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APPLICATION OF THE TRIGONAL CURVE TO THE BLASZAK–MARCINIAK LATTICE HIERARCHY

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We develop a method for constructing algebro-geometric solutions of the Blaszak–Marciniak (BM) lattice hierarchy based on the theory of trigonal curves. We first derive the BM lattice hierarchy associated with a discrete (3×3)-matrix spectral problem using Lenard recurrence relations. Using the characteristic polynomial of the Lax matrix for the BM lattice hierarchy, we introduce a trigonal curve with two infinite points, which we use to establish the associated Dubrovin-type equations. We then study the asymptotic properties of the algebraic function carrying the data of the divisor and the Baker–Akhiezer function near the two infinite points on the trigonal curve. We finally obtain algebro-geometric solutions of the entire BM lattice hierarchy in terms of the Riemann theta function.

Keywords: Blaszak–Marciniak lattice, algebro-geometric solution, trigonal curve

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1. Introduction

It is well known that constructing explicit algebro-geometric solutions for both continuous and discrete integrable systems is an important topic. In a wide literature on the subject [1]–[16], several systematic methods have been proposed, such as the algebro-geometric method, the inverse scattering transform for a periodic problem, and other methods based on the theory of hyperelliptic curves, using which algebro-geometric solutions for many soliton equations associated with (2×2)-matrix spectral problems have been obtained in recent decades. But in the case of the (3×3)-matrix spectral problems, the research becomes more complicated because it involves the theory of trigonal curves [17]–[23] instead of the theory of hyperelliptic curves in the case of second-order problems. Only a few papers [3], [10], [24]–[28] using the reduction theory of Riemann theta functions considered algebro-geometric solutions of the Boussinesq equation related to a third-order differential operator. In [29], [30], a universal approach leading to all algebro-geometric solutions of the entire Boussinesq hierarchy was proposed. In [31]–[33], a general method was developed for introducing the trigonal curve using the characteristic polynomial of the Lax matrix associated with the (3×3)-matrix spectral problem, from which the unified framework was successfully generalized to obtain algebro-geometric solutions for the modified Boussinesq, the Kaup–Kupershmidt, and the coupled mKdV hierarchies.

Here, we develop the method for generalizing the universal approach proposed in [29]–[33] to the discrete case and use it to obtain algebro-geometric solutions of the Blaszak–Marciniak (BM) lattice hierarchy

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associated with the discrete (3×3)-matrix spectral problem. A typical representative of the hierarchy is the BM lattice equation

$$u_t = w^+ - w^-, \qquad v_t = u^- w^- - uw, \qquad w_t = w(v - v^+), \tag{1.1}$$

which was first obtained by Blaszak and Marciniak [34] as an example of applying the r-matrix formalism to the algebra of shift operators. Various mathematical structures related to BM lattice equation (1.1) were found in [35]–[39]. For example, the bi-Hamiltonian structure, Bäcklund transformation, nonlinear superposition formula, master symmetries, infinitely many conservation laws, and many others were obtained.

The contents of this paper are as follows. In Sec. 2, using the Lenard recurrence relations and the zero-curvature equation, we derive the BM lattice hierarchy associated with the discrete (3×3)-matrix spectral problem. In Sec. 3, we introduce a trigonal curve $\mathcal{K}_{m-1}$ using the characteristic polynomial of the Lax matrix for the stationary BM lattice hierarchy, using which we find the stationary Baker–Akhiezer function and the associated meromorphic function. We show that the trigonal curve has two infinite points, one of which is a double branch point and the other is not a branch point. In Sec. 4, we present the explicit theta function representations of the potentials u, v, and w for the entire stationary BM lattice hierarchy. In Sec. 5, we then generalize the analyses in Secs. 3 and 4 to the time-dependent case. The analogues of the Baker–Akhiezer function, the meromorphic function, and the theta function representations in Sec. 4 are all extended to the time-dependent case.

2. The BM lattice hierarchy

We use the following assumption.

We assume that u, v, and w satisfy the conditions

$$u(\,\cdot\,,t), v(\,\cdot\,,t), w(\,\cdot\,,t) \in \mathbb{C}^{\mathbb{Z}}, \quad t \in \mathbb{R}, \quad u(n,\,\cdot\,), v(n,\,\cdot\,), w(n,\,\cdot\,) \in C^1(\mathbb{R}), \quad n \in \mathbb{Z},$$

where $\mathbb{C}^{\mathbb{Z}}$ denotes the set of all complex-valued sequences labeled by an index in $\mathbb{Z}$.

For convenience, we let $E^{\pm}$ denote the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n\in\mathbb{Z}}$ according to

$$(E^{\pm}f)(n) = f(n \pm 1), \qquad (E^k f)(n) = f(n+k), \quad n, k \in \mathbb{Z},$$

and define difference operators by $\triangle = E - 1$. Moreover, we use the notation

$$f^{\pm} = E^{\pm} f, \quad f \in \mathbb{C}^{\mathbb{Z}}.$$

We assume that $(E-1)^{-1}f(n)\big|_{n=n_0} = \alpha$ and $(E+1)^{-1}f(n)\big|_{n=n_0} = \beta$, where α and β are constants. We then define

$$(E-1)^{-1}f(n) = \begin{cases} \alpha + \sum_{n'=n_0}^{n-1} f(n'), & n \geq n_0 + 1, \\ \alpha, & n = n_0, \\ \alpha - \sum_{n'=n}^{n_0-1} f(n'), & n \leq n_0 - 1, \end{cases}$$

$$(E+1)^{-1}f(n) = \begin{cases} (-1)^{n-n_0}\beta - \sum_{n'=n_0}^{n-1} (-1)^{n-n'} f(n'), & n \geq n_0 + 1, \\ \beta, & n = n_0, \\ (-1)^{n-n_0}\beta + \sum_{n'=n}^{n_0-1} (-1)^{n-n'} f(n'), & n \leq n_0 - 1, \end{cases}$$

whence we can define an inverse operator $(E-E^{-1})^{-1}$ using $(E-E^{-1})^{-1}=E(E-1)^{-1}(E+1)^{-1}$.

In this section, we derive the BM lattice hierarchy associated with a discrete (3×3)-matrix spectral problem

$$E\psi=U\psi,\qquad \psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix},\qquad U=\begin{pmatrix}0&1&0\\ v+\lambda&u&1\\ w&0&0\end{pmatrix},\tag{2.1}$$

where u, v, and w are three potentials and λ is a constant spectral parameter. For this, we introduce two sets of Lenard recurrence relations

$$\begin{aligned}K_n\hat{g}_j&=J_n\hat{g}_{j+1},\qquad \hat{g}_j=(\hat{a}_j,\hat{b}_j,\hat{c}_j)^{\mathrm{T}},\quad j\ge 0,\\ K_n\check{g}_j&=J_n\check{g}_{j+1},\qquad \check{g}_j=(\check{a}_j,\check{b}_j,\check{c}_j)^{\mathrm{T}},\quad j\ge 0,\end{aligned}\tag{2.2}$$

with the conditions $\hat{g}_j\big|_{(u,v,w)=0}=\check{g}_j\big|_{(u,v,w)=0}=0$, $j\ge 1$, and the starting points

$$\hat{g}_0=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix},\qquad \check{g}_0=\begin{pmatrix}1\\ 0\\ -(E+1)^{-1}u\end{pmatrix},\tag{2.3}$$

where the initial conditions mean that the summation constants are set to zero, and we define two difference operators K_n and J_n as

$$K_n=\begin{pmatrix}u\triangle uE+EvE-v & EwE-E^{-1}w & u\triangle E\\ v\triangle uE+wE^2-E^{-1}w & E^{-1}uw-uwE & v(E^2-1)\\ w\triangle EuE & -w\triangle v & w(E^3-1)\end{pmatrix},$$

$$J_n=\begin{pmatrix}1-E^2&0&0\\ -\triangle uE&0&1-E^2\\ 0&w\triangle&0\end{pmatrix}.$$

Hence, $\hat{g}_j$ and $\check{g}_j$ are uniquely determined. For example, the first two members are

$$\begin{aligned}\hat{g}_1&=\begin{pmatrix}-w^-\\ -v\\ u^-w^-\end{pmatrix},\\ \check{g}_1&=\begin{pmatrix}(1-E^2)^{-1}[v^+-v+u^2-2u(E+1)^{-1}u]\\ -(E+1)^{-1}u\\ (E+1)^{-1}[u(E-E^{-1})^{-1}(v^+-v+u^2-2u(E+1)^{-1}u)-w^-]\end{pmatrix}.\end{aligned}\tag{2.4}$$

To generate a hierarchy of evolution equations associated with discrete spectral problem (2.1), we solve the stationary zero-curvature equation

$$(EV)U-UV=0,\qquad V=(V_{ij})_{3\times 3},\tag{2.5}$$

which is equivalent to the system

$$\begin{aligned}
&(v+\lambda)V_{12}^{+}+wV_{13}^{+}-V_{21}=0,\\
&V_{11}^{+}+uV_{12}^{+}-V_{22}=0,\\
&V_{12}^{+}-V_{23}=0,\\
&(v+\lambda)(V_{22}^{+}-V_{11})+wV_{23}^{+}-uV_{21}-V_{31}=0,\\
&V_{21}^{+}+u(V_{22}^{+}-V_{22})-(v+\lambda)V_{12}-V_{32}=0,\\
&V_{22}^{+}-V_{33}-(v+\lambda)V_{13}-uV_{23}=0,\\
&(v+\lambda)V_{32}^{+}+w(V_{33}^{+}-V_{11})=0,\\
&V_{31}^{+}+uV_{32}^{+}-wV_{12}=0,\\
&V_{32}^{+}-wV_{13}=0,
\end{aligned}\tag{2.6}$$

where each element $V_{ij}=V_{ij}(a,b,c)$ is a Laurent expansion in λ:

$$\begin{aligned}
&V_{11}=c, && V_{12}=a, && V_{13}=b,\\
&V_{21}=wb^{+}+(v+\lambda)a^{+}, && V_{22}=c^{+}+ua^{+}, && V_{23}=a^{+},\\
&V_{31}=w^{-}a^{-}-u^{-}w^{-}b^{-}, && V_{32}=w^{-}b^{-}, && V_{33}=c^{++}+\triangle ua^{+}-(v+\lambda)b,
\end{aligned}\tag{2.7}$$

$$a=\sum_{j\geq 0}a_j\lambda^{-j},\qquad b=\sum_{j\geq 0}b_j\lambda^{-j},\qquad c=\sum_{j\geq 0}c_j\lambda^{-j}.\tag{2.8}$$

A direct calculation shows that (2.6) and (2.7) imply the Lenard equation

$$K_nG=\lambda J_nG,\qquad G=(a,b,c)^{\mathrm{T}}.\tag{2.9}$$

Substituting (2.8) in (2.9) and collecting like powers of λ, we obtain the recurrence relation

$$K_nG_j=J_nG_{j+1},\qquad J_nG_0=0,\quad j\geq 0,\tag{2.10}$$

where $G_j=(a_j,b_j,c_j)^{\mathrm{T}}$. Because the equation $J_nG_0=0$ has a solution

$$G_0=\alpha_0\hat{g}_0+\beta_0\check{g}_0,\tag{2.11}$$

G_j can be expressed as

$$G_j=\alpha_0\hat{g}_j+\beta_0\check{g}_j+\cdots+\alpha_j\hat{g}_0+\beta_j\check{g}_0,\quad j\geq 0,\tag{2.12}$$

where α_j and β_j are arbitrary constants.

