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ALGEBRAIC AND GEOMETRIC STRUCTURES OF ANALYTIC PARTIAL DIFFERENTIAL EQUATIONS

O. V. Kaptsov*

We study the problem of the compatibility of nonlinear partial differential equations. We introduce the algebra of convergent power series, the module of derivations of this algebra, and the module of Pfaffian forms. Systems of differential equations are given by power series in the space of infinite jets. We develop a technique for studying the compatibility of differential systems analogous to the Gröbner bases. Using certain assumptions, we prove that compatible systems generate infinite manifolds.

Keywords: compatibility of differential equations, reduction, infinite-dimensional manifold, Gröbner basis

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1. Introduction

The compatibility question is one of the first to arise in the study of a system of equations. The compatibility criterion for linear algebraic equations is given by the Kronecker–Capelli theorem. There is no such simple criterion in the case of polynomial equations: the elimination theory is therefore used to study compatibility, and Gröbner bases are also used [1], [2]. It is much harder to obtain an answer for systems of nonlinear partial differential equations, where both local and global problems arise. Moreover, it is important to know the smoothness classes to which the equations belong. From the pioneering works of Riquier and Janet [3], [4] to the modern works [5]–[9], concepts fundamental for compatibility—involutivity, solvability—have been refined, and research methods have changed. The research focus has recently shifted toward computational algorithms. For example, algorithms have been implemented in the computer algebra system Maple [10] that should translate an original system of equations into some "standard" form. At the same time, we can state that there is no established definition of an involutive (passive, standard) system of partial differential equations. Here, we consider the passive systems introduced in [11], [12] and study their properties.

This paper is structured as follows. In Sec. 2, we consider the infinite-dimensional space $\mathbb{K}^T$ of maps of a countable set T to a field $\mathbb{K}$ that is complete under a nontrivial absolute value. The most important examples of such fields are the fields of real and complex numbers. The topology of the direct product and a Cartesian coordinate system is introduced in the space $\mathbb{K}^T$. With an arbitrary point $a \in \mathbb{K}^T$, we associate

*Institute of Computational Modeling, Siberian Branch, RAS, Krasnoyarsk, Russia; Siberian Federal University, Krasnoyarsk, Russia, e-mail: kaptsov@icm.krasn.ru.

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an algebra $\mathcal{F}_a$ of convergent power series. Each series in $\mathcal{F}_a$ depends on a finite number of variables, but the number of these variables can be arbitrarily large. Using the series in $\mathcal{F}_a$, we define analytic functions on open sets of $\mathbb{K}^T$ and analytic maps of this space. This allows defining analytic manifolds in $\mathbb{K}^T$. At the end of the section, we introduce the concept of a normalized system of generators of an ideal of the algebra $\mathcal{F}_a$. We show that the zeros of analytic functions corresponding to the normalized system define a manifold in $\mathbb{K}^T$.

In Sec. 3, we study derivations of the algebra $\mathcal{F}_a$, Pfaffian (differential) forms, and Lie derivatives of these forms. We prove that derivations are uniquely defined by the action on the generators of $\mathcal{F}_a$. We introduce invariant ideals and submodules and then consider the space of (infinite) jets $\mathbb{J} = \mathbb{K}^T$, where $T = \mathbb{N}_n \cup (\mathbb{N}_m \times \mathbb{N}^n)$, $\mathbb{N}_n = \{1, \dots, n\}$, and $\mathbb{N}$ is the set of integers. On the space $\mathbb{J}$, we introduce operators of total differentiation, canonical forms, contact differentiations, and symmetries of partial differential equations. We note that the problems of calculating the symmetries and conservation laws of a system of equation are closely related to the problem of testing whether an element of the algebra belongs to a given differential ideal.

At the beginning of Sec. 4, we prove that if differential systems generate the same differential ideal, then they give the same germs of zeros. We then define concepts such as the reduction of a series with respect to a differential system, compatibility conditions, and passive systems. We show that if a system $S \subset \mathcal{F}_a$ is passive, then a series $f \in \mathcal{F}_a$ belongs to the differential ideal generated by S if and only if f reduces to zero with respect to S. Our main result in the paper is a theorem stating that if a differential system satisfies certain conditions of weak solvability and compatibility, then it is passive at some point $a \in \mathbb{J}$ and defines an analytic manifold in a neighborhood of that point. As an example, we consider the Dubreil-Jacotin equation describing planar stationary flows of an inhomogeneous liquid and find an exact noninvariant solution depending on two parameters.

2. The algebra of convergent power series and analytic functions

Let $\mathbb{K}$ be a field complete under a nontrivial absolute value.. We fix the notation: $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{N}_m = \{1, \dots, m\}$, $\mathbb{R}_+$ is the set of positive real numbers, T is a countable set, and $\mathbb{K}^T$ is the space of maps from T to $\mathbb{K}$.

