}_0 + \overset{2}{B}{}''^i{}_0 - \varepsilon_{ijn}\overset{2}{B}{}'^j{}_n)\big] - \\ &\qquad - \partial_k\big[\overset{1}{\lambda}{}^{0i}(\partial_i\overset{2}{f}{}^0{}_0 - \overset{2}{B}{}''^i{}_0)\big] = -\frac{4\pi G}{c^4}\overset{1}{T}{}^{0k},\end{aligned} \tag{55}$$

the equations with the form at $\mu=k$ and $\alpha=0$

$$\begin{aligned}&\frac{1}{4}\,\partial_i\partial_i(\overset{3}{f}{}^0{}_k - \overset{3}{f}{}^k{}_0) - \frac{1}{4}\,\partial_k\partial_i(\overset{3}{f}{}^0{}_i - \overset{3}{f}{}^i{}_0) + \partial_k\partial_0\overset{2}{f}{}^0{}_0 - \\ &\quad - \frac{1}{2}\partial_i[\overset{1}{\lambda}{}^{0i}(\partial_k\overset{2}{f}{}^0{}_0 + \overset{2}{B}{}''^k{}_0 - \varepsilon_{kjn}\overset{2}{B}{}'^j{}_n) - \overset{1}{\lambda}{}^{0k}(\partial_i\overset{2}{f}{}^0{}_0 + \overset{2}{B}{}''^i{}_0 - \varepsilon_{ijn}\overset{2}{B}{}'^j{}_n)] = -\frac{4\pi G}{c^4}\overset{1}{T}{}^{0k},\end{aligned} \tag{56}$$

and the gauge field equations of the third order of smallness at $\mu=n$ and $a''=k$

$$\delta^n_k\overset{1}{\lambda}{}^{0i}(\partial_i\overset{2}{f}{}^0{}_0 - \overset{2}{B}{}''^i{}_0) + \overset{1}{\lambda}{}^{0k}(\partial_n\overset{2}{f}{}^0{}_0 + \overset{2}{B}{}''^n{}_0 - \varepsilon_{nij}\overset{2}{B}{}'^i{}_j) = 0. \tag{57}$$

Equation (57) implies the equality $\overset{1}{\lambda}{}^{0k} = 0$, by virtue of which Eq. (55) coincides with (56) and is

$$\frac{1}{4}\,\partial_i\partial_i(\overset{3}{f}{}^0{}_k - \overset{3}{f}{}^k{}_0) - \frac{1}{4}\,\partial_k\partial_i(\overset{3}{f}{}^0{}_i - \overset{3}{f}{}^i{}_0) + \partial_k\partial_0\overset{2}{f}{}^0{}_0 = -\frac{4\pi G}{c^4}\overset{1}{T}{}^{0k}. \tag{58}$$

Applying the operation ∂_k to Eqs. (58), taking (49) into account, and omitting the numerical factor $4\pi G/c^4$, we derive the relation

$$\partial_0\overset{0}{T}{}^{00} + \partial_k\overset{1}{T}{}^{0k} = 0, \tag{59}$$

which follows directly from (45b) and (45e) for a system of N point particles. We introduce the notation $\partial_\mu T^{0\mu}$ for the left-hand side of Eq. (59) and regard $\partial_\mu T^{0\mu} = 0$ as an equality up to quantities of the first order of smallness. Integrating over the region of the space–time going to infinity along spacelike directions and bounded by two hypersurfaces orthogonal to the time axis, we obtain the conservation law for the quantity

$$P^0 = \frac{1}{c}\iiint d^3x\,\overset{0}{T}{}^{00} = c\sum_{n=1}^{N} m_n, \tag{60}$$

i.e., up to the first order of smallness with respect to the parameter $\bar{v}/c$, the mass conservation law for a system of N point particles moving along trajectories $\mathbf{x}_n(x^0)$ with small velocities $\mathbf{v}_n(x^0)$ compared with the speed of light.