Let ψ satisfy discrete spectral problem (2.1) and an auxiliary problem

$$\psi_{t_r}=\widetilde{V}^{(r)}\psi,\qquad \widetilde{V}^{(r)}=(\widetilde{V}_{ij}^{(r)})_{3\times 3},\tag{2.13}$$

with the elements $\widetilde{V}_{ij}^{(r)}=V_{ij}(\tilde{a}^{(r)},\tilde{b}^{(r)},\tilde{c}^{(r)})$ and

$$\tilde{a}^{(r)}=\sum_{j=0}^{r}\tilde{a}_j\lambda^{r-j},\quad \tilde{b}^{(r)}=\sum_{j=0}^{r}\tilde{b}_j\lambda^{r-j},\quad \tilde{c}^{(r)}=\sum_{j=0}^{r}\tilde{c}_j\lambda^{r-j}.\tag{2.14}$$

Here, $\tilde{a}_j$, $\tilde{b}_j$, and $\tilde{c}_j$ are determined by $\widetilde{G}_j = (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j)^{\mathrm{T}}$, and

$$\widetilde{G}_j = \tilde{\alpha}_0 \hat{g}_j + \tilde{\beta}_0 \check{g}_j + \cdots + \tilde{\alpha}_j \hat{g}_0 + \tilde{\beta}_j \check{g}_0, \quad j \geq 0, \tag{2.15}$$

are also solutions of (2.10), where the constants $\{\tilde{\alpha}_j, \tilde{\beta}_j\}$ and $\{\alpha_j, \beta_j\}$ in (2.12) are independent of each other. The compatibility condition for (2.1) and (2.13) then yields the zero-curvature equation $U_{t_r} = (E\widetilde{V}^{(r)})U - U\widetilde{V}^{(r)}$, which is equivalent to the BM lattice hierarchy

$$(u_{t_r}, v_{t_r}, w_{t_r})^{\mathrm{T}} = \widetilde{X}_r, \quad r \geq 0, \tag{2.16}$$

where the vector fields

$$\widetilde{X}_j = X\big(u, v, w; \underline{\tilde{\alpha}}^{(j)}, \underline{\tilde{\beta}}^{(j)}\big) = K_n \widetilde{G}_j = J_n \widetilde{G}_{j+1}, \quad j \geq 0, \tag{2.17}$$

$\underline{\tilde{\alpha}}^{(j)} = (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_j)$, and $\underline{\tilde{\beta}}^{(j)} = (\tilde{\beta}_0, \ldots, \tilde{\beta}_j)$. The first nontrivial member of hierarchy (2.16) is

$$\begin{aligned} u_{t_0} &= \tilde{\alpha}_0(w^+ - w^-) + \tilde{\beta}_0[v^+ - v + u^2 - 2u(E+1)^{-1}u], \\ v_{t_0} &= \tilde{\alpha}_0(u^- w^- - uw) + \tilde{\beta}_0(w - w^-), \\ w_{t_0} &= \tilde{\alpha}_0 w(v - v^+) + \tilde{\beta}_0 w[2(E+1)^{-1}u - u]. \end{aligned} \tag{2.18}$$

For $\tilde{\alpha}_0 = 1$ and $\tilde{\beta}_0 = 0$, $t_0 = t$, Eq. (2.18) is just BM lattice equation (1.1). For $\tilde{\alpha}_0 = 0$ and $\tilde{\beta}_0 = 1$, $t_0 = t$, Eq. (2.18) reduces to

$$\begin{aligned} u_t &= v^+ - v + u^2 - 2u(E+1)^{-1}u, \\ v_t &= w - w^-, \\ w_t &= w[2(E+1)^{-1}u - u]. \end{aligned} \tag{2.19}$$

3. The stationary meromorphic function

We first introduce the trigonal curve $\mathcal{K}_{m-1}$ and then define the stationary Baker–Akhiezer function and the associated meromorphic function. We consider the stationary BM lattice hierarchy, $X_q = X\big(u, v, w; \underline{\alpha}^{(q)}, \underline{\beta}^{(q)}\big) = 0$, $\underline{\alpha}^{(q)} = (\alpha_0, \ldots, \alpha_q)$ and $\underline{\beta}^{(q)} = (\beta_0, \ldots, \beta_q)$, which is equivalent to the stationary zero-curvature equation

$$(EV^{(q)})U - UV^{(q)} = 0, \qquad V^{(q)} = (\lambda^q V)_+ = (V_{ij}^{(q)})_{3\times 3}, \tag{3.1}$$

with the elements $V_{ij}^{(q)} = V_{ij}(a^{(q)}, b^{(q)}, c^{(q)})$,

$$a^{(q)} = \sum_{j=0}^{q} a_j \lambda^{q-j}, \qquad b^{(q)} = \sum_{j=0}^{q} b_j \lambda^{q-j}, \qquad c^{(q)} = \sum_{j=0}^{q} c_j \lambda^{q-j}, \tag{3.2}$$

and a_j, b_j, and c_j are determined by (2.12). A direct calculation shows that $yI - V^{(q)}$ also satisfies the stationary zero-curvature equation. The characteristic polynomial $\mathcal{F}_m(\lambda, y) = \det(yI - V^{(q)})$ of the Lax matrix $V^{(q)}$ is then a constant independent of the variable n with the expansion

$$\det(yI - V^{(q)}) = y^3 - y^2 R_m(\lambda) + y S_m(\lambda) - T_m(\lambda), \tag{3.3}$$

where $R_m(\lambda)$, $S_m(\lambda)$ and $T_m(\lambda)$ are polynomials in λ with constant coefficients,

$$
\begin{aligned}
&R_m(\lambda) = \operatorname{tr} V^{(q)} = -\alpha_0\lambda^{q+1} - \alpha_1\lambda^q - \alpha_2\lambda^{q-1} + \dots, \\
&S_m(\lambda) = \sum_{1\le i<j\le 3} \begin{vmatrix} V_{ii}^{(q)} & V_{ij}^{(q)} \\ V_{ji}^{(q)} & V_{jj}^{(q)} \end{vmatrix} = -\beta_0^2\lambda^{2q+1} - 2\beta_0\beta_1\lambda^{2q} + \dots, \\
&T_m(\lambda) = \det V^{(q)} = \alpha_0\beta_0^2\lambda^{3q+2} + \beta_0(2\alpha_0\beta_1 + \beta_0\alpha_1)\lambda^{3q+1} + \dots.
\end{aligned}
\tag{3.4}
$$

Then $\mathcal{F}_m(\lambda, y) = 0$ naturally defines the trigonal curve of degree m

$$
\mathcal{K}_{m-1}: \quad \mathcal{F}_m(\lambda, y) = y^3 - y^2 R_m(\lambda) + y S_m(\lambda) - T_m(\lambda) = 0, \tag{3.5}
$$

where $m = 3q + 2$ for $\alpha_0\beta_0 \neq 0$. Below, we assume that $\alpha_0\beta_0 \neq 0$. It follows from (3.4) and (3.5) that the trigonal curve $\mathcal{K}_{m-1}$ is compactified by joining two different infinite points P_{∞_1} and P_{∞_2}, one of which is a double branch point and the other is not a branch point. Without loss of generality, we let P_{∞_2} be the double branch point. The compactification of the curve $\mathcal{K}_{m-1}$ is still denoted by the same symbol for convenience. The discriminant of (3.5) is

$$
\Delta(\lambda) = 4S_m^3 - R_m^2 S_m^2 + 4R_m^3 T_m - 18 R_m S_m T_m + 27 T_m^2 = -4\alpha_0^4\beta_0^2\lambda^{6q+5} + \dots. \tag{3.6}
$$

The Riemann–Hurwitz formula shows that the arithmetic genus of $\mathcal{K}_{m-1}$ is $3q + 1$ [40]. Therefore, $\mathcal{K}_{m-1}$ becomes a three-sheeted Riemann surface of arithmetic genus $m - 1$ if it is nonsingular and irreducible. Here, the meaning of nonsingular is that $(\partial\mathcal{F}_m(\lambda, y)/\partial\lambda, \partial\mathcal{F}_m(\lambda, y)/\partial y)\big|_{(\lambda,y)=(\lambda_0,y_0)} \neq (0, 0)$ at each point $P_0 = (\lambda_0, y_0) \in \mathcal{K}_{m-1}$.

We now define the stationary Baker–Akhiezer function as a solution of the set of equations

$$
\begin{aligned}
&E\psi(P, n, n_0) = U(u(n), v(n), w(n); \lambda(P))\psi(P, n, n_0), \\
&V^{(q)}(u(n), v(n), w(n); \lambda(P))\psi(P, n, n_0) = y(P)\psi(P, n, n_0), \\
&\psi_1(P, n_0, n_0) = 1, \qquad P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1},\ P_{\infty_2}\}, \quad n, n_0 \in \mathbb{Z}.
\end{aligned}
\tag{3.7}
$$

The meromorphic function $\phi(P, n)$ on $\mathcal{K}_{m-1}$ defined by

$$
\phi(P, n) = \frac{\psi_1^+(P, n, n_0)}{\psi_1(P, n, n_0)}, \quad P \in \mathcal{K}_{m-1}, \quad n \in \mathbb{Z}, \tag{3.8}
$$

such that

$$
\psi_1(P, n, n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} \phi(P, n'), & n \ge n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} \phi(P, n')^{-1}, & n \le n_0 - 1, \end{cases} \tag{3.9}
$$

is closely related to $\psi(P, n, n_0)$. From (3.7) after tedious calculations, we obtain

$$
\begin{aligned}
\phi &= \frac{yV_{23}^{(q)} + C_m}{yV_{13}^{(q)} + A_m} = \frac{F_{m-1}}{y^2 V_{23}^{(q)} - y(C_m + V_{23}^{(q)} R_m) + D_m} = \\
&= \frac{y^2 V_{13}^{(q)} - y(A_m + V_{13}^{(q)} R_m) + B_m}{E_{m-1}},
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
A_m &= V_{12}^{(q)} V_{23}^{(q)} - V_{13}^{(q)} V_{22}^{(q)}, \\
B_m &= V_{13}^{(q)} [V_{11}^{(q)} V_{33}^{(q)} - V_{13}^{(q)} V_{31}^{(q)}] + V_{12}^{(q)} [V_{11}^{(q)} V_{23}^{(q)} - V_{13}^{(q)} V_{21}^{(q)}], \\
C_m &= V_{13}^{(q)} V_{21}^{(q)} - V_{11}^{(q)} V_{23}^{(q)}, \\
D_m &= V_{23}^{(q)} [V_{22}^{(q)} V_{33}^{(q)} - V_{23}^{(q)} V_{32}^{(q)}] + V_{21}^{(q)} [V_{13}^{(q)} V_{22}^{(q)} - V_{12}^{(q)} V_{23}^{(q)}],
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
E_{m-1} &= (V_{13}^{(q)})^2 V_{32}^{(q)} + V_{12}^{(q)} V_{13}^{(q)} (V_{22}^{(q)} - V_{33}^{(q)}) - (V_{12}^{(q)})^2 V_{23}^{(q)}, \\
F_{m-1} &= (V_{23}^{(q)})^2 V_{31}^{(q)} + V_{21}^{(q)} V_{23}^{(q)} (V_{11}^{(q)} - V_{33}^{(q)}) - (V_{21}^{(q)})^2 V_{13}^{(q)}.
\end{aligned}
\tag{3.12}
$$

For later use, we introduce the polynomials

$$
\begin{aligned}
\mathcal{A}_m &= V_{13}^{(q)} V_{32}^{(q)} - V_{12}^{(q)} V_{33}^{(q)}, \\
\mathcal{B}_m &= V_{12}^{(q)} [V_{11}^{(q)} V_{22}^{(q)} - V_{12}^{(q)} V_{21}^{(q)}] + V_{13}^{(q)} [V_{11}^{(q)} V_{32}^{(q)} - V_{12}^{(q)} V_{31}^{(q)}].
\end{aligned}
\tag{3.13}
$$

It is easy see that there exist various relations between the polynomials A_m, B_m, C_m, D_m, $\mathcal{A}_m$, $\mathcal{B}_m$, E_{m-1}, F_{m-1}, R_m, S_m, and T_m, some of which we list:

$$
\begin{aligned}
& V_{13}^{(q)} F_{m-1} = V_{23}^{(q)} D_m - (V_{23}^{(q)})^2 S_m - C_m^2 - C_m V_{23}^{(q)} R_m, \\
& A_m F_{m-1} = (V_{23}^{(q)})^2 T_m + C_m D_m,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
& V_{23}^{(q)} E_{m-1} = V_{13}^{(q)} B_m - (V_{13}^{(q)})^2 S_m - A_m^2 - A_m V_{13}^{(q)} R_m, \\
& C_m E_{m-1} = (V_{13}^{(q)})^2 T_m + A_m B_m,
\end{aligned}
\tag{3.15}
$$

$$
E_{m-1} = V_{13}^{(q)} \mathcal{A}_m - V_{12}^{(q)} A_m, \qquad F_{m-1} = -w E_{m-1}^{+}, \tag{3.16}
$$

$$
\mathcal{A}_m = C_m^{-}, \qquad \mathcal{B}_m = D_m^{-}. \tag{3.17}
$$

From (3.1), (3.2), (3.12), and (3.16), we know that E_{m-1} and F_{m-1} are polynomials in λ of degree $m-1$. We can hence write them in the form

$$
\begin{aligned}
E_{m-1} &= \alpha_0^2 \beta_0 \prod_{j=1}^{m-1} (\lambda - \mu_j(n)), \\
F_{m-1} &= -\alpha_0^2 \beta_0 w(n) \prod_{j=1}^{m-1} (\lambda - \mu_j^{+}(n)).
\end{aligned}
\tag{3.18}
$$

