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AN INTEGRABLE HIERARCHY INCLUDING THE AKNS HIERARCHY AND ITS STRICT VERSION

G. F. Helminck*

We present an integrable hierarchy that includes both the AKNS hierarchy and its strict version. We split the loop space $\mathfrak{g}$ of gl_2 into Lie subalgebras $\mathfrak{g}_{\geq 0}$ and $\mathfrak{g}_{<0}$ of all loops with respectively only positive and only strictly negative powers of the loop parameter. We choose a commutative Lie subalgebra C in the whole loop space $\mathfrak{s}$ of sl_2 and represent it as $C = C_{\geq 0} \oplus C_{<0}$. We deform the Lie subalgebras $C_{\geq 0}$ and $C_{<0}$ by the respective groups corresponding to $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\geq 0}$. Further, we require that the evolution equations of the deformed generators of $C_{\geq 0}$ and $C_{<0}$ have a Lax form determined by the original splitting. We prove that this system of Lax equations is compatible and that the equations are equivalent to a set of zero-curvature relations for the projections of certain products of generators. We also define suitable loop modules and a set of equations in these modules, called the linearization of the system, from which the Lax equations of the hierarchy can be obtained. We give a useful characterization of special elements occurring in the linearization, the so-called wave matrices. We propose a way to construct a rather wide class of solutions of the combined AKNS hierarchy.

Keywords: AKNS equation, compatible Lax equations, AKNS hierarchy, strict version, zero curvature form, linearization, oscillating matrix, wave matrix, loop group, loop algebra

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Dedicated to the memory of Petr Petrovich Kulish

1. Introduction

We recall that integrable hierarchies can often be described as the evolution equations of deformed generators of a commutative subalgebra of a Lie algebra $\mathfrak{s}$, and many of them are named after the simplest nontrivial equation in the system. For the central hierarchy of this paper, these simplest nontrivial equations are the Ablowitz–Kaup–Newell–Segur (AKNS) equations, a system for two complex functions whose initial value problem was solved using the inverse scattering transform (see, e.g., [1]). It was shown in [2] that the AKNS equations are part of an integrable hierarchy. The AKNS hierarchy and its strict version are examples of hierarchies admitting a deformation description. We recall [3] that the relevant Lie algebra is the loop space $sl_2(R)[z, z^{-1})$ with at most a pole at infinity. Namely, it comprises elements of the form

$$X = \sum_{i=-\infty}^{N} X_i z^i, \quad X_i \in sl_2(R), \tag{1}$$

*Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands, e-mail: g.f.helminck@uva.nl.

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where R is a complex commutative algebra. We later specify R as a suitable algebra of complex functions depending on parameters. Let Q_0 be the matrix $\left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right)$. The commutative complex Lie subalgebra of $sl_2(R)[z, z^{-1})$ that is deformed at the AKNS hierarchy is

$$C_0 = \left\{ \sum_{m \geq 0}^{N} a_m Q_0 z^m \;\middle|\; a_m \in \mathbb{C} \right\}.$$

Because all powers of z are central, it suffices to look inside $sl_2(R)[z, z^{-1})$ at deformations Q of Q_0 of the form

$$Q = (\mathrm{Id} + Y_{<0}) Q_0 (\mathrm{Id} + Y_{<0})^{-1} = \sum_{j=0}^{\infty} Q_j z^{-j}, \quad Y_{<0} \in gl_2(R)[z, z^{-1})_{<0}, \tag{2}$$

where $gl_2(R)[z, z^{-1})_{<0} = \left\{ \sum_{i<0} Y_i z^i \;\middle|\; Y_i \in gl_2(R) \right\}$. It was shown in [3] that deformations of form (2) belong to $sl_2(R)[z, z^{-1})$. Further, let the algebra R be endowed with a set $\{\partial_m \mid m \geq 0\}$ of commuting $\mathbb{C}$-linear derivation operators $\partial_m \colon R \to R$, where each ∂_m is regarded as the derivation corresponding to the flow generated by $Q_0 z^m$. We assume that each derivation ∂_m acts coefficient-wise on the matrices in $gl_2(R)$, which defines a derivation of this algebra. The same holds for the extension to $gl_2(R)[z, z^{-1})$ defined by

$$\partial_m(X) := \sum_{j=-\infty}^{N} \partial_m(X_j) z^j.$$

We now seek deformations Q of form (2) such that the evolution with respect to $\{\partial_m\}$ satisfies the condition that for all $m \geq 0$,

$$\partial_m(Q) = [(Q z^m)_{\geq 0}, Q] = -[(Q z^m)_{<0}, Q], \tag{3}$$

where $X_{\geq 0}$ denotes the projection $\sum_{i=0}^{N} X_i z^i$ for any element $X = \sum_{i=-\infty}^{N} X_i z^i$ and $X_{<0}$ is the projection $\sum_{i=-\infty}^{-1} X_i z^i$. The second identity in (3) follows because all $\{Q z^m\}$ commute. Equations (3) are called the *Lax equations of the AKNS hierarchy*, and Q is called a *solution* of the AKNS hierarchy in the setting $(R, \{\partial_m \mid m \geq 0\})$. The trivial solution is $Q = Q_0$.

For the strict AKNS hierarchy, we consider deformations of the commutative complex Lie subalgebra

$$C_1 = \left\{ \sum_{m \geq 1}^{N} b_m Q_0 z^m \;\middle|\; b_m \in \mathbb{C} \right\}$$

but by a bigger group than $G_{<0} = \{\mathrm{Id} + Y \mid Y \in gl_2(R)[z, z^{-1})_{<0}\}$. As above, it suffices to consider deformations Z of $Q_0 z$ inside $sl_2(R)[z, z^{-1})$ of the form

$$\begin{gathered} Z = (Y_0 + Y_{<0}) Q_0 z (Y_0 + Y_0)^{-1} = \sum_{j=0}^{\infty} Z_j z^{1-j}, \\ Y_{<0} \in gl_2(R)[z, z^{-1})_{<0}, \qquad Y_0 \in gl_2(R)^*, \end{gathered} \tag{4}$$

where $gl_2(R)^*$ is the group of matrices in $gl_2(R)$ with an inverse in $gl_2(R)$. It was proved in [3] that deformations of form (4) also belong to $sl_2(R)[z, z^{-1})$. We again assume that the algebra R is endowed with a set $\{\partial_m \mid m \geq 1\}$ of commuting $\mathbb{C}$-linear derivation operators $\partial_m \colon R \to R$, where each ∂_m is regarded as the infinitesimal generator of the flow generated by $Q_0 z^m$, $m \geq 1$. We also assume that these derivations act analogously on elements of $gl_2(R)[z, z^{-1})$.

