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ZONE STRUCTURE OF THE RENORMALIZATION GROUP FLOW IN A FERMIONIC HIERARCHICAL MODEL

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The Gaussian part of the Hamiltonian of the four-component fermion model on a hierarchical lattice is invariant under the block-spin transformation of the renormalization group with a given degree of normalization (the renormalization group parameter). We describe the renormalization group transformation in the space of coefficients defining the Grassmann-valued density of a free measure as a homogeneous quadratic map. We interpret this space as a two-dimensional projective space and visualize it as a disk. If the renormalization group parameter is greater than the lattice dimension, then the unique attractive fixed point of the renormalization group is given by the density of the Grassmann delta function. This fixed point has two different (left and right) invariant neighborhoods. Based on this, we classify the points of the projective plane according to how they tend to the attracting point (on the left or right) under iterations of the map. We discuss the zone structure of the obtained regions and show that the global flow of the renormalization group is described simply in terms of this zone structure.

Keywords: renormalization group, fermion model, projective space, zone structure

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1. Introduction

In [1]–[6], we studied the transformation properties of the renormalization group (RG) in the framework of the fermionic hierarchical model. This model is a fermionic (anticommutative) analogue of the hierarchical boson Dyson φ^4 model. The transformation of the block Wilson RG in the framework of the bosonic hierarchical model reduces to a nonlinear integral operator in an infinite-dimensional space of free measure densities. The properties of this transformation in the neighborhood of a Gaussian fixed point were studied in the works of Blecher and Sinai and many other authors [7]–[9]. In contrast to the bosonic case, the RG transformation in the hierarchical fermion model can be calculated exactly and represented as a birational map in the two-dimensional space of coupling constants of the model. This allows describing all the fixed points and other dynamical properties and symmetries of the RG map explicitly. We also note that the fermionic hierarchical model has a natural continuous p-adic version if the size of the elementary block of the hierarchical lattice is equal to a prime p (see [6], [10], [11]). In this case, problems in quantum field theory such as ultraviolet divergences and the renormalization procedure have a natural interpretation in terms of the classical theory of dynamical systems.

In [4], we described some of the global dynamical properties of the RG flow in the plane of coupling constants of the model (r, g). Analyzing the RG flow with a computer, we found that almost all points of the upper half-plane $\{(r, g)\colon g > 0\}$, which is invariant under the RG map, under successive iterations of the RG map go to infinity "from the left" ($r \to -\infty$, $g \to \infty$) or "from the right" ($r \to \infty$, $g \to \infty$). We

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found that the sets of points leaving "to the left" or "to the right" have a complex fractal structure, and these results led to a description of the complex critical behavior in this model. To obtain a global picture of the dynamics of the RG, we used a representation of the model in the projective space. We recently investigated the RG transformation and developed a simple algorithm for analyzing global dynamics directly in the projective space [12], [13]. Here, we present the results of computer experiments obtained using this algorithm. We classify points of the projective plane according to how the iterations of the RG map of these points tend to a unique attractive fixed point. We describe the zone structure of the sets obtained as a result of this classification explicitly. Computer experiments show that the global dynamics of the RG has a beautiful description in terms of this structure. In particular, we clarify the global behavior of stable invariant curves for various fixed points. We note that the RG is investigated in Euclidean models in the lower-order perturbation theory in the neighborhood of a Gaussian fixed point. The hierarchical fermion model is interesting in that it provides a unique opportunity to describe a global RG flow in the entire plane of the coupling constants. An explicit description of the properties of the RG in the framework of the hierarchical fermion model generates a number of nontrivial hypotheses for hierarchical and Euclidean bosonic models [14].