We define $\{\hat{\mu}_j(n)\}_{j=1,\ldots,m-1} \subset \mathcal{K}_{m-1}$ and $\{\hat{\mu}_j^{+}(n)\}_{j=1,\ldots,m-1} \subset \mathcal{K}_{m-1}$ by

$$
\begin{aligned}
\hat{\mu}_j(n) &= \big(\mu_j(n), y(\hat{\mu}_j(n))\big) = \left(\mu_j(n), -\frac{A_m(\mu_j(n), n)}{V_{13}^{(q)}(\mu_j(n), n)}\right), \quad n \in \mathbb{Z}, \\
\hat{\mu}_j^{+}(n) &= \big(\mu_j^{+}(n), y(\hat{\mu}_j^{+}(n))\big) = \left(\mu_j^{+}(n), -\frac{C_m(\mu_j^{+}(n), n)}{V_{23}^{(q)}(\mu_j^{+}(n), n)}\right), \quad n \in \mathbb{Z}.
\end{aligned}
\tag{3.19}
$$

For convenience, we define three points P, P^*, and P^{**} on three different sheets of the same Riemann surface $\mathcal{K}_{m-1}$. For a fixed λ, let $y_i(\lambda)$, $i=0,1,2$, denote the three roots of $\mathcal{F}_m(\lambda,y)=0$, i.e.,

$$(y-y_0(\lambda))(y-y_1(\lambda))(y-y_2(\lambda))=y^3-y^2R_m+yS_m-T_m=0. \tag{3.20}$$

The points $(\lambda,y_0(\lambda))$, $(\lambda,y_1(\lambda))$, and $(\lambda,y_2(\lambda))$ are then on three different sheets of the Riemann surface $\mathcal{K}_{m-1}$. Let $P=(\lambda,y_i(\lambda))$, $i=0,1,2$, be an arbitrary one of the three points. Then the other two points are denoted by P^* and P^{**}.

From (3.20), we easily obtain the system

$$\begin{aligned}
&y_0+y_1+y_2=R_m,\\
&y_0y_1+y_0y_2+y_1y_2=S_m,\\
&y_0y_1y_2=T_m,\\
&y_0^2+y_1^2+y_2^2=R_m^2-2S_m,\\
&y_0^3+y_1^3+y_2^3=R_m^3-3R_mS_m+3T_m,\\
&y_0^2y_1^2+y_0^2y_2^2+y_1^2y_2^2=S_m^2-2R_mT_m.
\end{aligned} \tag{3.21}$$

The function $\phi(P,n)$ then satisfies the system

$$\begin{aligned}
&\phi(P,n)\phi(P^*,n)\phi(P^{**},n)=-\frac{F_{m-1}(\lambda,n)}{E_{m-1}(\lambda,n)},\\
&\phi(P,n)+\phi(P^*,n)+\phi(P^{**},n)=\frac{3B_m(\lambda,n)-2V_{13}^{(q)}(\lambda,n)S_m(\lambda)-A_m(\lambda,n)R_m(\lambda)}{E_{m-1}(\lambda,n)},\\
&\frac{1}{\phi(P,n)}+\frac{1}{\phi(P^*,n)}+\frac{1}{\phi(P^{**},n)}=\frac{3D_m(\lambda,n)-2V_{23}^{(q)}(\lambda,n)S_m(\lambda)-C_m(\lambda,n)R_m(\lambda)}{F_{m-1}(\lambda,n)}.
\end{aligned} \tag{3.22}$$

4. Algebro-geometric solutions of the stationary BM lattice hierarchy

In this section, we obtain explicit Riemann theta function representations for the meromorphic function ϕ, the Baker–Akhiezer function ψ_1, and the potentials u, v, and w of the entire stationary BM lattice hierarchy.

A direct calculation shows that $\phi(P,n)$ satisfies the Riccati-type equation

$$\phi^+(P,n)\phi(P,n)\phi^-(P,n)-u(n)\phi(P,n)\phi^-(P,n)-v(n)\phi^-(P,n)-w^-(n)=\lambda\phi^-(P,n). \tag{4.1}$$

Introducing a local coordinate $\zeta=1/\lambda$ near P_{∞_1}, substituting a power series ansatz in Riccati-type equation (4.1), and comparing like powers of ζ, we obtain

$$\phi=\sum_{j=1}^{\infty}\phi_j\zeta^j,\quad P\to P_{\infty_1}, \tag{4.2}$$

where

$$\begin{aligned}
&\phi_1=-w,\qquad \phi_2=wv^+,\qquad \phi_3=-w(u^+w^++(v^+)^2),\\
&\phi_j=\sum_{\substack{i+l+s=j-1\\ i,l,s\geq 1}}\phi_i^{++}\phi_l^+\phi_s-u^+\sum_{\substack{l+s=j-1\\ l,s\geq 1}}\phi_l^+\phi_s-v^+\phi_{j-1},\quad j\geq 4.
\end{aligned}$$

Similarly introducing a local coordinate $\xi = 1/\lambda^{1/2}$ near P_{∞_2}, substituting a power series ansatz in (4.1), and comparing like powers of ξ, we obtain

$$\phi = \sum_{j=-1}^{\infty} \kappa_j \xi^j, \quad P \to P_{\infty_2}, \tag{4.3}$$

where

$$\begin{aligned}
&\kappa_{-1} = 1, \qquad \kappa_0 = (E+1)^{-1} u, \qquad \kappa_1 = (E+1)^{-1}[v + (\kappa_0)^2],\\
&\kappa_2 = (E+1)^{-1}[w^- - \kappa_0 \triangle \kappa_1],\\
&\kappa_j = \sum_{\substack{i+l+s=j-2\\ i,l,s\geq -1}} \kappa_i^{++}\kappa_l^{+}\kappa_s - u^+ \sum_{\substack{l+s=j-2\\ l,s\geq -1}} \kappa_l^{+}\kappa_s - v^+\kappa_{j-2}, \quad j \geq 3.
\end{aligned}$$

We next let

$$\mathcal{D}_{P_1,\dots,P_{m-1}}\colon \begin{cases} \mathcal{K}_{m-1} \to \mathbb{N}_0, \quad \mathbb{N}_0 = \mathbb{N}\cup\{0\},\\ P \mapsto \mathcal{D}_{P_1,\dots,P_{m-1}}(P) = \begin{cases} k \text{ if } P \text{ occurs } k \text{ times in } \{P_1,\dots,P_{m-1}\},\\ 0 \text{ if } P \in \mathcal{K}_{m-1} \notin \{P_1,\dots,P_{m-1}\}, \end{cases} \end{cases}$$

denote positive divisors of $\mathcal{K}_{m-1}$ of degree $m-1$. In particular, we define the divisor (f) of a meromorphic function $f(P)$ on $\mathcal{K}_{m-1}$ by

$$(f)\colon \mathcal{K}_{m-1} \to \mathbb{Z}, \qquad P \mapsto \nu_f(P),$$

where $\nu_f(P)$ is the order of f at P, i.e., if $f(P)$ has the asymptotic expansion $f(P) = \sum_{j=m_0}^{\infty} d_j \zeta_P^j$ in terms of a local coordinate ζ_P for some $m_0 \in \mathbb{Z}$, then $\nu_f(P) = m_0$ [30]. From (4.2), (4.3), and (3.10), we therefore conclude that the divisor $(\phi(P,n))$ of $\phi(P,n)$ is given by

$$(\phi(P,n)) = \mathcal{D}_{P_{\infty_1},\hat{\mu}_1^+(n),\dots,\hat{\mu}_{m-1}^+(n)}(P) - \mathcal{D}_{P_{\infty_2},\hat{\mu}_1(n),\dots,\hat{\mu}_{m-1}(n)}(P). \tag{4.4}$$

Hence, $\phi(P,n)$ has m simple zeros at P_{∞_1}, $\hat{\mu}_1^+(n),\dots,\hat{\mu}_{m-1}^+(n)$, and m simple poles at P_{∞_2}, $\hat{\mu}_1(n),\dots,$ $\hat{\mu}_{m-1}(n)$. Moreover, using (4.2), (4.3), and (3.9), we obtain

$$\psi_1(P,n,n_0) \underset{\zeta\to 0}{=} \Gamma(n,n_0)\zeta^{n-n_0}(1+O(\zeta)), \quad P \to P_{\infty_1}, \quad \zeta = \frac{1}{\lambda}, \tag{4.5}$$

where

$$\Gamma(n,n_0) = \begin{cases} \prod_{n'=n_0}^{n-1}\big(-w(n')\big), & n \geq n_0+1,\\ 1, & n = n_0,\\ \prod_{n'=n}^{n_0-1}\big(-w(n')\big)^{-1}, & n \leq n_0 - 1, \end{cases}$$

and

$$\psi_1(P,n,n_0) \underset{\xi\to 0}{=} \xi^{n_0-n}(1+O(\zeta)), \quad P \to P_{\infty_2}, \quad \xi = \frac{1}{\lambda^{1/2}}. \tag{4.6}$$

As a result, we conclude that the divisor $(\psi_1(P,n,n_0))$ of $\psi_1(P,n,n_0)$ has the form

$$(\psi_1(P,n,n_0)) = \mathcal{D}_{\hat{\mu}_1(n),\dots,\hat{\mu}_{m-1}(n)} - \mathcal{D}_{\hat{\mu}_1(n_0),\dots,\hat{\mu}_{m-1}(n_0)} + (n-n_0)(\mathcal{D}_{P_{\infty_1}} - \mathcal{D}_{P_{\infty_2}}). \tag{4.7}$$

We equip the Riemann surface $\mathcal{K}_{m-1}$ with a canonical basis $\{\mathsf{a}_j, \mathsf{b}_j\}_{j=1}^{m-1}$, whose components are independent and have intersection numbers

$$\mathsf{a}_j \circ \mathsf{b}_k = \delta_{j,k}, \qquad \mathsf{a}_j \circ \mathsf{a}_k = 0, \qquad \mathsf{b}_j \circ \mathsf{b}_k = 0, \quad j,k = 1,\dots,m-1.$$

For now, we choose the set

$$\widetilde{\omega}_l(P) = \frac{1}{3y^2 - 2yR_m + S_m} \begin{cases} \lambda^{l-1} d\lambda, & 1 \le l \le 2q+1, \\ \left(y - \dfrac{R_m}{3}\right)\lambda^{l-2q-2} d\lambda, & 2q+2 \le l \le m-1, \end{cases} \tag{4.8}$$

of $m-1$ linearly independent homomorphic differentials on $\mathcal{K}_{m-1}$ as our basis. Using the cycles a_j and b_j, we can construct the period matrices A and B from

$$A_{ij} = \int_{\mathsf{a}_j} \widetilde{\omega}_i, \qquad B_{ij} = \int_{\mathsf{b}_j} \widetilde{\omega}_i. \tag{4.9}$$

It can be shown that A and B are invertible [40], [41]. We now define the matrices C and τ by $C = A^{-1}$ and $\tau = A^{-1}B$. The matrix τ can be shown to be symmetric, ($\tau_{ij} = \tau_{ji}$), and it has a positive-definite imaginary part ($\operatorname{Im}\tau > 0$). If we normalize $\widetilde{\omega}_l$ in the new basis ω_j, then we obtain

$$\omega_j = \sum_{l=1}^{m-1} C_{jl}\widetilde{\omega}_l, \quad j = 1,\dots,m-1, \tag{4.10}$$

and also

$$\int_{\mathsf{a}_k} \omega_j = \delta_{jk}, \qquad \int_{\mathsf{b}_k} \omega_j = \tau_{jk}, \quad 1 \le k, \quad j \le m-1. \tag{4.11}$$

Let $\omega^{(3)}_{Q_1,Q_2}$ be the normalized Abelian differential of the third kind holomorphic on $\mathcal{K}_{m-1} \setminus \{Q_1, Q_2\}$ with simple poles at Q_s with the residues $(-1)^{s+1}$, $s = 1,2$. Then

$$\int_{\mathsf{a}_j} \omega^{(3)}_{Q_1,Q_2} = 0, \qquad \int_{\mathsf{b}_j} \omega^{(3)}_{Q_1,Q_2} = 2\pi i \int_{Q_2}^{Q_1} \omega_j, \quad j = 1,\dots,m-1. \tag{4.12}$$

In particular, for $\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}$, we obtain

$$\begin{aligned} \omega^{(3)}_{P_{\infty_1},P_{\infty_2}} &\underset{\zeta\to 0}{=} (\zeta^{-1} + \omega_0^{\infty_1} + O(\zeta))d\zeta, && P \to P_{\infty_1}, \quad \zeta = \frac{1}{\lambda}, \\ \omega^{(3)}_{P_{\infty_1},P_{\infty_2}} &\underset{\xi\to 0}{=} (-\xi^{-1} + \omega_0^{\infty_2} + O(\xi))d\xi, && P \to P_{\infty_2}, \quad \xi = \frac{1}{\lambda^{1/2}}. \end{aligned} \tag{4.13}$$

We then have

$$\begin{aligned} \int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_1},P_{\infty_2}} &\underset{\zeta\to 0}{=} \log\zeta + e_1(Q_0) + \omega_0^{\infty_1}\zeta + O(\zeta^2), && P \to P_{\infty_1}, \\ \int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_1},P_{\infty_2}} &\underset{\xi\to 0}{=} -\log\xi + e_2(Q_0) + \omega_0^{\infty_2}\xi + O(\xi^2), && P \to P_{\infty_2}, \end{aligned} \tag{4.14}$$

where $\omega_0^{\infty_1}$, $\omega_0^{\infty_2}$, $e_1(Q_0)$, and $e_2(Q_0)$ are constants with Q_0 an appropriately chosen base point on $\mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}\}$.