Among the deformations Z of form (4), we seek Z such that the evolution of Z with respect to the $\{\partial_m\}$ satisfies the condition that for all $m \geq 1$,

$$\partial_m(Z) = [(Zz^{m-1})_{>0}, Z] = -[(Zz^{m-1})_{\leq 0}, Z], \tag{5}$$

where the projections $X_{>0}$ and $X_{\leq 0}$ of any $X \in gl_2(R)[z, z^{-1})$ are defined analogously to the projections $X_{\geq 0}$ considered above. The second identity in (5) is a direct consequence of the commutativity of all $\{Zz^{m-1}\}$. Because Eqs. (5) correspond to a strict cutoff, they are called the *Lax equations of the strict AKNS hierarchy*, and the deformation Z is called a *solution* of this hierarchy. In this case, there is at least one solution, $Z = Q_0 z$. It is called the *trivial* solution of the hierarchy.

Our goal here is to discuss a natural merger of these two systems. Section 2 is devoted to describing it. We show that the combined Lax equations form a compatible set, i.e., the projections of products of generators that occur in the Lax equations satisfy a set of zero-curvature relations. Further, we prove there that these zero-curvature relations also suffice for obtaining the Lax equations for both generators. Section 3 is devoted to describing the linearization of the system and discussing its properties. In Sec. 4, we present a construction yielding a large collection of solutions of the combined AKNS hierarchy.

2. The combined AKNS hierarchy

The commutative Lie subalgebra C on which the combined hierarchy is based is a complex algebra with the basis $\{Q_0 z^m \mid m \in \mathbb{Z}\}$. It is a Lie subalgebra of both $sl_2(R)[z, z^{-1})$ and $sl_2(R)[z^{-1}, z)$, where the latter Lie algebra comprises loops with at most a pole around zero:

$$sl_2(R)[z^{-1}, z) = \left\{ \sum_{i=-N}^{\infty} X_i z^i \;\middle|\; X_i \in sl_2(R) \right\}.$$

The algebra C can be split as $C = C_{\geq 0} \oplus C_{<0}$, where $C_{\geq 0}$ is spanned by $\{Q_0 z^m \mid m \geq 0\}$ and $C_{<0}$, by $\{Q_0 z^m \mid m < 0\}$. We deform both $C_{\geq 0}$ and $C_{<0}$, the first inside $sl_2(R)[z, z^{-1})$ and the second inside $sl_2(R)[z^{-1}, z)$. This might lead to the deformations of $C_{\geq 0}$ and $C_{<0}$ no longer commuting. Because the powers of z are central, it suffices to consider the deformations of the elements Q_0 and $Q_0 z^{-1}$. We deform the element Q_0 as in the AKNS case with an element of the group

$$G_{<0} = \{\mathrm{Id} + Y_{<0} \mid Y_{<0} \in gl_2(R)[z, z^{-1})_{<0}\}$$

and obtain a deformation $Q = Q(z) = \sum_{j\geq 0}^{\infty} Q_j z^{-j}$ as in (2). In contrast, we deform the element $Q_0 z^{-1}$ with an element from the group

$$G_{\geq 0} = \{X = X_0 + X_{\geq 1} \mid X_0 \in gl_2(R)^*,\ X_{\geq 1} \in gl_2(R)[z^{-1}, z)_{>0}\}$$

and obtain

$$S = S(z) := X Q_0 z^{-1} X^{-1} = \sum_{j=0}^{\infty} S_j z^{j-1} \in sl_2(R)[z^{-1}, z). \tag{6}$$

If we substitute $z \to z^{-1}$ in $S(z)$, then we obtain a deformation $Z(z) = S(1/z)$ as in (4) for the strict AKNS hierarchy. Hence, deforming the basis of C, we obtain a basis in two parts $\{Qz^m \mid m \geq 0\}$ and $\{Sz^{m+1} \mid m < 0\}$ that are each commutative but do not necessarily commute with each other.

Next, we discuss the Lax equations that the pair (Q, S) should satisfy. For this, we assume that the algebra R has a collection $\{\partial_m \mid m \in \mathbb{Z}\}$ of commuting $\mathbb{C}$-linear derivation operators $\partial_m \colon R \to R$, where each ∂_m can be regarded as an algebraic substitute for the derivation corresponding to the flow generated by each $Q_0 z^m$, $m \in \mathbb{Z}$. For $X \in gl_2(R)[z, z^{-1})$ or $X \in gl_2(R)[z^{-1}, z)$, we define the action of each ∂_m by

$$\partial_m(X) := \sum_j \partial_m(X_j) z^j,$$

where the action on $gl_2(R)$ is defined coefficient-wise. This defines a derivation of both algebras. Following the terminology in [4] and [3], we call the data $(R, \{\partial_m \mid m \in \mathbb{Z}\})$ a *setting* for the *combined AKNS hierarchy*.

Example 1. Examples of settings are for the moment the algebras of complex polynomials $\mathbb{C}[t_m]$ in the variables $\{t_m \mid m \in \mathbb{Z}\}$ or the formal power series $\mathbb{C}[[t_m]]$ in the same variables; both algebras are equipped with the derivations $\partial_m = \partial/\partial t_m$, $m \in \mathbb{Z}$. More sophisticated choices for R appear later when we construct solutions.

We now require that the pair (Q, S) satisfy the evolution equations

$$\partial_m(Q) = [(Qz^m)_{\geq 0}, Q], \qquad \partial_m(S) = [(Qz^m)_{\geq 0}, S], \qquad m \geq 0, \tag{7}$$

$$\partial_m(S) = [(Sz^{m+1})_{<0}, S], \qquad \partial_m(Q) = [(Sz^{m+1})_{<0}, Q], \quad m < 0. \tag{8}$$

We note that the first set of equations in (7) means that Q satisfies the Lax equations of the AKNS hierarchy with respect to $\{\partial_m \mid m \geq 0\}$ and the first set of equations in (8) means that if S is translated back to $gl_2(R)[z, z^{-1})$ by $Z(z) = S(1/z)$, then Z is a solution of the strict AKNS hierarchy with respect to $\{\partial_m \mid m < 0\}$. We therefore call Eqs. (7) and (8) the *Lax equations of the combined AKNS hierarchy* and a pair (Q, S) satisfying these equations a *solution of the combined AKNS hierarchy*. We note that the pair $(Q_0, Q_0 z^{-1})$ solves this system because all elements of the basis of C in the unperturbed situation commute and are moreover constants for all the derivations $\{\partial_m\}$. We call it the *trivial* solution.

System of Lax equations (7), (8) is also compatible because we have the following proposition.