Further assuming that

$$\overset{3}{f}{}^k{}_0 = -\overset{3}{f}{}^0{}_k, \tag{61}$$

we represent Eqs. (58) as

$$\frac{1}{2}\,\partial_i\partial_i\overset{3}{f}{}^0{}_k - \partial_k\left(\partial_0\overset{2}{f}{}^0{}_0 - \frac{1}{2}\,\partial_i\overset{3}{f}{}^0{}_i\right) = -\frac{4\pi G}{c^4}\overset{1}{T}{}^{0k}. \tag{62}$$

If the condition

$$\partial_0\overset{2}{f}{}^0{}_0 - \frac{1}{2}\,\partial_i\overset{3}{f}{}^0{}_i = 0 \tag{63}$$

is satisfied, then (62) is simplified and becomes

$$\partial_i\partial_i\overset{3}{f}{}^0{}_k = -\frac{4\pi G}{c^4}\,2\overset{1}{T}{}^{0k}. \tag{64}$$

We can write the solution of (64) vanishing at infinity as

$$\overset{3}{f}{}^0{}_k(x^0,\mathbf{x}) = 2\frac{G}{c^4}\iiint d^3x'\,\frac{\overset{1}{T}{}^{0k}(x^0,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{65}$$

Using relations (50), (65), (45b), and (45e), we verify condition (63). According to formulas (40b), (61), and (65), the metric components of the third order of smallness can be written as (see [7])

$$\overset{3}{g}_{0k}(x^0,\mathbf{x}) = 2\overset{3}{f}{}^0{}_k(x^0,\mathbf{x}) = 4\frac{G}{c^4}\iiint d^3x'\,\frac{\overset{1}{T}{}^{0k}(x^0,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{66}$$

5.3. Solving equations of the fourth order of smallness. In the system of the fourth order of smallness, we have the constraint equations (see (14))

$$\overset{4}{\Phi}_{ij} \equiv (\overset{2}{S}_{ij}{}^{m} + \overset{2}{S}_{im}{}^{j} - \overset{2}{S}_{jm}{}^{i})\overset{2}{S}_{m\rho}{}^{\rho} = 0,$$

which become identities for zero gauge fields of the second order of smallness, i.e., for $\overset{2}{B}{}'^{i}{}_{j} \equiv \overset{2}{B}{}''^{k}{}_{0} \equiv 0$. The rest of the gauge field equations of the fourth order of smallness by virtue of the equality $\overset{1}{\lambda}{}^{0i} \equiv 0$ become

$$\overset{2}{\lambda}{}^{ij}(\overset{2}{S}_{ij}{}^{k} + \overset{2}{S}_{ik}{}^{j} - \overset{2}{S}_{jk}{}^{i}) = 0$$

for $\mu = 0$ and $a'' = k$ and

$$\varepsilon_{mnk}\overset{2}{\lambda}{}^{ij}(\overset{2}{S}_{ij}{}^{k} + \overset{2}{S}_{ik}{}^{j} - \overset{2}{S}_{jk}{}^{i}) - 2\varepsilon_{mij}\overset{2}{\lambda}{}^{ij}\overset{2}{S}_{n\rho}{}^{\rho} = 0$$

for $\mu = n$ and $a' = m$. Hence, taking $\overset{2}{S}_{n\rho}{}^{\rho} = -(1/2)\partial_n \overset{2}{f}{}^{0}{}_{0} \neq 0$ into account, we obtain $\overset{2}{\lambda}{}^{ij}$=0. Using these results and the solutions of the equations of the second and third smallness orders, we represent the tetrad field equations of the fourth smallness order in the form

$$\frac{1}{2}\,\partial_i(\partial_i \overset{4}{f}{}^{j}{}_{j} - \partial_j \overset{4}{f}{}^{i}{}_{j}) - \overset{2}{f}{}^{0}{}_{0}\,\partial_i\partial_i \overset{2}{f}{}^{0}{}_{0} - \frac{1}{2}\,\partial_i \overset{2}{f}{}^{0}{}_{0}\,\partial_i \overset{2}{f}{}^{0}{}_{0} = -\frac{4\pi G}{c^4}\overset{2}{T}{}^{00} \tag{67}$$