Definition. We call maps $y_t : \mathbb{K}^T \to \mathbb{K}$ given by the formula

$$y_t(z) = z(t), \quad t \in T, \tag{2.1}$$

Cartesian coordinate functions on $\mathbb{K}^T$ and call the values $z(t)$ coordinates of the point $z \in \mathbb{K}^T$.

It is convenient to let z_t denote the value $z(t)$ by analogy with the finite-dimensional case.

On the space $\mathbb{K}^T$, we introduce the direct product topology by choosing points $a \in \mathbb{K}^T$ of the set

$$U(a_\tau, \rho) = \{z \in \mathbb{K}^T \colon\ |z_{t_i} - a_{t_i}| < \rho_i,\ i \in \mathbb{N}_n\} \tag{2.2}$$

as the fundamental system of neighborhoods; here, $t_i \in T$, $\rho_i \in \mathbb{R}_+$, $\rho = (\rho_1, \dots, \rho_n)$, $a_\tau = \{a_{t_1}, \dots, a_{t_n}\}$ is the set of n coordinates of the point a, and $z_{t_1}, \dots, z_{t_n}$ are the coordinates of the point z. We call set (2.2) a *parallelepiped.*

Let $\{X_t\}_{t \in T}$ be a set of symbols,

$$\tau = \{t_1, \dots, t_n\} \subset T, \qquad \rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n_+, \qquad a_\tau = \{a_{t_1}, \dots, a_{t_1}\} \subset \mathbb{K}.$$

We let $A(a_\tau, \rho)$ denote the set of power series of the form

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (X_{t_1} - a_{t_1})^{\alpha_1} \cdots (X_{t_n} - a_{t_n})^{\alpha_n}, \tag{2.3}$$

where $c_\alpha \in \mathbb{K}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, for which the quantity

$$\|f\|_\rho = \sum_{\alpha \in \mathbb{N}^n} |c_\alpha| \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n} \tag{2.4}$$

is finite. As follows from [13], $A(a_\tau, \rho)$ is a Banach algebra with norm (2.4).

We introduce a relation $\prec$ on $\mathbb{R}^n_+$. Let $\rho = (\rho_1, \dots, \rho_n)$ and $\rho^* = (\rho_1^*, \dots, \rho_n^*)$. We assume that $\rho \prec \rho^*$ if the difference $\rho_i^* - \rho_i$ is positive for all $i \in \mathbb{N}_n$.

Statement 2.1. *Let*

$$\tau = \{t_1, \dots, t_n\} \subset \tau' = \{t_1, \dots, t_m\} \subset T, \quad n < m,$$

$$\rho = (\rho_1, \dots, \rho_n) \prec \rho^* = (\rho_1^*, \dots, \rho_n^*), \qquad \rho' = (\rho_1, \dots, \rho_n, 1, \dots, 1) \in \mathbb{R}^m_+.$$

Then the algebra $A(a_\tau, \rho^*)$ is embedded in the algebra $A(a_\tau, \rho)$, the algebra $A(a_\tau, \rho)$ is embedded in the algebra $A(a_{\tau'}, \rho')$, and the inclusions

$$\frac{\partial}{\partial X_{t_i}} A(a_\tau, \rho^*) \subset A(a_\tau, \rho) \tag{2.5}$$

hold for all $i \in \mathbb{N}_n$.

Proof. The embeddings are obvious, and formula (2.5) follows from a similar formula in [13].

Definition. A power series f of type (2.3) is said to be *convergent* (in a neighborhood of $a \in \mathbb{K}^T$) if $f \in A(a_\tau, \rho)$ for some $\rho \in \mathbb{R}^n_+$ and $a_\tau = \{a_{t_1}, \dots, a_{t_n}\} \subset \mathbb{K}$, where $a_{t_1}, \dots, a_{t_n}$ are the coordinates of the point $a \in \mathbb{K}^T$.

For each point $a \in \mathbb{K}^T$ with a part of the coordinates $a_\tau = \{a_{t_1}, \dots, a_{t_n}\}$, we consider the union (not disjoint) of the algebras

$$\mathcal{F}_a = \bigcup_{\substack{\rho \in \mathbb{R}^n_+,\, n \in N_0, \\ a_\tau \subset \mathbb{K}}} A(a_\tau, \rho),$$

where $N_0 = \mathbb{N} \setminus \{0\}$. The set $\mathcal{F}_a$ is $\mathbb{K}$-algebra of convergent power series.

