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SOLVING EVOLUTIONARY-TYPE DIFFERENTIAL EQUATIONS AND PHYSICAL PROBLEMS USING THE OPERATOR METHOD

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We present a general operator method based on the advanced technique of the inverse derivative operator for solving a wide range of problems described by some classes of differential equations. We construct and use inverse differential operators to solve several differential equations. We obtain operator identities involving an inverse derivative operator, integral transformations, and generalized forms of orthogonal polynomials and special functions. We present examples of using the operator method to construct solutions of equations containing linear and quadratic forms of a pair of operators satisfying Heisenberg-type relations and solutions of various modifications of partial differential equations of the Fourier heat conduction type, Fokker–Planck type, Black–Scholes type, etc. We demonstrate using the operator technique to solve several physical problems related to the charge motion in quantum mechanics, heat propagation, and the dynamics of the beams in accelerators.

Keywords: inverse operator, exponential operator, inverse derivative, differential equation, Laguerre polynomial, Hermite polynomial, special function

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1. Introduction

Differential equations (DEs) are the most important mathematical tool for describing a wide range of physical processes. Their study of and in itself constitutes a serious mathematical problem, the importance of which can hardly be overestimated in view of physical applications of the obtained solutions. The use and development of computer techniques in the 21st century made obtaining DE solutions much easier. Exact analytic solutions, which allow an in-depth and comprehensive analysis of both the mathematical aspects of the problem and the physical features of the modeled process, are especially valuable. Obtaining exact analytic solutions is a complicated and not always solvable problem, even using the modern computer programs that have appeared in the last decade for analytic calculations. The analytic description of the behavior of physical systems has always been given much attention. This is indicated by the examples of recent studies of the phenomena of nonlinear effects in electrodynamics, the spectra and dynamics of atoms and the features of the motion and radiation from charges in magnetic fields [1]–[10], heat propagation not obeying the Fourier law [11]–[14], and financial and even some biological models [15], which allowed describing the behavior of complex systems partly or completely analytically.

Integral transformations were used to solve the Fokker–Planck equation in [16], and the results were presented in the form of convergent series. The method of inverse differential operators [17]–[22] is useful for obtaining the exact analytic solutions of linear DEs. It uses advanced forms of Hermite and Laguerre

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orthogonal polynomials [23] with many variables and indices [24], [25] and their operator representations [26]. Integral relations for tensor polynomials were investigated in [27].

Here, we generalize an operator technique that we developed for solving DEs to more complicated equations of the type of heat conduction, Black–Scholes and Fokker–Planck equations with additional terms. We show how using inverse differential operators combined with integral transformations and the operator representations of the extended forms of orthogonal polynomials allow quickly solving relatively complex problems.

2. Operator solution of ordinary differential equations of noninteger order

The traditional approach to solving inhomogeneous DEs consists in using the Green's function. Another approach consists in using inverse differential operators including those of noninteger order. We consider the DE

$$\big(\beta^2 - (d_x + \alpha)^2\big)^{\nu} F(x) = f(x), \tag{1}$$

where α and β are constants and ν is an arbitrary real parameter. To find a particular integral, we use the known operator identity [26]

$$\hat{q}^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\hat{q}t} t^{\nu-1}\, dt, \quad \min\{\mathrm{Re}(q), \mathrm{Re}(\nu)\} > 0, \tag{2}$$

and integral representation (see [28])

$$e^{\hat{p}^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 2\xi\hat{p}}\, d\xi = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 - 2\xi\hat{p}}\, d\xi, \tag{3}$$

where in our case $\hat{p} = \sqrt{t}\widetilde{D}$ and the shift and heat conduction operators have the respective forms [29]

$$\widehat{\Theta} f(x) = e^{\eta(\partial_x + \alpha)} f(x) = e^{\eta\alpha} f(x + \eta), \qquad \widehat{S} = e^{t\partial_x^2}. \tag{4}$$

The action of the last operator is easy to represent in the form of a Gauss–Weierstrass transformation. As part of the operator approach, it is expedient to define polynomials using operator relations (see [29] and [19], [30]). For the generalized forms of the Hermite and Laguerre polynomials $H_n(x, y)$ and $L_n(x, y)$, we have the respective equalities

$$H_n(x, y) = e^{y\,\partial^2/\partial x^2} x^n, \qquad L_n(x, y) = e^{-y\,\partial_x x\,\partial_x} \frac{(-x)^n}{n!}. \tag{5}$$

The Laguerre derivative ${}_{\mathrm{L}}D_x \equiv \partial_x x\,\partial_x = \partial/\partial D_x^{-1}$ [29] makes a commutator with the inverse derivative D_x^{-1}: $[{}_{\mathrm{L}}D_x, D_x^{-1}] = -1$, where

$$D_x^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi)\, d\xi, \quad n \in N = \{1, 2, \dots\}$$

(see [19]).

We demonstrate the use of inverse differential operators and orthogonal polynomials to solve DEs. For example, we consider noninteger-order ordinary DE (ODE) (1) with the monomial $f(x) = x^k$. We use the trivial operator relation to a shift of the inverse operator

$$(\psi(\partial_x + \alpha))^{-1} f(x) = e^{-\alpha x} (\psi(\partial_x))^{-1} e^{\alpha x} f(x) \tag{6}$$

and the relation $e^{y\,\partial_x^2} x^k e^{\alpha x} = e^{(\alpha x+\alpha^2 y)} H_k(x+2\alpha y, y)$ [21]. The particular solution of (1) with the initial condition $f(x) = x^k$ can then be represented as

$$F(x) = \big(\beta^2 - (d_x+\alpha)^2\big)^{-\nu} x^k = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-t(\beta^2-\alpha^2)} t^{\nu-1} H_k(x+2\alpha t, t)\, dt. \tag{7}$$

The integrand contains Hermite polynomials with a shifted argument. We note that solution (7) of Eq. (1) can be represented as a series of the convolution $\phi(x) = \Phi(x) * f(x)$ with the kernel $\Phi(x) = x^n$ and the weight given by Hermite polynomials:

$$\begin{aligned}
&F(x) = \sum_{n=0}^{\infty} \phi(x) C(\nu,\alpha,\beta),\\
&\phi(x) = \int_{-\infty}^{\infty} \Phi(x-\eta) f(\eta)\, d\eta \equiv \Phi(x) * f(x),\\
&\Phi(x-\eta) = (\eta - x)^n, \qquad \Phi(x) = (-x)^n,\\
&C(\nu,\alpha,\beta) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_0^\infty \tau^{2(\nu-1)} e^{-\beta^2\tau^2} \frac{1}{n!} H_n\Big(\alpha, -\frac{1}{4\tau^2}\Big) d\tau.
\end{aligned} \tag{8}$$

Moreover, because the expression for the generating exponential is $e^{xt+yt^2} = \sum_{n=0}^{\infty} (t^n/n!) H_n(x,y)$, we can easily show that the solution of (1) is given by an integral with the weight of the following convolution with a kernel of the Gaussian-type probability distribution $\Omega(x,\tau)$:

$$\begin{aligned}
&F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_0^\infty \tau^{2(\nu-1)} e^{-\beta^2\tau^2} \varpi(x,\tau) d\tau,\\
&\varpi(x,\tau) = \int_{-\infty}^{\infty} \Omega(x-\eta,\tau) f(\eta) d\eta \equiv \Omega(x,\tau) * f(x),\\
&\Omega(x-\eta,\tau) = \exp\Big(\alpha(\eta - x) - \frac{(\eta-x)^2}{4\tau^2}\Big), \qquad \Omega(x,\tau) = \exp\Big(-\alpha x - \frac{x^2}{4\tau^2}\Big).
\end{aligned} \tag{9}$$

