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NEW APPROACH TO CALCULATING THE SPECTRUM OF A QUANTUM SPACE–TIME

A. N. Starodubtsev*

We study the dynamics of a massive pointlike particle coupled to gravity in four space–time dimensions. It has the same degrees of freedom as an ordinary particle: its coordinates with respect to a chosen origin (observer) and the canonically conjugate momenta. The effect of gravity is that such a particle is a black hole: its momentum becomes spacelike at a distances to the origin less than the Schwarzschild radius. This happens because the phase space of the particle has a nontrivial structure: the momentum space has curvature, and this curvature depends on the position in the coordinate space. The momentum space curvature in turn leads to the coordinate operator in quantum theory having a nontrivial spectrum. This spectrum is independent of the particle mass and determines the accessible points of space–time.

Keywords: quantization of gravity, conformal field theory, space–time discreteness, Planck scale

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1. Introduction

It has long been hoped that a nonperturbative quantization of general relativity could ensure its ultraviolet regularization. Several arguments were advanced in favor of such a possibility. Among them are the discreteness of the area operator in loop quantum gravity [1], dimensional reduction at the Planck scale observed numerically in the framework of the approach based on causal dynamical triangulation [2], and the impossibility to make a measurement at sub-Planckian scales because of the formation of a black hole, first already noted in 1935 [3].

The fact that all three arguments presented above are interrelated is easiest to see in (2+1)-dimensional gravity where matter is represented by pointlike particles. Such a theory is exactly solvable [4]. The particle momentum is described by a Wilson loop of the Lorentz connection around the particle worldline [4], [5]. The space of such momenta is the Lorentz group manifold. It has one compact dimension: a spatial rotation canonically conjugate to the time coordinate. The first consequence of this is that the particle energy is bounded by the Planck value or, more precisely, a particle energy–momentum vector above the Planck value cannot be timelike. The second consequence is that the compactness of one direction in the momentum space leads to an analogue of Kaluza–Klein reduction at large energies. And, finally, the coordinate canonically conjugate to the compact momentum has a discrete spectrum in quantum theory.

Here, we continue an attempt started in [6] to generalize the abovementioned arguments to (3+1)-dimensional gravity. We derive an effective action of a particle coupled to gravity in 3+1 dimensions in terms of its coordinates and momenta. Just as in [6], the momentum space is the group manifold, but its curvature now depends on the radial coordinate in accordance with Newton's law. Therefore, the action becomes especially simple with a fixed radial coordinate. The curvature of the momentum space becomes

*St. Petersburg State University, St. Petersburg, Russia, e-mail: artemstarodubtsev@gmail.com.

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constant, as in (2+1)-dimensional gravity, and we can use the results obtained in that case. The time coordinate has a discrete spectrum, and the remaining two spatial coordinates are continuous. The issue of the spectrum of the radial coordinate remains open.

2. Action principle and boundary conditions

We choose a pure-gauge formulation of (3+1)-dimensional gravity where the gauge group also includes translations in addition to Lorentz transformations. This is the MacDowell–Mansouri formulation [7], which is very similar to the Chern–Simons formulation of (2+1)-dimensional gravity but is applicable only in the case of a nonzero cosmological constant (for definiteness, we here choose it initially positive but let it tend to zero as soon as this becomes possible).

Let A^{IJ} be an $SO(4,1)$ connection, where $I, J = 0, 1, \ldots, 4$, and v^I be a 0-form taking values in the space of $SO(4,1)$ vectors. A particle is introduced as an extrinsic charge of the gauge group (charges of $SO(4,1)$ are the energy–momentum and spin).

The action of gravity coupled to the particle has the form

$$S = \frac{l^2}{8\pi G} \int_{M^4} \epsilon_{IJKLM} F^{IJ}(A) v^K \wedge F^{LM}(A) + \lambda(v^I v_I - 1) + \int_\gamma \mathrm{Tr}(h^{-1}\, d_A h \mathbf{K}), \tag{1}$$

where $F^{IJ}(A)$ is the curvature 2-form of the connection A. The second term in (1) is the normalization condition for v^I introduced using a Lagrangian multiplier. The last term is the particle action, where $\mathbf{K}$ is a fixed element of the $so(4,1)$ algebra, h is an element of the group $SO(4,1)$ added to compensate the change of A under gauge transformations, M^4 is the four-dimensional space–time, γ denotes the particle worldline, and $l = 1/\sqrt{\Lambda}$ is the cosmological length.