Let $\mathcal{T}_{m-1}$ be the period lattice $\{\underline{z} \in \mathbb{C}^{m-1} \mid \underline{z} = \underline{N} + \underline{M}\tau, \underline{N}, \underline{M} \in \mathbb{Z}^{m-1}\}$. The complex torus $\mathcal{J}_{m-1} = \mathbb{C}^{m-1}/\mathcal{T}_{m-1}$ is called the Jacobian variety of $\mathcal{K}_{m-1}$. The Abel map $\underline{\mathcal{A}}\colon \mathcal{K}_{m-1} \to \mathcal{J}_{m-1}$ is defined by

$$\underline{\mathcal{A}}(P) = (\mathcal{A}_1(P), \ldots, \mathcal{A}_{m-1}(P)) = \left(\int_{Q_0}^{P} \omega_1, \ldots, \int_{Q_0}^{P} \omega_{m-1}\right) \pmod{\mathcal{T}_{m-1}} \tag{4.15}$$

with the natural extension to the quotient group $\mathrm{Div}(\mathcal{K}_{m-1})$ by linearity

$$\underline{\mathcal{A}}\Big(\sum l_k P_k\Big) = \sum l_k \underline{\mathcal{A}}(P_k).$$

We consider the nonspecial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n)} = \sum_{k=1}^{m-1} \hat{\mu}_k(n)$ and define

$$\underline{\rho}(n) = \underline{\mathcal{A}}\Big(\sum_{k=1}^{m-1} \hat{\mu}_k(n)\Big) = \sum_{k=1}^{m-1} \underline{\mathcal{A}}(\hat{\mu}_k(n)) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(n)} \underline{\omega}, \tag{4.16}$$

where $\underline{\rho}(n) = \big(\rho_1(n), \ldots, \rho_{m-1}(n)\big)$ and $\underline{\omega} = \big(\omega_1, \ldots, \omega_{m-1}\big)$.

Let $\theta(\underline{z})$ denote the Riemann theta function associated with $\mathcal{K}_{m-1}$:

$$\begin{gathered}\theta(\underline{z}) = \sum_{\underline{N} \in \mathbb{Z}^{m-1}} \exp\big\{2\pi i \langle \underline{N}, \underline{z}\rangle + \pi i \langle \underline{N}, \underline{N}\tau\rangle\big\},\\ \underline{z} = (z_1, \ldots, z_{m-1}) \in \mathbb{C}^{m-1}, \qquad \langle \underline{N}, \underline{z}\rangle = \sum_{i=1}^{m-1} N_i z_i, \qquad \langle \underline{N}, \underline{N}\tau\rangle = \sum_{i,j=1}^{m-1} \tau_{ij} N_i N_j.\end{gathered} \tag{4.17}$$

In what follows, it is convenient to use the abbreviation

$$\begin{aligned}&\theta(\underline{z}(P, \underline{\hat{\mu}}(n))) = \theta(\underline{\Lambda} - \underline{\mathcal{A}}(P) + \underline{\rho}(n)),\\ &P \in \mathcal{K}_{m-1}, \quad \underline{\widehat{\mu}}(n) = \{\hat{\mu}_1(n), \ldots, \hat{\mu}_{m-1}(n)\} \in \sigma^{m-1}\mathcal{K}_{m-1},\end{aligned} \tag{4.18}$$

where $\sigma^{m-1}\mathcal{K}_{m-1}$ denotes the $(m-1)$th symmetric power of $\mathcal{K}_{m-1}$ and $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_{m-1})$ is the vector of Riemann constants depending on the base point Q_0 as

$$\Lambda_j = \frac{1}{2}(1 + \tau_{jj}) - \sum_{\substack{i=1\\ i\neq j}}^{m-1} \int_{\mathsf{a}_i} \omega_i \int_{Q_0}^{P} \omega_j, \quad j = 1, \ldots, m-1.$$

Theorem 1. *We assume that the curve $\mathcal{K}_{m-1}$ is nonsingular. Let $P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}\}$, $(n, n_0) \in \mathbb{Z}^2$. If $\mathcal{D}_{\underline{\widehat{\mu}}(n)}$ is nonspecial for $n \in \mathbb{Z}$, then*

$$\phi(P, n) = \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\widehat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\widehat{\mu}}^+(n)))} \frac{\theta(\underline{z}(P, \underline{\widehat{\mu}}^+(n)))}{\theta(\underline{z}(P, \underline{\widehat{\mu}}(n)))} \exp\left(\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_1}, P_{\infty_2}} - e_2(Q_0)\right) \tag{4.19}$$

and

$$\begin{aligned}\psi_1(P, n, n_0) = {}& \frac{\theta(\underline{z}(P_{\infty_2}, \underline{\widehat{\mu}}(n_0)))}{\theta(\underline{z}(P_{\infty_2}, \underline{\widehat{\mu}}(n)))} \frac{\theta(\underline{z}(P, \underline{\widehat{\mu}}(n)))}{\theta(\underline{z}(P, \underline{\widehat{\mu}}(n_0)))} \times\\ & \times \exp\left((n - n_0)\left(\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_1}, P_{\infty_2}} - e_2(Q_0)\right)\right).\end{aligned} \tag{4.20}$$

The Abel map linearizes this auxiliary divisor $\mathcal{D}_{\underline{\widehat{\mu}}(n)}$ in the sense that

$$\underline{\rho}(n) = \underline{\rho}(n_0) + (n - n_0)\big(\underline{\mathcal{A}}(P_{\infty_2}) - \underline{\mathcal{A}}(P_{\infty_1})\big). \tag{4.21}$$

Proof. Applying Abel's theorem to (4.7) yields (4.21). It follows from (4.14) that

$$\begin{aligned}
&\exp\Big(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}-e_2(Q_0)\Big)\underset{\zeta\to 0}{=}\zeta\exp\big(e_1(Q_0)-e_2(Q_0)\big)+O(\zeta^2), && P\to P_{\infty_1},\\
&\exp\Big(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}-e_2(Q_0)\Big)\underset{\xi\to 0}{=}\xi^{-1}+O(1), && P\to P_{\infty_2}.
\end{aligned}\tag{4.22}$$

We let Φ denote the right-hand side of (4.19). From the expression for Φ, (4.22), and (4.4), we know that Φ and ϕ share the same local zeros and poles and identical essential singularities at P_{∞_s}, $s=1,2$. By the Riemann–Roch theorem [29], [30], [41], we conclude that $\Phi/\phi=\gamma$, where γ is a constant. Using (4.3) and (4.22), we compute

$$\frac{\Phi}{\phi}\underset{\xi\to 0}{=}\frac{(1+O(\xi))(\xi^{-1}+O(1))}{\xi^{-1}+O(1)}\underset{\xi\to 0}{=}1+O(\xi),\quad P\to P_{\infty_2},\tag{4.23}$$

which shows that $\gamma=1$. This proves (4.19). We obtain representation (4.20) using (3.9) and (4.19). ■

Theorem 2. *We assume that $\mathcal{K}_{m-1}$ is nonsingular and let $n\in\mathbb{Z}$. If $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ is nonspecial for $n\in\mathbb{Z}$, then*

$$u(n)=2\omega_0^{\infty_2}-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_2)}\partial_{z_j}\log\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{+}(n)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{-}(n)))},\tag{4.24}$$

$$v(n)=-\omega_0^{\infty_1}+\sum_{j=1}^{m-1}d_{j,0}^{(\infty_1)}\partial_{z_j}\log\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{-}(n)))},\tag{4.25}$$

$$w(n)=-\frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{-}(n)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^{+}(n)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n)))^2}\exp\big(e_1(Q_0)-e_2(Q_0)\big).\tag{4.26}$$

Proof. Applying Abel's theorem to (4.4), we obtain

$$\underline{\rho}^{+}(n)+\underline{\mathcal{A}}(P_{\infty_1})=\underline{\rho}(n)+\underline{\mathcal{A}}(P_{\infty_2}),\tag{4.27}$$

and we hence conclude that

$$\theta\big(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^{+}(n))\big)=\theta\big(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n))\big).\tag{4.28}$$

Because $y=V_{11}^{(q)}+V_{12}^{(q)}\phi+V_{13}^{(q)}(w^-/\phi^-)$, we obtain

$$\begin{aligned}
&y\underset{\zeta\to 0}{=}\zeta^{-q-1}[-\alpha_0-\alpha_1\zeta-\alpha_2\zeta^2+O(\zeta^3)], && P\to P_{\infty_1},\\
&y\underset{\xi\to 0}{=}\xi^{-2q-1}[\beta_0+\beta_1\xi^2+O(\xi^4)], && P\to P_{\infty_2},
\end{aligned}\tag{4.29}$$

with the local coordinates $\zeta=1/\lambda$ near P_{∞_1} and $\xi=1/\lambda^{1/2}$ near P_{∞_2}. From (4.8) and (4.10), we obtain

$$\omega_j=\sum_{l=1}^{m-1}C_{jl}\widetilde{\omega}_l=\sum_{l=1}^{2q+1}C_{jl}\frac{\lambda^{l-1}d\lambda}{3y^2-2yR_m+S_m}+\sum_{l=2q+2}^{m-1}C_{jl}\frac{(y-R_m/3)\lambda^{l-2q-2}d\lambda}{3y^2-2yR_m+S_m},\tag{4.30}$$

where $j=1,\dots,m-1$. By a direct calculation, we obtain the asymptotic expansions

$$\begin{aligned}\omega_j &\underset{\zeta\to 0}{=} \big(d_{j,0}^{(\infty_1)}+O(\zeta)\big)d\zeta, \quad P\to P_{\infty_1}, \quad j=1,\dots,m-1,\\ \omega_j &\underset{\xi\to 0}{=} \big(d_{j,0}^{(\infty_2)}+O(\xi)\big)d\xi, \quad P\to P_{\infty_2}, \quad j=1,\dots,m-1,\end{aligned} \tag{4.31}$$

where

$$d_{j,0}^{(\infty_1)}=\frac{2}{3\alpha_0}C_{j,m-1}-\frac{1}{\alpha_0^2}C_{j,2q+1}, \qquad d_{j,0}^{(\infty_2)}=-\frac{1}{3\beta_0}C_{j,m-1}-\frac{1}{\alpha_0\beta_0}C_{j,2q+1}.$$