With

$$\int_0^\infty \tau^{2(\nu-1)} \exp\Big(-(\beta\tau)^2 - \frac{(x-\eta)^2}{4\tau^2}\Big) d\tau = \Big(\frac{|x-\eta|}{2\beta}\Big)^{\nu-1/2} K_{\nu-1/2}(\beta|x-\eta|)$$

taken into account, where $K_n(x)$ is the Macdonald function, the solution becomes

$$F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_{-\infty}^{\infty} \Big(\frac{|x-\eta|}{2\beta}\Big)^{\nu-1/2} K_{\nu-1/2}(\beta|x-\eta|) e^{-\alpha(x-\eta)} f(\eta) d\eta. \tag{10}$$

We thus obtain the solution of noninteger order ODE (1) by the operator method in the form of a convolution:

$$\begin{aligned}
&F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \int_{-\infty}^{\infty} \chi(x-\eta) f(\eta) d\eta,\\
&\chi(x-\eta) = \Big(\frac{|x-\eta|}{2\beta}\Big)^{\nu-1/2} K_{\nu-1/2}(\beta|x-\eta|) e^{-\alpha(x-\eta)}
\end{aligned} \tag{11}$$

with the kernel χ. It can be written in abbreviated form as

$$F(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu)} \chi * f, \qquad \chi = \Big(\frac{|x|}{2\beta}\Big)^{\nu-1/2} K_{\nu-1/2}(\beta|x|) e^{-\alpha x}. \tag{12}$$

The method of inverse differential and exponential operators finds numerous applications for solving physical problems. Some of them were considered in [14], [17], [19]. Before turning to examples, we note that formula (6), despite its triviality, allows significant progress in solving certain DEs by the operator method. In accordance with relation (6), for example, for the equation $f(x,t) = \psi(\partial_x + \kappa)F(x,t)$, we can write the equality $e^{\kappa x}F(x,t) = \psi^{-1}(\partial_x)e^{\kappa x}f(x,t)$. For the function $f(x,t) = \partial_t F(x,t)$, for example, we then have $\psi(\partial_x)e^{\kappa x}F(x,t) = \partial_t e^{\kappa x}F(x,t)$. Introducing the notation $G(x,t) = e^{\kappa x}F(x,t)$, we obtain the equation $\psi(\partial_x)G(x,t) = \partial_t G(x,t)$ with the initial condition $g(x) = G(x,0) = e^{\kappa x}F(x,0) = e^{\kappa x}f(x)$. Hence, to obtain the sought solution $F(x,t) = e^{-\kappa x}G(x,t)$ of the equation $\psi(\partial_x + \kappa)F(x,t) = f(x,t)$ with the operator $\psi(\partial_x + \kappa)$ and the initial condition $F(x,0) = f(x)$, we must solve the equation with the operator $\psi(\partial_x)$ for the function $G(x,t)$ with the corresponding initial condition $g(x) = e^{\kappa x}f(x)$.

3. Operator solution of equations of the Black–Scholes type

To demonstrate the technique outlined above, we solve the DE of the Black–Scholes type:

$$\frac{1}{\rho}\frac{\partial}{\partial t}F(x,t) = \left[x^2\frac{\partial^2}{\partial x^2} + (2\alpha x^2 + \lambda x)\frac{\partial}{\partial x} + (\alpha x)^2 - \mu\right]F(x,t), \qquad f(x) = F(x,0), \tag{13}$$

where α, ρ, λ, and μ are constant coefficients and $f(x) = F(x,0)$ is a function of the initial condition. It is easy to see that (13) is an equation with a shifted derivative, and it reduces to an equation of the form

$$\frac{1}{\rho}\frac{\partial}{\partial t}G(x,t) = x^2\frac{\partial^2}{\partial x^2}G(x,t) + \lambda x\frac{\partial}{\partial x}G(x,t) - \mu G(x,t), \qquad g(x) = G(x,0), \tag{14}$$

by replacing $\partial_x \to \partial_x + \alpha$. The solution of Eq. (13) can then be written in terms of the solution $G(x,t)$ of Eq. (14) with the initial condition $g(x) = G(x,0) = e^{\alpha x}F(x,0)$:

$$F(x,t) = e^{-\alpha x}G(x,t), \qquad g(x) = G(x,0) = e^{\alpha x}F(x,0). \tag{15}$$

Equation (14) in turn was solved by the operator method (see [19]). We choose the initial condition for Eq. (13) of the Black–Scholes type with a shifted variable in the form $f(x) = e^{-ax}x^n$. Then $g(x) = x^n$, and with the equalities (see [19])

$$G(x,t) = \frac{e^{-\rho\varepsilon t}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\sigma^2+\sigma\gamma\lambda/2\rho}g(xe^{\sigma\gamma})\,d\sigma$$

taken into account, the sought solution of Eq. (13) takes the simple form

$$F(x,t)|_{f(x)=e^{-ax}x^n} = e^{-\alpha x}x^n e^{\rho t(n^2+\lambda n-\mu)}. \tag{16}$$

Another example is the generalization of the DE of the Black–Scholes type with the Laguerre derivative ${}_{\mathrm{L}}D_x$

$$\frac{1}{\rho}\frac{\partial}{\partial t}A(x,t) = (\partial_x x\,\partial_x)^2 A(x,t) + \lambda(\partial_x x\,\partial_x)A(x,t) - \mu A(x,t), \qquad g(x) = A(x,0), \tag{17}$$

where ρ, λ, and μ are constant coefficients. We note that Eq. (17) is essentially a generalization of two equations, the diffusion and the Laguerre heat conduction equations, previously discussed in [19] and [25]. Following [19], we write solution (17) in form $A(x,t) = e^{\rho t(({}_{\mathrm{L}}D_x+\lambda/2)^2-\mu-(\lambda/2)^2)}g(x)$, and using operator relation (3), we obtain the solution for $A(x,t)$

$$A(x,t) = \frac{e^{-\rho t(\mu+(\lambda/2)^2)}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\sigma(\sigma+\sqrt{\rho t}(\lambda-2\,{}_{\mathrm{L}}D_x))}g(x)\,d\sigma. \tag{18}$$

We choose the initial function in the form $g(x) = (-x)^n/n!$. With operator definition (5) of the Laguerre polynomials, we obtain

$$A(x,t) = \frac{e^{-\varepsilon\rho t/4}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(\sigma+\sqrt{\rho t}\lambda)} L_n(x, 2\sigma\sqrt{\rho t}) d\sigma. \tag{19}$$