By a change of notation, the first term in the right-hand side of (1) reduces to the action of gravity with a cosmological constant in the Einstein–Cartan form with an additional term containing the Euler characteristic. We introduce the connection ω^{IJ} with respect to which v^I is covariantly constant, $d_\omega v^I = 0$. Its curvature $R^{IJ}(\omega)$ is an $SO(3,1)$ curvature in the v^I stability subgroup because $R^{IJ}(\omega) v_I = 0$. We then introduce the tetrad

$$e^I = l\, d_A v^I.$$

This is indeed a tetrad because $e^I v_I = 0$. The $SO(4,1)$ connection can then be decomposed as

$$A^{IJ} = \omega^{IJ} - \frac{1}{l}(v^I e^J - v^J e^I),$$

which leads to the decomposition of the curvature

$$F^{IJ} = R^{IJ} + \frac{1}{l^2} e^I \wedge e^J + \frac{1}{l}(v^J\, d_\omega e^I - v^I\, d_\omega e^J). \tag{2}$$

Substituting this expression in (1), we see that the last term in (2), which contains torsion, does not enter the action. The first two terms give the gravity action; in particular, the cross term gives the Einstein–Cartan action.

To make the action functionally differentiable, we must add boundary conditions and the corresponding compensating boundary term to it. The first thing to do is to eliminate the Euler term

$$\int \epsilon_{IJKLM} R^{IJ}(\omega) R^{KL}(\omega) v^M$$

from the action because this term imposes the flatness condition $R^{IJ}(\omega) = 0$ on the boundary. As the boundary, we choose a sphere on which an observer (a test particle with negligible energy and momentum) is located. In the general case, this boundary is located at a finite distance from the massive particle, and the flatness condition therefore cannot be imposed at the boundary.

We follow the "solve constraints then quantize" strategy [8]: we first solve the constraints and then substitute the solution back in the action. To obtain the constraint equations, we must vary the action with respect to the time component of the connection A_t, and no boundary term should remain after the variation. For this, we must either fix A_t on the boundary or add a compensating term of the form $\int_{\partial M} \epsilon_{IJKLM} F^{IJ} v^K \wedge A_t^{LM}\, dx_0$. Either of these two choices breaks the gauge invariance on the boundary and leads to the appearance of the so-called "would be gauge" degrees of freedom [9], among which are the degrees of freedom of the considered particle.

We thus obtain the action

$$S = \frac{l^2}{8\pi G} \int_{M^4} \epsilon_{IJKLM} (2\, d_A v^I \wedge d_A v^J v^K \wedge F^{LM}(A) + d_A v^I \wedge d_A v^J v^K \wedge d_A v^L \wedge d_A v^M) + \\ + \int_{M^4} \lambda(v^I v_I - 1) - \frac{l^2}{4\pi G} \int_{\partial M^4} \epsilon_{IJKLM}\, d_A v^I \wedge d_A v^J v^K \wedge A_t^{LM}\, dx_0 + \int_\gamma \mathrm{Tr}(h^{-1}\, d_A h \mathbf{K}). \tag{3}$$

The boundary conditions for A_t are arbitrary, and for the spatial components of A, they follow from the choice of a particular solution of the constraint equations. Actions (1) and (3) are equivalent up to the boundary terms, and we can therefore use the simpler expression (1) to derive the equations of motion in the bulk.

3. Particle effective action

Varying action (1) with respect to the time components of A, we obtain the set of constraint equations

$$\frac{1}{8\pi G}\, d_A(\epsilon_{IJKLM} v^K F^{LM}) = (h\mathbf{K}h^{-1})_{IJ}\, \delta^3(x - x_p). \tag{4}$$

We consider a spinless particle. Hence, there exists a gauge in which $h\mathbf{K}h^{-1} = MT_{04}$ (here T_{04} is the generator of time translations and M is a parameter of the dimension of mass), and there exists a spherically symmetric solution. If we make an analogy with the Yang–Mills theory, then it turns out that the energy–momentum is an analogue of the magnetic charge, v^I plays the role of the Higgs field, and the solution of constraints (4) in its angular dependence is similar to the 't Hooft–Polyakov solution [10].

