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COSMOLOGICAL MODELS WITH HOMOGENEOUS AND ISOTROPIC SPATIAL SECTIONS

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The assumption that the universe is homogeneous and isotropic is the basis for the majority of modern cosmological models. We give an example of a metric all of whose spatial sections are spaces of constant curvature but the space–time is nevertheless not homogeneous and isotropic as a whole. We give an equivalent definition of a homogeneous and isotropic universe in terms of embedded manifolds.

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1. Introduction

Let a coordinate system x^α, $\alpha = 0, 1, 2, 3$, be given in a space–time $(\mathbf{M}, g)$, where $\mathbf{M}$ is a four-dimensional manifold and g is a metric of Lorentzian signature, $\operatorname{sign} g = (+ - - -)$. We assume that the zeroth-coordinate line is timelike: $(\partial_0, \partial_0) = g_{00} > 0$, where the parentheses denote the scalar product. The coordinate $x^0 := t$ is called time (here and hereafter, the sign $:=$ means "is equal by definition"). Space indices are denoted by Greek letters from the middle of the alphabet: $\mu, \nu, \dots = 1, 2, 3$. Then $\{x^\alpha\} = \{x^0, x^\mu\}$.

Modern observational data provides evidence that our universe is homogeneous and isotropic (the cosmological principle), at least in the first approximation. A recent discussion of the cosmological principle from observational and theoretical standpoints is presented in [1]. The possibility of it being violated is also given there.

The majority of cosmological models are based on the following statement.

Theorem 1.1. *Let a four-dimensional space–time be the topological product* $\mathbf{M} = \mathbf{R} \times \mathbf{S}$, *where* $t \in \mathbf{R}$ *is the time coordinate,* $x \in \mathbf{S}$, *and* $\mathbf{S}$ *is a three-dimensional space of constant curvature. We assume that a sufficiently smooth metric of Lorentzian signature is given on* $\mathbf{M}$. *If the space–time is homogeneous and isotropic, then in a neighborhood of each point, a coordinate system* t, x^μ *exists such that the metric has the form*

$$ds^2 = dt^2 + a^2 \overset{\circ}{g}_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{1}$$

where $a(t) > 0$ *is an arbitrary function of time (scale factor) and* $\overset{\circ}{g}_{\mu\nu}(x)$ *is a negative-definite metric on a constant-curvature space* $\mathbf{S}$ *depending only on the spatial coordinates* $x \in \mathbf{S}$.

The definition of a homogeneous and isotropic universe and also the proof of Theorem 1.1 is given in [2].

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Hence, the most general metric of a homogeneous and isotropic universe has form (1) up to a coordinate transformation. This theorem is not affected by the dimensionality of the manifold $\mathbf{M}$ nor the signature of the metric g. The first condition of the theorem can be replaced with the sentence "Let each section of a space–time $\mathbf{M}$ corresponding to a constant time $t \in \mathbf{R}$ be a space of constant curvature."

Theorem 1.1 is the basis of relativistic cosmology and hence very important. Standard references for metric (1) are [1]–[11]. We comment on the parts in those papers that relate to the form of the metric.

Friedmann was the first who considered metric (1) for constructing cosmological models in the framework of general relativity [3], [4]. He did not write about a homogeneous and isotropic universe and simply required that all spatial sections corresponding to constant time be constant-curvature spaces and required that the metric have form (1). Friedmann considered spatial sections of positive and negative curvatures respectively in [3] and [4].

Abbé Lemaître analyzed solutions of Einstein's equations describing a closed universe [5]. He did not formulate Theorem 1.1. A more general class of cosmological models was considered in [6], also without formulation of the theorem.

Robertson formulated the theorem in both [7] and [8] but did not prove it. Instead, he referred to [12] and [13]. The proof of the theorem consists of two parts. The first part was proved by Hilbert [12] in a general case. The second part was proved by Fubini [13] (also see Exercise 3 in Chap. 6 in [14]) in one direction. Namely, he proved that metric (1) is homogeneous and isotropic, but the converse statement that any homogeneous and isotropic space has a metric of this form was not proved. Metric (1) was obtained from other assumptions by considering a system of observers with given properties in [9]. The metric was homogeneous and isotropic by construction. But Robertson (before Eq. (2.1) in [9]) assumed that the spatial part of the metric describes spaces of constant curvature taking only discrete values $\pm 1, 0$, and the statement that metric (1) represents the most general form of the metric was therefore unproved. Metric (5) given below fits the construction but does not have form (1).