Integrating over $d\sigma$, we obtain the solution of generalized Black–Scholes equation (17) with the Laguerre derivative with the initial condition $g(x) = (-x)^n/n!$:

$$\begin{aligned} A(x,t) = \frac{e^{-\rho t\mu}}{\sqrt{\pi}} n! \sum_{r=0}^{n} \frac{(-x)^r (2\sqrt{\rho t})^{n-r}}{(n-r)!(r!)^2} \times \\ \times \left\{ \frac{e^{i(n-r)\pi}+1}{2} \Gamma\left(\frac{1+n-r}{2}\right) {}_1F_1\left(-\frac{n-r}{2}, \frac{1}{2}, -\left(\frac{\lambda\sqrt{\rho t}}{2}\right)^2\right) + \right. \\ \left. + \frac{e^{i(n-r)\pi}-1}{2} \lambda\sqrt{\rho t}\, \Gamma\left(1+\frac{n-r}{2}\right) {}_1F_1\left(\frac{1-(n-r)}{2}, \frac{3}{2}, -\left(\frac{\lambda\sqrt{\rho t}}{2}\right)^2\right) \right\}, \end{aligned} \tag{20}$$

where Γ is the gamma function and ${}_1F_1$ is the hypergeometric function. It is obvious that the solution for an initial condition in the form of a polynomial in x corresponds to the sum of expression (20), and for an initial condition function given as a series of Laguerre polynomials $g(x) = \sum_n a_n L_n(x)$, the solution can be written as a series in the form

$$A(x,t) = \frac{e^{-\varepsilon\rho t/4}}{\sqrt{\pi}} \sum_n a_n \int_{-\infty}^{\infty} e^{-\sigma(\sigma+\lambda\sqrt{\rho t})} L_n(x, 2\sigma\sqrt{\rho t}+1) d\sigma. \tag{21}$$

We introduce the transform $\varphi(x) = \int_0^\infty e^{-\kappa} g(x\kappa) d\kappa$ of the function $g(x) = \varphi(D_x^{-1})\{\mathbf{1}\}$ in the same way as described in [19]. The solution of Eq. (17) with the equalities $[{}_{\mathrm{L}}D_x, D_x^{-1}] = -1$ and $e^{-t\,\partial/\partial D_x^{-1}} \varphi(D_x^{-1}) = e^{-t\,\partial/\partial D_x^{-1}} g(x) = \varphi(D_x^{-1} - t)\{\mathbf{1}\}$ taken into account can be expressed by the integral of the inverse derivative operator D_x^{-1} under the condition that it converges:

$$A(x,t) = \frac{e^{-\varepsilon\rho t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(\sigma+\alpha\lambda)} g(x,t) d\sigma, \tag{22}$$

where

$$g(x,t) = \varphi(D_x^{-1} - 2\sigma\alpha)\{\mathbf{1}\} = e^{-2\sigma\alpha\,\partial/\partial D_x^{-1}} \varphi(D_x^{-1})\{\mathbf{1}\}. \tag{23}$$

For example, for $g(x) = W_0(-x^2, 2)$, where

$$W_n(x,m) = \sum_{r=0}^{\infty} \frac{x^r}{r!(mr+n)!}, \quad m \in \mathbb{N}, \quad n \in \mathbb{N}_0,$$

is a special case of the Bessel–Wright function [29], its image turns out to be a function $\varphi(x) = e^{-x^2}$, and applying operator relation (3), according to (23) (see [19]), we obtain the formula

$$g(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 + 4i\sigma\alpha\xi} C_0(2i\xi x) d\xi, \tag{24}$$

where $C_n(x)$ is the Bessel–Tricomi function [31] associated with the Bessel–Wright function and Bessel functions,

$$C_n(x) = W_n(-x,1) = x^{-n/2} J_n(2\sqrt{x}) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!(r+n)!}, \quad n \in \mathbb{N}_0.$$

Some modifications of the Black–Scholes equations were studied by the operator method in [32] and [33].

4. Operator solution of the extended Fourier heat conduction equation and Schrödinger-type equations

The dynamics of a charge passing a potential barrier in an electrostatic field is described by generalized heat conduction equation with a linear term

$$\partial_t G(x,t) = \alpha\, \partial_x^2 G(x,t) + \beta x G(x,t), \qquad G(x,0) = g(x), \tag{25}$$

and is essentially the Schrödinger equation with imaginary time. The Euclidean approach in quantum mechanics corresponding to this is described in detail, for example, in [34]. The tunneling effect of a particle in the region where its energy is lower than the potential energy of the barrier also seems important in the study of vacuum effects in quantum chromodynamics. In the context of heat propagation, the second term in the right-hand side of (25) describes radiation in an environment with a small temperature difference across a heat insulating gasket with a linearly varying thickness along the coordinate. Equation (25) was studied in [17], [18], and [20]. The solution was obtained in the form of the sequential action of the operators $\widehat{\widehat{S}}$ and $\widehat{\widehat{\Theta}}$ given by (4) on the initial condition function $G(x,0) = g(x)$,

$$G(x,t) = e^{\Phi(x,t;\alpha,\beta)}\, \widehat{\Theta}\widehat{S} g(x) = e^{\Phi(x,t;\alpha,\beta)} g(x + \alpha\beta\, t^2, t) =$$

$$= \frac{e^{\Phi(x,t;\alpha,\beta)}}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} e^{-(x+\alpha\beta t^2 - \xi)^2/4t\alpha} g(\xi) d\xi, \tag{26}$$

where $\widehat{\Theta} = e^{\alpha\beta t^2\, \partial_x}$, $\widehat{S} = e^{\alpha t\, \partial_x^2}$, and $\Phi(x,t;\alpha,\beta) = \alpha\beta^2 t^3/3 + \beta t x$. Solutions describing the evolution of a pulse have a special significance for the practical study of heat propagation because it is a well-established experimental method in investigating the heat conduction of materials. We consider Eq. (25) with the initial condition $g(x) = x^k e^{\delta x}$. It models an asymmetric pulse at $\delta < 0$, and an approximation of the experimental data by the series $\sum_{k,\delta} x^k e^{\delta x}$ allows studying the evolution of almost all pulses analytically. In accordance with (26) and (4), we obtain solution (25) at $g(x) = x^k e^{\delta x}$

$$G(x,t)|_{g(x)=x^k e^{\delta x}} = e^{\Phi + \delta(x+\delta\alpha t+\alpha\beta t^2)} H_k(x + 2t\alpha\delta + \alpha\beta t^2, \alpha t), \tag{27}$$

and for $g(x) = \sum_{k,\delta} x^k e^{\delta x}$, we have $G(x,t) = e^{\Phi} \sum_{k,\delta} e^{\delta(x+\delta\alpha t+\alpha\beta t^2)} H_k(x + 2t\alpha\delta + \alpha\beta t^2, \alpha t)$.

We consider a generalization of Eq. (25) of the form

$$\partial_t F(x,t) = \alpha \partial_x^2 F(x,t) + \varsigma\, \partial_x F(x,t) + \beta x F(x,t) + \gamma F(x,t), F(x,0) = f(x). \tag{28}$$

Following the general logic of solving equations with a shifted derivative (see the beginning of this section) and separating the square of the operator $\partial_x + \kappa$, where $\kappa = \varsigma/2\alpha$, we obtain the solution of Eq. (28)

$$F(x,t) = e^{t(\gamma - \alpha\kappa^2 - \kappa x)} G(x,t), \tag{29}$$

where the function $G(x,t)$ satisfies (25) with the initial condition $g(x) = e^{\kappa x} f(x)$. We choose the initial condition for (28) in the form of a monomial $f(x) = x^k$. Then $g(x) = x^k e^{\kappa x}$, and substituting solution (27) of Eq. (25) in (29), we immediately obtain the sought solution of Eq. (28) written in terms of Hermite polynomials:

$$F(x,t)|_{f(x)=x^k} = \exp\!\left(\frac{\alpha\beta^2 t^3}{3} + \frac{\varsigma\beta t^2}{2} + \beta t x + t\gamma\right) H_k(x + \varsigma t + \alpha\beta t^2, \alpha t). \tag{30}$$