for $\mu = 0$ and $\alpha = 0$ and in the form

$$\frac{1}{4}\Big[\partial_i\partial_i(\overset{4}{f}{}^{n}{}_{k} + \overset{4}{f}{}^{k}{}_{n}) - \partial_k\partial_i(\overset{4}{f}{}^{n}{}_{i} + \overset{4}{f}{}^{i}{}_{n}) - \partial_n\partial_i(\overset{4}{f}{}^{k}{}_{i} + \overset{4}{f}{}^{i}{}_{k})\Big] + \frac{1}{2}\,\partial_n\partial_k \overset{4}{f}{}^{i}{}_{i} + \frac{1}{2}\,\partial_n\partial_k \overset{4}{f}{}^{0}{}_{0} +$$
$$+\,\delta^k_n\Big[\partial_0\partial_0 \overset{2}{f}{}^{0}{}_{0} + \overset{2}{f}{}^{0}{}_{0}\,\partial_i\partial_i \overset{2}{f}{}^{0}{}_{0} + \frac{1}{2}\,\partial_i \overset{2}{f}{}^{0}{}_{0}\,\partial_i \overset{2}{f}{}^{0}{}_{0} - \frac{1}{2}\,\partial_i\partial_i \overset{4}{f}{}^{0}{}_{0} - \frac{1}{2}\,\partial_i(\partial_i \overset{4}{f}{}^{j}{}_{j} - \partial_j \overset{4}{f}{}^{j}{}_{i})\Big] -$$
$$-\,\overset{2}{f}{}^{0}{}_{0}\,\partial_n\partial_k \overset{2}{f}{}^{0}{}_{0} - \frac{1}{2}\,\partial_0\partial_k \overset{3}{f}{}^{0}{}_{n} - \frac{1}{2}\,\partial_0\partial_n \overset{3}{f}{}^{0}{}_{k} = -\frac{4\pi G}{c^4}\,\overset{2}{T}{}^{kn} \tag{68}$$

for $\mu = k$ and $\alpha = n$. The seven Eqs. (67) and (68) have seven unknowns: $\overset{4}{f}{}^{0}{}_{0}$ and the symmetric components $\overset{4}{f}{}^{i}{}_{j}$. Calculating the contraction of Eqs. (68) over the indices k and n and taking (63) into account, we obtain the equation

$$\frac{1}{2}\,\partial_i(\partial_i \overset{4}{f}{}^{j}{}_{j} - \partial_j \overset{4}{f}{}^{i}{}_{j}) + \partial_i\partial_i \overset{4}{f}{}^{0}{}_{0} - \partial_0\partial_0 \overset{2}{f}{}^{0}{}_{0} - 2\overset{2}{f}{}^{0}{}_{0}\,\partial_i\partial_i \overset{2}{f}{}^{0}{}_{0} - \frac{3}{2}\partial_i \overset{2}{f}{}^{0}{}_{0}\,\partial_i \overset{2}{f}{}^{0}{}_{0} = \frac{4\pi G}{c^4}\overset{2}{T}{}^{ii},$$

which after (67) is subtracted yields the equations for determining $\overset{4}{f}{}^{0}{}_{0}$:

$$\partial_i\partial_i \overset{4}{f}{}^{0}{}_{0} - \overset{2}{f}{}^{0}{}_{0}\,\partial_i\partial_i \overset{2}{f}{}^{0}{}_{0} - \partial_i \overset{2}{f}{}^{0}{}_{0}\,\partial_i \overset{2}{f}{}^{0}{}_{0} = \frac{4\pi G}{c^4}(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}) + \partial_0\partial_0 \overset{2}{f}{}^{0}{}_{0}. \tag{69}$$