Using the convergent power series $\mathcal{F}_a$, we introduce analytic functions on open sets of the space $\mathbb{K}^T$. Each series $f \in A(a_\tau, \rho)$ of type (2.3) generates a function $\tilde{f}$ as follows. Let $\tau = \{t_1, \dots, t_n\} \subset T$, $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n_+$, and $a_\tau = \{a_{t_1}, \dots, a_{t_n}\} \subset \mathbb{K}$. According to the definition of $A(a_\tau, \rho)$, the series f converges in the polycylinder

$$\Pi(a_\tau, \rho) = \{(z_{t_1}, \dots, z_{t_n}) \in \mathbb{K}^n \colon\ |z_{t_i} - a_{t_i}| < \rho_i,\ i \in \mathbb{N}_n\}. \tag{2.6}$$

This polycylinder uniquely corresponds to a parallelepiped $U(a_\tau, \rho)$ of type (2.2). For any point $z \in U(a_\tau, \rho)$, the function $\tilde{f}$ can then be given by the formula

$$\tilde{f}(z) = f(z_\tau) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z_{t_1} - a_{t_1})^{\alpha_1} \cdots (z_{t_n} - a_{t_n})^{\alpha_n}, \tag{2.7}$$

where $z_\tau = (z_{t_1}, \dots, z_{t_n}) \in \Pi(a_\tau, \rho)$. The function $\tilde{f}$ thus depends on a finite number of variables. We say that the constructed function $\tilde{f}$ is locally analytic. By analogy with the finite-dimensional case [14], this allows introducing analytic functions on open sets.

Definition. Let U be an open set in $\mathbb{K}^T$. A function $h: U \to \mathbb{K}$ is said to be *analytic* in U if for each point $z \in U$, there exist a parallelepiped $U(a_\tau, \rho)$ and a locally analytic function $\tilde{f}$ in this parallelepiped such that $h(z) = \tilde{f}(z)$ for all $z \in U(a_\tau, \rho)$.

Remark. In what follows, for $E \subset \mathcal{F}_a$,

$$\widetilde{E} = \{\tilde{f}\colon f \in E\} \tag{2.8}$$

denotes the set of locally analytic functions $\tilde{f}$ corresponding to a series $f \in \mathcal{F}_a$. It can be shown following [15] that a function locally analytic in a parallelepiped $U(a_\tau, \rho)$ is analytic in $U(a_\tau, \rho)$.

Definition. Let U be an open set in $\mathbb{K}^T$. A map $\phi: U \to \mathbb{K}^T$ with the components ϕ_t, $t \in T$, is said to be *analytic* in U if each function ϕ_t is analytic in U. A map ϕ is said to be *bianalytic* if it has an analytic inverse map ϕ^{-1}.

Definition. The set

$$C_S = \{z \in \mathbb{K}^T\colon z(t) = 0 \text{ for all } t \in S \subset T\}$$

is called a *coordinate subspace* in $\mathbb{K}^T$.

The following definition of a manifold is a direct generalization of a finite structure.

Definition. A set $M \subset \mathbb{K}^T$ is called a *manifold* in $\mathbb{K}^T$ if for any point $z \in M$ there exist open sets U and U' in $\mathbb{K}^T$ and $z \in U$ and a bianalytic map $\phi: U \to U'$ such that

$$\phi(U \cap M) = U' \cap C_S,$$

where C_S is a coordinate subspace of $\mathbb{K}^T$. This restriction $\bar{\phi} = \phi|_{M \cap U}$ of ϕ to $U \cap M$ is called a *local coordinate system* on $U \cap M$, and the set of variables (Cartesian coordinate functions on C_S) on which the inverse map $\bar{\phi}^{-1}$ depends is called the *parameter set* of the manifold.

Analytic functions and manifolds in the coordinate subspaces are introduced similarly.

We let $\mathrm{iv}(f)$ denote the set of symbols on which the series $f \in \mathcal{F}_a$ depends. If $E \subset \mathcal{F}_a$, then

$$\mathrm{iv}(E) = \{\mathrm{iv}(f)\colon f \in E\}. \tag{2.9}$$

Definition. Let $\mathcal{R}$ be a subalgebra of $\mathcal{F}_a$ and I be an ideal in $\mathcal{R}$. A system of generators $\mathcal{B}$ of the ideal I is said to be *normalized* if

1. any element $f \in \mathcal{B}$ has the form $f = X_s + g$, the elements X_s form a subset $L \subset \{X_t\}_{t \in T}$, and
2. if $f_1 = X_t + g_1 \in \mathcal{B}$ and $f_2 = X_t + g_2 \in \mathcal{B}$, then $g_1 = g_2$.

In this case, L is called the set of *principal variables* of the system $\mathcal{B}$.

Let $T' \subset T$. Then the set

$$C(T') = \{z \in \mathbb{K}^T\colon\ z(t) = 0 \text{ for all } t \in T \setminus T'\} \tag{2.10}$$

is a coordinate subspace of $\mathbb{K}^T$. The topology in $C(T')$ is induced by the topology in $\mathbb{K}^T$. Analytic functions on open sets of the subspace $C(T')$ are determined by the power series and formula (2.7). The set of series

$$\mathcal{F}_a(T') = \{f \in \mathcal{F}_a\colon\ \mathrm{iv}(f) \subseteq \{X_t\}_{t \in T'}\} \tag{2.11}$$

forms a subalgebra of $\mathcal{F}_a$.