Expanding the ratios of the Riemann theta functions in (4.19), we obtain

$$\begin{aligned}\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))} &= \frac{\theta(\underline{\Lambda}-\underline{\mathcal{A}}(P)+\underline{\rho}^+(n))}{\theta(\underline{\Lambda}-\underline{\mathcal{A}}(P)+\underline{\rho}(n))}=\\ &=\frac{\theta(\underline{\Lambda}-\underline{\mathcal{A}}(P_{\infty_1})+\underline{\rho}^+(n)+\int_P^{P_{\infty_1}}\underline{\omega})}{\theta(\underline{\Lambda}-\underline{\mathcal{A}}(P_{\infty_1})+\underline{\rho}(n)+\int_P^{P_{\infty_1}}\underline{\omega})} \underset{\zeta\to 0}{=}\\ &\underset{\zeta\to 0}{=} \frac{\theta(\dots,\Lambda_j-\mathcal{A}_j(P_{\infty_1})+\rho_j^+(n)-d_{j,0}^{(\infty_1)}\zeta+O(\zeta^2),\dots)}{\theta(\dots,\Lambda_j-\mathcal{A}_j(P_{\infty_1})+\rho_j(n)-d_{j,0}^{(\infty_1)}\zeta+O(\zeta^2),\dots)} \underset{\zeta\to 0}{=}\\ &\underset{\zeta\to 0}{=} \frac{\theta_1^+-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\theta_1^+\zeta+O(\zeta^2)}{\theta_1-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\theta_1\zeta+O(\zeta^2)} \underset{\zeta\to 0}{=}\\ &\underset{\zeta\to 0}{=} \frac{\theta_1^+}{\theta_1}\left(1-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\log\frac{\theta_1^+}{\theta_1}\zeta+O(\zeta^2)\right), \quad P\to P_{\infty_1},\end{aligned} \tag{4.32}$$

where $\theta_1=\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n)))$ and $\theta_1^+=\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^+(n)))$. Similarly, we obtain

$$\frac{\theta(\underline{z}(P,\underline{\hat{\mu}}^+(n)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(n)))} \underset{\xi\to 0}{=} \frac{\theta_2^+}{\theta_2}\left(1-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_2)}\frac{\partial}{\partial z_j}\log\frac{\theta_2^+}{\theta_2}\xi+O(\xi^2)\right), \quad P\to P_{\infty_2}, \tag{4.33}$$

where $\theta_2=\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}(n)))$ and $\theta_2^+=\theta(\underline{z}(P_{\infty_2},\underline{\hat{\mu}}^+(n)))$. Therefore,

$$\begin{aligned}\phi(P,n) &\underset{\zeta\to 0}{=} \frac{\theta_2\theta_1^+}{\theta_2^+\theta_1}\exp\big(e_1(Q_0)-e_2(Q_0)\big)\times\\ &\quad\times\left[\zeta+\left(\omega_0^{\infty_1}-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_1)}\frac{\partial}{\partial z_j}\log\frac{\theta_1^+}{\theta_1}\right)\zeta^2+O(\zeta^3)\right], \quad P\to P_{\infty_1},\\ \phi(P,n) &\underset{\xi\to 0}{=} \xi^{-1}+\omega_0^{\infty_2}-\sum_{j=1}^{m-1}d_{j,0}^{(\infty_2)}\frac{\partial}{\partial z_j}\log\frac{\theta_2^+}{\theta_2}+O(\xi), \qquad P\to P_{\infty_2}.\end{aligned} \tag{4.34}$$

On the other hand, from (4.2) and (4.3), we have

$$\begin{aligned}\phi(P,n) &\underset{\zeta\to 0}{=} -w\big(\zeta-v^+\zeta^2+O(\zeta^3)\big), \qquad P\to P_{\infty_1},\\ \phi(P,n) &\underset{\xi\to 0}{=} \xi^{-1}+(E+1)^{-1}u+O(\zeta), \quad P\to P_{\infty_2}.\end{aligned} \tag{4.35}$$

Comparing (4.34) and (4.35) and using (4.28), we obtain representations (4.26) for $w(n)$, (4.25) for $v(n)$, and (4.24) for $u(n)$. ■

The b-period of the differential $\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}$ is denoted by

$$\underline{U}^{(3)} = (U^{(3)}_1, \ldots, U^{(3)}_{m-1}), \qquad U^{(3)}_j = \frac{1}{2\pi i}\int_{\mathsf{b}_j} \omega^{(3)}_{P_{\infty_1},P_{\infty_2}}, \quad j = 1, \ldots, m-1. \tag{4.36}$$

Combining (4.12), (4.21), (4.36), and (4.24)–(4.26) shows the remarkable linearity of the theta function representations for $u(n)$, $v(n)$, and $w(n)$ with respect to $n \in \mathbb{Z}$. In fact, we can rewrite (4.24)–(4.26) as

$$u(n) = 2\omega_0^{\infty_2} - \sum_{j=1}^{m-1} d^{(\infty_2)}_{j,0} \frac{\partial}{\partial z_j} \log \frac{\theta(\underline{K} - \underline{U}^{(3)} - \underline{U}^{(3)} n)}{\theta(\underline{K} + \underline{U}^{(3)} - \underline{U}^{(3)} n)}, \tag{4.37}$$

$$v(n) = -\omega_0^{\infty_1} + \sum_{j=1}^{m-1} d^{(\infty_1)}_{j,0} \frac{\partial}{\partial z_j} \log \frac{\theta(\underline{K} - \underline{U}^{(3)} n)}{\theta(\underline{K} + \underline{U}^{(3)} - \underline{U}^{(3)} n)}, \tag{4.38}$$

$$w(n) = -\frac{\theta(\underline{K} + \underline{U}^{(3)} - \underline{U}^{(3)} n)\theta(\underline{K} - \underline{U}^{(3)} - \underline{U}^{(3)} n)}{\theta(\underline{K} - \underline{U}^{(3)} n)^2} \exp\big(e_1(Q_0) - e_2(Q_0)\big), \tag{4.39}$$

where $\underline{K} = \underline{\Lambda} - \underline{\mathcal{A}}(P_{\infty_1}) + \underline{\rho}(n_0) + \underline{U}^{(3)} n_0$.

5. Algebro-geometric solutions of the BM lattice hierarchy

In this section, we extend the results in Secs. 3 and 4 for the stationary BM lattice hierarchy to the time-dependent case. By analogy with (3.7), we introduce the time-dependent Baker–Akhiezer function

$$\begin{aligned}
&\psi(P, n, n_0, t_r, t_{0,r}) = U(u(n, t_r), v(n, t_r), w(n, t_r); \lambda(P))\psi(P, n, n_0, t_r, t_{0,r}),\\
&\psi_{t_r}(P, n, n_0, t_r, t_{0,r}) = \widetilde{V}^{(r)}(u(n, t_r), v(n, t_r), w(n, t_r); \lambda(P))\psi(P, n, n_0, t_r, t_{0,r}),\\
&V^{(q)}(u(n, t_r), v(n, t_r), w(n, t_r); \lambda(P))\psi(P, n, n_0, t_r, t_{0,r}) = y(P)\psi(P, n, n_0, t_r, t_{0,r}),\\
&\psi_1(P, n_0, n_0, t_{0,r}, t_{0,r}) = 1, \quad P \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1}, P_{\infty_2}\}, \quad (n, t_r), (n_0, t_{0,r}) \in \mathbb{Z} \times \mathbb{R}.
\end{aligned} \tag{5.1}$$

The trigonal curve $\mathcal{K}_{m-1}$ used here is defined below. The compatibility conditions for the first three expressions in (5.1) yield

$$U_{t_r} - (E\widetilde{V}^{(r)})U + U\widetilde{V}^{(r)} = 0, \tag{5.2}$$

$$(EV^{(q)})U - UV^{(q)} = 0, \tag{5.3}$$

$$V^{(q)}_{t_r} - [\widetilde{V}^{(r)}, V^{(q)}] = 0. \tag{5.4}$$

A direct calculation shows that $yI - V^{(q)}$ satisfies (5.3) and (5.4). The characteristic polynomial of Lax matrix $V^{(q)}$ for the BM lattice hierarchy is then a constant independent of variables n and t_r with the expansion

$$\det(yI - V^{(q)}) = y^3 - y^2 R_m(\lambda) + yS_m(\lambda) - T_m(\lambda),$$

where $R_m(\lambda)$, $S_m(\lambda)$, and $T_m(\lambda)$ are defined as in (3.4). The time-dependent BM lattice curve $\mathcal{K}_{m-1}$ is then determined by

$$\mathcal{K}_{m-1}\colon \quad \mathcal{F}_m(\lambda, y) = y^3 - y^2 R_m(\lambda) + yS_m(\lambda) - T_m(\lambda) = 0.$$

The meromorphic function $\phi(P,n,t_r)$ on $\mathcal{K}_{m-1}$ defined by

$$\phi(P,n,t_r)=\frac{\psi_1^+(P,n,n_0,t_r,t_{0,r})}{\psi_1(P,n,n_0,t_r,t_{0,r})},\quad P\in\mathcal{K}_{m-1},\quad (n,t_r)\in\mathbb{Z}\times\mathbb{R}, \tag{5.5}$$

is closely related to $\psi(P,n,n_0,t_r,t_{0,r})$. By (5.1), Eq. (5.5) implies that

$$\begin{aligned}\phi(P,n,t_r)&=\frac{yV_{23}^{(q)}(\lambda,n,t_r)+C_m(\lambda,n,t_r)}{yV_{13}^{(q)}(\lambda,n,t_r)+A_m(\lambda,n,t_r)}=\\&=\frac{F_{m-1}(\lambda,n,t_r)}{y^2V_{23}^{(q)}(\lambda,n,t_r)-y[C_m(\lambda,n,t_r)+V_{23}^{(q)}(\lambda,n,t_r)R_m(\lambda)]+D_m(\lambda,n,t_r)}=\\&=\frac{y^2V_{13}^{(q)}(\lambda,n,t_r)-y[A_m(\lambda,n,t_r)+V_{13}^{(q)}(\lambda,n,t_r)R_m(\lambda)]+B_m(\lambda,n,t_r)}{E_{m-1}(\lambda,n,t_r)},\end{aligned} \tag{5.6}$$

where $P=(\lambda,y)\in\mathcal{K}_{m-1}$, $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, and where $A_m(\lambda,n,t_r)$, $B_m(\lambda,n,t_r)$, $C_m(\lambda,n,t_r)$, $D_m(\lambda,n,t_r)$, $E_{m-1}(\lambda,n,t_r)$, and $F_{m-1}(\lambda,n,t_r)$ are defined as in (3.11) and (3.12) and $\mathcal{A}_m(\lambda,n,t_r)$ and $\mathcal{B}_m(\lambda,n,t_r)$ are introduced as in (3.13). Hence, Eqs. (3.14)–(3.17) also hold in the present context. Similarly, we write

$$E_{m-1}(\lambda,n,t_r)=\alpha_0^2\beta_0\prod_{j=1}^{m-1}\big(\lambda-\mu_j(n,t_r)\big), \tag{5.7}$$

$$F_{m-1}(\lambda,n,t_r)=-\alpha_0^2\beta_0 w(n,t_r)\prod_{j=1}^{m-1}\big(\lambda-\mu_j^+(n,t_r)\big) \tag{5.8}$$

and define $\{\hat{\mu}_j(n,t_r)\}_{j=1,\ldots,m-1}\subset\mathcal{K}_{m-1}$ and $\{\hat{\mu}_j^+(n,t_r)\}_{j=1,\ldots,m-1}\subset\mathcal{K}_{m-1}$ by

$$\begin{aligned}\hat{\mu}_j(n,t_r)&=\big(\mu_j(n,t_r),y(\hat{\mu}_j(n,t_r))\big)=\\&=\left(\mu_j(n,t_r),-\frac{A_m(\mu_j(n,t_r),n,t_r)}{V_{13}^{(q)}(\mu_j(n,t_r),n,t_r)}\right),\quad (n,t_r)\in\mathbb{Z}\times\mathbb{R},\end{aligned} \tag{5.9}$$

$$\begin{aligned}\hat{\mu}_j^+(n,t_r)&=\big(\mu_j^+(n,t_r),y(\hat{\mu}_j^+(n,t_r))\big)=\\&=\left(\mu_j^+(n,t_r),-\frac{C_m(\mu_j^+(n,t_r),n,t_r)}{V_{23}^{(q)}(\mu_j^+(n,t_r),n,t_r)}\right),\quad (n,t_r)\in\mathbb{Z}\times\mathbb{R}.\end{aligned} \tag{5.10}$$

It follows from (5.6) that the divisor $(\phi(P,n,t_r))$ of $\phi(P,n,t_r)$ has the form

$$(\phi(P,n,t_r))=\mathcal{D}_{P_{\infty_1},\hat{\mu}_1^+(n,t_r),\ldots,\hat{\mu}_{m-1}^+(n,t_r)}(P)-\mathcal{D}_{P_{\infty_2},\hat{\mu}(n,t_r),\ldots,\hat{\mu}_{m-1}(n,t_r)}(P), \tag{5.11}$$

i.e., P_{∞_1}, $\hat{\mu}_1^+(n,t_r),\ldots,\hat{\mu}_{m-1}^+(n,t_r)$ are the m simple zeros of $\phi(P,n,t_r)$ and P_{∞_2} and $\hat{\mu}_1(n,t_r),\ldots,$ $\hat{\mu}_{m-1}(n,t_r)$ are its m simple poles.