Proposition 1. *Let (Q, S) be a solution of the combined AKNS hierarchy. Then the projections $\{B_m := (Qz^m)_{\geq 0} \mid m \geq 0\}$ and $\{C_m := (Sz^{m+1})_{<0} \mid m < 0\}$ in the Lax equations of this hierarchy satisfy the zero-curvature relations*

$$\partial_{m_1}(B_{m_2}) - \partial_{m_2}(C_{m_1}) - [C_{m_1}, B_{m_2}] = 0, \quad m_1 < 0, \;\; m_2 \geq 0, \tag{9}$$

$$\partial_{m_1}(B_{m_2}) - \partial_{m_2}(B_{m_1}) - [B_{m_1}, B_{m_2}] = 0, \quad m_1 \geq 0, \;\; m_2 \geq 0, \tag{10}$$

$$\partial_{m_1}(C_{m_2}) - \partial_{m_2}(C_{m_1}) - [C_{m_1}, C_{m_2}] = 0, \quad m_1 < 0, \;\; m_2 < 0. \tag{11}$$

Proof. We prove only mixed relation (9); the proof of the other two relations is similar to the corresponding proof presented in [3]. The main idea of the proof is to show that the left-hand side of the equation in (9) belongs to both $sl_2(R)[z, z^{-1})_{\geq 0}$ and $sl_2(R)[z, z^{-1})_{<0}$ and must therefore be equal to zero.

Because all powers of z are central and the second set of equations in (8) holds for Q, the equality

$$\partial_{m_1}(Qz^{m_2}) = [(Sz^{m_1+1})_{<0}, Qz^{m_2}] = [C_{m_1}, Qz^{m_2}]$$

is satisfied for any $m_2 \geq 0$ and any $m_1 < 0$, whence using the substitution $B_{m_2} = Qz^{m_2} - (Qz^{m_2})_{<0}$, we obtain

$$\partial_{m_1}(B_{m_2}) - [C_{m_1}, B_{m_2}] = -\partial_{m_1}((Qz^{m_2})_{<0}) + [C_{m_1}, (Qz^{m_2})_{<0}],$$

whose right-hand side obviously belongs to $sl_2(R)[z,z^{-1})_{<0}$. Because $\partial_{m_2}(C_{m_1})$ also belongs to this Lie subalgebra, the whole left-hand side of (9) belongs to $sl_2(R)[z,z^{-1})_{<0}$.

To obtain the other inclusion, we use the second set of Lax equations in (7) for S. For the same reason as above, we then find that for any $m_1 < 0$ and any $m_2 \geq 0$, we have

$$\partial_{m_2}(Sz^{m_1+1}) = [(Qz^{m_2})_{\geq 0}, Sz^{m_1+1}] = [B_{m_2}, Sz^{m_1+1}].$$

We combine this expression with the substitution $C_{m_1} = Sz^{m_1+1} - (Sz^{m_1+1})_{\geq 0}$ and obtain

$$-\partial_{m_2}(C_{m_1}) - [C_{m_1}, B_{m_2}] = \partial_{m_2}((Sz^{m_1+1})_{\geq 0}) + [(Sz^{m_1+1})_{\geq 0}, B_{m_2}].$$

The right-hand side of the equation obviously belongs to the Lie subalgebra $sl_2(R)[z,z^{-1})_{\geq 0}$. The same holds for the term $\partial_{m_1}(B_{m_2})$, which proves the second inclusion. ■

The converse statement also holds.

Proposition 2. *Let a deformation Q of type* (2) *and a deformation S of form* (6) *be given. We assume that the projections $\{B_m := (Qz^m)_{\geq 0} \mid m \geq 0\}$ and the projections $\{C_m := (Sz^{m+1})_{<0} \mid m < 0\}$ satisfy zero-curvature relations* (9)–(11)*. Then the pair (Q, S) is a solution of the combined AKNS hierarchy.*

Proof. Following [3], we can prove that the first set of Lax equations in (7) follows from zero-curvature relations (10). Also, the first set of Lax equations in (8) follows from zero-curvature relations (11). We suppose that the first of the remaining Lax equations for Q does not hold. Then there exists $\ell_1 < 0$ such that

$$\partial_{\ell_1}(Q) - [(Sz^{\ell_1+1})_{<0}, Q] = \partial_{\ell_1}(Q) - [C_{\ell_1}, Q] = \sum_{k \leq k_2} A_k z^k, \quad A_{k_2} \neq 0.$$

It hence follows that for any $\ell \geq 0$,

$$\partial_{\ell_1}(Qz^\ell) - [C_{\ell_1}, Qz^\ell] = \sum_{k \leq k_2} A_k z^{k+\ell}. \tag{12}$$

We let ℓ tend to infinity. Then nonzero terms with arbitrarily high powers of z appear in expression (12). On the other hand, we can split the expression and substitute identity (9) with $m_1 = \ell_1$ and $m_2 = \ell$, which yields

$$\begin{aligned}\partial_{\ell_1}(Qz^\ell) - [C_{\ell_1}, Qz^\ell] &= \partial_{\ell_1}(B_\ell) - [C_{\ell_1}, B_\ell] + \partial_{\ell_1}((Qz^\ell)_{<0}) - [C_{\ell_1}, (Qz^\ell)_{<0}] = \\ &= \partial_\ell(C_{\ell_1}) + \partial_{\ell_1}(Qz^\ell_{<0}) - [C_{\ell_1}, Qz^\ell_{<0}],\end{aligned}$$

and this last expression has only negative powers of z. This contradicts the unlimited growth of these powers. Therefore, all the Lax equations for Q must be satisfied. We suppose that one of the remaining Lax equations for S is violated. Let there be an $s_1 \geq 0$ such that

$$\partial_{s_1}(S) - [(Qz^{s_1})_{\geq 0}, S] = \partial_{s_1}(S) - [B_{s_1}, S] = \sum_{k \geq k_1} D_k z^k, \quad D_{k_1} \neq 0.$$

Similarly, we find that for any $s < 0$,

$$\partial_{s_1}(Sz^{s+1}) - [B_{s_1}, Sz^{s+1}] = \sum_{k \geq k_1} D_k z^{k+s+1}, \tag{13}$$

and we see that in the limit $s \to -\infty$, there is no lower bound for the powers of z in expression (13) with nonzero coefficients. We split this expression and substitute relation (9) with $m_1 = s$ and $m_2 = s_1$. We obtain

$$\partial_{s_1}(Sz^{s+1}) - [B_{s_1}, Sz^{s+1}] = \partial_{s_1}(C_s) - [B_{s_1}, C_s] + \partial_{s_1}((Sz^{s+1})_{\geq 0}) - [B_{s_1}, (Sz^{s+1})_{\geq 0}] =$$
$$= \partial_s(B_{s_1}) + \partial_{s_1}((Sz^{s+1})_{\geq 0}) - [B_{s_1}, (Sz^{s+1})_{\geq 0}],$$

where the last expression contains only positive powers of z. We thus again obtain a contradiction, and all the Lax equations for S must therefore be satisfied. ■

3. The linearization of the combined AKNS hierarchy

The zero-curvature form of the combined AKNS hierarchy suggest the possible existence of a linear system for which the zero-curvature equations form the compatibility conditions. In [3], we proposed linearizations for both the AKNS hierarchy and its strict version. We adapt these linearizations taking the presence of the additional variables into account.