Let p be a natural number, $p > 1$. A hierarchical lattice T_p^d can be defined as a lattice Z^d of d-dimensional integer vectors endowed with a hierarchical distance $d_p(i,j)$. Let $k = (k_1, \dots, k_d) \in Z^d$, $s \in \mathbb{N}$, $V_{k,s} = \{j \in Z^d \colon (k_l - 1)p^s < j_l \le k_l p^s,\ l = 1, \dots, d\}$. The hierarchical distance $d_p(i,j)$, $i, j \in Z^d$, is defined as $d_p(i,j) = p^{s(i,j)}$ if $i \ne j$,

$$s(i,j) = \min\{s \colon \text{there exists } k \text{ such that } i \in V_{k,s},\ j \in V_{k,s}\}.$$

In the case where p is prime, the lattice T_p^d can be represented as a lattice of purely fractional d-dimensional p-adic vectors with a p-adic distance between them [6]. The fermion field is determined by the set of four-component spins $\psi^*(i) = (\bar\psi_1(i), \psi_1(i), \bar\psi_2(i), \psi_2(i))$, $i \in T_p^d$, whose components are generators of the Grassmann algebra, and the model is given by the Hamiltonian

$$H(\psi^*;\alpha) = H_0(\psi^*;\alpha) + \sum_{i \in T_p^d} L(\psi^*; r, g).$$

Here, the Gaussian part of the Hamiltonian is defined as

$$H_0(\psi^*;\alpha) = \sum_{i,j \in T_p^d} d_0(i,j)[\bar\psi_1(i)\psi_1(j) + \bar\psi_2(i)\psi_2(j)],$$

$$d_0(i,j) = \frac{1 - p^{\alpha-d}}{1 - p^{-\alpha}}\, d_p^{-\alpha}(i,j), \quad i \ne j, \qquad d_0(i,i) = \frac{1 - p^{-d}}{1 - p^{-\alpha}}.$$

The non-Gaussian part of the Hamiltonian is determined by the Lagrangian

$$L(\psi^*(i); r, g) = r\bigl(\bar\psi_1(i)\psi_1(i) + \bar\psi_2(i)\psi_2(i)\bigr) + g\bar\psi_1(i)\psi_1(i)\bar\psi_2(i)\psi_2(i),$$

where r and g are real-valued coupling constants and α is a real-valued model parameter. In the case where $\alpha = 2 + d$, the Hamiltonian $H_0(\psi^*; 2+d)$ is a hierarchical version of the Gaussian Hamiltonian given by the Laplace operator. In the physical literature, Euclidean models usually consider Hamiltonians of this type, and the correlation functions of the non-Gaussian fields are analyzed in terms of formal expansions in the parameter $\epsilon = 4 - d$, where d is the spatial dimension. But for the completeness and mathematical clarity of the picture, it is natural to consider free parameters α and d. The block-spin Wilson RG transformation is defined as

$$r(\alpha)\psi^*(i) = p^{-\alpha/2} \sum_{j \in V_{i,1}} \psi^*(j),$$

where α is the RG parameter. The RG transformation does not change the structure of the Hamiltonian and reduces to the transformation $R(\alpha)$ in the plane of the coupling constants $R(\alpha)(r,g) = (r', g')$:

$$r' = \lambda\biggl(\frac{(r+1)^2 - g}{(r+1)^2 - g/n}(r+1) - 1\biggr), \qquad g' = \frac{\lambda^2}{n}\biggl(\frac{(r+1)^2 - g}{(r+1)^2 - g/n}\biggr)^2 g,$$

where $\lambda = p^{\alpha-d}$ and $n = p^d$ is the number of vertices in the unit cell of the hierarchical lattice T_p^d.

In what follows, we describe the transformation of the RG in the space of coefficients of the expansion of the Grassmann-valued "density" of the free measure

$$f(\psi^*) = e^{-L(\psi^*;r,g)}.$$

In the general case, this density is defined as

$$f(\psi^*; c) = c_0 + c_1(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) + c_2\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2,$$

$c = (c_0, c_1, c_2) \in \mathbb{R}^3$. In the regular case where $c_0 \neq 0$, the coupling constants r and g are related to the coefficients c by

$$r(c) = -\frac{c_1}{c_0}, \qquad g(c) = \frac{c_1^2 - c_0c_2}{c_0^2}.$$

If $c_0 = 0$ (e.g., in the case where the density is a Grassmann δ-function $\delta(\psi^*) = \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$), then an exponential representation of the density is impossible.