Tolman obtained line element (1) from different assumptions [10], [11], [15]. In particular, he assumed spherical symmetry, geodesic time coordinate lines, and satisfaction of Einstein's equations. He did not discuss the homogeneity and isotropy of a universe in his papers.

In [16] (see Sec. 10), Walker proved Theorem 1.1 in one direction: metric (1) is homogeneous and isotropic. But he did not prove that any homogeneous and isotropic metric has this form. Indeed, metric (5) below satisfies Eq. (52) in [16] but does not have form (1).

At first glance, if all spatial sections of space–time are constant-curvature spaces, i.e., homogeneous and isotropic, then the universe is homogeneous and isotropic as a whole. But this is not the case [17]. In what follows, we give an example of a space–time all of whose spatial sections are constant-curvature spaces but the universe is not homogeneous and isotropic as a whole.

2. Cosmological metric with homogeneous and isotropic sections

Explicit form (1) of the Friedmann metric for a homogeneous and isotropic universe depends on coordinates chosen on spaces of constant curvature. In stereographic coordinates, the Friedmann metric is diagonal:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2 \eta_{\mu\nu}}{(1 + b_0 x^2)^2} \end{pmatrix}, \tag{2}$$

where $b_0 = -1, 0, 1$, $\eta_{\mu\nu} := \operatorname{diag}(- - -)$ is the negative-definite Euclidean metric, and $x^2 := \eta_{\mu\nu} x^\mu x^\nu \leq 0$. The values $b_0 = -1, 0, 1$ correspond to respective spaces of positive, zero, and negative constant curvature because we choose a negative-definite metric on spatial sections. For positive and zero curvature, the stereographic coordinates are defined on the whole Euclidean space $\mathbf{R}^3$. For spaces of negative curvature, the stereographic coordinates are defined inside a ball $|x^2| < 1/b_0$.

We perform the coordinate transformation $x^\mu \mapsto x^\mu/a$. The metric then becomes nondiagonal, and the conformal factor disappears:

$$g = \begin{pmatrix} 1 + \dfrac{\dot{b}^2 x^2}{4b^2(1+bx^2)^2} & \dfrac{\dot{b}x_\nu}{2b(1+bx^2)^2} \\ \dfrac{\dot{b}x_\mu}{2b(1+bx^2)^2} & \dfrac{\eta_{\mu\nu}}{(1+bx^2)^2} \end{pmatrix}, \tag{3}$$

where

$$b(t) := \frac{b_0}{a^2(t)} \tag{4}$$

and a dot denotes the time derivative.

We see that the metric of a homogeneous and isotropic universe can be nondiagonal and not contain the scale factor. In addition, the scalar curvature of spatial sections, which is proportional to $b(t)$, depends explicitly on time.

We now simply drop the nondiagonal terms, set $g_{00} = 1$, and add the scale factor. The metric then becomes

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2\eta_{\mu\nu}}{(1+bx^2)^2} \end{pmatrix}. \tag{5}$$

This metric contains two independent functions of time: $a(t) > 0$ and $b(t)$. It is nondegenerate for all values of b including $b = 0$. All sections $t = \text{const}$ of the corresponding space–time are obviously constant-curvature spaces and therefore homogeneous and isotropic. This metric is interesting because it allows analyzing solutions that pass through the zeros $b = 0$ in the general case. If such solutions exist, then spatial sections change curvature from positive to negative and vice versa during time evolution.

An arbitrary function $b(t)$ cannot be eliminated by a coordinate transformation without nondiagonal terms appearing.

