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A BIANCHI TYPE-II DARK-ENERGY COSMOLOGY WITH A DECAYING Λ-TERM IN THE BRANS–DICKE THEORY OF GRAVITY

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We present a sequence of anisotropic Bianchi type-II dark-energy models in the framework of the Brans–Dicke theory of gravity with a variable equation of state (EoS) parameter and a constant deceleration parameter. We use power-law relations between the scalar field ϕ and the scale factor A and between the average Hubble parameter H and the average scale factor A to obtain most of the analytic solutions. The dark-energy EoS parameter ω and its range admitted by the models agrees well with the most recent observational data. It has been observed that the cosmological constant Λ is decreasing with time, which is consistent with recent cosmological observations. We study the dynamical stability and physical features of the models.

Keywords: Bianchi type-II space–time, anisotropic dark energy, Brans–Dicke theory of gravity, equation of state parameter, Λ-term

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1. Introduction

The Universe currently experiences an unexpected accelerated expansion due to a mysterious dark energy (DE) according to observational data obtained from high red-shift surveys of Type-Ia supernovas (SNeIa) [1], [2]. The experimental data [3] about late-time acceleration has attracted much attention in recent years. The recent results for cosmic microwave background (CMB) fluctuations [4] and large-scale structure (LSS) [5] suggest an accelerated expansion of the Universe. The DE seems the best candidate for explaining the cosmic acceleration. The most important evidence is related to the SNeIa data [6]–[10], CMB [4], [11], [12], and baryon acoustic oscillations [13], [14]. These data confirm that about 76% of the energy density in the current Universe consists of DE. It is now believed that 96% of the energy of the Universe consists of DE and dark matter (76% DE + 20% dark matter). Dark energy is the most popular way to explain recent observations that the Universe is expanding at an accelerating rate. The simplest DE candidate is the cosmological constant Λ with the equation of state (EoS) parameter $\omega = -1$ because it fits well with the observational data. Currently, Λ with its dynamical behavior is preferred as a constant [15]–[17]. Chaplygin gas and $f(R)$, where R is the scalar curvature in the Einstein–Hilbert Lagrangian, have also been considered as possible DE models because of negative pressure [18]–[29].

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The EoS for the evolution of DE is conventionally characterized by the EoS parameter $\omega(t) = p/\rho$, which is not necessarily constant, where p is the fluid pressure and ρ is the energy density [30]. The observational limits obtained from SNeIa data [31] and SNeIa data combined with CMB radiation (CMBR) anisotropy and galaxy clustering statistics [5] are respectively $-1.67 < \omega < -0.62$ and $-1.33 < \omega < -0.79$. In 2011, the latest results were obtained from a combination of cosmological data sets coming from CMBR anisotropies, luminosity distances of high red-shift type-Ia supernovas, and galaxy clustering, which constrain the DE EoS to $-1.44 < \omega < -0.92$ at the 68% confidence level [12], [32]. Various time-dependent EoS parameters were recently used for models with a variable Λ [33], [34], and several DE models with a variable ω were recently obtained in [35] and also in [36].

The isotropic models can play an important role in studying the LSS model of the Universe, but it is believed that the late-time cosmic acceleration and DE along with the modified gravity theories have become more popular for many researchers. Brans and Dicke proposed a scalar–tensor generalization of general relativity popularly known as the Brans–Dicke (BD) theory of gravity [37]. In scalar–tensor theories, gravity is modified by one or more scalar fields. Here, we consider just one field with the action

$$S = \frac{1}{16\pi}\int \sqrt{-g}\left\{\phi R - \frac{\varpi}{\phi}\phi_{,i}\phi^{,i} + L_m\right\}d^4x, \tag{1}$$

where $\sqrt{-g}\,d^4x$ denotes the four-dimensional volume, ϖ is the BD coupling constant, L_m is obtained from the Lagrangian of the flat space–time field theory using the equivalence principle, and ϕ denotes the scalar field. The BD field equations can now be easily obtained by varying Eq. (1) in g_{ij} and ϕ as

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\frac{8\pi}{\phi}T_{ij} - \frac{\varpi}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) - \frac{1}{\phi}(\phi_{i;j} - g_{ij}\phi^{;k}_{;k}), \tag{2}$$

$$\Box\phi = \phi^{;k}_{;k} = g^{ik}\phi_{i;k} = \frac{8\pi}{(3+2\varpi)\phi}T, \tag{3}$$

where $T = g^{ij}T_{ij}$ is the trace of the energy–momentum tensor, $\Box$ is the d'Alembertian operator, and ϖ is a dimensionless coupling constant. We here note that general relativity is recovered in the limit case as $\varpi \to \infty$. We can therefore compare our results with the experimental data for significantly large values of ϖ. A comma and semicolon respectively denote partial and covariant differentiation. The Bianchi type-II model is a spatially homogeneous and totally anisotropic cosmological model, which plays a significant role in describing the Universe at early stages of its evolution. The study of Bianchi-type models in the context of BD theory and also various other alternative theories of gravity have attracted many researchers in recent years [38]–[46].