We now consider the identity $\overset{2}{f}{}^{0}{}_{0}\,\partial_i\partial_i \overset{2}{f}{}^{0}{}_{0} + \partial_i \overset{2}{f}{}^{0}{}_{0}\,\partial_i \overset{2}{f}{}^{0}{}_{0} \equiv (1/2)\partial_i\partial_i(\overset{2}{f}{}^{0}{}_{0})^2$ and set

$$\psi \equiv \overset{4}{f}{}^{0}{}_{0} - \frac{1}{2}(\overset{2}{f}{}^{0}{}_{0})^2. \tag{70}$$

We can finally write (69) as

$$\partial_i\partial_i\psi = \frac{4\pi G}{c^4}(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}) + \partial_0\partial_0 \overset{2}{f}{}^{0}{}_{0}. \tag{71}$$

The solution of Eq. (71) is (see (49) and (50))

$$\psi(x^0,\mathbf{x}) = -\iiint d^3x' \, \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left\{\frac{1}{4\pi}\,\partial_0\partial_0 \overset{2}{f}{}^0{}_0(x^0,\mathbf{x}') + \frac{G}{c^4}[\overset{2}{T}{}^{00}(x^0,\mathbf{x}') + \overset{2}{T}{}^{ii}(x^0,\mathbf{x}')]\right\}. \tag{72}$$

The quantity $\overset{4}{f}{}^0{}_0$ is determined using (70), (72), and (50).

According to (45c) and (45h), in the case of a single particle of mass m_{p}, the combination of energy–momentum tensor components in (72) has the meaning of the kinetic energy density of the gravitational field source:

$$\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} = \frac{m_{\mathrm{p}}\mathbf{v}_{\mathrm{p}}^2}{2}\,\delta^3(\mathbf{x}-\mathbf{x}_{\mathrm{p}}(x^0)).$$

Using relations (30b), (70), and (72), we obtain

$$\overset{4}{g}_{00} = (\overset{2}{f}{}^0{}_0)^2 + 2\overset{4}{f}{}^0{}_0 = 2(\overset{2}{f}{}^0{}_0)^2 + 2\psi. \tag{73}$$

We note that for solving the equations of motion in the first post-Newtonian approximation, we do not need the quantities $\overset{4}{f}{}^i{}_j$ (see (40)). The TGTG metric found in the Newtonian and first post-Newtonian approximations coincides with the metric found using the same approximations for GR (see [7], where the signature differs from the signature used here). From relations (50), (54), (66), (72), and (73) we find Christoffel symbols (38) and (39), which are identified with the affine connection in GR. In view of coinciding metrics in the two theories, the Christoffel symbols also coincide, and the trajectories of particle motions consequently coincide. The TGTG used to describe systems like the Solar System agrees with observations, but the interpretations of gravity differ essentially in these theories. The gauge fields in the Newtonian and first post-Newtonian approximations are absent, and the space–time is flat in these approximations. In the TGTG, the space–time torsion is responsible for gravity. Torsion tensor (44) in the Newtonian approximation is

$$\overset{2}{S}_{0i}{}^0 = -\overset{2}{S}_{i0}{}^0 = \frac{1}{2}\,\partial_i \overset{2}{f}{}^0{}_0, \qquad \overset{2}{S}_{ij}{}^k = \frac{1}{2}(\delta_i^k\,\partial_j \overset{2}{f}{}^0{}_0 - \delta_j^k\,\partial_i \overset{2}{f}{}^0{}_0). \tag{74}$$

Finding the other components of the torsion tensor requires considering the second post-Newtonian approximation. In particular, we must determine the gauge field components of the third and fourth orders of smallness because the post-Newtonian expansion procedure is associated with expanding the equations of motion and they do not include gauge fields, which are coupled only to gravity.