Let a pair (Q, S) be a potential solution of the combined AKNS hierarchy, i.e., Q be an element of $sl_2(R)[z, z^{-1})$ of form (2) and S be an element of $sl_2(R)[z^{-1}, z)$ of form (6). As in the preceding section, we associate projections $\{B_m := (Qz^m)_{\geq 0} \mid m \geq 0\}$ and $\{C_m := (Sz^{m+1})_{<0} \mid m < 0\}$ with each such pair. Then the *linearization of the combined AKNS hierarchy* is the system

$$Q\psi = \psi Q_0, \qquad \begin{array}{ll} \partial_m(\psi) = B_m\psi & \text{for } m \geq 0, \\ \partial_m(\psi) = C_m\psi & \text{for } m < 0, \end{array} \tag{14}$$

$$S\varphi = \varphi Q_0 z^{-1}, \qquad \begin{array}{ll} \partial_m(\varphi) = C_m\varphi & \text{for } m < 0, \\ \partial_m(\varphi) = B_m\varphi & \text{for } m \geq 0. \end{array} \tag{15}$$

Without specifying ψ and φ, we first show what is needed to pass from (14) and (15) to the Lax equations for Q and S. We describe all the manipulations only for Q; the procedure is similar for S. We first act with ∂_m, $m \geq 0$, on the first equation in (14) and use the first two equations:

$$\partial_m(Q\psi - \psi Q_0) = \partial_m(Q)\psi + Q\partial_m(\psi) - \partial_m(\psi)Q_0 = 0 =$$
$$= \partial_m(Q)\psi + QB_m\psi - B_m\psi Q_0 = \{\partial_m(Q) - [B_m, Q]\}\psi = 0. \tag{16}$$

We do the same with ∂_m, $m < 0$, and using the first and third equations in (14), we obtain

$$\partial_m(Q\psi - \psi Q_0) = \partial_m(Q)\psi + Q\,\partial_m(\psi) - \partial_m(\psi)Q_0 = 0 =$$
$$= \partial_m(Q)\psi + QC_m\psi - C_m\psi Q_0 = \{\partial_m(Q) - [C_m, Q]\}\psi = 0. \tag{17}$$

If we can eliminate ψ both from (16) and from (17), then we obtain the required Lax equations for Q. Hence, we first need a left action of elements such as Q, B_m, and C_m to be defined. Next, there should be a right action of Q_0 and an appropriate left action of all the ∂_m, $m \in \mathbb{Z}$, that satisfies the Leibniz rule with respect to the action of the elements from $sl_2(R)[z, z^{-1})$. Finally, it must be possible to eliminate ψ from the equations. This can all be realized using a choice of a suitable ψ in an appropriate $gl_2(R)[z, z^{-1})$-module. Similarly, we can derive the Lax equations for S from (15) if φ is a suitable vector in a certain $gl_2(R)[z^{-1}, z)$-module.

To understand how to determine the abovementioned modules, we first consider the linearization for the trivial solutions $Q = Q_0$ and $S = Q_0 z^{-1}$. In this case, the projections are $B_m = Q_0 z^m$, $m \geq 0$, and $C_m = Q_0 z^m$, $m < 0$, and the linearization equations are

$$Q_0\psi_0 = \psi_0 Q_0, \qquad \begin{aligned} \partial_m(\psi_0) &= Q_0 z^m \psi_0 \quad \text{for } m \geq 0, \\ \partial_m(\psi_0) &= Q_0 z^m \psi_0 \quad \text{for } m < 0, \end{aligned} \tag{18}$$

$$Q_0 z^{-1}\varphi_0 = \varphi_0 Q_0 z^{-1}, \qquad \begin{aligned} \partial_m(\varphi) &= Q_0 z^m \varphi \quad \text{for } m < 0, \\ \partial_m(\varphi) &= Q_0 z^m \varphi \quad \text{for } m \geq 0. \end{aligned} \tag{19}$$

Assuming that each derivation ∂_m is equal to $\partial/\partial t_m$ and using t as a brief notation for all parameters $\{t_m \mid m \in \mathbb{Z}\}$, we obtain the solution (ψ_0, φ_0) for (18) and (19)

$$\psi_0 = \psi_0(t,z) = \exp\left(\sum_{m\in\mathbb{Z}} t_m Q_0 z^m\right) = \varphi_0(t,z) = \varphi_0.$$

In the general case, the functions ψ should be $gl_2(R)[z,z^{-1})$-perturbations of ψ_0, i.e., they should belong to

$$\mathcal{M}_{\geq 0} = \left\{\{g(z)\}\psi_0 = \left\{\sum_{i=-\infty}^{N} g_i z^i\right\}\psi_0 \;\Big|\; g(z) = \sum_{i=-\infty}^{N} g_i z^i \in gl_2(R)[z,z^{-1})\right\}, \tag{20}$$

and φ should be $gl_2(R)[z^{-1},z)$-perturbations of φ_0, i.e., they should belong to

$$\mathcal{M}_{<0} = \left\{\{h(z)\}\varphi_0 = \left\{\sum_{i=-N}^{\infty} h_i z^i\right\}\varphi_0 \;\Big|\; h(z) = \sum_{i=-N}^{\infty} h_i z^i \in gl_2(R)[z^{-1},z)\right\}, \tag{21}$$

where the products $\{g(z)\}\psi_0$ and $\{h(z)\}\varphi_0$ are understood formally and both factors must be separated to avoid convergence issues. We can define the required actions on both $\mathcal{M}_{\geq 0}$ and $\mathcal{M}_{<0}$: for $k_1(z) \in gl_2(R)[z,z^{-1})$ and $k_2(z) \in gl_2(R)[z^{-1},z)$, we respectively set

$$k_1(z)\cdot\{g(z)\}\psi_0 := \{k_1(z)g(z)\}\psi_0 \qquad \text{and} \qquad k_2(z)\cdot\{h(z)\}\varphi_0 := \{k_2(z)h(z)\}\varphi_0.$$

We define the right action of Q_0 on $\mathcal{M}_{\geq 0}$ and of $Q_0 z^{-1}$ on $\mathcal{M}_{<0}$ as

$$\{g(z)\}\psi_0 Q_0 := \{g(z)Q_0\}\psi_0 \qquad \text{and} \qquad \{h(z)\}\varphi_0 Q_0 z^{-1} := \{h(z)Q_0 z^{-1}\}\varphi_0,$$

and the action of each ∂_m as

$$\partial_m(\{g(z)\}\psi_0) = \left\{\sum_{i=-\infty}^{N} \partial_m(g_i) z^i + \left\{\sum_{i=-\infty}^{N} g_i Q_0 z^{i+m}\right\}\right\}\psi_0,$$

$$\partial_m(\{h(z)\}\varphi_0) = \left\{\sum_{i=-N}^{\infty} \partial_m(h_i) z^i + \left\{\sum_{i=-N}^{\infty} h_i Q_0 z^{i+m}\right\}\right\}\varphi_0.$$