There is an interesting situation. On one hand, all spatial sections of metric (5) are homogeneous and isotropic. On the other hand, any homogeneous and isotropic metric must have form (1). The answer to the question of how these two statements can be compatible is that metric (5) is not homogeneous and isotropic as a whole. Indeed, each section $t = \text{const}$ of the space–time $\mathbf{M}$ is a constant-curvature space, and the spatial (μ, ν) components of the Killing equations

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0 \tag{6}$$

are satisfied, but mixed $(0, \mu)$ components are not. The six independent Killing vectors of spatial sections in stereographic coordinates are

$$\begin{aligned} \widehat{K}_{0\mu} &= (1 + bx^2)\partial_\mu - \frac{2}{b}x_\mu x^\nu \partial_\nu, \\ \widehat{K}_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu, \end{aligned} \tag{7}$$

where the indices $\mu, \nu = 1, 2, 3$ enumerate Killing vector fields. The first three Killing vectors generate translations at the origin of the coordinate system, and the last three Killing vectors generate rotations. We see that the first three Killing vectors depend explicitly on time through the function $b(t)$, and it is easily verified that the mixed $(0, \mu)$ components of Killing equations (6) are not satisfied.

There is another way to verify that metric (5) is not homogeneous and isotropic. Straightforward calculations yield the expression

$$R = -\frac{24b}{a^2} + 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{1}{1+bx^2}\left(4\frac{\dot{a}\dot{b}x^2}{a} + \ddot{b}x^2\right) + 3\frac{\dot{b}^2x^4}{(1+bx^2)^2}\right]$$

for the scalar curvature. It depends explicitly on x and is therefore not homogeneous and isotropic.

This example shows that homogeneity and isotropy of spatial sections does not provide sufficient conditions for the homogeneity and isotropy of the whole four-dimensional metric. The equivalent definition is as follows.

Definition. A space–time is said to be *homogeneous and isotropic* if

1. all sections of constant time $t = \text{const}$ are constant-curvature spaces $\mathbf{S}$ and
2. the extrinsic curvature of hypersurfaces $\mathbf{S} \hookrightarrow \mathbf{M}$ is homogeneous and isotropic.

The definition of extrinsic curvature can be found, for example, in [18], [19]. In our notation, the extrinsic curvature $K_{\mu\nu}$ for the block diagonal metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & h_{\mu\nu} \end{pmatrix} \tag{8}$$

is proportional to the time derivative of the spatial part of the metric:

$$K_{\mu\nu} = -\frac{1}{2}\dot{h}_{\mu\nu}. \tag{9}$$

The last definition of a homogeneous and isotropic space–time is equivalent to the definition given in [2]. Indeed, the first requirement means that the space–time is the topological product $\mathbf{M} = \mathbf{R} \times \mathbf{S}$. The metric can then be transformed to block-diagonal form (8). The second requirement then means that the time derivative of the spatial part of the metric must be proportional to the metric itself [2]. This implies satisfaction of the equation

$$\dot{h}_{\mu\nu} = f h_{\mu\nu}, \tag{10}$$

where $f(t)$ is a sufficiently smooth function of time.

If $f = 0$, then there is nothing to be proved, and metric already has form (1) for $a = \text{const}$.

Let $f \neq 0$. We then introduce the new time coordinate $t \mapsto t'$ defined by the differential equation

$$dt' = f(t)\,dt.$$

Equation (10) then becomes

$$\frac{dh_{\mu\nu}}{dt'} = h_{\mu\nu}.$$

It has the general solution

$$h_{\mu\nu}(t', x) = Ce^{t'}\overset{\circ}{g}_{\mu\nu}(x), \quad C = \text{const} \neq 0,$$

where $\overset{\circ}{g}_{\mu\nu}(x)$ is a constant-curvature metric on $\mathbf{S}$, which is independent of time. This yields representation (1).

We note that the second requirement in the definition of a homogeneous and isotropic universe is necessary because metric (5) provides a counterexample.

3. Equations of motion

Although metric (5) is not homogeneous and isotropic, it is meaningful to consider possibilities of constructing cosmological models with such a metric, particularly in connection with the question whether Einstein's equations admit solutions describing a spatial topology changing with time.

In a space–time with metric (5), there is a spherical $\mathbf{S}\mathbb{O}(3)$ symmetry around the origin of the coordinate system $x^\mu = 0$. Straightforward calculations yield the Christoffel symbols

$$\begin{aligned} \Gamma_{0\mu}{}^{\nu} &= \Gamma_{\mu 0}{}^{\nu} = -K_\mu{}^\nu, \qquad \Gamma_{\mu\nu}{}^0 = K_{\mu\nu}, \\ \Gamma_{\mu\nu}{}^{\rho} &= -\frac{2b}{(1+bx^2)}(x_\mu \delta_\nu^\rho + x_\nu \delta_\mu^\rho - x^\rho \eta_{\mu\nu}), \end{aligned} \tag{11}$$