Similarly, $\phi(P,n,t_r)$ satisfies the Riccati-type equation

$$\begin{aligned}&\phi^+(P,n,t_r)\phi(P,n,t_r)\phi^-(P,n,t_r)-u(n,t_r)\phi(P,n,t_r)\phi^-(P,n,t_r)-\\&\qquad-v(n,t_r)\phi^-(P,n,t_r)-w^-(n,t_r)=\lambda\phi^-(P,n,t_r)\end{aligned} \tag{5.12}$$

and also

$$\phi(P,n,t_r)\phi(P^*,n,t_r)\phi(P^{**},n,t_r) = -\frac{F_{m-1}(\lambda,n,t_r)}{E_{m-1}(\lambda,n,t_r)},$$

$$\begin{aligned}&\phi(P,n,t_r)+\phi(P^*,n,t_r)+\phi(P^{**},n,t_r) = \\ &\qquad = \frac{3B_m(\lambda,n,t_r)-2V_{13}^{(q)}(\lambda,n,t_r)S_m(\lambda)-A_m(\lambda,n,t_r)R_m(\lambda)}{E_{m-1}(\lambda,n,t_r)}, \\ &\frac{1}{\phi(P,n,t_r)}+\frac{1}{\phi(P^*,n,t_r)}+\frac{1}{\phi(P^{**},n,t_r)} = \\ &\qquad = \frac{3D_m(\lambda,n,t_r)-2V_{23}^{(q)}(\lambda,n,t_r)S_m(\lambda)-C_m(\lambda,n,t_r)R_m(\lambda)}{F_{m-1}(\lambda,n,t_r)}.\end{aligned} \tag{5.13}$$

Differentiating (5.5) with respect to t_r and using (5.1), we obtain

$$\begin{aligned}\phi_{t_r} &= \left(\frac{\psi_1^+}{\psi_1}\right)_{t_r} = \frac{\psi_{1,t_r}^+\psi_1-\psi_1^+\psi_{1,t_r}}{\psi_1^2} = \frac{\psi_1^+}{\psi_1}\left(\frac{\psi_{1,t_r}^+}{\psi_1^+}-\frac{\psi_{1,t_r}}{\psi_1}\right) = \\ &= \phi\triangle\frac{\psi_{1,t_r}}{\psi_1} = \phi\triangle\left(\widetilde{V}_{11}^{(r)}+\widetilde{V}_{12}^{(r)}\phi+\widetilde{V}_{13}^{(r)}\frac{w^-}{\phi^-}\right).\end{aligned} \tag{5.14}$$

Hence,

$$\frac{\phi(P,n,t_r)_{t_r}}{\phi(P,n,t_r)} = \triangle\left(\widetilde{V}_{11}^{(r)}(\lambda,n,t_r)+\widetilde{V}_{12}^{(r)}(\lambda,n,t_r)\phi+\widetilde{V}_{13}^{(r)}(\lambda,n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)}\right). \tag{5.15}$$

We can describe the dynamics of the zeros $\{\mu_j(n,t_r)\}_{j=1,\dots,m-1}$ of $E_{m-1}(\lambda,n,t_r)$ by Dubrovin-type equations.

Lemma 1. *If the zeros $\{\mu_j(n,t_r)\}_{j=1,\dots,m-1}$ of $E_{m-1}(\lambda,n,t_r)$ remain distinct for $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, then $\{\mu_j(n,t_r)\}_{j=1,\dots,m-1}$ satisfy the system of differential equations*

$$\begin{aligned}\mu_{j,t_r}(n,t_r) &= [\widetilde{V}_{13}^{(r)}(\mu_j(n,t_r),n,t_r)V_{12}^{(q)}(\mu_j(n,t_r),n,t_r) - \\ &\quad - \widetilde{V}_{12}^{(r)}(\mu_j(n,t_r),n,t_r)V_{13}^{(q)}(\mu_j(n,t_r),n,t_r)]\times \\ &\quad \times\frac{[3y^2(\hat{\mu}_j(n,t_r))-2y(\hat{\mu}_j(n,t_r))R_m(\mu_j(n,t_r))+S_m(\mu_j(n,t_r))]}{\alpha_0^2\beta_0\prod_{\substack{k=1\\k\neq j}}^{m-1}(\mu_j(n,t_r)-\mu_k(n,t_r))}, \quad 1\le j\le m-1.\end{aligned} \tag{5.16}$$

Proof. Using (3.11)–(3.13) and (5.4), we obtain

$$\begin{aligned}E_{m-1,t_r}(\lambda,n,t_r) &= \big((V_{13}^{(q)})^2V_{32}^{(q)}+V_{12}^{(q)}V_{13}^{(q)}(V_{22}^{(q)}-V_{33}^{(q)})-(V_{12}^{(q)})^2V_{23}^{(q)}\big)_{t_r} = \\ &= (3\widetilde{V}_{11}^{(r)}-\widetilde{R}^{(r)})E_{m-1}-\big(\widetilde{V}_{12}^{(r)}A_m-\widetilde{V}_{13}^{(r)}\mathcal{A}_m\big)R_m + \\ &\quad + 3\big(\widetilde{V}_{12}^{(r)}B_m-\widetilde{V}_{13}^{(r)}\mathcal{B}_m\big)-2\big(\widetilde{V}_{12}^{(r)}V_{13}^{(q)}-\widetilde{V}_{13}^{(r)}V_{12}^{(q)}\big)S_m = \\ &= (3\widetilde{V}_{11}^{(r)}-\widetilde{R}^{(r)})E_{m-1}+3\widetilde{V}_{12}^{(r)}\big(V_{23}^{(q)}\mathcal{A}_m-V_{22}^{(q)}A_m\big) - \\ &\quad + 3\widetilde{V}_{13}^{(r)}\big(V_{32}^{(q)}A_m-V_{33}^{(q)}\mathcal{A}_m\big)+2\big(\widetilde{V}_{12}^{(r)}A_m-\widetilde{V}_{13}^{(r)}\mathcal{A}_m\big)R_m + \\ &\quad + \big(\widetilde{V}_{12}^{(r)}V_{13}^{(q)}-\widetilde{V}_{13}^{(r)}V_{12}^{(q)}\big)S_m,\end{aligned} \tag{5.17}$$

where $\widetilde{R}^{(r)} = \widetilde{V}_{11}^{(r)} + \widetilde{V}_{22}^{(r)} + \widetilde{V}_{33}^{(r)}$. By virtue of (5.7), (5.9), and (3.16), we obtain

$$\frac{A_m}{V_{13}^{(q)}}\Big|_{\lambda=\mu_j(n,t_r)} = \frac{\mathcal{A}_m}{V_{12}^{(q)}}\Big|_{\lambda=\mu_j(n,t_r)} = -y(\hat{\mu}_j(n,t_r)). \tag{5.18}$$

Then

$$\begin{aligned}
&\widetilde{V}_{12}^{(r)}(V_{23}^{(q)}\mathcal{A}_m - V_{22}^{(q)}A_m)\big|_{\lambda=\mu_j(n,t_r)} = y^2(\hat{\mu}_j(n,t_r))\widetilde{V}_{12}^{(r)}V_{13}^{(q)}\big|_{\lambda=\mu_j(n,t_r)},\\
&\widetilde{V}_{13}^{(r)}(V_{32}^{(q)}A_m - V_{33}^{(q)}\mathcal{A}_m)\big|_{\lambda=\mu_j(n,t_r)} = y^2(\hat{\mu}_j(n,t_r))\widetilde{V}_{13}^{(r)}V_{12}^{(q)}\big|_{\lambda=\mu_j(n,t_r)},\\
&(\widetilde{V}_{12}^{(r)}A_m - \widetilde{V}_{13}^{(r)}\mathcal{A}_m)\big|_{\lambda=\mu_j(n,t_r)} = -y(\hat{\mu}_j(n,t_r))\widetilde{V}_{12}^{(r)}V_{13}^{(q)} - \widetilde{V}_{13}^{(r)}V_{12}^{(q)}\big|_{\lambda=\mu_j(n,t_r)}.
\end{aligned} \tag{5.19}$$

Hence,

$$\begin{aligned}
E_{m-1,t_r}(\lambda,n,t_r)\big|_{\lambda=\mu_j(n,t_r)} &= (\widetilde{V}_{12}^{(r)}V_{13}^{(q)} - \widetilde{V}_{13}^{(r)}V_{12}^{(q)})\big|_{\lambda=\mu_j(n,t_r)} \times\\
&\times \big[3y^2(\hat{\mu}_j(n,t_r)) - 2y(\hat{\mu}_j(n,t_r))R_m(\mu_j(n,t_r)) + S_m(\mu_j(n,t_r))\big].
\end{aligned} \tag{5.20}$$

On the other hand, from (5.7), we obtain

$$E_{m-1,t_r}(\lambda,n,t_r)\big|_{\lambda=\mu_j(n,t_r)} = -\alpha_0^2\beta_0\mu_{j,t_r}(n,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}(\mu_j(n,t_r) - \mu_k(n,t_r)), \tag{5.21}$$

which together with (5.20) leads to (5.16). ∎

From the first two expression in (5.1), we obtain

$$\begin{aligned}
\psi_1(P,n,n_0,t_r,t_{0,r}) = \exp\bigg(\int_{t_{0,r}}^{t_r}\Big[&\widetilde{V}_{11}^{(r)}(\lambda,n_0,t') +\\
&+ \widetilde{V}_{12}^{(r)}(\lambda,n_0,t')\phi(P,n_0,t') + \widetilde{V}_{13}^{(r)}(\lambda,n_0,t')\frac{w^-(n_0,t')}{\phi^-(P,n_0,t')}\Big]dt'\bigg) \times\\
&\times \begin{cases} \prod_{n'=n_0}^{n-1}\phi(P,n',t_r), & n \geq n_0+1,\\ 1, & n = n_0,\\ \prod_{n'=n}^{n_0-1}\phi(P,n',t_r)^{-1}, & n \leq n_0-1. \end{cases}
\end{aligned} \tag{5.22}$$

We note that

$$\psi_1(P,n,n_0,t_r,t_{0,r}) = \psi_1(P,n,n_0,t_r,t_r)\psi_1(P,n_0,n_0,t_r,t_{0,r}), \tag{5.23}$$

where $P = (\lambda,y) \in \mathcal{K}_{m-1} \setminus \{P_{\infty_1},\ P_{\infty_2}\}, (n,t_r), (n_0,t_{0,r}) \in \mathbb{Z}\times\mathbb{R}$.

Analyzing the integrand in (5.22), we introduce a function $I_r(P,n,t_r)$ by

$$I_r(P,n,t_r) = \widetilde{V}_{11}^{(r)}(\lambda,n,t_r) + \widetilde{V}_{12}^{(r)}(\lambda,n,t_r)\phi(P,n,t_r) + \widetilde{V}_{13}^{(r)}(\lambda,n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)} \tag{5.24}$$

whose homogeneous cases are denoted by

$$\hat{I}_r^{(s)}(P,n,t_r)=\widehat{\widetilde{V}}_{11}^{(r,s)}(\lambda,n,t_r)+\widehat{\widetilde{V}}_{12}^{(r,s)}(\lambda,n,t_r)\phi(P,n,t_r)+ \\ +w^-(n,t_r)\frac{\widehat{\widetilde{V}}_{13}^{(r,s)}(\lambda,n,t_r)}{\phi^-(P,n,t_r)},\quad s=1,2, \tag{5.25}$$

and

$$\begin{aligned}\widehat{\widetilde{V}}_{1j}^{(r,1)}&=\widetilde{V}_{1j}^{(r)}\big|_{\tilde{\alpha}_0=1,\beta_0=0,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0},\\ \widehat{\widetilde{V}}_{1j}^{(r,2)}&=\widetilde{V}_{1j}^{(r)}\big|_{\tilde{\alpha}_0=0,\beta_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0},\end{aligned}\quad j=1,2,3.$$

Hence,

$$I_r(P,n,t_r)=\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\hat{I}_l^{(1)}(P,n,t_r)+\sum_{l=0}^{r}\tilde{\beta}_{r-l}\hat{I}_l^{(2)}(P,n,t_r). \tag{5.26}$$

Lemma 2. *Let $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, and let $\zeta=1/\lambda$ and $\xi=1/\lambda^{1/2}$ be the respective local coordinates near P_{∞_1} and P_{∞_2}. Then*

$$\begin{aligned}\hat{I}_r^{(1)}(P,n,t_r)&\underset{\zeta\to0}{=}-\zeta^{-r-1}+\hat{b}_{r+1}(n,t_r)+O(\zeta),\quad &P\to P_{\infty_1},\\ \hat{I}_r^{(1)}(P,n,t_r)&\underset{\xi\to0}{=}O(\xi),\quad &P\to P_{\infty_2},\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\hat{I}_r^{(2)}(P,n,t_r)&\underset{\zeta\to0}{=}\check{b}_{r+1}(n,t_r)+O(\zeta),\quad &P\to P_{\infty_1},\\ \hat{I}_r^{(2)}(P,n,t_r)&\underset{\xi\to0}{=}\xi^{-2r-1}+O(\xi),\quad &P\to P_{\infty_2}.\end{aligned} \tag{5.28}$$