Following the terminology used in the scalar case (see [5]), we call the elements of $\mathcal{M}_{\geq 0}$ *oscillating matrices at infinity* and those of $\mathcal{M}_{<0}$ *oscillating matrices at zero*. We note that $\mathcal{M}_{\geq 0}$ is a free $gl_2(R)[z,z^{-1})$-module and $\mathcal{M}_{<0}$ is a free $gl_2(R)[z^{-1},z)$-module with the respective generators ψ_0 and φ_0 because for each $k_1(z) \in gl_2(R)[z,z^{-1})$ and $k_1(z) \in gl_2(R)[z^{-1},z)$, we have

$$k_1(z).\psi_0 = k_1(z)\cdot\{1\}\psi_0 = \{k_1(z)\}\psi_0 \qquad \text{and} \qquad k_2(z)\cdot\varphi_0 = k_2(z).\{1\}\psi_0 = \{k_2(z)\}\varphi_0.$$

Hence, to legitimately eliminate both $\psi = \{k_1(z)\}\psi_0$ and $\varphi = \{k_2(z)\}\varphi_0$, it suffices to find oscillating matrices such that $k_1(z)$ is invertible in $gl_2(R)[z, z^{-1})$ and $k_2(z)$ is invertible in $gl_2(R)[z^{-1}, z)$. We define a set of such elements, which we need for constructing solutions of the hierarchy.

Let a matrix $\delta(m) \in gl_2(R)[z, z^{-1}) \cap gl_2(R)[z^{-1}, z)$ for $m = (m_1, m_2) \in \mathbb{Z}^2$ have the form

$$\delta(m) = \begin{pmatrix} z^{m_1} & 0 \\ 0 & z^{m_2} \end{pmatrix}.$$

Then $\delta(m)$ has $\delta(-m) \in gl_2(R)[z, z^{-1}) \cap gl_2(R)[z^{-1}, z)$ as its inverse, and the collection $\Delta = \{\delta(m) \mid m \in \mathbb{Z}^2\}$ forms a group. An element $\psi \in \mathcal{M}_{\geq 0}$ is called an *oscillating matrix at infinity of type* $\delta(m)$ if it has the form

$$\psi = \{k_1(z)\delta(m)\}\psi_0, \quad k_1(z) \in G_{<0}, \tag{22}$$

and is an example of a generator of $\mathcal{M}_{\geq 0}$. Similarly, an element $\varphi \in \mathcal{M}_{<0}$ is called an *oscillating matrix at zero of type* $\delta(m)$ if it has the form

$$\varphi = \{k_2(z)\delta(m)\}\varphi_0, \quad k_2(z) \in G_{\geq 0}, \tag{23}$$

and such a φ generates $\mathcal{M}_{<0}$. Hence, eliminating ψ and φ is possible for any pair $(\psi, \varphi) \in \mathcal{M}_{\geq 0} \times \mathcal{M}_{<0}$ with ψ of form (22) and φ of form (23).

We now assume that (Q, S) is a potential solution of the combined AKNS hierarchy and that (ψ, φ) is a pair in $\mathcal{M}_{\geq 0} \times \mathcal{M}_{<0}$ with ψ of form (22) and φ of form (23) for which the linearization equations are (14) and (15). Then all actions needed for obtaining Lax equations (7) and (8) are meaningful. Hence, (Q, S) is a solution of the combined AKNS hierarchy, and we call the pair (ψ, φ) a set of *wave matrices of the combined AKNS hierarchy of type* $\delta(m)$. In particular, the pair (ψ, φ) totally determines the solution (Q, S) because the first equations in (14) and (15) respectively imply

$$\begin{aligned} &Qk_1(z)\delta(m) = k_1(z)Q_0\delta(m) \quad \Rightarrow \quad Q = k_1(z)Q_0k_1(z)^{-1}, \\ &Sk_2(z)\delta(m) = k_2(z)Q_0z^{-1}\delta(m) \quad \Rightarrow \quad S = k_2(z)Q_0z^{-1}k_2(z)^{-1}. \end{aligned}$$

A weaker condition for pairs of oscillating matrices of a certain type to be a set of wave matrices of the combined AKNS hierarchy is expressed in the following proposition.

Proposition 3. *Let $\psi = \{k_1(z)\delta(m)\}\psi_0$ be an oscillating matrix of type $\delta(m)$ in $\mathcal{M}_{\geq 0}$ and $\varphi = \{k_2(z)\delta(m)\}\varphi_0$ be such a matrix in $\mathcal{M}_{<0}$. Let*

$$(Q, S) := (k_1(z)Q_0k_1(z)^{-1}, k_2(z)Q_0z^{-1}k_2(z)^{-1})$$

denote the corresponding potential solution of the combined AKNS hierarchy. If an element $M_m \in gl_2(R)[z, z^{-1})_{\geq 0}$ exists for each $m \geq 0$ such that

$$\partial_m(\psi) = M_m\psi \qquad \text{and} \qquad \partial_m(\varphi) = M_m\varphi$$

and if an element $N_m \in gl_2(R)[z^{-1}, z)_{<0}$ exists for each $m < 0$ such that

$$\partial_m(\psi) = N_m\psi \qquad \text{and} \qquad \partial_m(\varphi) = N_m\varphi,$$

then each M_m is equal to $(Qz^m)_{\geq 0}$, each N_m is equal to $(Sz^{m+1})_{<0}$, and the pair (ψ, φ) is a set of wave matrices for the combined AKNS hierarchy of type $\delta(m)$.

Proof. For $m \geq 0$, we use the fact that $\mathcal{M}_{\geq 0}$ is a free module with the generator ψ_0. This property allows translating the equation $\partial_m(\psi) = M_m\psi$ into an equation in $gl_2(R)[z, z^{-1}]$,

$$\partial_m(k_1(z)) + k_1(z)Q_0z^m = M_mk_1(z) \quad \Rightarrow \quad \partial_m(k(z))k_1(z)^{-1} + Qz^m = M_m,$$

and projecting this onto $gl_2(R)[z, z^{-1}]_{\geq 0}$, we obtain the formula $M_m = (Qz^m)_{\geq 0}$. For $m < 0$, we use the fact that $\mathcal{M}_{<0}$ is a free module with the generator φ_0. We translate the equation $\partial_m(\varphi) = N_m\varphi$ into an equation in $gl_2(R)[z^{-1}, z)$:

$$\partial_m(k_2(z)) + k_2(z)Q_0z^m = N_mk_2(z) \quad \Rightarrow \quad \partial_m(k_2(z))k_2(z)^{-1} + Sz^{m+1} = N_m.$$

Projecting the right-hand side on $gl_2(R)[z^{-1}, z)_{<0}$, we obtain the sought identity $(Sz^{m+1})_{<0} = N_m$. ■

It can happen that different pairs of wave matrices of type $\delta(m)$ yield the same solution of the combined AKNS hierarchy. To see how this can occur, we consider a solution (Q, S) that corresponds to both sets of wave matrices of type δ_m, (ψ_1, φ_1) and (ψ_2, φ_2), i.e.,

$$\begin{aligned} \psi_i &= u_i\delta(m)\psi_0 \qquad \text{and} \qquad && \phi_i = p_i\delta(m)\varphi_0, \quad && i = 1, 2, \\ Q &= u_iQ_0u_i^{-1}, && S = p_iQ_0z^{-1}p_i^{-1}, && i = 1, 2, \end{aligned}$$