We naturally interpret the triplet (c_0, c_1, c_2) as a point of a two-dimensional real projective space RP^2 because two triples differing by a common nonzero factor give the same Gibbs state.

We let the RG transformation in the space of coefficients c be denoted the same as $R(\alpha)$:

$$R(\alpha)(c_0, c_1, c_2) = (c_0', c_1', c_2'), \qquad c_0' = \biggl((c_1 - c_0)^2 + \frac{1}{n}(c_0c_2 - c_1^2)\biggr),$$

$$c_1' = \lambda\biggl((c_1 - c_0)(c_2 - c_1) + \frac{1}{n}(c_0c_2 - c_1^2)\biggr), \qquad c_2' = \lambda^2\biggl((c_2 - c_1)^2 + \frac{1}{n}(c_0c_2 - c_1^2)\biggr).$$

In the (r,g) coordinates. the transformation $R(\alpha)$ has a trivial (Gaussian) fixed point $(0,0)$ and two non-Gaussian fixed points $(r_+(\alpha), g_+(\alpha))$ (the "positive" fixed point) and $(r_-(\alpha), g_-(\alpha))$ (the "negative" fixed point) under the condition $\alpha \neq d$:

$$r_+(\alpha) = \frac{p^{d/2} - p^{\alpha-d}}{1 - p^{d/2}}, \qquad g_+(\alpha) = \frac{r_+(\alpha)(1 + r_+(\alpha))^2}{1 + r_+(\alpha) + p^{-d/2}},$$

$$r_-(\alpha) = \frac{-p^{d/2} - p^{\alpha-d}}{1 + p^{d/2}}, \qquad g_-(\alpha) = \frac{r_-(\alpha)(1 + r_-(\alpha))^2}{1 + r_-(\alpha) - p^{-d/2}}.$$

We note that a positive fixed point is a fermionic analogue of the Euclidean non-Gaussian fixed point, which is constructed in the framework of the formal $(4-d)$-expansion. In the c coordinates, the Gaussian fixed point is given by the vector $(1,0,0)$, and we can see one more fixed point $(0,0,1)$, which corresponds to the density of the free measure given by the Grassmann δ-function $\delta(\psi^*) = \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2$. We let δ denote this fixed point. The map $R(\alpha)$ with $\alpha \neq d$ does not have other fixed points. In what follows, we assume that $\alpha > d$, and we can show that δ is the only attractive fixed point under this condition. The map $R(\alpha)$ from RP^2 to RP^2 is defined everywhere except the point $(1,1,1)$, which we say is singular.

2. Zone structure of the RG flow

We consider the realization of a projective c space in the form of a hemisphere

$$S = \{(c_0, c_1, c_2)\colon c_0^2 + c_1^2 + c_2^2 = 1,\ c_0 \geq 0\},$$

where opposite points of the boundary circle $c_1^2 + c_2^2 = 1$ are identified. To obtain a flat (two-dimensional) picture, we use the orthogonal projection S on the disk $D = \{(c_1, c_2)\colon c_1^2 + c_2^2 \leq 1\}$. A regular point (r, g) corresponds to $\big(c_1(r, g), c_2(r, g)\big)$,

$$c_1(r, g) = -\frac{r}{\sqrt{1 + r^2 + (r^2 - g)^2}}, \qquad c_2(r, g) = \frac{r^2 - g}{\sqrt{1 + r^2 + (r^2 - g)^2}},$$

belonging to the inner part of D. The trivial fixed point $(r, g) = (0, 0)$ is also represented on the disk D by $(0, 0)$. The fixed point δ in the (c_1, c_2) coordinates is given by $(0, 1)$. The RG-invariant line $g = 0$ in the (c_1, c_2) coordinates corresponds to the curve $l_0 = \big\{\big(c_1(r, 0), c_2(r, 0)\big);\ r \in R\big\}$. Augmenting the curve l_0 with the limit point $(0, 1)$, we obtain a closed curve l. The lower half-plane $\{(r, g)\colon g < 0\}$ is defined on the disc D by the domain D_1 bounded by the curve l (see Fig. 1). The upper half-plane $\{(r, g)\colon g > 0\}$ is mapped onto the domain $D \setminus D_1$. We set