Proof. We use induction to prove (5.27). For convenience, we introduce the notation

$$\hat{a}^{(r)}=\tilde{a}^{(r)}\big|_{\tilde{\alpha}_0=1,\tilde{\beta}_0=0,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0},$$

where $\tilde{a}^{(r)}$ is defined in (2.14). From (5.1), (5.24) and (5.25), we easily obtain

$$\hat{I}_r^{(1)}(P,n,t_r)=\widehat{\widetilde{V}}_{11}^{(r,1)}(\lambda,n,t_r)+\widehat{\widetilde{V}}_{12}^{(r,1)}(\lambda,n,t_r)\phi(P,n,t_r)+\widehat{\widetilde{V}}_{13}^{(r,1)}(\lambda,n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)}= \\ =\hat{c}^{(r)}(\lambda,n,t_r)+\hat{a}^{(r)}(\lambda,n,t_r)\phi(P,n,t_r)+\hat{b}^{(r)}(\lambda,n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)}. \tag{5.29}$$

Using (4.2) and (4.3), we obtain

$$\begin{aligned}\hat{I}_0^{(1)}(P,n,t_r)&=\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)}\underset{\zeta\to0}{=}-\zeta^{-1}-v(n,t_r)+O(\zeta)\underset{\zeta\to0}{=}\\ &\underset{\zeta\to0}{=}-\zeta^{-1}+\hat{b}_1(n,t_r)+O(\zeta),\quad &P\to P_{\infty_1},\\ \hat{I}_0^{(1)}(P,n,t_r)&=\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)}\underset{\xi\to0}{=}O(\xi),\quad &P\to P_{\infty_2}.\end{aligned} \tag{5.30}$$

Therefore, (5.27) holds for $r=0$. We suppose that (5.27) has the expansions

$$\begin{aligned}
\hat{I}_r^{(1)}(P,n,t_r) &= -\zeta^{-r-1} + \sum_{j=0}^{\infty} \sigma_j(n,t_r)\zeta^j, \quad P \to P_{\infty_1}, \\
\hat{I}_r^{(1)}(P,n,t_r) &= \sum_{j=1}^{\infty} \delta_j(n,t_r)\xi^j, \qquad\qquad\quad P \to P_{\infty_2},
\end{aligned} \tag{5.31}$$

for some coefficients $\{\sigma_j(n,t_r)\}_{j\in\mathbb{N}_0}$ and $\{\delta_j(n,t_r)\}_{j\in\mathbb{N}}$ to be determined. From (5.15), (5.24), and (5.25), we obtain

$$\phi(P,n,t_r)_{t_r} = \phi(P,n,t_r)\triangle \hat{I}_r^{(1)}(P,n,t_r). \tag{5.32}$$

Using (4.2) and comparing like powers of ζ in (5.32), we obtain

$$\phi_{j,t_r} = \phi_1 \triangle\sigma_{j-1} + \phi_2\triangle\sigma_{j-2} + \cdots + \phi_j\triangle\sigma_0, \quad j \geq 1. \tag{5.33}$$

Explicitly, it follows from (2.16) that

$$\begin{aligned}
\triangle\sigma_0 &= \frac{1}{\phi_1}\phi_{1,t_r} = \frac{1}{w}w_{t_r} = \triangle\hat{b}_{r+1}, \\
\triangle\sigma_1 &= \frac{1}{\phi_1}\phi_{2,t_r} - \frac{\phi_2}{\phi_1}\triangle\sigma_0 = -v^+_{t_r} = \triangle(u^+\hat{a}^{++}_{r+1} + \hat{c}^{++}_{r+1} + \hat{c}^{+}_{r+1}).
\end{aligned} \tag{5.34}$$

Similarly, using (4.3) and comparing like powers of ξ in (5.32), we obtain

$$\kappa_{j,t_r} = \kappa_{-1}\triangle\delta_{j+1} + \kappa_0\triangle\delta_j + \cdots + \kappa_{j-1}\triangle\delta_1, \quad j \geq 0. \tag{5.35}$$

Explicitly, it follows from (2.16) that

$$\begin{aligned}
\triangle\delta_1 &= \kappa_{0,t_r} = (E+1)^{-1}u_{t_r} = \triangle\big(-\hat{a}_{r+1}\big), \\
\triangle\delta_2 &= \kappa_{1,t_r} - \kappa_0\triangle\delta_1 = \\
&= (E+1)^{-1}[v_{t_r} + 2((E+1)^{-1}u)(E+1)^{-1}u_{t_r}] - \\
&\quad - ((E+1)^{-1}u)(E+1)^{-1}u_{t_r} = \triangle\big(-\hat{c}_{r+1} - \hat{a}_{r+1}(E+1)^{-1}u\big).
\end{aligned} \tag{5.36}$$

Hence, from (5.34) and (5.36), we obtain

$$\begin{aligned}
&\sigma_0(n,t_r) = \hat{b}_{r+1}(n,t_r), \\
&\sigma_1(n,t_r) = u^+(n,t_r)\hat{a}^{++}_{r+1}(n,t_r) + \hat{c}^{++}_{r+1}(n,t_r) + \hat{c}^{+}_{r+1}(n,t_r), \\
&\delta_1(n,t_r) = -\hat{a}_{r+1}(n,t_r), \\
&\delta_2(n,t_r) = -\hat{c}_{r+1}(n,t_r) - \hat{a}_{r+1}(n,t_r)(E+1)^{-1}u(n,t_r),
\end{aligned} \tag{5.37}$$

where the summation constants are set equal to zero if we take into account that there are no arbitrary constants in the expansions of $\phi(P,n,t_r)$ near P_{∞_s}, $s=1,2$ nor in the homogeneous coefficients $\hat{a}_{r+1}$, $\hat{b}_{r+1}$, and $\hat{c}_{r+1}$ under the condition that $\triangle\triangle^{-1} = \triangle^{-1}\triangle = 1$. Therefore,

$$
\begin{aligned}
\hat{I}^{(1)}_{r+1}(P,n,t_r) = \widehat{\widehat{V}}^{(r+1,1)}_{11}&(\lambda,n,t_r) + \widehat{\widehat{V}}^{(r+1,1)}_{12}(\lambda,n,t_r)\phi(P,n,t_r) + \\
&+ \widehat{\widehat{V}}^{(r+1,1)}_{13}(\lambda,n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)} \underset{\zeta\to 0}{=} \\
&\underset{\zeta\to 0}{=} \zeta^{-1}\hat{I}^{(1)}_r(P,n,t_r) + \hat{c}_{r+1}(n,t_r) + \hat{a}_{r+1}(n,t_r)\phi(P,n,t_r) + \\
&+ \hat{b}_{r+1}(n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)} \underset{\zeta\to 0}{=} \\
&\underset{\zeta\to 0}{=} -\zeta^{-r-2} + \hat{b}_{r+2}(n,t_r) + O(\zeta), \qquad P \to P_{\infty_1}, \\
\hat{I}^{(1)}_{r+1}(P,n,t_r) &\underset{\xi\to 0}{=} \xi^{-2}\hat{I}^{(1)}_r(P,n,t_r) + \hat{c}_{r+1}(n,t_r) + \hat{a}_{r+1}(n,t_r)\phi(P,n,t_r) + \\
&+ \hat{b}_{r+1}(n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)} \underset{\xi\to 0}{=} O(\xi), \qquad P \to P_{\infty_2}.
\end{aligned}
\tag{5.38}
$$

The proof of (5.27) is complete. Relation (5.28) can be proved similarly. ■

Using Lemma 2 and (5.26), we obtain

$$
\begin{aligned}
I_r(P,n,t_r) &\underset{\zeta\to 0}{=} -\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\zeta^{-l-1} - \tilde{\alpha}_{r+1} + \tilde{b}_{r+1}(n,t_r) + O(\zeta), \quad P \to P_{\infty_1}, \\
I_r(P,n,t_r) &\underset{\xi\to 0}{=} \sum_{l=0}^{r}\tilde{\beta}_{r-l}\xi^{-2l-1} + O(\xi), \qquad P \to P_{\infty_2}.
\end{aligned}
\tag{5.39}
$$

Let $\omega^{(2)}_{P_{\infty_s},j}$, $j \geq 2$, be the normalized differential of the second kind holomorphic on $\mathcal{K}_{m-1} \setminus \{P_{\infty_s}\}$ with a pole of order j at P_{∞_s}, $s = 1,2$,

$$
\begin{aligned}
\omega^{(2)}_{P_{\infty_1},j} &\underset{\zeta\to 0}{=} \big(\zeta^{-j} + O(1)\big)d\zeta, \quad P \to P_{\infty_1}, \quad \zeta = \frac{1}{\lambda}, \\
\omega^{(2)}_{P_{\infty_2},j} &\underset{\xi\to 0}{=} \big(\xi^{-j} + O(1)\big)d\xi, \quad P \to P_{\infty_2}, \quad \xi = \frac{1}{\lambda^{1/2}},
\end{aligned}
\tag{5.40}
$$

with vanishing a-periods: $\int_{\mathsf{a}_k}\omega^{(2)}_{P_{\infty s,j}} = 0$, $s = 1,2$, $k = 1,\ldots,m-1$. Moreover, we define

$$
\widetilde{\Omega}^{(2)}_r = -\sum_{l=0}^{r}\tilde{\alpha}_{r-l}(l+1)\omega^{(2)}_{P_{\infty_1},l+2} + \sum_{l=0}^{r}\tilde{\beta}_{r-l}(2l+1)\omega^{(2)}_{P_{\infty_2},2l+2}. \tag{5.41}
$$

Integrating (5.41) yields

$$
\begin{aligned}
\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_r &\underset{\zeta\to 0}{=} \sum_{l=0}^{r}\tilde{\alpha}_{r-l}\zeta^{-l-1} + \widetilde{e}^{(2)}_1(Q_0) + O(\zeta), \qquad P \to P_{\infty_1}, \\
\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_r &\underset{\xi\to 0}{=} -\sum_{l=0}^{r}\tilde{\beta}_{r-l}\xi^{-2l-1} + \widetilde{e}^{(2)}_2(Q_0) + O(\xi), \quad P \to P_{\infty_2},
\end{aligned}
\tag{5.42}
$$

where $\widetilde{e}^{(2)}_1(Q_0)$ and $\widetilde{e}^{(2)}_2(Q_0)$ are constants.

Using these preparatory results, we write the representations of $\phi(P,n,t_r)$ and $\psi_1(P,n,n_0,t_r,t_{0,r})$ in terms of theta functions in the following forms.