$\partial_j(\psi_i) = B_j\psi_i$ and $\partial_j(\phi_i) = B_j\phi_i$ for $j \geq 0$, and $\partial_j(\psi_i) = C_j\psi_i$ and $\partial_j(\phi_i) = C_j\phi_i$ for $j < 0$, where $B_j = (Qz^j)_{\geq 0}$ and $C_j = (Sz^{j+1})_{<0}$. First, we can see that the element $\psi_1^{-1}\psi_2 = (\psi_0)^{-1}\delta(-m)u_1^{-1}u_2\delta(m)\psi_0$ commutes with Q_0 and the same then holds for $u_1^{-1}u_2$. Therefore, we obtain

$$u_1^{-1}u_2 = \mathrm{Id} + \sum_{i<0} d_iz^i = \psi_1^{-1}\psi_2, \qquad d_i = \begin{pmatrix} a(i) & 0 \\ 0 & d(i) \end{pmatrix}.$$

Similarly, the element $\varphi_1^{-1}\varphi_2 = (\varphi_0)^{-1}\delta(-m)p_1^{-1}p_2\delta(m)\varphi_0$ commutes with Q_0z^{-1} and hence also with Q_0. Consequently, the same holds for the element $p_1^{-1}p_2$, and we therefore obtain

$$p_1^{-1}p_2 = \sum_{i\geq 0} d_iz^i = \varphi_1^{-1}\varphi_2, \qquad d_i = \begin{pmatrix} a(i) & 0 \\ 0 & d(i) \end{pmatrix}, \quad a(0)d(0) \in R^*.$$

Hence, the oscillating matrices are related by

$$\psi_2 = \psi_1\Big(\mathrm{Id} + \sum_{i<0} d_iz^i\Big) \qquad \text{and} \qquad \varphi_2 = \varphi_1\Big(\sum_{i\geq 0} d_iz^i\Big). \tag{24}$$

Regarding the t-dependence of the factors $\mathrm{Id} + \sum_{i<0} d_iz^i$ and $\sum_{i\geq 0} d_iz^i$, we apply ∂_j for each $j \geq 0$ to both equations in (24) and obtain

$$\begin{aligned} \partial_j(\psi_2) &= \partial_j\Big(\psi_1\Big(\mathrm{Id} + \sum_{i<0} d_iz^i\Big)\Big) = B_j\psi_1\Big(\mathrm{Id} + \sum_{i<0} d_iz^i\Big) + \psi_1\Big(\sum_{i<0} \partial_j(d_i)z^i\Big) = B_j\psi_2, \\ \partial_j(\varphi_2) &= \partial_j\Big(\varphi_1\Big(\sum_{i\geq 0} d_iz^i\Big)\Big) = B_j\varphi_1\Big(\sum_{i\geq 0} d_iz^i\Big) + \varphi_1\Big(\sum_{i\geq 0} \partial_j(d_i)z^i\Big) = B_j\varphi_2, \end{aligned}$$

whence it follows that

$$\psi_1\Big(\sum_{i<0}\partial_j(d_i)z^i\Big)=u_1\Big(\sum_{i<0}\partial_j(d_i)z^i\Big)\delta(m)\gamma=0,$$

$$\varphi_1\Big(\sum_{i\geq 0}\partial_j(d_i)z^i\Big)=p_1\Big(\sum_{i\geq 0}\partial_j(d_i)z^i\Big)\delta(m)\gamma=0,$$

i.e., all $d(i)$ are constant for $\{\partial_j \mid j\geq 0\}$. The same can also be obtained for $\{\partial_j \mid j<0\}$: we replace B_j with C_j in the equations obtained above. Taking the above into account, we obtain the following statement.

Corollary 1. *If (ψ_1,φ_1) and (ψ_2,φ_2) are two pairs of wave matrices of type $\delta(m)$ for the combined AKNS hierarchy that lead to the same solution (Q,S) of this hierarchy, then we have*

$$\psi_2=\psi_1\Big(\mathrm{Id}+\sum_{i<0}d_iz^i\Big)\qquad\text{and}\qquad \varphi_2=\varphi_1\Big(\sum_{i\geq 0}d_iz^i\Big),$$

where all $d(i)$ are diagonal matrices in $sl_2(R)$ that are constant for all $\{\partial_j \mid j\in\mathbb{Z}\}$, i.e., $\partial_j(d_i)=0$.

This statement concludes the presentation of the algebraic framework of the linearization of the combined AKNS hierarchy. In the next section, we present an analytic context from which we can construct sets of wave matrices of this hierarchy in which the products are not formal but real.

4. A construction of solutions of the hierarchy

In this section, we show how to construct a wide class of solutions of the combined AKNS hierarchy. For this, we follow the technique in [6]. We first define the loop group in which we work. For each $0<r<1$, let A_r be the annulus

$$\Big\{z \;\Big|\; z\in\mathbb{C},\ r\leq |z|\leq \frac{1}{r}\Big\}.$$

Following [7], we let $L_{\mathrm{an}}GL_2(\mathbb{C})$ denote the collection of holomorphic maps from some annulus A_r into $GL_2(\mathbb{C})$. It is a group under pointwise multiplication and naturally contains the subgroup $GL_2(\mathbb{C})$ as the collection of constant maps into $GL_2(\mathbb{C})$. Other examples of elements in $L_{\mathrm{an}}GL_2(\mathbb{C})$ are the elements of Δ. But $L_{\mathrm{an}}GL_2(\mathbb{C})$ is more than just a group: it is an infinite-dimensional Lie group. Its manifold structure is determined by its Lie algebra $L_{\mathrm{an}}gl_2(\mathbb{C})$, comprising all holomorphic maps $\gamma\colon U\to gl_2(\mathbb{C})$, where U is an open neighborhood of some annulus A_r, $0<r<1$. Because $gl_2(\mathbb{C})$ is a Lie algebra, the space $L_{\mathrm{an}}gl_2(\mathbb{C})$ becomes a Lie algebra under the pointwise commutator. Topologically, the space $L_{\mathrm{an}}gl_2(\mathbb{C})$ is the direct limit of all the spaces $L_{\mathrm{an},r}gl_2(\mathbb{C})$, where this last space consists of all γ corresponding to the fixed annulus A_r. We endow each $L_{\mathrm{an},r}gl_2(\mathbb{C})$ with the topology of uniform convergence, and with that topology, it becomes a Banach space. Thus, $L_{\mathrm{an}}gl_2(\mathbb{C})$ becomes a Fréchet space. The point-wise exponential map defines a local diffeomorphism around zero in $L_{\mathrm{an}}gl_2(\mathbb{C})$ (see, e.g., [8]).