We now consider the two-dimensional heat conduction equation

$$\partial_t F(x,y,t) = \{(\alpha\, \partial_x^2 + \varepsilon\, \partial_x\, \partial_y + \gamma\, \partial_y^2) + bx + cy\} F(x,y,t), \quad \min(\alpha,\varepsilon,\gamma) > 0, \tag{31}$$

with the initial condition $F(x,y,0) = f(x,y)$, which also describes the two-dimensional dynamics of a charge in an electric field passing under a potential barrier. The operator solution of Eq. (31) can be obtained by analogy with the solution of one-dimensional equation (25) (see [17], [20]) or by using the Baker–Campbell–Hausdorff formula. It is a sequential action of the two-dimensional analogue $\widehat{S}$ of the heat conduction operator

$$\widehat{\mathrm{E}} = e^{t(\alpha\,\partial_x^2+\varepsilon\,\partial_x\,\partial_y+\gamma\,\partial_y^2)} \tag{32}$$

on the initial condition $\widehat{\mathrm{E}}f(x,y) = f(x,y,t)$ and the action of the coordinate shift operators $\widehat{\Theta}_x = e^{t^2(\alpha b+\varepsilon c/2)\partial_x}$ and $\widehat{\Theta}_y = e^{t^2(\gamma c+\varepsilon b/2)\partial_y}$ on $f(x,y,t)$. As a result, we obtain a solution

$$F(x,y,t) = e^{\Psi}\,\widehat{\Theta}_x\widehat{\Theta}_y\widehat{\mathrm{E}}f(x,y) \propto f\left(x+t^2\left(\alpha b+\frac{\varepsilon c}{2}\right), y+t^2\left(\gamma c+\frac{\varepsilon b}{2}\right), t\right), \tag{33}$$

where $\Psi = (\alpha b^2+\gamma c^2+\varepsilon bc)t^3/3 + t(bx+cy)$ is the phase. The action of $\widehat{\mathrm{E}}$ has the form of a double Gauss-type integral (see [22]). At $\varepsilon = 0$ in (31), instead of the heat conduction operator $\widehat{\mathrm{E}}$, we have the product $\widehat{S}_x\widehat{S}_y$ of heat conduction operators (4) for each of the coordinates, and the solution hence becomes

$$F(x,y,t;\varepsilon=0) = e^{\Psi}\widehat{\Theta}_x\widehat{\Theta}_y\widehat{S}_x\widehat{S}_y f(x,y) \propto f(x+t^2\alpha b, y+t^2\gamma c, t). \tag{34}$$

Operator definition (32) allows obtaining a solution of Eq. (31), for example, for a power-law function of the initial condition $f(x,y) = x^m y^n$. The action of the heat conduction operator $\widehat{\mathrm{E}}$ on f gives the expression

$$\widehat{\mathrm{E}}\{x^m y^n\} = H_{m,n}(x,t\alpha,y,t\gamma|t\varepsilon), \tag{35}$$

where $H_{m,n}(x,t\alpha,y,t\gamma|t\varepsilon)$ are Hermite polynomials in three variables and parameters with two indices (see [19], [22], [26]). The subsequent shift by the operators $\widehat{\Theta}_x\widehat{\Theta}_y$ leads to the solution of two-dimensional heat conduction equation (31) with $f(x,y) = x^m y^n$:

$$F(x,t) = e^{\Psi}H_{m,n}\left(x+t^2\left(\alpha b+\frac{\beta c}{2}\right), t\alpha; y+t^2\left(\gamma c+\frac{\beta b}{2}\right), t\gamma\middle| t\varepsilon\right). \tag{36}$$

It is obvious that the obtained solution (36) of two-dimensional heat conduction equation (31) is a direct generalization of solutions (27) with $\delta = 0$ for the one-dimensional analogue (25) of Eq. (31). A further generalization to three dimensions is not difficult.

5. Propagation of a δ-pulse in the Fourier heat conduction

The experimental value of heat conduction of different materials is usually determined by measuring heat pulses [35]. The characteristic relaxation time is measured for this purpose [36]. An ultrashort laser pointlike pulse is often used as the initial pulse. It is modeled by the Dirac δ-function. We model such a pulse mathematically in the equation of Fourier heat conduction in two dimensions, $f(x,y) = \delta(x,y)$. The action of an operator $\widehat{S}$ on the δ-function yields the Gaussian-type distribution

$$f(x,y,t) = \widehat{S}_x\widehat{S}_y\delta(x,y) = \frac{\exp[-(x^2/\alpha+y^2/\gamma)/4t]}{4\pi t\sqrt{\alpha\gamma}}, \tag{37}$$

and the shift operators $\widehat{\Theta}_{x,y}$ in the presence of linear terms in the Fourier heat conduction equation leads to a solution with the initial function $f(x,y) = \delta(x,y)$:

$$F(x,y,t)|_{f=\delta(x,y)} \equiv \chi(x,y,t) = e^{\Psi}\,\widehat{\Theta}_x\widehat{\Theta}_y\widehat{S}_x\widehat{S}_y\delta(x,y) = \\ = e^{\Psi}\frac{\exp\bigl(-[(x+t^2\alpha b)^2/\alpha+(y+t^2\gamma c)^2/\gamma]/4t\bigr)}{4\pi t\sqrt{\alpha\gamma}}. \tag{38}$$

Its one-dimensional analogue in the case of Eq. (25) at $f(x) = \delta(x)$ has the form

$$F(x,t)|_{f=\delta(x)} \equiv \chi(x,t) = e^{\Phi}\frac{\exp\bigl(-(x+\alpha\beta t^2)^2/4t\alpha\bigr)}{2\sqrt{\pi t\alpha}}, \tag{39}$$

where Φ and Ψ are defined above (see (26) and (33)). By direct substitution, we can easily verify that the solutions obtained above satisfy Eqs. (25) and (31).

We now find a solution of extended heat conduction equation (28) with the initial condition $f(x) = \delta(x)$. We distinguish $\partial_x + \kappa$, where $\kappa = \varsigma/2\alpha$, and use result (29). The solution for the function $G(x,t)$ with the initial condition $g(x) = e^{\kappa x}\delta(x)$ is easy to find from (26):

$$\begin{aligned} G(x,t)|_{g(x)=e^{\kappa x}\delta(x)} &= \frac{e^{\Phi(x,t;\alpha,\beta)}}{2\sqrt{\pi\alpha t}}\int_{-\infty}^{\infty} e^{-(x+\alpha\beta t^2-\xi)^2/4t\alpha}e^{\kappa\xi}\delta(\xi)d\xi = \\ &= \frac{e^{\Phi(x,t;\alpha,\beta)}}{2\sqrt{\pi\alpha t}}e^{-(x+\alpha\beta t^2)^2/4t\alpha}. \end{aligned} \tag{40}$$

The sought solution of Eq. (28) with $f(x) = \delta(x)$ becomes

$$F(x,t)|_{f(x)=\delta(x)} = \exp\biggl(\frac{1}{3}\alpha\beta^2t^3 + t\biggl(\beta x + \gamma - \frac{\varsigma^2}{4\alpha}\biggr) - \frac{\varsigma}{2\alpha}x\biggr)\frac{1}{2\sqrt{\pi\alpha t}}e^{-(x+\alpha\beta t^2)^2/4t\alpha}. \tag{41}$$

We note that the linear term appears in the equations of various models of heat conduction (see, e.g., [11]). The example of solution (41) of extended Fourier equation (28) shows that the initial δ-function decays, but there is then an unlimited exponential growth of the solution with a shift with respect to the maximum point $x = 0$, where the maximum is located at the initial instant (see Fig. 1, where $\alpha = 0.3$ and $\beta = \gamma = -\varsigma = 1$). The behavior of solution (39) of Eq. (25) at $\alpha = 0.3$ and $\beta = 3$ is qualitatively similar to the behavior shown in Fig. 1; we omit the plot for brevity. Choosing the parameter values α, β, γ, and ς, we can ensure that the solution almost completely decays in a wide area (see Fig. 2). We note the weak asymmetry of solution (39) with respect to the initial maximum point $x = 0$ at small times t due to the linear term $\beta \neq 0$.