Here, we use a power-law relation between the scalar field ϕ and the scale factor A to find solutions. The EoS parameter ω and its admissible range for the DE models agrees well with recent observations of SNeIa data, SNeIa data + CMBR anisotropy and galaxy clustering statistics, and also the combination of cosmological data sets coming from CMB anisotropy, luminosity distances of high red-shift type-Ia supernovas, and galaxy clustering. The cosmological constant Λ in these models is found to be a positive function of time that approaches small positive values in the current epoch, which are supported by recent cosmological observations. The motivation behind our work is to examine the time variation of the cosmological constant Λ from the dynamics of the investigated models keeping recent observations related to the accelerated expansion of the Universe in mind. We also discuss the physical behaviors and the dynamical stability of the model.

2. Metric and field equations

We define the Bianchi type-II line element as

$$ds^2 = dt^2 - a^2(t)(dx - z\,dy)^2 - b^2(t)\,dy^2 - c^2(t)\,dz^2, \tag{4}$$

where $a(t)$, $b(t)$, and $c(t)$ are cosmic scale factors.

The energy–momentum tensor of the anisotropic DE fluid is taken as

$$T^i_j = \mathrm{diag}[\rho, -p_x, -p_y, -p_z] = \mathrm{diag}[1, -\omega_x, -\omega_y, -\omega_z]\rho, \tag{5}$$

where ρ is the energy density of the fluid, p_x, p_y, and p_z are the directional pressures, and ω_x, ω_y, and ω_z are the directional EoS parameters of the fluid ω along the x, y, and z axes. We now parameterize Eq. (5) by choosing $\omega_y = \omega_z = \omega$ and introducing the skewness parameter δ, which is the deviation of the EoS parameter ω on the x axis from plane (y, z). Here, ω and δ are not necessarily constant and are functions of the cosmic time t

$$T^i_j = \mathrm{diag}\,[1, -(\omega+\delta), -\omega, -\omega]\rho. \tag{6}$$

For Bianchi type-II space–time (4), field equations (2) become

$$\frac{\dot a}{a}\frac{\dot b}{b} + \frac{\dot b}{b}\frac{\dot c}{c} + \frac{\dot c}{c}\frac{\dot a}{a} - \frac{1}{4}\frac{a^2}{b^2c^2} - \frac{\dot\phi}{\phi}\left(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c} - \frac{\dot\phi}{2\phi}\varpi\right) = -\frac{8\pi}{\phi}\rho, \tag{7}$$

$$\frac{\ddot b}{b} + \frac{\ddot c}{c} + \frac{\dot b}{b}\frac{\dot c}{c} - \frac{3}{4}\frac{a^2}{b^2c^2} - \frac{\dot\phi}{\phi}\left(\phi\ddot\phi + \frac{\dot b}{b} + \frac{\dot c}{c} + \frac{\dot\phi}{2\phi}\varpi\right) = \frac{8\pi}{\phi}(\omega+\delta)\rho, \tag{8}$$

$$\frac{\ddot a}{a} + \frac{\ddot c}{c} + \frac{\dot a}{a}\frac{\dot c}{c} + \frac{1}{4}\frac{a^2}{b^2c^2} - \frac{\dot\phi}{\phi}\left(\phi\ddot\phi + \frac{\dot a}{a} + \frac{\dot c}{c} + \frac{\dot\phi}{2\phi}\varpi\right) = \frac{8\pi}{\phi}\omega\rho, \tag{9}$$

$$\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\dot a}{a}\frac{\dot b}{b} + \frac{1}{4}\frac{a^2}{b^2c^2} - \frac{\dot\phi}{\phi}\left(\phi\ddot\phi + \frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot\phi}{2\phi}\varpi\right) = \frac{8\pi}{\phi}\omega\rho. \tag{10}$$

The energy conservation equation $T^{ij}_{;j} = 0$ becomes

$$\dot\rho + 3(1+\omega)\rho H + \rho(\delta H_x) = 0, \tag{11}$$

where the dot denotes the derivative with respect to the cosmic time t. The directional Hubble parameters expressing the expansion rates of the Universe along the directions of the x, y, and z axes can be defined as

$$H_x = \frac{\dot a}{a}, \qquad H_y = \frac{\dot b}{b}, \qquad H_z = \frac{\dot c}{c}. \tag{12}$$

The generalized mean Hubble parameter H, which expresses the volumetric expansion rate of the Universe, is given by

$$H = \frac{1}{3}(H_x + H_y + H_z). \tag{13}$$

The average scale factor A and the volume scale factor V are defined as

$$V = A^3 = abc. \tag{14}$$

Using Eqs. (12)–(14), we obtain

$$H = \frac{1}{3}\frac{\dot V}{V} = \frac{1}{3}(H_x + H_y + H_z) = \frac{\dot A}{A}. \tag{15}$$

The mean anisotropic expansion parameter Δ plays a crucial role in deciding whether the model is isotropic or anisotropic and is defined as

$$\Delta = \frac{1}{3}\frac{\sigma^2}{H^2} = \frac{1}{3}\sum_{i=x}^{z}\left(\frac{H_i - H}{H}\right)^2, \tag{16}$$

where σ is the shear scalar and Δ is the measure of the deviation from isotropic expansion. The Universe expands isotropically for $\Delta = 0$. The model approaches isotropy continuously if $V \to \infty$ and $\Delta = 0$ as $t \to \infty$ [47]. For physically realistic models, we expect that the energy density of the DE fluid is positive as $t \to \infty$.