In a gauge where the fields are time-independent, the solution in the limit $l \to \infty$ has the form (the time components of the connection are not included, because they are not in constraint equations (4))

$$\begin{aligned} &A_S^{ij} = (1 - N)(n^i\, dn^j - n^j\, dn^i), \qquad A_S^{i0} = 0, \\ &lA_S^{i4} = \left(1 - \frac{1}{N}\right) n^i\, dr + (1 - N) r\, dn^i, \\ &lv^i = rn^i, \qquad v^0 = 0, \qquad v^4 = 1 + O\left(\frac{r^2}{l^2}\right), \end{aligned} \tag{5}$$

where $N = \sqrt{1 - 2MG/r}$ is the familiar expression for the lapse function and

$$n^i = \frac{r^i}{|r|} \tag{6}$$

is a normalized vector field normal to the two-sphere called a "hedgehog." Solution (5) is just the Schwarzschild solution written in an unusual gauge, and the mass in the right-hand side of (4) coincides with the

Schwarzschild mass M observed at infinity. We note that the field n^i in (5) is not necessarily spherically symmetric as in (6). The only requirement is that it be normalized $n^i n_i = 1$ and have the winding number one on the sphere,

$$\int_{S^2} \epsilon_{ijk} n^i \, dn^j \wedge dn^k = 4\pi. \tag{7}$$

For any such n^i, expression (5) is a solution of (4). We use this freedom in what follows.

In the next step, we transfer solution (5) into an arbitrary gauge $A = g^{-1}(d + A_S)g$ and substitute it back in action (3). As a result, we obtain a total derivative in the Lagrangian, and the action finally reduces to an integral over the boundary. This is analogous to how the Chern–Simons action in three dimensions reduces to the Wess–Zumino–Witten action on the boundary (see, e.g., [8], [11]). This action (again in the limit $l \to \infty$) has the form

$$S = \frac{1}{4\pi G} \int_{\partial M^4} \epsilon_{abcd} x^a \, dx^b \wedge d(\dot{g} g^{-1})^{cd} \, dt + \frac{M}{2\pi} \int_{\partial M^4} \epsilon_{ijkLM} n^i \, dn^j \wedge dn^k (\dot{g} g^{-1})^{LM} \, dt, \tag{8}$$

where x^a is the translational part of g, $a, b = 0.3$, and n^i is the hedgehog field in solution (5). The first term in (8) is a three-dimensional analogue of the Wess–Zumino–Witten action, and the second term is a source contribution.

We have obtained a field theory with infinitely many degrees of freedom. We are interested in only a finite number of them, those that are the particle degrees of freedom. We therefore naturally choose a spherically symmetric ansatz for the group field g:

$$g^{0i} \sim g^{4i} \sim g^{jk} \epsilon^{ijk} \sim n^i, \tag{9}$$

where n^i is hedgehog (6) and the angular degrees of freedom of the particle can now be identified with the coordinates on which n^i depends.

Our action is now invariant under simultaneous continuous norm-preserving transformations of all the fields in (9). For further simplification of the action, it seems tempting to use this invariance to make x^a (and n^i) constant on the sphere ("comb the hedgehog"), pull them outside the integral in (8), and again obtain a total derivative in the integrand. If this were possible, then the result in (8) would be zero because the sphere has no boundary. But the transformation from a hedgehog to a constant n^i is impossible to realize on the whole sphere because it changes the winding number from 1 to 0. Nevertheless, this transformation can be made continuous on any open subset of the sphere. Hence, we can cut the sphere in half and comb the hedgehog on each of the hemispheres [12] (this is the transformation of a 't Hooft–Polyakov monopole into a Wu–Yang monopole). As a result, the three-dimensional field theory reduces to a two-dimensional field theory on the cut $S^1 \times \mathbb{R}^1$.

On the sphere, there is a preferred direction of the field x^a, which must remain fixed under deformations of the hedgehog. This is x_0^a, the coordinates of the origin (observer) with respect to the particle. We make the field x^a constant and equal to x_0^a on one hemisphere. On the other hemisphere, it is natural to choose the direction opposite to x_0^a. We introduce $g_\perp x_0 = x_0$, the subgroup of the Lorentz group that leaves x_0^a fixed and $x_\perp \perp x_0$. As a result, we obtain the two-dimensional action

$$S = \frac{1}{4\pi G} \int_{\mathbb{R}^1} dt \, x_0^a \int_{S^1} \epsilon_{abcd} \, dx_\perp^b \, (\dot{g}_\perp g_\perp^{-1})^{cd} + \\ + \frac{M}{2\pi} \int_{\mathbb{R}^1} dt \frac{x_0^i}{R} \int_{S^1} \epsilon_{ijkLM} m^j \, dm^k \, (\dot{g}_\perp g_\perp^{-1})^{LM},$$

where R is the radius of the sphere (the absolute value of x_0^a), $m^i n_i = 0$, and $m^i m_i = 1$.