The obtained solution (39) of ordinary Fourier equation (25) with $\beta = 0$ decays as $t \to \infty$ (see Fig. 3). Hence, although the influence of the linear term in Fourier equation (25) is insignificant at small times (see Fig. 1–3 for $t < 0.1$), it becomes dominant at large times (see Fig. 1 for $t > 0.1$ and Fig. 2 for $t > 2.5$).

Moreover, the solution $F(x,t)$ of extended Fourier equation (28) reaches its maximum at the border $t = T$ of the considered area $x \subset [-l, L]$, $t \subset [T_0, T]$ (see Figs. 1 and 2), violating the maximum principle, which guarantees the uniqueness and stability of solutions. The failure to satisfy the maximum principle and the fact that the solution growth over time is unlimited in the context of heat transfer do not correspond to the second law of thermodynamics, which assumes that $\beta = 0$.

Returning to the quantum mechanical interpretation of Eqs. (25) and (31), we note that by replacing $t \to -i\tau$, $\alpha \to -\alpha$, and $\gamma \to -\gamma$ in (25) and (31), we obtain the ordinary Schrödinger equation for a charge in an electrostatic field, and its solution can be obtained by respectively replacing as above in (26), (34) and in (38), (39) for the initial δ-function. The obtained solutions $F(x,y,t)$ can be regarded as the amplitude of the transition probability from the initial point $x = 0$ of the particle at $t = 0$ to the point with the coordinate x during the time interval $t > 0$. We note that in this case, the phases Φ and Ψ become complex and do not matter from the physical standpoint, because the probability is determined by $|F(x,y,t)|^2$; the increase of this probability over time due to the phase in the same way as shown in Fig. 1 does not happen in this case. A solution $F(x,y,t)$ of Eqs. (25) and (31) can also be understood as the concentration with each particle moving independently of the others. The initial function $f(x) = \delta(x,y)$ means that all

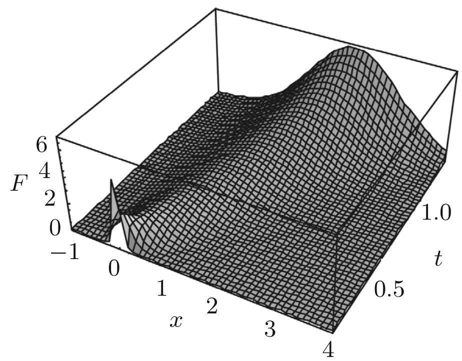

Fig. 1. Evolution of the initial function $\delta(x)$ in the extended Fourier heat conduction equation with $\alpha = 0.3$ and $\beta = \gamma = -\varsigma = 1$, $t \in [0.01, 1.43]$.

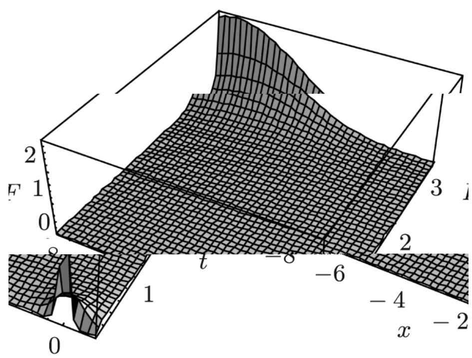

Fig. 2. Evolution of the initial function $\delta(x)$ in the extended Fourier heat conduction equation with $\alpha = 1$ and $\beta = \gamma = \varsigma = -1$, $t \in [0.01, 3.2]$.

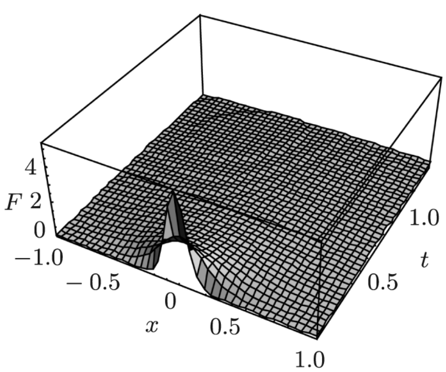

Fig. 3. Evolution of the initial function $\delta(x)$ in the extended Fourier heat conduction equation with $\alpha = 0.3$ and $\beta = \gamma = \varsigma = 0$, $t \in [0.01, 1.43]$.

particles were at the point (x, y) at the moment $t = 0$, and the solutions of Eqs. (25) and (31) describe the particle concentration during the time interval t. All the results, up to a factor equal to the number of diffusing particles, then relate to the particle concentration under the condition that we neglect the mutual interaction of the diffusing particles.

We also note that solutions (26) and (34) of Eqs. (25) and (31) for an arbitrary function of the initial condition is a convolution of the initial condition function f with the solution of the initial Dirac δ-function. Using the operator method, we thus obtain a solution of the Schrödinger equation (of heat conduction) of a particle in an electrostatic field diffusing under a potential barrier by a convolution with the kernel χ:

$$F(x,y,t)=\int_{-\infty}^{\infty}\chi(x-\eta,y-\rho)\big|_{\substack{t\to -i\tau,\\ \alpha\to-\alpha,\\ \gamma\to-\gamma}} f(\eta,\rho)d\eta\,d\rho\equiv\chi * f, \tag{42}$$

where the Gauss-type kernel χ is a solution of the original equation with $F(x,y,0)=\delta(x,y)$ and is defined by formula (38) with the replacements $t\to -i\tau$, $\alpha\to-\alpha$, and $\gamma\to-\gamma$. A one-dimensional analogue of convolution (42) turns out to be with kernel (39) by the same replacement.

Although the Fourier equation well describes the transfer of heat in homogeneous nondeformable solids under normal conditions, it is inapplicable at ultralow temperatures and also to low-dimensional systems, such as graphene, nanofibers, etc., and to materials with substantial internal inhomogeneity. The study of heat transfer processes in such cases is nontrivial and requires special consideration. We will turn to this in future publications.

6. Operator solution of equations of the Fokker–Planck type

Equations of the Fokker–Planck type are encountered in modeling the propagation of electron beams in accelerators and undulators. An operator solution of the Fokker–Planck equation

$$\partial_t F(x,t)=\alpha\,\partial_x^2F(x,t)+\beta x\,\partial_x F(x,t),\qquad F(x,0)=f(x), \tag{43}$$

is analogous to the solution of the Schrödinger equation:

$$F(x)=\widehat{U}f(x),\qquad \widehat{U}f(x)=e^{t\alpha\,\partial_x^2+t\beta x\,\partial_x}f(x)=e^{\sigma\,\partial_x^2}f(e^{\beta t}x), \tag{44}$$

where $\sigma=(1-e^{-2\beta t})\alpha/\beta$ (see, e.g., [17]). In contrast to the Schrödinger equation, in the solution of a Fokker–Planck-type equation, the initial function f is transformed by only one heat conduction operator $\widehat{S}$ (compare (44) with (26)).