We now find the expansion and shear scalar for metric (4). The expansion scalar is given by

$$\Theta = 3H, \tag{17}$$

and the shear scalar is given by

$$\sigma^2 = \frac{1}{2}\left[\sum_{i=x}^{z} H_i^2 - 3H^2\right]. \tag{18}$$

The dimensionless mean deceleration parameter q in cosmology is a measure of the cosmic acceleration of the Universe and is defined as

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1. \tag{19}$$

The Universe exhibits accelerating volumetric expansion for $q < 0$, decelerating volumetric expansion for $q > 0$, and constant acceleration at $q = 0$. We here note that q was initially believed to be positive, but recent observations from the supernova data *inter alia* suggest that it is negative. The behavior of the models thus depends on the sign of q. A positive deceleration parameter corresponds to a decelerating model, and a negative value indicates acceleration.

We consider a power-law relation between the average scale factor A and the scalar field ϕ [38] of the form

$$\phi = lA^{\alpha}, \tag{20}$$

where l is a proportionality constant and α is the power index. We also use a well-known relation [48] between the average Hubble parameter H and the average scale factor A given by

$$H = \chi A^{-n}, \tag{21}$$

where $\chi > 0$ and $n \geq 0$. This is an important relation because it yields a constant value for the deceleration parameter. From Eqs. (15) and (21), we obtain

$$\dot{A} = \chi A^{-n+1}, \tag{22}$$

and using this equation, we find that the deceleration parameter is constant, i.e.,

$$q = n - 1. \tag{23}$$

Integrating (22), we obtain

$$A = (n\chi t + k_1)^{1/n}, \quad n \neq 0, \tag{24}$$

$$A = k_2 e^{\chi t}, \qquad n = 0, \tag{25}$$

where k_1 and k_2 are integration constants. We thus obtain two forms for the average scale factor corresponding to two different models of the Universe.

3. The DE model of the Universe for $n \neq 0$

We now discuss the model of the Universe for $n \neq 0$, i.e., $A = (n\chi t + k_1)^{1/n}$. For this model, ϕ becomes

$$\phi = l(n\chi t + k_1)^{\alpha/n}. \tag{26}$$

Field equations (7)–(11) constitute a system of five independent equations with seven unknown parameters a, b, c, ω, δ, ρ, and ϕ. Therefore, some additional constraint equations relating these parameters are required to obtain explicit solutions of the system of equations. We assume that

$$b = a^r, \tag{27}$$

$$c = a^s, \tag{28}$$

where r and s are arbitrary constants. Using Eq. (6), we obtain

$$\Box\phi = \ddot{\phi} + \dot{\phi}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) = \frac{8\pi(1 - 3\omega - \delta)\rho}{(3 + 2\varpi)\phi}. \tag{29}$$

Subtracting Eq. (9) from (10), we obtain

$$\frac{\ddot{c}}{c} - \frac{\ddot{b}}{b} + \left(\frac{\dot{c}}{c} - \frac{\dot{b}}{b}\right)\left(\frac{\dot{a}}{a} - \frac{\dot{\phi}}{\phi}\right) = 0. \tag{30}$$

Solving this equation using (26), we obtain the expressions for the metric coefficients:

$$a = \left\{\frac{(r + s + 1)(n\chi t + k_1)^{\alpha/n+1}}{\chi(\alpha + n)}\right\}^{1/(r+s+1)}. \tag{31}$$

The physical parameters, i.e., the directional Hubble parameters H_i, the mean generalized Hubble parameter H, the mean anisotropy parameter $\triangle$, the spatial volume V, the expansion scalar Θ, and the shear scalar σ^2 are given by

$$H_x = \frac{\chi(\alpha + n)}{(r + s + 1)(n\chi t + k_1)}, \qquad H_y + H_z = (r + s)H_x, \tag{32}$$

$$H = \frac{\chi(\alpha + n)}{3(n\chi t + k_1)}, \tag{33}$$

$$\triangle = \frac{3(r^2 + s^2 + 1)}{(r + s + 1)^2} - 1, \tag{34}$$

$$V = \frac{(r + s + 1)(n\chi t + k_1)^{\alpha/n+1}}{\chi(\alpha + n)}, \tag{35}$$

$$\Theta = \frac{\chi(\alpha + n)}{(n\chi t + k_1)}, \tag{36}$$

$$\sigma^2 = \frac{\chi^2(\alpha + n)^2}{2(n\chi t + k_1)^2}\left[\frac{(r^2 + s^2 + 1)}{(r + s + 1)^2} - \frac{1}{3}\right]. \tag{37}$$

We note that the Universe in this cosmological model exhibits an initial point-type singularity at $t = -k_1/n\chi$. The space–time is well behaved in the range $-k_1/n\chi < t < \infty$. At the initial instant

$t = -k_1/n\chi$, the parameters H, Θ, and σ^2 diverge. Hence, the Universe starts from an initial singularity with infinite shear and expansion. Moreover H, Θ, and σ^2 tend to zero as $t \to \infty$. Therefore, the expansion and shear decay. Here, H, Θ, and σ^2 are monotonically decreasing quantities for t in the range $-k_1/n\chi < t < \infty$. The shear tends to zero much faster than the expansion. The proper volume V tends to zero at the initial singularity. As time proceeds, the Universe approaches infinite volume in the limit as $t \to \infty$. The expansion scalar Θ is always positive in the interval $-k_1/n\chi < t < \infty$. Therefore, the models describe expanding cosmological models in the presence of DE. The model describes an accelerating model ($q < 0$) for $n < 1$ and a decelerating model ($q > 0$) for $n > 1$. The ratio σ^2/Θ^2 of the model is constant. This shows that the model does not approach isotropy as the Universe expands. The mean anisotropy parameter Δ is uniform throughout the evolution of the Universe because it is independent of t. In this model, particle horizons exist because the integral

$$\int_{t_0}^{t} \frac{d\acute{t}}{V(\acute{t})} = \left[\frac{-(\alpha+n)}{\alpha(r+s+1)(n\chi\acute{t}+k_1)^{\alpha/n}}\right]_{t_0}^{t} \tag{38}$$

converges.