Theorem 3. *We assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $P=(\lambda,y)\in\mathcal{K}_{m-1}\setminus\{P_{\infty_1},P_{\infty_2}\}$, $(n,n_0,t_r,t_{0,r})\in\mathbb{Z}^2\times\mathbb{R}^2$. If $\mathcal{D}_{\underline{\widehat{\mu}}(n,t_r)}$ is nonspecial for $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, then*

$$\phi(P,n,t_r)=\frac{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}^+(n,t_r)))}\frac{\theta(\underline{z}(P,\underline{\widehat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n,t_r)))}\times$$
$$\times\exp\Big(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}-e_2(Q_0)\Big) \tag{5.43}$$

and

$$\psi_1(P,n,n_0,t_r,t_{0,r})=\frac{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n,t_r)))}\frac{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n_0,t_{0,r})))}\times$$
$$\times\exp\Big((n-n_0)\Big(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}-e_2(Q_0)\Big)+$$
$$+(t_r-t_{0,r})\Big(\widetilde{e}^{(2)}_2(Q_0)-\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_r\Big)\Big). \tag{5.44}$$

Proof. As in Theorem 1, we conclude that $\phi(P,n,t_r)$ has form (5.43) and that $\psi_1(P,n,n_0,t_r,t_{0,r})$ for $t_{0,r}=t_r$ has the form

$$\psi_1(P,n,n_0,t_r,t_r)=\frac{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n_0,t_r)))}{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n,t_r)))}\frac{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n,t_r)))}{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n_0,t_r)))}\times$$
$$\times\exp\Big((n-n_0)\Big(\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_1},P_{\infty_2}}-e_2(Q_0)\Big)\Big). \tag{5.45}$$

It remains to use (5.22) to investigate

$$\psi_1(P,n_0,n_0,t_r,t_{0,r})=\exp\Big(\int_{t_{0,r}}^{t_r}I_r(P,n_0,t')dt'\Big). \tag{5.46}$$

Letting $\Psi_1(P,n_0,n_0,t_r,t_{0,r})$ denote the right-hand side of (5.44) for $n=n_0$, i.e.,

$$\Psi_1(P,n_0,n_0,t_r,t_{0,r})=\frac{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_2},\underline{\widehat{\mu}}(n_0,t_r)))}\frac{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n_0,t_r)))}{\theta(\underline{z}(P,\underline{\widehat{\mu}}(n_0,t_{0,r})))}\times$$
$$\times\exp\Big((t_r-t_{0,r})\Big(\widetilde{e}^{(2)}_2(Q_0)-\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_r\Big)\Big), \tag{5.47}$$

we must prove that

$$\psi_1(P,n_0,n_0,t_r,t_{0,r})=\Psi_1(P,n_0,n_0,t_r,t_{0,r}). \tag{5.48}$$

Using (3.17), (5.6), (5.24), and Lemma 1, we obtain

$$I_r(P,n,t_r) = \widetilde{V}_{11}^{(r)}(\lambda,n,t_r) + \widetilde{V}_{12}^{(r)}(\lambda,n,t_r)\phi(P,n,t_r) + \widetilde{V}_{13}^{(r)}(\lambda,n,t_r)\frac{w^-(n,t_r)}{\phi^-(P,n,t_r)} =$$

$$= \widetilde{V}_{11}^{(r)} + \widetilde{V}_{12}^{(r)}\frac{y^2V_{13}^{(q)} - y(A_m + V_{13}^{(q)}R_m) + B_m}{E_{m-1}} -$$

$$- \widetilde{V}_{13}^{(r)}\frac{y^2V_{12}^{(q)} - y(\mathcal{A}_m + V_{12}^{(q)}R_m) + \mathcal{B}_m}{E_{m-1}} =$$

$$= \frac{1}{E_{m-1}}\Bigg[\frac{1}{3}E_{m-1,t_r} + \frac{1}{3}\widetilde{R}^{(r)}E_{m-1} +$$

$$+ (\widetilde{V}_{12}^{(r)}V_{13}^{(q)} - \widetilde{V}_{13}^{(r)}V_{12}^{(q)})\left(y^2 - yR_m + \frac{2}{3}S_m\right) -$$

$$- (\widetilde{V}_{12}^{(r)}A_m - \widetilde{V}_{13}^{(r)}\mathcal{A}_m)\left(y - \frac{1}{3}R_m\right)\Bigg] \underset{\lambda\to\mu_j(n,t_r)}{=}$$

$$\underset{\lambda\to\mu_j(n,t_r)}{=} -\frac{\mu_{j,t_r}(n,t_r)}{\lambda-\mu_j(n,t_r)} + O(1) \underset{\lambda\to\mu_j(n,t_r)}{=} \partial_{t_r}\log(\lambda-\mu_j(n,t_r)) + O(1).$$

Then

$$\psi_1(P,n_0,n_0,t_r,t_{0,r}) = \exp\left(\int_{t_{0,r}}^{t_r}\partial_{t'}\log(\lambda-\mu_j(n_0,t'))dt'\right) =$$

$$= \frac{\lambda-\mu_j(n_0,t_r)}{\lambda-\mu_j(n_0,t_{0,r})}O(1) =$$

$$= \begin{cases} (\lambda-\mu_j(n_0,t_r))O(1), & P \text{ near } \hat{\mu}_j(n_0,t_r) \neq \hat{\mu}_j(n_0,t_{0,r}), \\ O(1), & P \text{ near } \hat{\mu}_j(n_0,t_r) = \hat{\mu}_j(n_0,t_{0,r}), \\ (\lambda-\mu_j(n_0,t_{0,r}))^{-1}O(1), & P \text{ near } \hat{\mu}_j(n_0,t_{0,r}) \neq \hat{\mu}_j(n_0,t_r), \end{cases} \tag{5.49}$$

where $O(1) \neq 0$. Hence, all zeros and poles of $\psi_1(P,n_0,n_0,t_r,t_{0,r})$ and $\Psi_1(P,n_0,n_0,t_r,t_{0,r})$ on $\mathcal{K}_{m-1} \setminus \{P_{\infty_1},P_{\infty_2}\}$ are simple and coincident. Moreover, using (5.39) and (5.42), we can easily see that the essential singularities of $\psi_1(P,n_0,n_0,t_r,t_{0,r})$ and $\Psi_1(P,n_0,n_0,t_r,t_{0,r})$ at P_{∞_1} and P_{∞_2} are identical. We can therefore apply the uniqueness result for Baker–Akhiezer functions to conclude that (5.48) holds because $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ is nonspecial for all $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$. We hence obtain (5.44). ■

The b period of the differential $\widetilde{\Omega}_r^{(2)}$ is denoted by

$$\underline{\widetilde{U}}_r^{(2)} = (\widetilde{U}_{r,1}^{(2)},\ldots,\widetilde{U}_{r,m-1}^{(2)}), \qquad \widetilde{U}_{r,k}^{(2)} = \frac{1}{2\pi i}\int_{\mathsf{b}_k}\widetilde{\Omega}_r^{(2)}, \quad k = 1,\ldots,m-1. \tag{5.50}$$

Theorem 4 (Straightening the flow)**.** *We have the relation*

$$\underline{\rho}(n,t_r) = \underline{\rho}(n_0,t_{0,r}) - \underline{U}^{(3)}(n-n_0) + \underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}) \pmod{\mathcal{T}_{m-1}}. \tag{5.51}$$

Proof. We introduce a meromorphic differential

$$\Omega(n,n_0,t_r,t_{0,r}) = \frac{\partial}{\partial\lambda}\log(\psi_1(P,n,n_0,t_r,t_{0,r}))d\lambda. \tag{5.52}$$

From representation (5.44), we obtain

$$\Omega(n,n_0,t_r,t_{0,r}) = (n-n_0)\omega^{(3)}_{P_{\infty_1},P_{\infty_2}} - (t_r - t_{0,r})\widetilde{\Omega}^{(2)}_r + \sum_{j=1}^{m-1} \omega^{(3)}_{\hat{\mu}_j(n,t_r),\hat{\mu}_j(n_0,t_{0,r})} + \overline{\omega},$$

where $\overline{\omega}$ denotes a holomorphic differential on $\mathcal{K}_{m-1}$, i.e., $\overline{\omega} = \sum_{j=1}^{m-1} \overline{e}_j\omega_j$ for some $\overline{e}_j \in \mathbb{C}$, $j = 1,\ldots,m-1$. Because $\psi_1(P,n,n_0,t_r,t_{0,r})$ is single-valued on $\mathcal{K}_{m-1}$, all a and b periods of Ω are integer multiples of $2\pi i$, and hence

$$2\pi i M_k = \int_{\mathsf{a}_k} \Omega(n,n_0,t_r,t_{0,r}) = \int_{\mathsf{a}_k} \overline{\omega} = \overline{e}_k, \quad k = 1,\ldots,m-1, \tag{5.53}$$

for some $M_k \in \mathbb{Z}$. Similarly, for some $N_k \in \mathbb{Z}$,

$$\begin{aligned}
2\pi i N_k = & \int_{\mathsf{b}_k} \Omega(n,n_0,t_r,t_{0,r}) = (n-n_0)\int_{\mathsf{b}_k} \omega^{(3)}_{P_{\infty_1},P_{\infty_2}} - (t_r - t_{0,r})\int_{\mathsf{b}_k} \widetilde{\Omega}^{(2)}_r + \\
& + \sum_{j=1}^{m-1} \int_{\mathsf{b}_k} \omega^{(3)}_{\hat{\mu}_j(n,t_r),\hat{\mu}_j(n_0,t_{0,r})} + \int_{\mathsf{b}_k} \overline{\omega} = \\
= & \ 2\pi i(n-n_0)U^{(3)}_k - 2\pi i(t_r - t_{0,r})\widetilde{U}^{(2)}_{r,k} + \\
& + 2\pi i \sum_{j=1}^{m-1} \int_{\hat{\mu}_j(n_0,t_{0,r})}^{\hat{\mu}_j(n,t_r)} \omega_k + 2\pi i \sum_{j=1}^{m-1} M_j \int_{b_k} \omega_j = \\
= & \ 2\pi i(n-n_0)U^{(3)}_k - 2\pi i(t_r - t_{0,r})\widetilde{U}^{(2)}_{r,k} + \\
& + 2\pi i\Big[\sum_{j=1}^{m-1} \int_{Q_0}^{\hat{\mu}_j(n,t_r)} \omega_k - \sum_{j=1}^{m-1} \int_{Q_0}^{\hat{\mu}_j(n_0,t_{0,r})} \omega_k\Big] + \\
& + 2\pi i \sum_{j=1}^{m-1} M_j \tau_{j,k}, \quad k = 1,\ldots,m-1.
\end{aligned} \tag{5.54}$$

Therefore, we have

$$\underline{N} = (n-n_0)\underline{U}^{(3)} - (t_r - t_{0,r})\underline{\widetilde{U}}^{(2)}_r + \sum_{j=1}^{m-1} \int_{Q_0}^{\hat{\mu}_j(n,t_r)} \underline{\omega} - \sum_{j=1}^{m-1} \int_{Q_0}^{\hat{\mu}_j(n_0,t_{0,r})} \underline{\omega} + \underline{M}\tau, \tag{5.55}$$

where $\underline{N} = (N_1,\ldots,N_{m-1})$ and $\underline{M} = (M_1,\ldots,M_{m-1}) \in \mathbb{Z}^{m-1}$. Hence, (5.55) is equivalent to (5.51). ∎

Theorem 5. *We assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$. If $\mathcal{D}_{\underline{\widehat{\mu}}(n,t_r)}$ is nonspecial for $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$, then*

$$\begin{aligned}
u(n,t_r) &= 2\omega_0^{\infty_2} - \sum_{j=1}^{m-1} d^{(\infty_2)}_{j,0} \frac{\partial}{\partial z_j} \log \frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^-(n,t_r)))}, \\
v(n,t_r) &= -\omega_0^{\infty_1} + \sum_{j=1}^{m-1} d^{(\infty_1)}_{j,0} \frac{\partial}{\partial z_j} \log \frac{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}(n,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^-(n,t_r)))}, \\
w(n,t_r) &= -\frac{\theta(\underline{z}(P_{\infty_1},\underline{\widehat{\mu}}^-(n,t_r)))\theta(\underline{z}(P_{\infty_1},\underline{\hat{\mu}}^+(n,t_r)))}{\theta(\underline{z}(P_{\infty_1},\underline{\widehat{\mu}}(n,t_r)))^2} \exp\big(e_1(Q_0) - e_2(Q_0)\big).
\end{aligned} \tag{5.56}$$

Proof. The proof is similar to the proof of Theorem 2 in the stationary case. ■

Combining (5.51) and (5.56) shows the remarkable linearity of the theta function representations for $u(n,t_r)$, $v(n,t_r)$, and $w(n,t_r)$ with respect to $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$. In fact, we can rewrite (5.56) as

$$\begin{aligned} u(n,t_r) &= 2\omega_0^{\infty_2} - \sum_{j=1}^{m-1} d_{j,0}^{(\infty_2)} \frac{\partial}{\partial z_j}\log\frac{\theta(\underline{\widetilde{K}} - \underline{U}^{(3)} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)}{\theta(\underline{\widetilde{K}} + \underline{U}^{(3)} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)},\\ v(n,t_r) &= -\omega_0^{\infty_1} + \sum_{j=1}^{m-1} d_{j,0}^{(\infty_1)} \frac{\partial}{\partial z_j}\log\frac{\theta(\underline{\widetilde{K}} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)}{\theta(\underline{\widetilde{K}} + \underline{U}^{(3)} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)},\\ w(n,t_r) &= -\frac{\theta(\underline{\widetilde{K}} + \underline{U}^{(3)} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)\theta(\underline{\widetilde{K}} - \underline{U}^{(3)} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)}{\theta(\underline{\widetilde{K}} - \underline{U}^{(3)}n + \underline{\widetilde{U}}_r^{(2)}t_r)^2}\times\\ &\qquad\times\exp\big(e_1(Q_0)-e_2(Q_0)\big), \end{aligned} \tag{5.57}$$

where $\underline{\widetilde{K}} = \underline{\Lambda} - \underline{\mathcal{A}}(P_{\infty_1}) + \underline{\rho}(n_0,t_{0,r}) + \underline{U}^{(3)}n_0 - \underline{\widetilde{U}}_r^{(2)}t_{0,r}$.

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