We now consider the generalization of the Fokker–Planck equation

$$\partial_t F(x,t)=\{\alpha\,\partial_x^2+(\beta x+2\alpha\delta)\partial_x+\beta\delta x+\gamma\}F(x,t). \tag{45}$$

For Fokker–Planck-type equations, it makes sense to consider a Gauss-type initial condition $f(x)=e^{-x^2}$ because it is most common for beams in accelerators. Separating the operator $\partial_x+\delta$, we see that the generalized solution of Fokker–Planck-type equation (45) reduces to the solution of Eq. (43) for the function G with the initial condition $g(x)=G(x,0)=e^{\alpha x}f(x)=e^{\delta x-x^2}$. The solution $G(x,t)$ is obtained using the Gauss-type transformation

$$G(x,t)=\frac{1}{\sqrt{2\pi\rho}}\int_{-\infty}^{\infty}g(\xi)\exp\left[-\left(\frac{e^{\beta t}x-\xi}{\sqrt{2\rho}}\right)^2\right]d\xi,$$

where $\rho(t)=(\alpha/\beta)(e^{2\beta t}-1)$. We then have $F(x,t)=e^{\gamma-\alpha\delta^2}e^{-\delta x}G(x,t)$ and finally obtain the sought solution of Eq. (45) in the form

$$F(x,t)\big|_{f(x)=e^{-x^2}}=\frac{e^{\gamma-(\delta/2)^2(4\alpha-1)-\delta x}}{\sqrt{1+2\rho(t)}}\exp\left(-\frac{(e^{\beta t}x-\delta/2)^2}{1+2\rho(t)}\right). \tag{46}$$

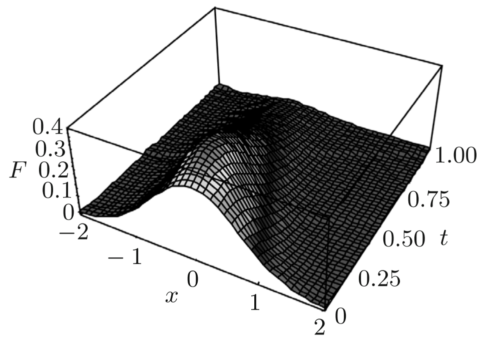

Fig. 4. Plot of the solution of the Fokker–Planck-type equation $\partial_t F(x,t) = \{\alpha\,\partial_x^2 + (\beta x + 2\alpha\delta)\partial_x + \beta\delta x + \gamma\}F(x,t)$ for $F(x,0) = e^{-x^2}$, $F(x,\infty) < \infty$ with $\alpha = 1$, $\beta = 4$, $\gamma = 3$, and $\delta = 2$.

A plot of the solution at $\alpha = 1$, $\beta = 4$, $\gamma = 3$, and $\delta = 2$ is shown in Fig. 4. We note the asymmetry of the solution due to the nonzero γ and δ. It is also important that the solution reaches a maximum inside considered area and violates the maximum principle, which guarantees the uniqueness and stability of the solution.

Another modification of Fokker–Planck equation (43) consists in adding a second-order differential operator ∂_t^2 with respect to time in the left-hand side of the equation,

$$\left(\frac{\partial^2}{\partial t^2} + \varepsilon\,\frac{\partial}{\partial t}\right)F(x,t) = \left(\alpha\,\frac{\partial^2}{\partial x^2} + \beta x\,\partial_x\right)F(x,t), \quad \alpha,\beta,\varepsilon = \text{const}, \tag{47}$$

similar to the hyperbolic heat conduction equation of Cattaneo [37], who proposed a relaxation heat conduction model qualitatively describing the low-temperature process due to heat transfer by the phonon mechanism in the equation $(\tau\,\partial_t^2 + \partial_t)T = k_T\nabla^2 T$. Studying this equation is beyond the scope of our paper; we will return to it in future publications. We use the operator technique to solve Eq. (47). For equations of the type

$$\left(\frac{\partial^2}{\partial t^2} + \hat{\varepsilon}(x)\frac{\partial}{\partial t}\right)F(x,t) = \widehat{D}(x)F(x,t) \tag{48}$$

containing the operator $\hat{\varepsilon}(x)$, the operator solution with the initial condition $F(x,0) = f(x)$ and the final condition $F(x,\infty) = 0$, or at least a bounded solution as $t \to \infty$, $F(x,\infty) < \infty$, can be obtained using the Laplace transformation

$$e^{-t\sqrt{V}} = \frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-t^2/4\xi - \xi V}\,\frac{d\xi}{\xi\sqrt{\xi}}, \quad t > 0, \tag{49}$$

as an integral

$$F(x,t) = e^{(-t/2)\hat{\varepsilon}(x)}\frac{t}{4\sqrt{\pi}}\int_0^\infty e^{-t^2/16\xi}e^{-\xi\hat{\varepsilon}^2(x)}e^{-4\xi\widehat{D}(x)}f(x)\frac{d\xi}{\xi\sqrt{\xi}} \tag{50}$$

under the condition that it converges. In the general case of $\widehat{D}(x)$ and $\hat{\varepsilon}(x)$, Eq. (48) describes a very wide range of physical processes such as diffusion, heat propagation, evolution of packets of charged particles, etc. The solution depends entirely on the explicit form of $\widehat{D}(x)$ and $\hat{\varepsilon}(x)$ and on the value of commutator $[\hat{\varepsilon}^2, \widehat{D}]$. In our case, $\varepsilon = \text{const}$, and commutator with $\widehat{D}(x) = \alpha\,\partial_x^2 + \beta x\,\partial_x$ is equal to zero. We then have

$$F(x,t) = e^{(-t/2)\varepsilon}\frac{t}{4\sqrt{\pi}}\int_0^\infty e^{-t^2/16\xi - \xi\varepsilon^2}e^{-a\,\partial_x^2 - bx\,\partial_x}f(x)\frac{d\xi}{\xi\sqrt{\xi}}, \tag{51}$$

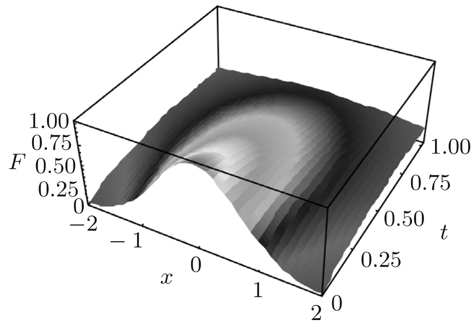

Fig. 5. Plot of the solution of the generalized Fokker–Planck equation $(\tau\,\partial_t^2+\partial_t)F=(A\,\partial_x^2+Bx\,\partial_x)F$ for $F(x,0)=e^{-x^2}$, $F(x,\infty)<\infty$ at $\tau=2/3$ and $B=8/3$ ($\alpha=1$, $\beta=4$, $\varepsilon=1.5$).