Solving field equation (7) using Eqs. (26) and (31), we obtain

$$\rho = \frac{l(n\chi t+k_1)^{\alpha/n-2}}{8\pi}[\rho_1+\rho_2], \tag{39}$$

where

$$\rho_1 = \chi^2(\alpha+n)\left\{\alpha - \frac{(r+rs+s)(\alpha+n)}{(r+s+1)^2}\right\},$$

$$\rho_2 = \frac{(n\chi t+k_1)^2}{4}\left\{\frac{(r+s+1)(n\chi t+k_1)^{\alpha/n+1}}{\chi(\alpha+n)}\right\}^{-2+4/(r+s+1)} - \frac{\varpi\chi^2\alpha^2}{2}.$$

Solving field equation (9) using (26), (31), and (39), we obtain

$$\omega = -1 + \frac{l(n\chi t+k_1)^{\alpha/n-2}}{8\pi\rho}[\omega_1+\omega_2-\omega_3], \tag{40}$$

where

$$\omega_1 = \frac{\chi^2(\alpha+n)}{(r+s+1)^2}\{\alpha(r^2+s^2+1) - 2n(r+rs+s)\},$$

$$\omega_2 = \frac{(n\chi t+k_1)^2}{2}\left\{\frac{(r+s+1)(n\chi t+k_1)^{\alpha/n+1}}{\chi(\alpha+n)}\right\}^{-2+4/(r+s+1)},$$

$$\omega_3 = \chi^2\{\alpha(\alpha-n) - \varpi\alpha^2\}.$$

Subtracting field equation (8) from (10) and using Eqs. (26) and (31), we obtain

$$\delta = -\frac{l(n\chi t+k_1)^{\alpha/n-2}}{8\pi\rho}[\delta_1+\delta_2], \tag{41}$$

where

$$\delta_1 = \frac{2rs\chi^2(\alpha+n)^2}{(r+s+1)^2(n\chi t+k_1)^2},$$

$$\delta_2 = (n\chi t+k_1)^2\left\{\frac{(r+s+1)(n\chi t+k_1)^{\alpha/n+1}}{\chi(\alpha+n)}\right\}^{-2+4/(r+s+1)}.$$

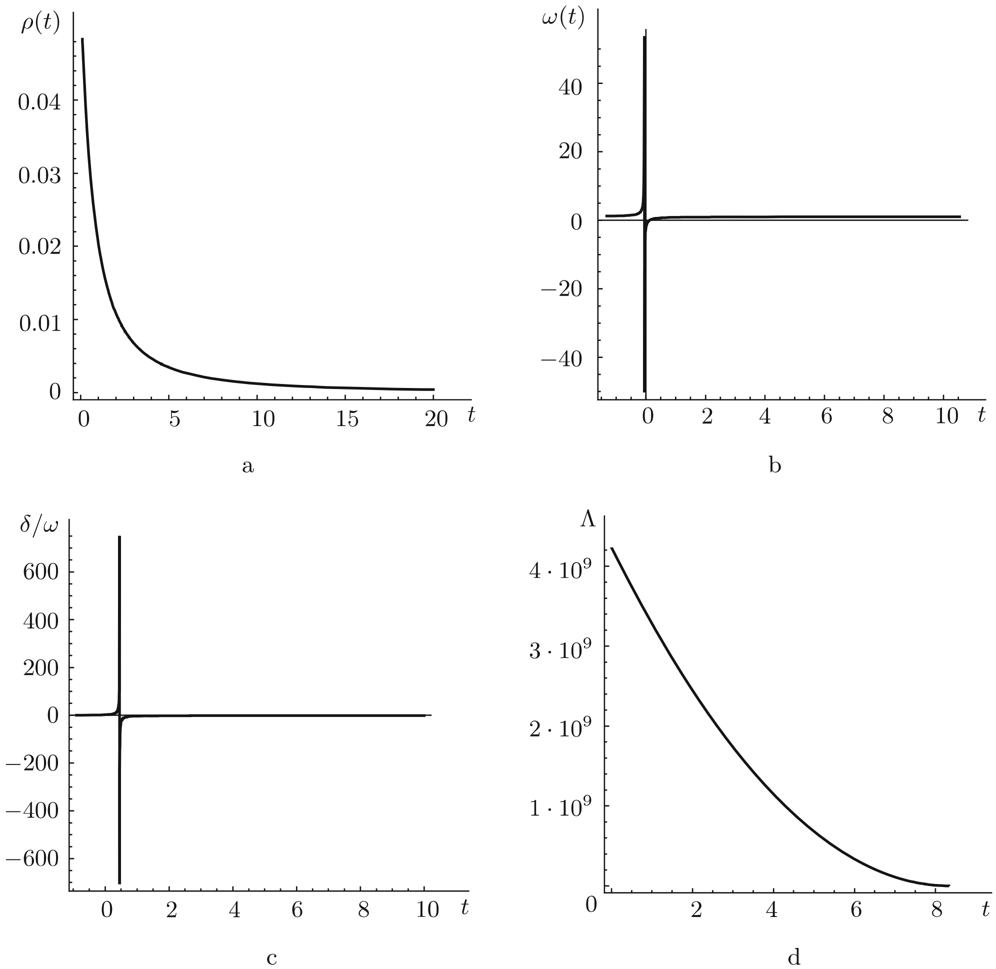

Fig. 1. Plots of (a) the energy density ρ versus the cosmic time t, (b) the EoS parameter ω versus t, (c) the anisotropy δ/ω of the DE fluid versus t, and (d) the cosmological constant Λ versus t.