Statement 2.2. *Let $T' \subseteq T$, $\mathcal{B}$ be a normalized system of generators of an ideal of the subalgebra $\mathcal{F}_a(T')$ given by* (2.11), *$\widetilde{\mathcal{B}}$ be the corresponding set of analytic functions in some open set V of coordinate subspace* (2.10), *L be the set of principal variables of the system $\mathcal{B}$, $S = \{s \in T' \colon X_s \in L\}$, and $T'' = T' \setminus S$. Then the set*

$$Z(\widetilde{\mathcal{B}}) = \{z \in V \colon \tilde{f}(z) = 0 \text{ for all } f \in \widetilde{\mathcal{B}}\}$$

is a manifold in the subspace $C(T')$, and the set of Cartesian coordinate functions $\{y_t\}_{t \in T''}$ forms a set of manifold parameters.

Proof. The map ϕ is given by the formulas $y'_t = y_t + g_t$ and $y'_s = y_s$, where y_t corresponds to the symbol $X_t \in L$ and $\mathrm{iv}(g_t), y_s \in \{y_q\}_{q \in T''}$. Therefore, the inverse map ϕ^{-1} has the form $y_t = y'_t - g_t$, $y_s = y'_s$. Obviously, the restriction of ϕ to $Z(\widetilde{\mathcal{B}})$ is a projection $Z(\widetilde{\mathcal{B}})$ on $V \cap C(T'')$. Therefore, the set $Z(\widetilde{\mathcal{B}})$ is a manifold.

Remark. It is important that in Statement 2.2, all the functions in $\widetilde{\mathcal{B}}$ must be given on the same open set V.

3. Derivations and local systems

We recall that a K-linear map $\mathcal{D} \colon A \to A$ for which $\mathcal{D}(ab) = a\mathcal{D}(b) + b\mathcal{D}(a)$ is called a derivation of a commutative algebra A over the field K.

Lemma 3.1. *An arbitrary derivation $\mathcal{D}$ of an algebra $\mathcal{F}_a$ is uniquely determined by the values on the coordinate functions y_t and is given for any $f \in \mathcal{F}_a$ by the formula*

$$\mathcal{D}(f) = \sum_{t \in T} \mathcal{D}(y_t) \frac{\partial f}{\partial y_t}.$$

Proof. Without loss of generality, we can assume that $a = 0 \in \mathbb{K}^T$. We let y^α denote the monomial $y_{t_1}^{\alpha_1} \cdots y_{t_n}^{\alpha_n}$. Let the polynomial $p = \sum_{\alpha \in A} c_\alpha y^\alpha$, where $c_\alpha \in \mathbb{K}$, and A be finite subset of $\mathbb{N}^n$. By the definition of a derivation, we have the formulas

$$\begin{aligned} \mathcal{D}(y^\alpha) &= \sum_{i=1}^n \alpha_i \mathcal{D}(y_{t_i}) y_{t_1}^{\alpha_1} \cdots y_{t_i}^{\alpha_i - 1} \cdots y_{t_n}^{\alpha_n} = \sum_{i=1}^n \mathcal{D}(y_{t_i}) \frac{\partial y^\alpha}{\partial y_{t_i}}, \\ \mathcal{D}(p) &= \sum_{\alpha \in A} c_\alpha \mathcal{D}(y^\alpha) = \sum_{\alpha \in A} c_\alpha \sum_{i=1}^n \mathcal{D}(y_{t_i}) \frac{\partial y^\alpha}{\partial y_{t_i}} = \sum_{i=1}^n \mathcal{D}(y_{t_i}) \frac{\partial p}{\partial y_{t_i}}. \end{aligned} \tag{3.1}$$

We prove that the derivation $\mathcal{D}$ extends uniquely from the algebra of polynomials to the algebra $\mathcal{F}_0$. We suppose that there is a different derivation $\mathcal{D}_0$ of $\mathcal{F}_0$ coinciding with $\mathcal{D}$ on the polynomials. Then the derivation $\mathcal{D}^* = \mathcal{D} - \mathcal{D}_0$ vanishes on any polynomial.

We suppose that there exists a series $f \in \mathcal{F}_0$ such that $\mathcal{D}^*(f) \neq 0$. We recall [16] that the smallest positive integer q such that the homogeneous part of f to the power q is nonzero is called the order of the series $f \neq 0$ (denoted by $\mathrm{ord}(f)$).