where we write only nonzero components, and extrinsic curvature (9) is

$$\begin{aligned} K_{\mu\nu} &= \left[-\frac{a\dot{a}}{(1+bx^2)^2} + \frac{a^2\dot{b}x^2}{(1+bx^2)^3}\right]\eta_{\mu\nu}, \\ K_\mu{}^\nu &:= g^{\nu\rho}K_{\mu\rho}, \qquad g^{\mu\nu} = \frac{(1+bx^2)^2}{a^2}\eta^{\mu\nu}. \end{aligned}$$

All components of Einstein's tensor $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ are nonzero:

$$\begin{aligned} G_0{}^0 &= \frac{12b}{a^2} - 3\left[\frac{\dot{a}}{a} - \frac{\dot{b}x^2}{1+bx^2}\right]^2, \qquad G_0{}^\mu = -\frac{4\dot{b}x^\mu}{a^2}, \\ G_\mu{}^\nu &= \left[\frac{4b}{a^2} - \frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{6\dot{a}\dot{b}x^2}{a(1+bx^2)} + \frac{2\ddot{b}x^2}{1+bx^2} - \frac{5\dot{b}^2x^4}{(1+bx^2)^2}\right]\delta_\mu^\nu. \end{aligned} \tag{12}$$

These components are equal to the components of Einstein's tensor for the Friedmann universe for $\dot{b} = 0$.

As matter, we choose a scalar field $\varphi(t, x^2)$ and a continuous medium with an energy density $\mathcal{E}(t, x^2)$, a pressure $\mathcal{P}(t, x^2)$, and a velocity $u = \{u^0, u^\mu\}$, $u^2 = 1$. The energy–momentum tensor then consists of two parts:

$$\begin{aligned} T_{\alpha\beta} &= T_{1\alpha\beta} + T_{2\alpha\beta}, \\ T_{1\alpha\beta} &= \epsilon\, \partial_\alpha\varphi\, \partial_\beta\varphi - g_{\alpha\beta}\left(\frac{\epsilon}{2}\partial\varphi^2 - V(\varphi)\right), \qquad T_{2\alpha\beta} = (\mathcal{E} + \mathcal{P})u_\alpha u_\beta - \mathcal{P}g_{\alpha\beta}, \end{aligned}$$

where $\partial\varphi^2 := g^{\alpha\beta}\, \partial_\alpha\varphi\, \partial_\beta\varphi$ and $V(\varphi)$ is a scalar field potential including a mass term. The constant ϵ takes two values: $\epsilon = 1$ corresponds to a positive-definite kinetic term in the Hamiltonian, and $\epsilon = -1$ corresponds to a negative-definite term (ghost). We assume that the matter distribution is spherically symmetric, i.e.,

$$u^0 = u^0(t, x^2), \qquad u^\mu = v\frac{(1+bx^2)^2}{a^2}x^\mu,$$

where $v(t, x^2)$ is a function of time and the radius squared x^2. In addition, the scalar field must satisfy the wave equation

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi + V(\varphi) = 0, \tag{13}$$

and the energy–momentum tensor of the continuous medium must satisfy the covariant conservation law and the equation of state,

$$\nabla_\beta T_{2\alpha}{}^\beta = 0, \qquad \mathcal{P} = \mathcal{P}(\mathcal{E}). \tag{14}$$

Hence, the full system of equations consists of Einstein's equations, which in our notation have the form

$$G_\alpha{}^\beta = -\frac{1}{2} T_\alpha{}^\beta, \tag{15}$$

and Eqs. (13) and (14) for unknown functions $a(t)$, $b(t)$, $\varphi(t, x^2)$, $\mathcal{E}(t, x^2)$, $\mathcal{P}(t, x^2)$, and $v(t, x^2)$. Of course, the first equation in (14) is the consistency requirement for Einstein's equations (15). The obtained system of equations is complicated, and solutions with $\dot{b} \neq 0$ are not known at present. For

$$\dot{b} = 0, \qquad \varphi = \varphi(t), \qquad \mathcal{E} = \mathcal{E}(t), \qquad \mathcal{P} = \mathcal{P}(t), \qquad v = 0,$$

it reduces to the Friedmann equations.