This action becomes especially simple if the radial coordinate does not change in time, $\dot{R} = 0$, but $\dot{x}_\perp \neq 0$. In this case, we obtain the Wess–Zumino–Witten action for $g_\perp$ in the form in which it is obtained from (2+1)-dimensional gravity with pointlike particles [5], [11] but with the coupling constant G/R:

$$S = \frac{R}{4\pi G}\int_{\mathbb{R}^1\times S^1} dt\,\epsilon_{bcd}\,dx_\perp^b(\dot{g}_\perp g_\perp^{-1})^{cd} + \frac{M}{2\pi}\int_{\mathbb{R}^1\times S^1} dt\,d\phi\,\mathrm{Tr}(T^{04}\dot{g}_\perp g_\perp^{-1}). \tag{10}$$

Hence, following [5], [11], we can derive a particle action with a finite number of degrees of freedom depending only on the particle coordinates and momenta. The momentum is given by the Wilson loop of the Lorentz group around the spatial circle S^1 in (10):

$$u = g_\perp \exp\biggl(T^{12}\pi\biggl(1-\sqrt{1-\frac{2MG}{R}}\biggr)\biggr)g_\perp^{-1}. \tag{11}$$

This loop can also be obtained from connection (5), but defined only for $R > 2MG$. It satisfies the constraint equation fixing its conjugacy class:

$$\mathrm{Tr}(u) = \cos\biggl(\pi\biggl(1-\sqrt{1-\frac{2MG}{R}}\biggr)\biggr) = -\cos\biggl(\pi\sqrt{1-\frac{2MG}{R}}\biggr). \tag{12}$$

We note that the last equation can naturally be continued inside the Schwarzschild radius, $R > 2MG$. In this region, the argument of the cosine in the right-hand side of (12) becomes imaginary, and the cosine becomes hyperbolic. This means that the Lorentz transformation u cannot be reduced to a pure rotation by a similarity transformation, and static solutions do not exist.

If the distance to the origin is less than the Schwarzschild radius, then the Wilson loop around S^1 gives an elliptic Lorentz transformation, and the particle momentum is timelike. Inside the Schwarzschild radius, the Wilson loop around S^1 gives a hyperbolic Lorentz transformation, and the particle momentum is spacelike. This transition occurs because the time direction is compact in the momentum space.

The final particle action has the form

$$S_p = \int dt\,\epsilon_{abc}x^a(u^{-1}\dot{u})^{bc} + \lambda\biggl(\mathrm{Tr}(u) + \cos\biggl(\pi\sqrt{1-\frac{2MG}{R}}\biggr)\biggr).$$

Here, the kinetic term has the standard form in the case where half the canonical variables are group-valued [5].

4. Quantization

Here, we list the results in [5] that can be applied to our case.

Our momentum space is the $SO(2,1)$ group manifold or the AdS_3 space. Quantization of $x_\perp$ is done based on harmonic analysis on the AdS_3 space. If we work in the momentum representation, then wave functions are functions on AdS_3. Coordinate operators act as left-invariant derivatives,

$$\hat{x}^a|u\rangle = \frac{l_P^2}{R}\mathcal{L}^a|u\rangle.$$

The interval operator is the Beltrami–Laplace operator

$$\hat{S}^2|u\rangle = \hat{x}^a\hat{x}_a|u\rangle = \frac{l_P^4}{R^2}\mathcal{L}^a\mathcal{L}_a|u\rangle.$$

The eigenstates and eigenvalues are

$$\hat{x}^0|t\rangle = \frac{l_P^2}{R}t|t\rangle, \quad t \in \mathbb{Z},$$

$$\widehat{S}^2|t,\lambda\rangle = \frac{l_P^4}{R^2}(\lambda^2+1)|t,\lambda\rangle, \qquad \widehat{S}^2|t,l\rangle = -\frac{l_P^4}{R^2}l(l-2)|t,\lambda\rangle,$$

where $\lambda \in \mathbb{R}^+$, $l \in \mathbb{Z}^+$, $2 \le l \le |t|$. The spacelike part of the spectrum of the interval operator is continuous, while the timelike part is discrete. We note that the time coordinate is quantized not in Planck units but in l_P^2/R units, where R is the distance to the origin introduced in (10). The interval between eigenvalues of the time operator tends to zero at large distances.

5. Conclusion

If we compare the results in this paper with the results obtained in various versions of loop gravity, then we see more similarity with the results in the Lorentz-covariant version [13], while there are contradictions with earlier results [1]. In our case, the results can be explained based on phenomena well known in classical gravity, such as gravitational collapse, which adds to their verisimilitude.

We have not yet quantized the radial variable. The complication here is that it is a canonical coordinate and, on the other hand, defines the curvature of the momentum space. It is therefore unclear in which representation to work. But the radial variable was quantized in a different approach [14], where it was the only variable. It would be interesting to connect these results.

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