5.4. Tensor equality describing the variation of the energy–momentum vector. In the framework of the TGTG, we consider the variation of the energy–momentum tensor for the gravity sources due to their coupling to the gravitational field. Using relation (4), which expresses the connection $\Gamma^\lambda_{\mu\nu}$ in terms of the Christoffel symbol $G^\lambda_{\mu\nu}$ and the torsion tensor $S_{\mu\nu}{}^\lambda$, we find the covariant derivative of an arbitrary symmetric rank-2 tensor in the Riemann–Cartan space,

$$T^{\mu\nu}{}_{;\mu} = \partial_\mu T^{\mu\nu} + G^\mu_{\mu\rho}T^{\rho\nu} + G^\nu_{\mu\rho}T^{\mu\rho} - 2S_{\rho\mu}{}^\mu T^{\rho\nu} + 2S^\nu{}_{\mu\rho}T^{\mu\rho}, \tag{75}$$

which implies the tensor equality

$$T^{\mu\nu}{}_{;\mu} + 2S_{\rho\mu}{}^\mu T^{\rho\nu} - 2S^\nu{}_{\mu\rho}T^{\mu\rho} = \partial_\mu T^{\mu\nu} + G^\mu_{\mu\rho}T^{\rho\nu} + G^\nu_{\mu\rho}T^{\mu\rho}, \tag{76}$$

whose left-hand side is obviously a tensor, and the right-hand side is therefore also a tensor. We note that the right-hand side of (76) has the form of a covariant derivative, where the Christoffel symbols play the role of the connection. Essentially, the second and third terms in the left-hand side of this equality remove everything related to the torsion from the covariant divergence of the tensor $T^{\mu\nu}$, and only the metric part hence remains. This corresponds to the fact that not only the gauge fields but also the tetrad fields in explicit form do not appear in the equations of motion for the gravity sources. Only the combination of the tetrad fields in the form of the metric are included in the equations. Using definition (10) of the symmetric energy–momentum tensor $T^{\mu\nu}$ for a system of particles and equations of motion (16) for particles, we derive the equality

$$\partial_\mu T^{\mu\nu} + G^{\mu}_{\mu\rho} T^{\rho\nu} + G^{\nu}_{\mu\rho} T^{\mu\rho} = 0, \tag{77}$$

which is a tensor equality in both TGTG and GR according to (76). If there is no gravity, then (77) reduces to $\partial_\mu T^{\mu\nu} = 0$, whence the conservation of the energy–momentum vector for the system of particles follows according to the Noether theorem [9]. According to Eq. (77), the energy–momentum vector for the gravity sources is not conserved in presence of gravity, and its variation due to the coupling to gravity is described by this equation.

According to the Noether theorem, the energy–momentum tensor is generated by the translation symmetries of the Minkowski space. In the presence of gravity, there are no space–time translation symmetries and therefore no energy–momentum tensor for gravity and consequently no energy–momentum tensor for the whole system with gravity. But this does not prevent finding the variation of the energy–momentum vector for the gravitational sources due to their coupling to the gravitational field. We recall that the energy–momentum tensor for the gravitational sources in presence of a gravitational field can be found from the energy–momentum tensor without gravity, i.e., from the energy–momentum tensor in a locally inertial reference frame, via its contraction with the tetrad components, as for any other tensor. If the energy–momentum tensor existed, then it would also be nonzero in the locally inertial reference frame. This would contradict the meaning of this system and is impossible for a formal reason: the tensor transformation rule is linear and homogeneous in the tensor components; therefore, a nonzero tensor cannot be obtained from a zero tensor using such a transformation. The absence of the energy–momentum tensor of the gravitational field follows because the gravitational field (unlike all other material objects) can be removed (at least locally) by passing to a locally inertial frame of reference.

We once more emphasize that Eq.(77) is a tensor equality in the Riemann–Cartan space–time (see (76)), but it does not “know” anything about torsion, although torsion defines gravity in the TGTG.

To avoid an enormous extension of this paper, we only briefly mention the results of using Eq. (77). In the first order with respect to the small parameter v/c, we obtain the mass conservation (see (60)); in the second order, we find the momentum variation for the system of particles under the action of forces generated by the gravitational field; in the third order, we find the variation of the energy for the system of particles due to the variation of the gravity field energy; after similar transformations in the fourth order, we obtain an equality that has no such simple physical interpretation.