Each $\gamma\in L_{\mathrm{an}}gl_2(\mathbb{C})$ can be expanded in a Fourier series

$$\gamma=\sum_{k=-\infty}^{\infty}\gamma_kz^k,\quad \gamma_k\in gl_2(\mathbb{C}),\tag{25}$$

which converges absolutely on the annulus where it is defined,

$$\sum_{k=-\infty}^{\infty}\|\gamma_k\|r^{-|k|}<\infty.$$

We use the Fourier expansion to obtain the corresponding decomposition of the Lie algebra $L_{\mathrm{an}}gl_2(\mathbb{C})$. Namely, we consider the subspaces

$$L_{\mathrm{an}}gl_2(\mathbb{C})_{\geq 0} := \left\{\gamma \,\Big|\, \gamma \in L_{\mathrm{an}}gl_2(\mathbb{C}),\ \gamma = \sum_{k=0}^{\infty} \gamma_k z^k \right\},$$

$$L_{\mathrm{an}}gl_2(\mathbb{C})_{<0} := \left\{\gamma \,\Big|\, \gamma \in L_{\mathrm{an}}gl_2(\mathbb{C}),\ \gamma = \sum_{k=-\infty}^{-1} \gamma_k z^k \right\}.$$

Both are Lie subalgebras of $L_{\mathrm{an}}gl_2(\mathbb{C})$, and their direct sum coincides with the whole Lie algebra. The first Lie algebra comprises the elements in $L_{\mathrm{an}}gl_2(\mathbb{C})$ that extend to holomorphic maps defined on a disk $\{z \in \mathbb{C} \mid |z| \leq 1/r\}$, $0 < r < 1$. The second Lie algebra corresponds to the maps in $L_{\mathrm{an}}gl_2(\mathbb{C})$ that have a holomorphic extension to a disk around infinity $\{z \in \mathbb{P}^1(\mathbb{C}) \mid |z| \geq r\}$, $0 < r < 1$, and are zero at infinity. A subgroup of $L_{\mathrm{an}}GL_2(\mathbb{C})$ belongs to each of these two Lie subalgebras. The pointwise exponential map applied to elements of $L_{\mathrm{an}}gl_2(\mathbb{C})_{<0}$ yields elements of

$$U_- = \left\{\gamma \,\Big|\, \gamma \in L_{\mathrm{an}}gl_2(\mathbb{C}),\ \gamma = \mathrm{Id} + \sum_{k=-\infty}^{-1} \gamma_k z^k \right\},$$

and the exponential map applied to elements of $L_{\mathrm{an}}gl_2(\mathbb{C})_{\geq 0}$ maps them into

$$P_+ = \left\{\gamma \,\Big|\, \gamma \in L_{\mathrm{an}}gl_2(\mathbb{C}),\ \gamma = \gamma_0 + \sum_{k=1}^{\infty} \gamma_k z^k,\ \text{where } \gamma_0 \in GL_2(\mathbb{C}) \right\}.$$

It is easy to verify that U_- and P_+ are subgroups of $L_{\mathrm{an}}GL_2(\mathbb{C})$, and because the direct sum of their Lie algebras is $L_{\mathrm{an}}gl_2(\mathbb{C})$, their product

$$\Omega = U_- P_+ \tag{26}$$

is open in $L_{\mathrm{an}}GL_2(\mathbb{C})$ and, as in the finite-dimensional case, is called the *big cell* with respect to U_- and P_+.

The next subgroup of $L_{\mathrm{an}}SL_2(\mathbb{C})$ corresponds to the exponential factor in the linearization of the combined AKNS hierarchy. The commuting group relevant for this hierarchy is

$$\Gamma = \left\{\gamma(t) = \exp\left(\sum_{i \in \mathbb{Z}} t_i Q_0 z^i\right) \,\Big|\, \gamma \in L_{\mathrm{an}}SL_2(\mathbb{C}) \right\}.$$

The group Δ commutes with Γ and contains two subgroups of interest: the subgroup

$$\Delta_c = \{\delta^k \mid \delta = \delta((1,1)),\ k \in \mathbb{Z}\},$$

which is central in $L_{\mathrm{an}}GL_2(\mathbb{C})$, and the subgroup

$$\Delta_1 = \Delta \cap L_{\mathrm{an}}SL_2(\mathbb{C}) = \{\delta_1^k \mid \delta_1 = \delta((1,-1)),\, k \in \mathbb{Z}\}.$$

According to the Birkhoff theorem (see [7]), the product $\Delta_1\Gamma$ forms the centralizer of Γ in $L_{\mathrm{an}}SL_2(\mathbb{C})$.

We now have all ingredients for describing the construction of solutions of the combined AKNS hierarchy. In the product $L_{\mathrm{an}}GL_2(\mathbb{C}) \times \Delta$, we take a collection $\mathcal{S}$ of pairs $(g, \delta(m))$ such that there exists $\gamma(t)$, $\gamma \in \Gamma$, satisfying

$$\delta(m)\gamma(t)g\gamma(t)^{-1}\delta(-m) \in \Omega = U_- P_+. \tag{27}$$

For each pair $(g, \delta(m))$ in $\mathcal{S}$, we find the set $\Gamma(g, \delta(m))$ of all $\gamma(t)$ satisfying condition (27). It is a nonempty open subset of Γ. Let $R(g, \delta(m))$ be the algebra of analytic functions $\Gamma(g, \delta(m)) \to \mathbb{C}$. This is the algebra of functions R that we associate with the point $(g, \delta(m)) \in \mathcal{S}$. As the commuting derivations of $R(g, \delta(m))$, we choose

$$\partial_i := \frac{\partial}{\partial t_i}, \quad i \in \mathbb{Z}.$$

By property (27), we find that for all $\gamma(t) \in \Gamma(g, \delta(m))$,

$$\delta(m)\gamma(t)g\gamma(t)^{-1}\delta(-m) = u_-(g, \delta(m))(t)^{-1}p_+(g, \delta(m))(t), \tag{28}$$

where $u_-(g, \delta(m))(t) \in U_-$ and $p_+(g, \delta(m))(t) \in P_+$. All the matrix coefficients in the Fourier expansions of the elements $u_-(g, \delta(m))$ and $p_+(g, \delta(m))$ then belong to the algebra $R(g, \delta(m))$. From (28), we can obtain two oscillating matrices of type $\delta(m)$, one $\Psi_{g,\delta(m)} \in \mathcal{M}_{\geq 0}$ and the other $\Phi_{g,\delta(m)} \in \mathcal{M}_{<0}$. Namely, we set

$$\Psi_{g,\delta(m)}(t) := u_-(g, \delta(m))(t)\delta(m)\gamma(t), \tag{29}$$

$$\Phi_{g,\delta(m)}(t) := p_+(g, \delta(m))(t)\delta(m)\gamma(t). \tag{30}$$

We note that all the products between the different factors are well defined. By virtue of relation (28), these two oscillating matrices of type $\delta(m)$ are related by

$$\Psi_{g,\delta(m)}(t) = \Phi_{g,\delta(m)}(t)g^{-1}. \tag{31}$$

It follows directly from relation (28) that if $(g, \delta(m)) \in \mathcal{S}$, then also $(g, \delta(m)\delta^k) \in \mathcal{S}$ for any $k \in \mathbb{Z}$, and the sets of oscillating matrices are related by

$$\Psi_{g,\delta(m)\delta^k} = \Psi_{g,\delta(m)}\delta^k, \qquad \Phi_{g,\delta(m)\delta^k} = \Phi_{g,\delta(m)}\delta^k.$$