From Eq. (40), we see that ω is a function of the cosmic time t, and it can therefore equally well be a function of the red-shift z or the scale factor A. The parameterization concerning the scale-factor dependence of ω is given by

$$\omega(A) = \omega_0 + \omega_A(1 - A), \tag{42}$$

where ω_0 is the present value at $A = 1$ and ω_A is the measure of the time variation ω' [49]. Comparing this result with the experimental results, we conclude that the limit of ω fits well within the accepted range of the EoS parameter as described in Sec. 1. We note that ω vanishes at $t = t_c$, where t_c is the critical time, which is obtained by Eq. (40) and has the value $t_c = 0.02948$ (see Fig. 1b). This model represents a dusty Universe at $t = t_c$, the matter-dominated phase of the Universe for $t \geq t_c$, where $\omega \geq 0$, and the DE-dominated phase of the Universe at $t < t_c$, where $\omega < 0$ as shown in Fig. 1b. For this model, the value of the EoS parameter ω is restricted to the observational limits

- $-1.67 < \omega < -0.62$, which agrees well with the results obtained from SNeIa data [31],

- $-1.33 < \omega < -0.79$, which agrees well with the results obtained from SNeIa data combined with CMBR anisotropies and galaxy clustering statistics [5], and

- $-1.44 < \omega < -0.92$, which agrees well with the results obtained from combining cosmological data sets from CMB anisotropies, luminosity distances of high red-shift type-Ia supernovas, and galaxy clustering in 2009 at the 68% confidence level [12], [32].

Figure 1b also shows the variation of the EoS parameter ω with the cosmic time t in two evolutionary phases of the Universe: acceleration ($q < 0$) and deceleration ($q > 0$) with the appropriate choice of integration constants and other physical parameters. We note that ω increases rapidly in the initial stage and attains the maximum positive value $\omega \simeq 53.44$ at the epoch $t \simeq 0$, which is closer to the early evolutionary stages of the Universe. It decreases sharply to the minimum negative value $\omega \simeq -50.61$ at the epoch $t \simeq 0$ and then passes through the various phases of DE model of the Universe:

- the phantom phase for $\omega < -1$,
- the quintom phase, which inherits both the properties of quintessence and the phantom phase by the phantom divide line $\omega = -1$,
- the cosmological constant Λ, which is regarded as the simplest case of DE at $\omega = -1$,
- the quintessence phase for $-1 < \omega < -1/3$ [30],
- probably the dusty Universe as ω tends to zero at the epoch $t \simeq 0.02948$, and
- the later stage, where ω increases and attains the maximum value $\omega \simeq 0.8924$ in the interval $0 < t < 12$.

Therefore, we say that the earlier matter-dominated phase of the Universe is converted into the DE-dominated phase, which is converted again into a matter-dominated phase at late times. The energy density ρ of the fluid is a decreasing function of time and tends to $\rho \simeq 0$ at $t \simeq 20$, as shown in Fig. 1a. The anisotropy of the DE fluid can be measured by δ/ω, and it starts with $\delta/\omega \simeq 4.939$ at $t \simeq -0.153$, attains $(\delta/\omega)_{\max} \simeq 745.3$ at $t \simeq 0.4464$, then reduces to $(\delta/\omega)_{\min} \simeq -710.4$ at $t \simeq 0.4839$, and tends asymptotically to $\delta/\omega \simeq 0$ in the interval $0.7462 \leq t \leq 10.07$, as shown in Fig. 1c. Hence, we can see that the anisotropic DE fluid disappears in the interval $0.7462 \leq t \leq 10.07$ in this model.

In absence of any curvature, the matter energy density Ω_m and the DE density Ω_Λ are related by

$$\Omega_m + \Omega_\Lambda = 1, \tag{43}$$

where $\Omega_m = \rho/3H^2$ and $\Omega_\Lambda = \Lambda/3H^2$. Hence, Eq. (42) reduces to

$$\Lambda = 3H^2 - \rho. \tag{44}$$

Using (33), (39), and (44), we obtained the cosmological constant Λ

$$\Lambda = \frac{\chi^2(\alpha+n)^2}{3(n\chi t+k_1)^2} - \frac{l(n\chi t+k_1)^{\alpha/n-2}}{8\pi}(\rho_1+\rho_2). \tag{45}$$

We note that the Λ-term is always positive if

$$8\pi\chi^2(\alpha+n)^2 > 3l(n\chi t+k_1)^{\alpha/n}. \tag{46}$$

This Λ-term is a decreasing function of time in the interval $0 \leq t \leq 8.5$ with $\Lambda_{\min} \simeq 0$ (see Fig. 1d), which is consistent with the recent cosmological observations that suggest the existence of a positive Λ-term with the magnitude $\Lambda(G\hbar/c^3) \simeq 10^{-123}$. Our DE model thus shows an accelerating expansion of the Universe, which