6. Schwarzschild metric

6.1. Standard form of the static isotropic metric. In the TGTG, we consider the problem of constructing a metric tensor representing a static isotropic gravitational field and determined by the expression for the interval [7]

$$d\tau^2 = B(r)(dx^0)^2 - A(r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2), \tag{78}$$

where the conditions $B(r) > 0$ and $A(r) > 0$ are satisfied. A particular case of (78) is the metric of an empty space, which implies that $B(r) = 1$ and $A(r) = 1$. Expressed in terms of the tetrad coefficients, interval (78) is

$$d\tau^2 = \eta_{\alpha\beta}\, e^{\alpha}{}_{0}\, e^{\beta}{}_{0} (dx^0)^2 + \eta_{\alpha\beta}\, e^{\alpha}{}_{1}\, e^{\beta}{}_{1}\, dr^2 + \eta_{\alpha\beta}\, e^{\alpha}{}_{2}\, e^{\beta}{}_{2}\, d\theta^2 + \eta_{\alpha\beta}\, e^{\alpha}{}_{3}\, e^{\beta}{}_{3}\, d\varphi^2. \tag{79}$$

Among the four Lorentz vectors $e_0(r)$ and $e_k(r,\theta,\varphi)$, the vector $e_0(r)$ is timelike, and others are spacelike. According to (78) and (79), these vectors can be taken in the forms

$$\begin{aligned} e_0(r) &= ({e^0}_0, 0,0,0) \equiv (B^{1/2}(r),0,0,0), \\ e_k(r,\theta,\varphi) &= (0, {e^1}_k(r,\theta,\varphi), {e^2}_k(r,\theta,\varphi), {e^3}_k(r,\theta,\varphi)). \end{aligned} \tag{80}$$

The explicit dependence of the vector components $e_k(r,\theta,\varphi)$ on the coordinates r, θ, and φ is obvious and is not presented here to save room. In the considered case, if there are no gauge fields, then the nonzero components of the torsion tensor are (hereafter, the prime on B denotes the derivative with respect to r)

$$ {S_{10}}^0 = -{S_{01}}^0 = -\frac{1}{4}B^{-1}B', \qquad {S_{12}}^2 = -{S_{21}}^2 = {S_{13}}^3 = -{S_{31}}^3 = \frac{1}{2}r^{-1}(1-A^{1/2}). \tag{81}$$

It hence follows that there is a unique nonzero contraction of the torsion tensor ${S_{1\rho}}^\rho = B^{-1}B'/4 + r^{-1}(1-A^{1/2})$. There are also two nonzero Christoffel symbols:

$$ G^\lambda_{\lambda 1} = \frac{1}{2}(B^{-1}B' + A^{-1}A' + 4r^{-1}), \qquad G^\lambda_{\lambda 2} = \sin^{-1}\theta\cos\theta. \tag{82}$$

6.2. Solving equations describing the static isotropic metric. According to the above arguments, we find that the constraint equations have the forms of identities $0 \equiv 0$ in the framework of the considered problem, i.e., they impose no constraint on the considered quantities. Therefore, there are also no Lagrange multipliers. It is easy to verify that if there are no gauge fields, then the gauge field equations also become identities. Hence, only the tetrad field equations remain with all terms containing gauge fields or Lagrange multipliers omitted. In this case, the equations for the tetrad fields outside the gravitational source become (see (17))

$$ \partial_\nu {Q_\alpha}^{\mu\nu} + G^\rho_{\rho\nu}{Q_\alpha}^{\mu\nu} - {Z^\mu}_\rho\, {e_\alpha}^\rho + \frac{2\pi G}{c^3} L_{\mathrm{G}} {e_\alpha}^\mu = 0. \tag{83}$$