For any polynomial $p \in \mathcal{F}_0$, we have the equality $\mathcal{D}^*(f) = \mathcal{D}^*(f - p)$, and therefore

$$\mathrm{ord}(\mathcal{D}^*(f)) = \mathrm{ord}(\mathcal{D}^*(f - p)). \tag{3.2}$$

Moreover, because of formulas (3.1), the inequality $\mathrm{ord}(\mathcal{D}^*(f)) \geq \mathrm{ord}(f) - 1$ holds. Hence, choosing a polynomial p that "annihilates the smallest terms" of the series f, we can make $\mathrm{ord}(\mathcal{D}^*(f - p))$ be an arbitrarily large number. But this contradicts (3.2) because the order of the series $\mathcal{D}^*(f)$ is a finite number. The lemma is proved.

Obviously, the set of derivations of an algebra $\mathcal{F}_a$ forms a module over $\mathcal{F}_a$. We let Der_a denote it. It is well known [16] that the set of all derivations of a commutative algebra over a field forms a Lie algebra with the commutator

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1.$$

Hence, Der_a is a Lie algebra over the field $\mathbb{K}$. According to Lemma 3.1, derivations of $\mathcal{F}_a$ can be also called local vector fields in a neighborhood of $a \in \mathbb{K}^T$.

Following [16], we call the linear map $df : \ Der_a \to \mathcal{F}_a$, acting by the formula

$$df(\mathcal{D}) = \mathcal{D}(f)$$

the *differential* of the series $f \in \mathcal{F}_a$. Obviously, the set of differentials of $f \in \mathcal{F}_a$ generates a module over $\mathcal{F}_a$. This module is denoted by Der_a^* and is called a module of Pfaffian forms. The elements of this module are finite sums of the form $\sum g_t\, df_t$, where $g_t, f_t \in \mathcal{F}_a$.

Statement 3.1. *The module of Pfaffian forms* Der_a^* *is generated by differentials of the Cartesian coordinate functions, i.e., by the elements* $\{dy_t\}_{t\in T}$.

Proof. It suffices to show that the differential df can be given by the classical formula

$$df = \sum_{t\in T} \frac{\partial f}{\partial y_t}\, dy_t. \tag{3.3}$$

The summation in the right-hand side here is finite because f depends on a finite number of variables. We take an arbitrary derivation $\mathcal{D} \in \mathrm{Der}_a$ and compare the values of the left- and right-hand sides of (3.3) on $\mathcal{D}$:

$$df(\mathcal{D}) = \mathcal{D}(f) = \sum_t \mathcal{D}(y_t)\frac{\partial f}{\partial y_t}, \qquad \sum_t \frac{\partial f}{\partial y_t}\, dy_t(\mathcal{D}) = \sum_t \frac{\partial f}{\partial y_t}\mathcal{D}(y_t).$$

We see that these values coincide.

Definition. A *Lie derivative* generated by differentiating $\mathcal{D} \in \mathrm{Der}_a$ is a $\mathbb{K}$-linear map $\mathcal{L}_\mathcal{D} : \ \mathrm{Der}_a^* \to \mathrm{Der}_a^*$ satisfying

$$\mathcal{L}_\mathcal{D}(g\, df) = \mathcal{D}(g)\, df + g\, d\mathcal{D}(f), \quad f, g \in \mathcal{F}_a. \tag{3.4}$$

Statement 3.2. *The set of Lie derivatives forms a Lie algebra over the field* $\mathbb{K}$*, and the equalities*

$$k_1\mathcal{L}_{\mathcal{D}_1} + k_2\mathcal{L}_{\mathcal{D}_2} = \mathcal{L}_{k_1\mathcal{D}_1 + k_2\mathcal{D}_2}, \tag{3.5}$$

$$[\mathcal{L}_{\mathcal{D}_1}, \mathcal{L}_{\mathcal{D}_2}] = \mathcal{L}_{[\mathcal{D}_1, \mathcal{D}_2]} \tag{3.6}$$

hold for all $k_1, k_2 \in \mathbb{K}$ *and any* $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Der}_a$.

Proof. To prove the statement, it suffices to verify formulas (3.5) and (3.6) on an arbitrary Pfaffian form $\omega = \sum_{t\in\tau} f_t\, dy_t$, where τ is a finite subset of T. The equality of the left- and right-hand sides of formula (3.5) can be elementarily verified by acting on the form ω.

We show that

$$\mathcal{L}_{\mathcal{D}_1}\mathcal{L}_{\mathcal{D}_2}(\omega) - \mathcal{L}_{\mathcal{D}_2}\mathcal{L}_{\mathcal{D}_1}(\omega) = \mathcal{L}_{[\mathcal{D}_1, \mathcal{D}_2]}(\omega).$$