$$V_{\mathrm{L}} = \left\{(c_1, c_2) \in D \setminus D_1\colon c_1 \leq 0,\ c_2 \geq \frac{4}{\sqrt{21}}\right\},$$

$$V_{\mathrm{R}} = \left\{(c_1, c_2) \in D \setminus D_1\colon c_1 \geq 0,\ c_2 \geq \frac{3 + b_0}{\sqrt{1 + (3 + b_0)^2}}\right\}, \qquad b_0 = -\frac{\lambda n - \lambda}{\lambda n - 1}.$$

The domains V_{L} and V_{R} have a common point $(0, 1)$, which corresponds to the fixed point δ. In [13], we proved that both these domains are invariant under the RG map and are in the attraction basin of δ. Based on this, we proposed an algorithm (see [13] for details) that allows classifying almost all points of the domain $D \setminus D_1$ according to how they tend to δ under iterations of the RG map. If a point (c_1, c_2) enters the domain V_{L} in a finite number of iterations of the RG map, then we color it dark gray and say that it tends to δ from the left. If (c_1, c_2) enters the domain V_{R} in a finite number of iterations of the RG map, then we color it gray and say that it tends to δ from the right.

In Fig. 1, we show the disc D and the RG-invariant attraction basins of the fixed point $\delta = (0, 1)$ in the case where $\alpha = 1,7$, $p = 2$, and $d = 1$. The Gaussian fixed point is labeled by 0, positive and negative fixed points are labeled by 1 and 2, and the singular point is labeled by 3. The lower half-plane $\{(r, g)\colon g < 0\}$ in the c coordinates maps to the domain D_1, which is colored light gray; we do not discuss the RG dynamics in this domain. The upper half-plane $\{(r, g)\colon g > 0\}$ correspondingly maps to the domain $D \setminus D_1$. Almost all points of the domain $D \setminus D_1$ under iterations of the RG map tend to the fixed point δ. The points colored dark gray under iterations of the RG map tend to δ from the left (remaining in the domain $D \setminus D_1$), and the points colored gray tend to δ from the right. In the dark gray domain, we see a large area $A(0)$ and a sequence of smaller disjoint areas $A(1), A(2), A(3), \ldots$. These areas are located one after another, and the larger the area number, the closer this area is to the main area $A(0)$.

We call the areas $A(i)$, $i = 1, 2, \ldots$, satellite areas of $A(0)$. In turn, each of the areas $A(i)$ has its own countable series of disjoint satellite areas $A(i, j)$, $j = 1, 2, \ldots$. Further, the areas $A(i, j)$ have their own series of satellite areas $A(i, j, k)$, $k = 1, 2, \ldots$, and so on (see Figs. 2 and 3). At first sight, the area $A(0)$ looks like the union of two different connected domains, but $A(0)$ is in fact a connected domain because the boundary segments of these areas on the circle $c_1^2 + c_2^2 = 1$ are located exactly opposite each other and

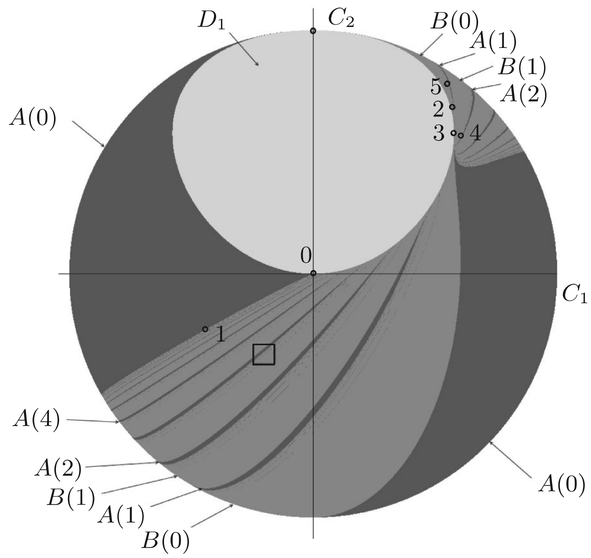

Fig. 1. The disc D and RG-invariant attraction basins of the fixed point $\delta = (0, 1)$ for $\alpha = 1,7$, $p = 2$, and $d = 1$.