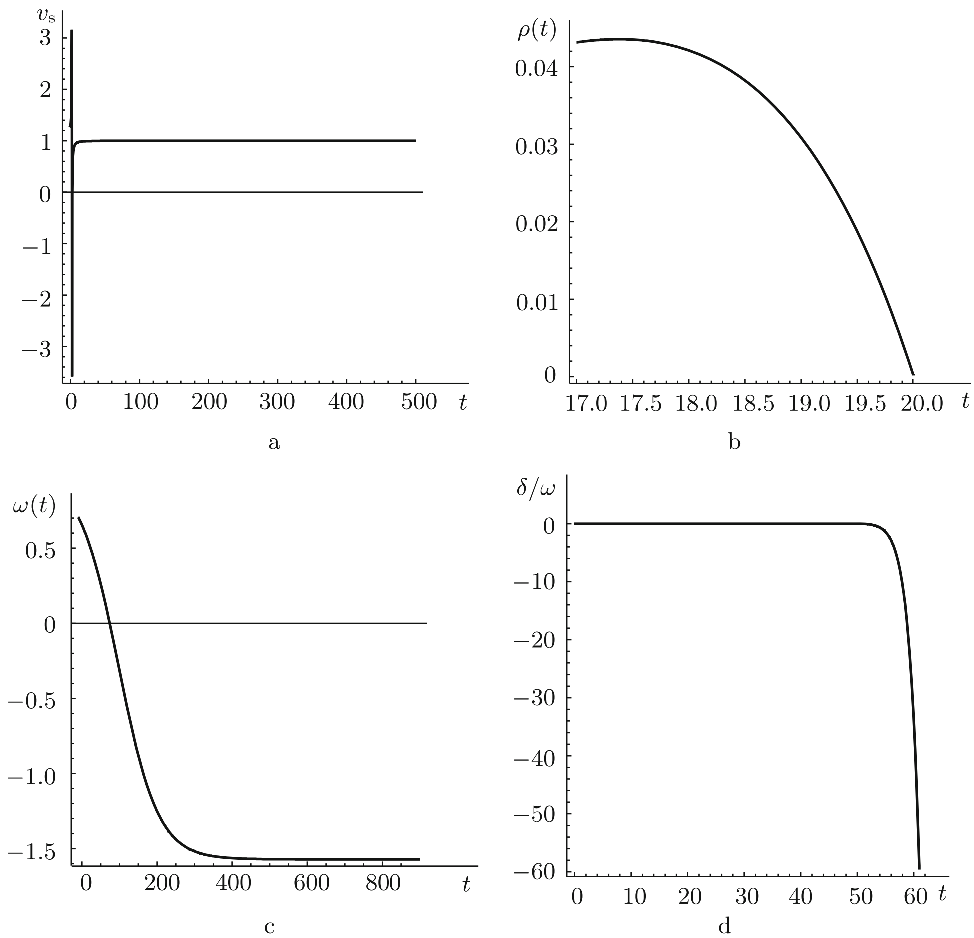

Fig. 2. Plots of (a) the speed of sound v_s versus the cosmic time t, (b) the energy density ρ versus t, (c) the EoS parameter ω versus t, and (d) the DE fluid anisotropy δ/ω versus t.

can arise as a result of the contribution to the vacuum energy from the EoS parameter in a time-dependent background.

The model is physically acceptable if the speed of sound v_s is less than the speed of light [50], i.e.,

$$0 \leq v_s = \frac{dp}{d\rho} < 1. \tag{47}$$

The speed of sound is given by

$$v_s = -1 + \left[\frac{l\chi(\alpha - 2n)(n\chi t + k_1)^{\alpha/n-3}(w_1 + w_2 - w_3) + 4l(n\chi t + k_1)^{\alpha/n-2} w_2 X}{l(n\chi t + k_1)^{\alpha/n-2}(2\rho_2 + \varpi\chi^2\alpha^2)X + l\chi(\rho_1 + \rho_2)(\alpha - 2n)(n\chi t + k_1)^{\alpha/n-3}} \right], \tag{48}$$

where

$$X = \frac{1}{(n\chi t + k_1)} \left\{ n\chi + \frac{\chi(\alpha + n)(1 - r - s)}{(r + s + 1)} \right\}.$$

In Fig. 2a, we show the behavior of the speed of sound v_s with respect to the cosmic time t. Choosing a suitable parameter, we find the stable region in the interval $16.49 < t < 500$, where the speed of sound is $v_s \simeq 0.994$. Therefore, the model is physically acceptable in the interval $1.51 < t < 500$, where the speed of sound varies in the range $0.0552 < v_s < 0.994$.