For this, we calculate $\mathcal{L}_{\mathcal{D}_1}\mathcal{L}_{\mathcal{D}_2}(\omega)$ and $\mathcal{L}_{\mathcal{D}_2}\mathcal{L}_{\mathcal{D}_1}(\omega)$. We obtain

$$\mathcal{L}_{\mathcal{D}_1}\mathcal{L}_{\mathcal{D}_2}(\omega) = \mathcal{L}_{\mathcal{D}_1}\mathcal{L}_{\mathcal{D}_2}\left(\sum_t f_t\, dy_t\right) = \mathcal{L}_{\mathcal{D}_1}\left(\sum_t \mathcal{D}_2(f_t)\, dy_t + \sum_t f_t\, d\mathcal{D}_2(y_t)\right) =$$

$$= \sum_t \mathcal{D}_1\mathcal{D}_2(f_t)\, dy_t + \sum_t \mathcal{D}_2(f_t)\, d\mathcal{D}_1(y_t) + \sum_t \mathcal{D}_1(f_t)\, d\mathcal{D}_2(y_t) + \sum_t f_t\, d\mathcal{D}_1\mathcal{D}_2(y_t),$$

$$\mathcal{L}_{\mathcal{D}_2}\mathcal{L}_{\mathcal{D}_1}(\omega) = \sum_t \mathcal{D}_2\mathcal{D}_1(f_t)\, dy_t + \sum_t \mathcal{D}_1(f_t)\, d\mathcal{D}_2(y_t) + \sum_t \mathcal{D}_2(f_t)\, d\mathcal{D}_1(y_t) + \sum_t f_t\, d\mathcal{D}_2\mathcal{D}_1(y_t).$$

Therefore, the equalities

$$(\mathcal{L}_{\mathcal{D}_1}\mathcal{L}_{\mathcal{D}_2} - \mathcal{L}_{\mathcal{D}_2}\mathcal{L}_{\mathcal{D}_1})(\omega) = \sum_t \big(\mathcal{D}_1\mathcal{D}_2(f_t) - \mathcal{D}_2\mathcal{D}_1(f_t)\big)\, dy_t + \sum_t f_t\, d\big(\mathcal{D}_1\mathcal{D}_2(y_t) - \mathcal{D}_2\mathcal{D}_1(y_t)\big) =$$

$$= \sum_t [\mathcal{D}_1, \mathcal{D}_2](f_t)\, dy_t + \sum_t f_t\, d([\mathcal{D}_1, \mathcal{D}_2](y_t)) = \mathcal{L}_{[\mathcal{D}_1, \mathcal{D}_2]}(\omega)$$

hold.

Definition. An ideal I of the algebra $\mathcal{F}_a$ is said to be *invariant* under the derivation $\mathcal{D} \in \mathrm{Der}_a$ if $\mathcal{D}(I) \subset I$. A submodule $\mathcal{M}$ of the module Der_a^* of Pfaffian forms is said to be *invariant* under the derivation $\mathcal{D} \in \mathrm{Der}_a$ if $\mathcal{L}_{\mathcal{D}}(\mathcal{M}) \subset \mathcal{M}$.

Statement 3.3. *The set of derivations leaving the ideal $I \subset \mathcal{F}_a$ or the submodule $\mathcal{M} \subset \mathrm{Der}_a^*$ invariant forms a Lie algebra over the field $\mathbb{K}$.*

Proof. Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Der}_a$ and $\mathcal{D}_i(I) \subset I$. Then we have $\mathcal{D}_1\mathcal{D}_2(I) \subset I$ and $\mathcal{D}_2\mathcal{D}_1(I) \subset I$. Therefore, $[\mathcal{D}_1, \mathcal{D}_2] \subset I$. Similar arguments apply to the submodule $\mathcal{M}$.

In what follows, we assume that $T = \mathbb{N}_n \cup (\mathbb{N}_m \times \mathbb{N}^n)$. In this case, the space $\mathbb{K}^T$ is called a *jet space* and is denoted by $\mathbb{J}$. Cartesian coordinate functions on $\mathbb{J}$ are denoted by x_i and u^j_α, where $i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$, and $\alpha \in \mathbb{N}^n$. The set Y of Cartesian coordinate functions is partitioned into two subsets

$$X = \{x_1, \ldots, x_n\}, \qquad U = \{u^j_\alpha\}^{j\in\mathbb{N}_m}_{\alpha\in\mathbb{N}^n}. \tag{3.7}$$

We introduce n maps $D_1, \ldots, D_n$ from Y to Y:

$$D_k(x_k) = 1, \qquad D_k(x_i) = 0 \quad \text{for } i \neq k, \qquad D_k(u^j_\alpha) = u^j_{\alpha+e_k},$$

where $k \in \mathbb{N}_n$ and $e_1 = (1, 0, \ldots, 0)$, …, $e_n = (0, \ldots, 0, 1)$ are elements of $\mathbb{N}^n$. According to Lemma 3.1, these maps then uniquely continue to derivations of the algebra $\mathcal{F}_a$ and are given by

$$D_k(f) = \frac{\partial f}{\partial x_k} + \sum_{\substack{j\in\mathbb{N}_m,\\ \alpha\in\mathbb{N}^n}} \frac{\partial f}{\partial u^j_\alpha} u^j_{\alpha+e_k} \tag{3.8}$$

for arbitrary $f \in \mathcal{F}_a$. The derivations D_k are often called *total differentiation operators* [17] because formula (3.8) is the chain rule.