Further, using Proposition 3, we show that each pair $(\Psi_{g,\delta(m)}, \Phi_{g,\delta(m)})$ is a set of wave matrices of the combined AKNS hierarchy. For this, we compute $\partial_j(\Psi_{g,\delta(m)})$, $j \geq 0$, in two different ways, using first (29) and then (30) and (31). On one hand, we obtain

$$\begin{aligned}\partial_j(\Psi_{g,\delta(m)}) &= \{\partial_j(u_-(g, \delta(m))) + u_-(g, \delta(m))Q_0 z^j\}\delta(m)\gamma = \\ &= \{\partial_j(u_-(g, \delta(m)))u_-(g, \delta(m))^{-1} + u_-(g, \delta(m))Q_0 z^j u_-(g, \delta(m))^{-1}\}\Psi_{g,\delta(m)}\end{aligned}$$

and, on the other hand,

$$\begin{aligned}\partial_j(\Psi_{g,\delta(m)}) &= \{\partial_j(p_+(g, \delta(m))) + p_+(g, \delta(m))Q_0 z^j\}\delta(m)\gamma g^{-1} = \\ &= \{\partial_j(p_+(g, \delta(m)))p_+(g, \delta(m))^{-1} + p_+(g, \delta(m))Q_0 z^j p_+(g, \delta(m))^{-1}\}\Psi_{g,\delta(m)}.\end{aligned}$$

Comparing the coefficients of $\Phi_{g,\delta(m)}$ in these expressions, we can see that

$$M_j := \partial_m(u_-(g, \delta(m)))u_-(g, \delta(m))^{-1} + u_-(g, \delta(m))Q_0 z^j u_-(g, \delta(m))^{-1}$$

belongs to $gl_2(R)[z, z^{-1})_{\geq 0}$. Because of relation (31), for $\Phi_{g,\delta(m)}$, we have the equality

$$\partial_j(\Phi_{g,\delta(m)}) = M_j\Phi_{g,\delta(m)}$$

for all $j \geq 0$. We similarly compute $\partial_j(\Phi_{g,\delta(m)})$ for all $j < 0$. We obtain the relations

$$\partial_j(\Phi_{g,\delta(m)}) = \{\partial_j(p_+(g,\delta(m))) + p_+(g,\delta(m))Q_0 z^j\}\delta(m)\gamma g^{-1} =$$
$$= \{\partial_j(p_+(g,\delta(m)))p_+(g,\delta(m))^{-1} + p_+(g,\delta(m))Q_0 z^j p_+(g,\delta(m))^{-1}\}\Phi_{g,\delta(m)}$$

and

$$\partial_j(\Phi_{g,\delta(m)}) = \{\partial_j(u_-(g,\delta(m))) + u_-(g,\delta(m))Q_0 z^j\}\delta(m)\gamma g =$$
$$= \{\partial_j(u_-(g,\delta(m)))u_-(g,\delta(m))^{-1} + u_-(g,\delta(m))Q_0 z^j u_-(g,\delta(m))^{-1}\}\Phi_{g,\delta(m)}.$$

Comparing coefficients of $\Phi_{g,\delta(m)}$ in these expressions, we see that

$$N_j := \partial_j(p_+(g,\delta(m)))p_+(g,\delta(m))^{-1} + p_+(g,\delta(m))Q_0 z^j p_+(g,\delta(m))^{-1}$$

belongs to $gl_2(R)[z^{-1}, z)_{<0}$. Because of relation (31), for $\Psi_{g,\delta(m)}$, we also have the equality

$$\partial_j(\Psi_{g,\delta(m)}) = N_j \Psi_{g,\delta(m)}$$

for all $j < 0$. We have thus shown that all the conditions in Proposition 3 are satisfied, which means that we have the following theorem.

Theorem 1. *We consider the product space* $\Pi := L_{\mathrm{an}}GL_2(\mathbb{C}) \times \Delta$ *and its subset* $\mathcal{S}$ *defined by* (27). *For each point* $(g,\delta(m)) \in \mathcal{S}$, *we define a pair of oscillating matrices* $(\Psi_{g,\delta(m)}, \Phi_{g,\delta(m)})$ *in* $\mathcal{M}_{\geq 0} \times \mathcal{M}_{<0}$ *by relations* (29) *and* (30). *This pair is a set of wave matrices for the combined AKNS hierarchy. In particular, the pair of deformations* $(Q_{g,\delta(m)}, S_{g,\delta(m)})$ *defined by*

$$Q_{g,\delta(m)} = u_-(g,\delta(m))Q_0 u_-(g,\delta(m))^{-1},$$
$$S_{g,\delta(m)} = p_+(g,\delta(m))Q_0 z^{-1} p_+(g,\delta(m))^{-1}$$

is a solution of the combined AKNS hierarchy. This solution does not change if $\delta(m)$ *is replaced with* $\delta(m)\delta^k$, $k \in \mathbb{Z}$.

REFERENCES

1. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, “The inverse scattering transform–Fourier analysis for nonlinear problems,” *Studies Appl. Math.*, **53**, 249–315 (1974).
2. H. Flaschka, A. C. Newell, and T. Ratiu, “Kac–Moody Lie algebras and soliton equations: II. Lax equations associated with $A_1^{(1)}$,” *Phys. D*, **9**, 300–323.
3. G. F. Helminck, “The strict AKNS hierarchy: Its structure and solutions,” *Adv. Math. Phys.*, **2016**, 3649205 (2016).
4. G. F. Helminck, A. G. Helminck, and E. A. Panasenko, “Integrable deformations in the algebra of pseudodifferential operators from a Lie algebraic perspective,” *Theor. Math. Phys.*, **174**, 134–153 (2013).
5. E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, “Transformation groups for soliton equations,” in: *Nonlinear Integrable Systems: Classical Theory and Quantum Theory* (Kyoto, 13–16 May 1981, M. Jimbo and T. Miwa, eds.), World Scientific, Singapore (1983), pp. 39–119.
6. G. Segal and G. Wilson, “Loop groups and equations of KdV type,” *Publ. Math. IHES*, **61**, 5–65 (1985).
7. A. Pressley and G. Segal, *Loop Groups*, Oxford Univ. Press, New York (1986).
8. R. S. Hamilton, “The inverse function theorem of Nash and Moser,” *Bull. Amer. Math. Soc.*, n.s., **7**, 65–222 (1982).