4. The DE model of the Universe at $n = 0$

The average scale factor for this model of the Universe is $A = le^{\chi t}$, and ϕ hence becomes

$$\phi = lk_2^{\alpha} e^{\chi\alpha t}. \tag{49}$$

Solving Eq. (30) using Eqs. (27), (28), and (49), we obtain the expressions for the metric coefficients

$$a = \left(\frac{r+s+1}{\alpha\chi}\right)^{1/(r+s+1)} \exp\left(\frac{\alpha\chi t}{r+s+1}\right). \tag{50}$$

The physical parameters, such as the directional Hubble parameters H_i, the mean generalized Hubble parameter H, the mean anisotropy parameter $\triangle$, the spatial volume V, the expansion scalar Θ, and the shear scalar σ^2, are given by

$$H_x = \frac{\chi\alpha}{r+s+1}, \qquad H_y + H_z = (r+s)H_x, \tag{51}$$

$$H = \frac{\chi\alpha}{3}, \tag{52}$$

$$\Delta = \frac{3(r^2+s^2+1)}{(r+s+1)^2} - 1, \tag{53}$$

$$V = \left(\frac{r+s+1}{\alpha\chi}\right) e^{\alpha\chi t}, \tag{54}$$

$$\Theta = \alpha\chi, \tag{55}$$

$$\sigma^2 = \frac{\alpha^2\chi^2}{2}\left[\frac{r^2+s^2+1}{(r+s+1)^2} - \frac{1}{3}\right]. \tag{56}$$

In this case, we note that it is an exponential model of the Universe with the average scale factor $A = k_2 e^{\chi t}$. Therefore, the model is nonsingular because an exponential function is never zero. The physical parameters H_i, H, $\triangle$, Θ, and σ^2 are all finite. The volume scale factor V increases exponentially with time, which indicates that the Universe starts its expansion with zero volume from the infinite past. The expansion scalar Θ is always positive for $\chi, \alpha > 0$, and the deceleration parameter $q < 0$ at $n = 0$. Therefore, the DE model describes an accelerating expansion of the Universe. The ratio σ^2/Θ^2 of the model is constant. This shows that the model does not approach isotropy at the time of the evolution of the Universe. The mean anisotropic parameter Δ is uniform throughout the evolution of the Universe because it is independent of t. In this model, a particle horizon exists because the integral

$$\int_{t_0}^{t} \frac{d\acute{t}}{V(\acute{t})} = \frac{e^{-\chi\alpha t_0} - e^{-\chi\alpha t}}{r+s+1} \tag{57}$$

converges.

Solving field equation (7) using (49) and (50), we obtain

$$\rho = \frac{lk_2^{\alpha} e^{\alpha\chi t}}{8\pi}[\rho_1 + \rho_2], \tag{58}$$

where

$$\rho_1 = \alpha^2\chi^2\left\{1 - \frac{(r+rs+s)}{(r+s+1)^2} - \frac{\varpi}{2}\right\},$$

$$\rho_2 = \frac{1}{4}\left(\frac{r+s+1}{\alpha\chi}\right)^{-2+4/(r+s+1)} \exp\left\{\frac{2\alpha\chi t(1-r-s)}{(r+s+1)}\right\}.$$

Solving field equation (9) using (49), (50), and (58), we obtain

$$\omega = -1 + \frac{lk_2^{\alpha} e^{\alpha\chi t}}{8\pi\rho}[\omega_1 + \omega_2], \tag{59}$$

where

$$\omega_1 = \alpha^2\chi^2\left\{\frac{(r^2+s+1)}{(r+s+1)^2} - 1 - \varpi\right\},$$

$$\omega_2 = \frac{1}{2}\left(\frac{r+s+1}{\alpha\chi}\right)^{-2+4/(r+s+1)} \exp\left\{\frac{2\alpha\chi t(1-r-s)}{(r+s+1)}\right\}.$$

Subtracting field equation (10) from (8) and using Eqs. (49), (50), and (58), we obtain

$$\delta = -\frac{lk_2^{\alpha}\big((r+s+1)/\alpha\chi\big)^{-2+4/(r+s+1)} \exp\big\{\alpha\chi t\big((3-r-s)/(r+s+1)\big)\big\}}{8\pi\rho}. \tag{60}$$

Figure 2c shows the variation of the EoS parameter ω with the cosmic time t. Comparing our results with the abovementioned experimental data, we can see that the limit of ω given by Eq. (59) fits well into the accepted range of the EoS parameter. We note the critical time $t = t_c \simeq 75.72$ where ω vanishes. We also see that the earlier real matter at $t \le t_c \simeq 75.72$, where $\omega \ge 0$ is later converted into the DE-dominated phase of the Universe at $t > t_c \simeq 75.72$, where $\omega < 0$. This model starts with a finite past and attains the positive value $\omega \simeq 0.6993$, which is closer to the early stages of evolution of the Universe. It decreases sharply to the minimum negative value $\omega \simeq -1.57$ at the epoch $t \simeq 405$. The curve passes through the various phases of DE model of the Universe:

- the phantom phase for $\omega < -1$,
- the quintom phase, which inherits both the properties of quintessence and phantom phase by the phantom divide line $\omega = -1$,
- the cosmological constant Λ, which is regarded as the simplest case of DE at $\omega = -1$,
- the quintessence phase for $-1 < \omega < -1/3$ [30],
- the phase where ω tends to zero at the epoch $t \simeq 75.72$, which can be suitable for describing a dusty Universe at this instant, and
- the later stage where $\omega \simeq -1.57$ is nearly stable in the interval $405 < t < 900$.