Therefore, the algebra $\mathcal{F}_a$ of convergent power series with n total differentiation operators is a differential algebra. The product of the operators $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ is denoted by D^α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$. The differential ideal of $\mathcal{F}_a$ generated by $S \subset \mathcal{F}_a$ is denoted by $\langle\!\langle S \rangle\!\rangle$.

We recall that a triple of sets $(\mathbb{K}, Y, S)$, where S is a finite subset of $\mathcal{F}_a$ was called a *local analytic system of partial differential equations* in [11]. For brevity, we call a finite set $S \subset \mathcal{F}_a$ a local differential system.

We consider a countable set $\Omega = \{\omega_\alpha^i\}_{\alpha \in \mathbb{N}^n}^{i \in \mathbb{N}_m}$ of canonical Pfaffian forms

$$\omega_\alpha^i = du_\alpha^i - \sum_{j=1}^n u_{\alpha+e_j}^i \, dx_j.$$

We let $\mathcal{P}_a$ denote the submodule of Der_a^* generated by these canonical forms. We recall that $\mathcal{L}_\mathcal{D}$ denotes the Lie derivative generated by differentiating $\mathcal{D}$.

Definition. A derivation $\mathcal{D} \in \mathrm{Der}_a$ is called a *contact derivation* if the condition $\mathcal{L}_\mathcal{D}(\mathcal{P}_a) \subseteq \mathcal{P}_a$ is satisfied.

Obviously, the total differentiation operators $D_1, \ldots, D_n$ given by (3.8) are contact derivations. If $\mathcal{D}$ is a contact derivation then the operators

$$\mathcal{D} - \sum_{j=1}^n \mathcal{D}(x_j) D_j$$

are also contact derivations. Therefore, without loss of generality, we can consider contact derivations of the form

$$\mathcal{D} = \sum_{\substack{i \in \mathbb{N}_m, \\ \alpha \in \mathbb{N}^n}} \mathcal{D}(u_\alpha^i) \frac{\partial}{\partial u_\alpha^i}. \tag{3.9}$$

Such derivations are often said to be vertical [18].

In fact, repeating well-known arguments [17], [18], we can show that derivation (3.9) is a contact derivation if and only if $[\mathcal{D}, D_i] = 0$ for all $i \in \mathbb{N}_n$. This means that the coefficients $\mathcal{D}(u_\alpha^i)$ in (3.9) are given by the formula $\mathcal{D}(u_\alpha^i) = D^\alpha \mathcal{D}(u^i)$.

Definition. A contact derivation $\mathcal{D} \in \mathrm{Der}_a$ is called an (*infinitesimal*) *symmetry* of the local differential system

$$S = \{f_1, \ldots, f_k\} \subset \mathcal{F}_a$$

if the differential ideal $I = \langle\!\langle S \rangle\!\rangle$ is invariant under $\mathcal{D}$, i.e., $\mathcal{D}(I) \subseteq I$.

We note that it suffices to verify the invariance of the ideal $\langle\!\langle S \rangle\!\rangle$ for any differential system of generators. More precisely, the following statement holds.

Statement 3.4. *Let S be a local differential system and $\mathcal{D}$ be a contact derivation such that $\mathcal{D}(S) \subseteq \langle\!\langle S \rangle\!\rangle$. Then $\mathcal{D}$ is a symmetry of the system S.*

Proof. Let $f \in \langle\!\langle S \rangle\!\rangle$. Then

$$f = \sum_{\substack{i \in \mathbb{N}_k, \\ \alpha \in A}} a_\alpha^i D^\alpha f_i,$$

where $a_\alpha^i \in \mathcal{F}_a$, A is a finite subset in $\mathbb{N}^n$, and $f_i \in S$. We must prove that $\mathcal{D}(f) \in \langle\!\langle S \rangle\!\rangle$. By the definition of a derivation, we have

$$\mathcal{D}(f) = \sum \mathcal{D}(a_\alpha^i) D^\alpha f_i + \sum a_\alpha^i \mathcal{D} D^\alpha f_i.$$

Without loss of generality, we can assume that the derivation is vertical, and this equality can hence be rewritten in the form

$$\mathcal{D}(f) = \sum \mathcal{D}(a^i_\alpha) D^\alpha f_i + \sum a^i_\alpha D^\alpha \mathcal{D}(f_i).$$

Each term in the right-hand side of this equality is in the ideal $\langle\langle S \rangle\rangle$. Consequently, the left-hand side belongs to the ideal.

Remark. The definition of symmetry presented above can be found formulated from other positions in [18].

The concepts in which differential ideals are used are applicable to conservation laws and various defining equations [19]. We can introduce a conservation law as follows.