where $a=4\xi\alpha>0$, $b=4\xi\beta>0$, and $a/b=\alpha/\beta$. Therefore, the exponential $\widehat{U}=e^{-a\,\partial_x^2-bx\,\partial_x}$ differs from the exponential in the solution of the ordinary Fokker–Planck equation by the sign and the time-independence. Using the identity

$$e^{\hat{A}+\hat{B}}=e^{\hat{A}(1-e^{-m})/m}e^{\hat{B}},$$

which holds for $[\hat{A},\hat{B}]=m\hat{A}$ (in our case, $\hat{A}=-a\,\partial_x^2$, $\hat{B}=-bx\,\partial_x$, and $m=-2b$), and the relation

$$e^{-bx\,\partial_x}f(x)=f(e^{-b}x),$$

according to [19] and [21], we have $\widehat{U}f(x)=\widehat{S}f(y)$, where $\widehat{S}=e^{(-\rho/2)\partial_y^2}$ and $y=e^{-b}x$. Hence, the sought solution of Eq. (47) transforms into the integral of the action of the operator $\widehat{S}=e^{(-\rho/2)\partial_y^2}$ on the function $f(y)$:

$$F(x,t)=\frac{te^{-t\varepsilon/2}}{4\sqrt{\pi}}\int_0^\infty e^{-t^2/16\xi-\xi\varepsilon^2}\widehat{S}f(y)\frac{d\xi}{\xi\sqrt{\xi}}. \tag{52}$$

As the initial function, we choose the Gaussian function $f(x)=e^{-x^2}$, which under the action of the evolution operator transforms into

$$\widehat{U}f(x)=\widehat{S}f(y)=\frac{1}{\sqrt{1-2\rho}}\exp\biggl[-\frac{e^{-2b}x^2}{1-2\rho}\biggr], \tag{53}$$

where $\rho=(a/b)(1-e^{-2b})=(\alpha/\beta)(1-e^{-8\beta\xi})$. Hence, the evolution of the initial Gaussian distribution $f(x)=e^{-x^2}$ under the condition that the solution of Eq. (47) is bounded is given by

$$F(x,t)|_{f(x)=e^{-x^2}}=\frac{te^{-\varepsilon t/2}}{4\sqrt{\pi}}\int_0^\infty \exp\biggl(-\frac{t^2}{16\xi}-\xi\varepsilon^2-\frac{e^{-8\beta\xi}x^2}{1-2\alpha(1-e^{-8\beta\xi})/\beta}\biggr)\times$$
$$\times\frac{d\xi}{\xi\sqrt{\xi}\sqrt{1-2\alpha(1-e^{-8\xi\beta})/\beta}}. \tag{54}$$

Equation (47) can also be written as

$$(\tau\,\partial_t^2+\partial_t)F(x,t)=(A\,\partial_x^2+Bx\,\partial_x)F(x,t), \tag{55}$$

where $\tau=1/\varepsilon$, $A=\alpha/\varepsilon$, and $B=\beta/\varepsilon$. Plots of the obtained solution (54) are shown in Figs. 5–7.

We note the rapidly decaying solution at small values of the parameter τ, i.e., at large ε (cf. Figs. 5 and 7). The solution is symmetric in x, and the contribution of the term with $\beta\neq 0$ can be seen in the

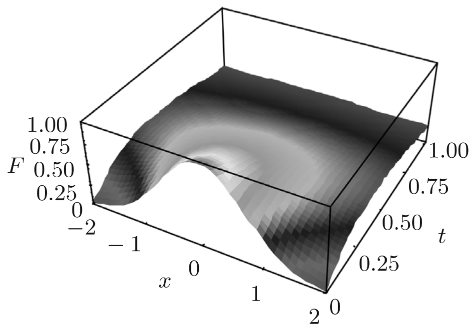

Fig. 6. Plot of the solution of the generalized Fokker–Planck equation $(\tau\,\partial_t^2+\partial_t)F = (A\,\partial_x^2+Bx\,\partial_x)F$ for $F(x,0) = e^{-x^2}$, $F(x,\infty) < \infty$ at $\tau = 2/3$, $A = 2/3$, and $B = 32/3$ ($\alpha = 1$, $\beta = 16$, $\varepsilon = 1.5$).

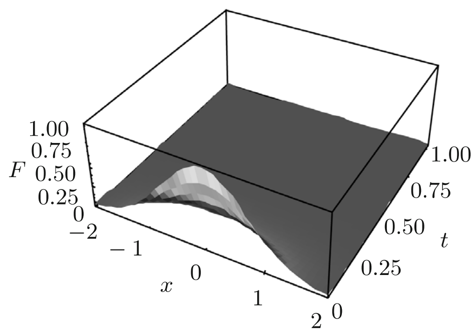

Fig. 7. Plot of the solution of the generalized Fokker–Planck equation $(\tau\,\partial_t^2+\partial_t)F = (A\,\partial_x^2+Bx\,\partial_x)F$ for $F(x,0) = e^{-x^2}$, $F(x,\infty) < \infty$ at $\tau = 1/15$, $A = 2/3$, and $B = 8/3$ ($\alpha = 10$, $\beta = 40$, $\varepsilon = 15$).

overall rise and convexity (i.e., the sign of the derivative $\partial_t^2 F$) of the constructed solution with respect to t for $t > 0$ independently of the value of x (see Fig. 6) compared with the solution of the ordinary Fokker–Planck equation $\partial_t F = (A\,\partial_x^2 + Bx\,\partial_x)F$, shown in Fig. 8 (cf. Fig. 5).

We note that for small values of the parameter β, the constructed solution of extended Fokker–Planck equation (47) is close to the solution of the hyperbolic Cattaneo heat conduction equation [37], which can be obtained at $\beta = 0$ and qualitatively describes such phenomena as the second sound in liquid helium [38] and solid crystals [39], conductivity at low temperatures less than 25 K, and other phenomena.

For contrast, we give the operator solution of an equation similar to (47) involving a mixed derivative with respect to the coordinate and time:

$$\left(\frac{\partial^2}{\partial t^2} + \varepsilon\,\frac{\partial^2}{\partial x\,\partial t}\right)F(x,t) = \left(\frac{\partial^2}{\partial x^2} + \beta x\,\partial_x\right)F(x,t), \qquad \varepsilon, \beta = \text{const}, \tag{56}$$

which contains an operator $\hat{\varepsilon} = \varepsilon\,\partial_x$ commuting with the second derivative with respect the coordinate in $\widehat{D}$. Acting similarly to solving Fokker–Planck-type equation (47), we obtain the equality

$$F(x,t) = \frac{t}{4\sqrt{\pi}}\int_0^\infty e^{-t^2/16\xi}\,\widehat{\Theta}\widehat{S}f(y)\frac{d\xi}{\xi\sqrt{\xi}}, \tag{57}$$

where

$$\widehat{S} = e^{-(\rho/2)\partial_y^2}, \qquad \widehat{\Theta} = e^{(-t/2)\varepsilon\,\partial_x},$$

$$\rho = \frac{a}{b}\left(1 - e^{-2b}\left(1 - \xi\varepsilon^2\frac{2b}{a}\right)\right) = \frac{\alpha}{\beta}\left(1 - e^{-8\beta\xi}\left(1 - \xi\varepsilon^2\frac{2\beta}{\alpha}\right)\right).$$

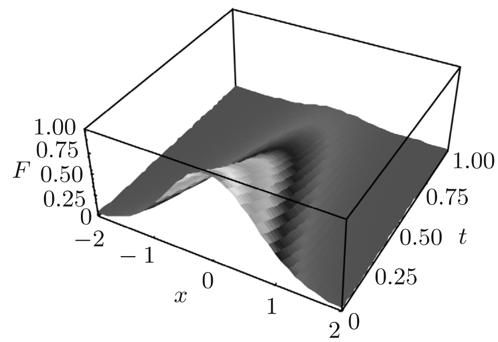

Fig. 8. Plot of the solution of the ordinary Fokker–Planck equation $\partial_t F = (A\,\partial_x^2 + Bx\,\partial_x)F$, $F(x,0) = e^{-x^2}$ for $F(x,\infty) < \infty$ at $A = 2/3$ and $B = 8/3$.