Contracting Eqs. (83) with ${e_\alpha}^\lambda$ and selecting the total derivative, we obtain

$$ \partial_\nu({Q_\alpha}^{\mu\nu}{e_\alpha}^\lambda) - {Q_\alpha}^{\mu\nu}\partial_\nu {e_\alpha}^\lambda + G^\rho_{\rho\nu}{Q_\alpha}^{\mu\nu}{e_\alpha}^\lambda - {Z^\mu}_\lambda + \frac{2\pi G}{c^3} L_{\mathrm{G}}\delta^\mu_\lambda = 0. \tag{84}$$

To analyze these equations further, we must choose particular values for the indices μ and λ. It turns out that only four equations with coinciding μ and λ differ from the identities $0 \equiv 0$. Using relations (11), (19), (22), (81), and (82), taking the derivatives, and collecting like terms, we obtain the respective equations at $\mu = \lambda = 0$ and $\mu = \lambda = 1$

$$ A^{-1}A' - r^{-1}(1-A) = 0, \qquad B^{-1}B' + r^{-1}(1-A) = 0. \tag{85}$$

Equations corresponding to $\mu = \lambda = 2$ and $\mu = \lambda = 3$ coincide with each other and become identities on solutions of Eqs. (85). The solutions of Eqs. (85) are

$$ A = \left(1 - \frac{r_{\mathrm{g}}}{r}\right)^{-1}, \qquad B = A^{-1} = 1 - \frac{r_{\mathrm{g}}}{r}, \tag{86}$$

where r_{g} is a constant. We thus obtain the final form of the sought metric:

$$ d\tau^2 = \left(1 - \frac{r_{\mathrm{g}}}{r}\right)(dx^0)^2 - \left(1 - \frac{r_{\mathrm{g}}}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{87}$$

It follows from comparing (87) with (52) and (53) that $r_{\mathrm{g}} = 2r_0$.

The considered problem of finding the metric tensor representing a static isotropic gravitational field has the same solution (Schwarzschild metric [2], [7]) both in GR and the TGTG. Considering a book where the classic experiments verifying GR were considered [7], we conclude that the experimental status of the tetrad (purely gravitational) sector in the TGTG is the same as the experimental status of GR.

7. Conclusion

We have shown that using the tetrad formalism allows constructing a physically acceptable gauge theory based on the noncompact symmetry group (local Lorentz group). The tetrad sector of the constructed theory describes pure gravity, and the gauge sector describes the fields coupled only to gravity. This allows preliminarily interpreting these fields as the fields corresponding to the only presently known physical objects coupled only to gravity: dark energy and dark matter. Despite the principal difference in the mathematical formalisms of the TGTG and GR, applying them to systems of gravitational bodies like the Solar System yields the same results. The two theories also yield the same solution for the metric of a static isotropic gravitational field (Schwarzschild metric). But the geometric interpretation of gravity is completely different in these theories. Space–time curvature provides gravity in GR, while space–time torsion provides gravity in the TGTG. As a result, the space–times differ essentially in these theories.

All locally inertial reference frames related to a given point in space–time are equivalent in GR, but because the metric used is invariant under choosing the reference frame as a dynamical variable, the consequences of this invariance are beyond the scope of GR, which therefore "knows" nothing about the gauge fields. The mathematical formalism of the TGTG is much more complicated than the GR formalism, but it takes the equivalence of all locally inertial reference frames naturally into account, which automatically leads to the gauge theory describing the gauge fields coupled only to gravity. In this case, pure gravity (the tetrad sector in the TGTG) is a nongauge phenomenon.

The gauge fields in the TGTG not only contribute to the space–time torsion but also define its curvature and in addition affect the motion of ordinary matter only through gravity rather than directly. The space–time in the TGTG is thus a more complicated object than the space–time in GR not only mathematically but also physically.

In view of proposed TGTG, several questions arise that cannot be answered in a single paper. The gauge fields coupled only to gravity are particularly interesting in the TGTG. To clarify their physical nature, we must solve equations describing these fields and their coupling to the tetrad fields. These are subjects for forthcoming papers.

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