The velocity time component is found from the restriction $u^2 = 1$:

$$u^0 = u_0 = \sqrt{1 - v^2 \frac{(1 + bx^2)^2}{a^2} x^2},$$

where we choose the solution with $u^0 > 0$ (flow lines of the continuous media are future directed).

For metric (5), the components of the energy–momentum tensor have the forms

$$T_0{}^0 = \epsilon \dot{\varphi}^2 - \left(\frac{\epsilon}{2} \, \partial\varphi^2 - V \right) + (\mathcal{E} + \mathcal{P}) u_0^2 - \mathcal{P}, \tag{16}$$

$$T_0{}^\mu = [2\epsilon \dot{\varphi} \varphi' + (\mathcal{E} + \mathcal{P}) u_0 v] \frac{(1 + bx^2)^2}{a^2} x^\mu, \tag{17}$$

$$T_\mu{}^\nu = [4\epsilon \varphi'^2 + (\mathcal{E} + \mathcal{P}) v^2] \frac{(1 + bx^2)^2}{a^2} x_\mu x^\nu - \left[\frac{\epsilon}{2} \, \partial\varphi^2 - V + \mathcal{P} \right] \delta_\mu^\nu, \tag{18}$$

where a prime denotes differentiation with respect to x^2.

Einstein's tensor (12), which is proportional to the Kronecker symbol, is in the left-hand side of Eqs. (15). The right-hand side of Eqs. (18) contains two tensor structures: $x_\mu x^\nu$ and δ_μ^ν. Therefore, they must be separately equal to zero. The equation proportional to $x_\mu x^\nu$ contains a restriction on possible matter fields:

$$4\epsilon (\varphi')^2 + (\mathcal{E} + \mathcal{P}) v^2 = 0. \tag{19}$$

For ordinary matter, this equality cannot be satisfied, because both terms are nonnegative. Therefore, we must, for example, admit the validity of $\mathcal{E} + \mathcal{P} < 0$ (violation of the isotropic energy dominance condition in Friedmann models of the universe) or the wrong sign of the kinetic term for the scalar field (ghost, $\epsilon = -1$). Models violating the isotropic energy dominance condition have attracted much interest in recent years. Their drawbacks and merits were considered, for example, in [20], [21]. We see that one matter type is not sufficient to satisfy equality (19). This is why we chose two matter types to compensate each other in Einstein's equations.

The equations with mixed components with equality (19) become

$$\frac{4\dot{b}}{a^2} = \epsilon \varphi' \left[\dot{\varphi} - 2\varphi' \frac{u_0}{v} \right] \frac{(1 + bx^2)^2}{a^2}. \tag{20}$$

We note that (19) is solved for a cosmological constant Λ if it is regarded as matter. In this case, $\mathcal{E} = -\mathcal{P} = \Lambda$, and the scalar field is independent of x^2. The equality $b = \text{const}$ then follows from (20), and we obtain the Friedmann equation.

In the considered case, the equation for scalar field (13) becomes

$$\ddot{\varphi} + 3\frac{\dot{a}}{a} \, \dot{\varphi} - 3 \frac{\dot{b} x^2}{1 + bx^2} \, \dot{\varphi} + \frac{(1 + bx^2)^2}{a^2} \, \eta^{\mu\nu} \, \partial^2_{\mu\nu} \varphi - 2 \frac{1 + bx^2}{a^2} b x^\mu \, \partial_\mu \varphi + V(\varphi) = 0.$$

The whole system of equations is complicated, and we leave it for future investigation.

4. Conclusion

We have constructed an example of a space–time all of whose spatial sections are constant-curvature spaces but the metric is not homogeneous and isotropic as a whole. We defined a homogeneous and isotropic universe in terms of embedded manifolds.

A separate investigation is needed to check how modern observational data provide evidence for the homogeneity and isotropy of the extrinsic curvature of the embedding $\mathbf{S} \hookrightarrow \mathbf{M}$.

This paper is a continuation of the investigation started in [17]. We obtained a complete system of equations for models of a universe with homogeneous and isotropic spatial sections, generalizing the Friedmann system of equations. We showed that for the self-consistency of the system of equations, at least two types of matter are needed. This system of equations is complicated and deserves future investigation.

The methods and approaches used in this paper were considered in [22]–[26].

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