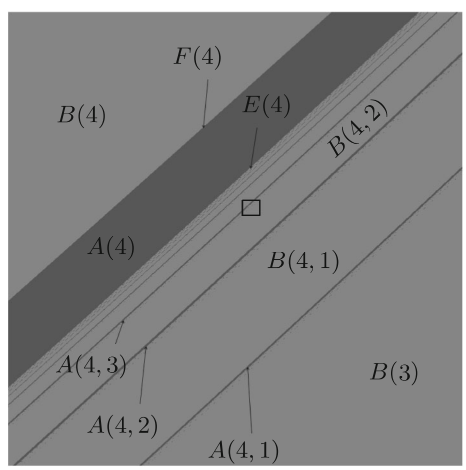

Fig. 2. An enlarged image of the square marked in Fig. 1.

therefore represent the same subset of points in the projective plane. The same is true for all satellite areas. In the gray zone, we see the same picture: the main area $B(0)$ and a countable series of disjoint satellite areas $B(1), B(2), B(3), \ldots,$ their satellite series $B(i, j)$, $i = 1, 2, \ldots,$ $j = 1, 2, \ldots,$ and so on. In Figs. 2 and 3, it can be seen that the areas $A(i_1, i_2, \ldots, i_k)$ and $B(i_1, i_2, \ldots, i_k)$ have a common boundary line, denoted by $F(i_1, i_2, \ldots, i_k)$. This type of boundary line can be characterized as a "clear" boundary because any sufficiently small disk centered at any point of such a boundary line is divided by this line into gray and dark gray areas of the same size. The areas $A(i_1, i_2, \ldots, i_k)$ and $B(i_1, i_2, \ldots, i_k)$ look like strips locally, and we see that these strips also have boundary lines of a "diffuse" type. This means that any sufficiently small disk centered at any point of a diffuse boundary line is divided by this line into a one-color region and a two-color region. We let $E(i_1, i_2, \ldots, i_k)$ and $G(i_1, i_2, \ldots, i_k)$ denote the diffuse boundary lines of the respective areas $A(i_1, i_2, \ldots, i_k)$ and $B(i_1, i_2, \ldots, i_k)$.

An enlarged image of the square marked in Fig. 1 is shown in Fig. 2, where we can see the areas $A(4)$,

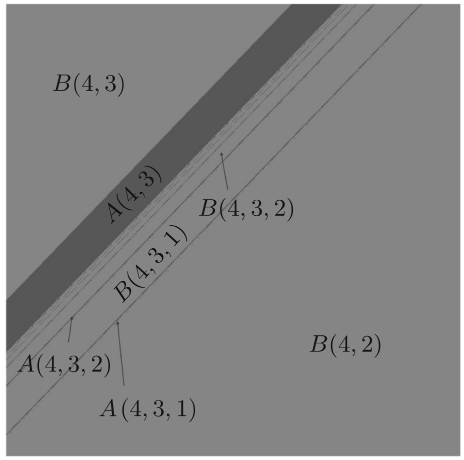

Fig. 3. An enlarged image of the square marked in Fig. 2.