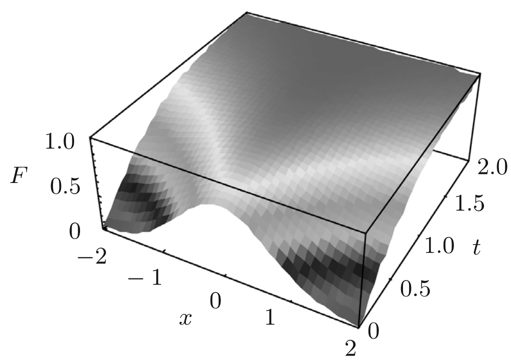

Fig. 9. Plot of a bounded solution of the equation $(\partial_t^2 + \varepsilon\,\partial_{x,t}^2)F(x,t) = (\alpha\,\partial_x^2 + \beta x\,\partial_x)F(x,t)$ with the initial function $f(x) = e^{-x^2}$ for $\alpha = 1$, $\beta = 5$, and $\varepsilon = 0$.

For the equation with an initial condition in the form of a Gaussian function $f(x) = e^{-x^2}$, we obtain a solution bounded at $t = \infty$ in the form

$$F(x,t)|_{f(x)=e^{-x^2}} = \frac{t}{4\sqrt{\pi}} \int_0^\infty \exp\biggl(-\frac{t^2}{16\xi} - \frac{e^{-8\xi\beta}(x - t\varepsilon/2)^2}{1 - 2\alpha(1 - e^{-8\beta\xi}(1 - \xi\varepsilon^2 2\beta/\alpha))/\beta}\biggr) \times$$
$$\times \frac{d\xi}{\xi\sqrt{\xi}\sqrt{1 - 2\alpha(1 - e^{-8\beta\xi}(1 - \xi\varepsilon^2 2\beta/\alpha))/\beta}}. \tag{58}$$

With $\varepsilon = 0$, the result is simplified to

$$F(x,t;\varepsilon = 0)|_{f(x)=e^{-x^2}} = \frac{t}{4\sqrt{\pi}} \int_0^\infty \exp\biggl(-\frac{t^2}{16\xi} - \frac{e^{-8\xi\beta}x^2}{1 - 2\alpha(1 - e^{-8\xi\beta})/\beta}\biggr) \times$$
$$\times \frac{d\xi}{\xi\sqrt{\xi}\sqrt{1 - 2\alpha(1 - e^{-8\xi\beta})/\beta}}. \tag{59}$$

A plot of the function $F(x,t)|_{f(x)=e^{-x^2}}$ with $\alpha = 1$, $\beta = 5$, and $\varepsilon = 0$ is shown in Fig. 9. The solution increases with time and saturates for $t > 2$, unlike solution (54) of Eq. (47), which decays. An example of the solution with $\alpha = 1$, $\beta = 5$, and $\varepsilon = 4$ is shown in Fig. 10. Nonzero values of ε break the monotonicity of the growth and lead to the appearance of a local maximum (see Fig. 10). In both cases, the maximum principle does not hold, as shown in Figs. 9 and 10.

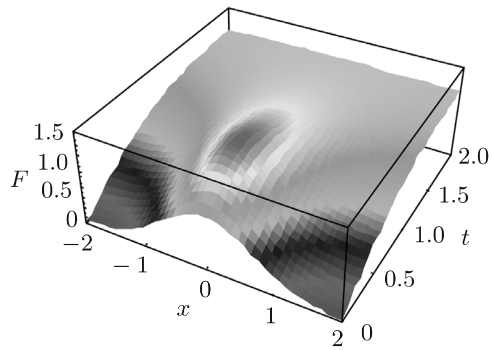

Fig. 10. Plot of a bounded solution of the equation $(\partial_t^2 + \varepsilon\,\partial_{x,t}^2)F(x,t) = (\alpha\,\partial_x^2 + \beta x\,\partial_x)F(x,t)$ with the initial function $f(x) = e^{-x^2}$ for $\alpha = 1$, $\beta = 5$, and $\varepsilon = 4$.

7. Conclusions

Using the operator method, we have obtained solutions for various classes of DEs. We used inverse differential operators and exponential operators and widely used operator identities and integral transformations. In particular, an operator solution of an ODE with a real noninteger-order derivative in the form of a convolution including the Macdonald function (second-kind Bessel function with an imaginary argument). We obtained an operator solution of an extended Black–Scholes-type equation with additional terms. We solved a Black–Scholes-type equation with the Laguerre derivative, i.e., an equation with fourth-order derivatives with respect to the coordinate.

We constructed the solution of the Schrödinger equation for a charge in an electric field passing through a potential barrier in the two-dimensional case. The quantum mechanical interpretation of the obtained solution of the Schrödinger equation for a particle in an electrostatic field is the amplitude of the probability that a particle located at the initial point $x = 0$ at the initial instant appears at the point with the coordinate x at the instant t. For an arbitrary initial condition f, the solution can be written in the form of its convolution with the solution for the initial Dirac δ-function. We constructed a solution of the extended Fourier heat conductivity equation and demonstrated the propagation of a δ-shaped impulse, simulating the most widespread technique for experimentally measuring heat conduction using an ultrashort initial laser impulse. We obtained exact analytic solutions of the Fourier heat conductivity equation in two dimensions. We analyzed the propagation of the initial δ-function and an exponential-power function that allows modeling almost any kind of impulses.

Using the operator method, we solved different variations of the Fokker–Planck equation, simulating the propagation of electron beams, particularly for free electron lasers. We showed and compared plots of the solutions. All solutions were obtained exactly in explicit analytic form. The maximum principle, which ensures the uniqueness and stability of the solution holds for the Fourier heat conductivity equation only when additional terms are absent from the equation. Based on the considered examples of equations of this type, the maximum principle does not hold for solutions of Eqs. (25), (28), and (31) in the presence of linear and other terms, because the maximum of the solution in the space–time rectangle $R = \{0 \leq x \leq 1, 0 \leq t \leq T\}$ is achieved not at the initial time ($t = 0$) and not at extreme values of the coordinates ($x = 0$ or $x = 1$). For Fokker–Planck equation (56), the maximum principle also does not hold. This is possible because the considered equation is neither elliptic nor parabolic. In obtaining solutions, we used generalized forms of Laguerre and Hermite polynomials, which allowed writing them as a series expansion of polynomials of the above types. Using the operator definitions and representations made it easy to use them to solve mathematical problems arising in modeling physical processes of heat propagation and particle beams and charges in classical and quantum mechanics.

We showed that the inversion of differential operators and the use of the inverse derivative often paves the way for directly obtaining analytic solutions and enables progress in solving complicated mathematical problems and associated physical processes. Our study showed that the operator approach, combined with integral transformations, using the extended forms of orthogonal polynomials and special functions and operator relations is a powerful tool for studying a wide range of physical problems.

The operator technique developed above is applicable to the solution of other DEs describing a wide range of different physical processes. They will be discussed in subsequent publications.

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