Therefore, we can say that the earlier matter-dominated phase of the Universe is later converted into the DE-dominated phase. The energy density ρ of the fluid is a decreasing function of time in acceleration phase of the Universe, as shown in Fig. 2b. The anisotropy of the DE fluid can be measured by δ/ω and starts with $\delta/\omega \simeq 0.0997$ at $t \simeq 0$, remains stable in the interval $0 < t < 52.06$, and then decreases to $(\delta/\omega)_{\min} \simeq -60$ at $t \simeq 61$, as shown in Fig. 2d. We hence note that this model is filled with an isotropic DE fluid in the range $0 < t < 52.06$. In absence of any curvature, the cosmological constant Λ is given by

$$\Lambda = \frac{\chi^2\alpha^2}{3} - \frac{lk_2^{\alpha} e^{\alpha\chi t}}{8\pi}(\rho_1 + \rho_2). \tag{61}$$

We see that the Λ-term is always positive if

$$8\pi\chi^2\alpha^2 > 3lk_2^{\alpha} e^{\chi\alpha t}(\rho_1 + \rho_2). \tag{62}$$

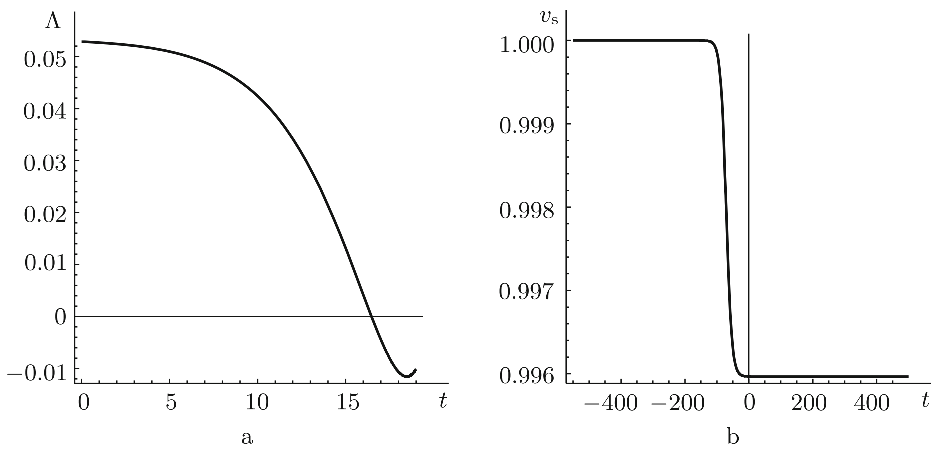

Fig. 3. Plots of (a) the cosmological constant Λ versus the cosmic time t and (b) the speed of sound v_s versus t.

In this case, the Λ-term is a decreasing function of time in the interval $0 < t < 18$. It starts with $\Lambda \simeq 0.05284$ at $t \simeq 0$ and attains $\Lambda_{\min} \simeq -0.01$ at $t \simeq 18$. The Λ-term is zero at $t \simeq 18$. Therefore, the model is consistent with the recent cosmological observations that suggest the existence of a positive Λ-term with the magnitude $\Lambda(G\hbar/c^3) \simeq 10^{-123}$, as shown in Fig. 3a. Hence, this DE model shows an accelerating expansion of the Universe. The model is physically acceptable if speed of sound v_s is less than the speed of light [50], i.e.,

$$v_s = 1 + \frac{2lk_2^{\alpha}\alpha\chi(w_1 + w_2)e^{\alpha\chi t}}{lk_2^{\alpha}(1 - r - s)\big((r + s + 1)/\alpha\chi\big)^{-3+4/(r+s+1)}\exp\big\{\alpha\chi t(3 - r - s)/(r + s + 1)\big\}}. \tag{63}$$

The behavior of the speed of sound v_s with respect to the cosmic time t can be seen in Fig. 3b. Choosing a suitable parameter, we find the stable region in the interval $13.16 < t < 504.2$, where the speed of sound is $v_s \simeq 0.994$. The model is hence physically acceptable in the range $0 < t < 500$.

5. Discussion and conclusions

We have explored solutions of Bianchi type-II cosmological models in BD theory of gravity in the background of anisotropic DE. We used a power-law relation between ϕ and R to find the solution. Assuming a constant deceleration parameter leads to two models of the Universe, i.e., a power-law model and an exponential model.

- The power-law model of the Universe corresponding to $n \neq 0$ with an average scale factor $A = (n\chi t + k_1)^{1/n}$ exhibits an initial point-type singularity at $t = -k_1/n\chi$. The physical parameters H_x, H_y, H_z, H, Θ, and σ^2 are all infinite at this point, but the volume scale factor V vanishes here. Hence, the Universe starts from an initial singularity with an infinite rate of shear and expansion. The isotropy condition $\sigma^2/\Theta \to 0$ as $t \to \infty$ is also satisfied. We hence conclude from these observations that the model starts its expansion with a zero volume and continues to expand for $0 < n < 1$.

- The exponential model of the Universe corresponds to $n = 0$ with an average scale factor $A = k_2e^{\chi t}$. The physical singularity does not exist in this model, because the exponential function is asymptotic at the infinite past. The physical parameters H_i, H, $\triangle$, Θ, and σ^2 are all finite because the metric functions are nonzero for this model. The volume scale factor V increases exponentially with time, which indicates that the Universe starts its expansion with a zero volume from the infinite past. The expansion scalar Θ and the cosmological constant Λ are always positive. Therefore, the DE model describes an accelerating expansion of the Universe in this case.

- Both DE models are totally anisotropic and exhibit an accelerating expansion of the Universe. A particle horizon exists in both models. The cosmological solutions in both DE models are consistent with recent cosmological observations.

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