$B(4)$, and $B(3)$ and their satellite areas $A(4,1)$, $A(4,2)$, $B(4,1)$, and $B(4,2)$, and so on. We can also see the clear boundary line $F(4)$ between $A(4)$ and $B(4)$ and the diffuse boundary line $E(4)$. An enlarged image of the square marked in Fig. 2 is shown in Fig. 3, where we can see the areas $A(4,3)$ and $B(4,2)$ and their satellites $A(4,3,1)$, $A(4,3,2)$, $B(4,3,1)$, and $B(4,3,2)$, and so on. Computer experiments show that if a point belongs to the area $A(i_1,i_2,\dots,i_k)$ and $i_1 > 1$, then one RG iteration sends it into the area $A(i_1-1,i_2,\dots,i_k)$. If $i_1 = 1$, then the point maps into the area $A(i_2,i_3,\dots,i_k)$. A point in $A(i)$, $i = 1,2,\dots$, maps into $A(i-1)$ in one RG iteration. Schematically, we can describe the dynamics of the RG in the dark gray domain by the diagram

$$A(i_1,i_2,\dots,i_k) \to A(i_1-1,i_2,\dots,i_k) \to \dots \to A(1,i_2,\dots,i_k) \to$$
$$\to A(i_2,\dots,i_k) \to \dots \to A(i_3,i_4,\dots,i_k) \to \dots \to A(1) \to A(0).$$

The diagram of RG dynamics in the gray zone has the same form:

$$B(i_1,i_2,\dots,i_k) \to B(i_1-1,i_2,\dots,i_k) \to \dots \to B(1,i_2,\dots,i_k) \to$$
$$\to B(i_2,\dots,i_k) \to \dots \to B(i_3,i_4,\dots,i_k) \to \dots \to B(1) \to B(0).$$

Finally, the RG dynamics on the boundary lines $F(i_1,i_2,\dots,i_k)$, $E(i_1,i_2,\dots,i_k)$, and $G(i_1,i_2,\dots,i_k)$ have a similar description. Computer experiments show that lines $F(i_1,i_2,\dots,i_k)$ are parts of a stable RG-invariant curve for the fixed point δ. The lines $G(i_1,i_2,\dots,i_k)$ are parts of the stable RG-invariant curve for the negative fixed point, the lines $E(i_1,i_2,\dots,i_k)$ are parts of a stable RG-invariant curve for the positive fixed point for $\alpha > 3d/2$ and a stable RG-invariant curve for the Gaussian fixed point for $3d/2 \geq \alpha > d$. Experiments show that almost all points of the region $D \setminus D_1$ are consequently colored gray or dark gray. But there is also a discrete (finite or infinite) set of uncolored points. For example, we can describe an explicit nontrivial cycle of a second-order RG map. Let

$$t_\pm(\alpha) = (\lambda^{-1/2}+\lambda^{1/2})^{-2}\big[\big((\lambda^{-1}+\lambda n^{-1})(\lambda+\lambda^{-1}n^{-1})\big)^{1/2} \mp (1+n^{-1})\big)\big],$$
$$s(\alpha) = \left(\frac{\lambda+n^{-1}\lambda^{-1}}{\lambda^{-1}+\lambda n^{-1}}\right)^{1/2},$$

where $\lambda = p^{\alpha-d}$ and $n = p^d$. Let

$$r_4(\alpha) = \frac{1 + t_+(\alpha)}{s(\alpha)t_+(\alpha) - 1}, \qquad r_5(\alpha) = \frac{1 - t_-(\alpha)}{s(\alpha)t_-(\alpha) - 1},$$

$$g_4(\alpha) = \frac{t_+(\alpha)(s(\alpha)^{1/2} + s(\alpha)^{-1/2})^2}{(t_+(\alpha)s(\alpha) - 1)^2}, \qquad g_5(\alpha) = \frac{t_-(\alpha)(s(\alpha)^{1/2} - s(\alpha)^{-1/2})^2}{(t_-(\alpha)s(\alpha) - 1)^2}.$$

The points $(r_4(\alpha), g_4(\alpha))$ and $(r_5(\alpha), g_5(\alpha))$ then form a nontrivial second-order cycle for the map $R(\alpha)$ in the space (r, g). In Fig. 1, these points in the space (c_1, c_2) are labeled by 4 and 5 in the case where $\alpha = 1,7$, $p = 2$, and $d = 1$. Computer experiments showed that points 4 and 5 do not belong to sets of type A, B, E, F, or G.

Numerous experiments showed that the same zone structure and the same behavior of the global RG flow hold for other values $\alpha > d$, $p > 1$, and $d > 1$.

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