Definition. A *conservation law* of a differential system $S \subset \mathcal{F}_a$ is a tuple $(g_1, \dots, g_n) \in \mathcal{F}_a^n$ such that

$$D_1 g_1 + \dots + D_n g_n \in \langle\langle S \rangle\rangle.$$

A question about verifying that an element belongs to an ideal arises. In the algebra of polynomials, it is helpful to use the Gröbner basis of the ideal [1] for this. In our case, the analogue of a Gröbner basis is a passive system. This concept and its application are considered in the next section.

4. Manifolds generated by systems

In this section, we prove the main results in the paper, but we first use notation (2.8) to formulate a statement that turns out to be very useful in what follows.

Statement 4.1. *Let $S = \{f_1, \dots, f_k\} \subset \mathcal{F}_a$ be a local differential system. Then there exists a parallelepiped $U(a_\tau, \rho)$ of type* (2.2) *such that the functions $\widetilde{D^\alpha f_i}$ are analytic in this parallelepiped for all $i \in \mathbb{N}_k$ and any $\alpha \in \mathbb{N}^n$.*

Proof. Obviously, there exists a polycylinder $\Pi(a_\tau, \rho^*)$ of type (2.6) in which the series $f_1, \dots, f_k$, converge i.e., $S \subset A(a_\tau, \rho^*)$. According to derivation formula (3.8), we have

$$D_i(f_s) = \frac{\partial f_s}{\partial x_i} + \sum_{\substack{j \in \mathbb{N}_m,\\ \alpha \in \mathbb{N}^n}} \frac{\partial f_s}{\partial u^j_\alpha} u^j_{\alpha + e_i}, \quad s \in \mathbb{N}_k,$$

where the sum in the right-hand side contains a finite number of terms. Moreover, by virtue of Statement 2.1, the partial derivatives $\partial f_s / \partial x_i$ and $\partial f_s / \partial u^j$ belong to the algebra $A(a_\tau, \rho)$ for any $\rho \prec \rho^*$. Therefore, the corresponding functions $\widetilde{\partial f_s / \partial x_i}$ and $\widetilde{\partial f_s / \partial u^j}$ are analytic in the parallelepiped $U(a_\tau, \rho)$. Because ρ is an arbitrary tuple satisfying $\rho \prec \rho^*$, repeating the arguments presented above, we see that $\widetilde{D^\alpha f_i}$ is an analytic function in $U(a_\tau, \rho)$ for all $i \in \mathbb{N}_k$ and any $\alpha \in \mathbb{N}^n$. The statement is proved.

Let $S = \{f_1, \dots, f_k\} \subset \mathcal{F}_a$ be a local differential system and $W = U(a_\tau, \rho)$ be a parallelepiped in which the functions $\widetilde{D^\alpha f_i}$ are analytic for all $i \in \mathbb{N}_k$ and any $\alpha \in \mathbb{N}^n$. In the space $\mathbb{J}$, we consider the set

$$Z_W(S) = \{z \in W\colon\ \widetilde{D^\alpha f_i}(z) = 0 \text{ for all } i \in \mathbb{N}_k,\ \alpha \in \mathbb{N}^n\} \tag{4.1}$$

and a point $a \in W$. In what follows, $\mathbb{Z}_a(S)$ denotes the germ of the set $Z_W(f)$ (see [20]).

Statement 4.2. *If two local differential systems*

$$f = \{f_1, \dots, f_k\} \subset \mathcal{F}_a, \qquad g = \{g_1, \dots, g_s\} \subset \mathcal{F}_a$$

generate the same differential ideal $\mathcal{F}_a$, then they define identical germs, i.e.,

$$\mathbb{Z}_a(f) = \mathbb{Z}_a(g). \tag{4.2}$$

Proof. Because the sets f and g generate the same differential ideal, any series $g_i \in g$ can be represented as

$$g_j = \sum_{i,\alpha} c_\alpha^{i,j}\, D^\alpha f_i, \quad c_\alpha^{i,j} \in \mathcal{F}_a. \tag{4.3}$$

We consider the sets $Z_W(f)$ and $Z_W(g)$ in the parallelepiped $U(a_\tau, \rho)$. Obviously, the inclusion

$$\mathbb{Z}_a(g) \subseteq \mathbb{Z}_a(f) \tag{4.4}$$

follows from (4.3). Using the above arguments, we obtain the representation

$$f_i = \sum_{j,\alpha} b_\alpha^{j,i} D^\alpha g_j, \quad b_\alpha^{j,i} \in \mathcal{F}_a,$$

and the inclusion

$$\mathbb{Z}_a(f) \subseteq \mathbb{Z}_a(g). \tag{4.5}$$

Equality (4.2) follows from formulas (4.4) and (4.5).

Below, we present concepts and results from [11], [12] that are needed for proving the main theorem.

Definition. The *orbit of a subset* S in $\mathcal{F}_a$ is a set

$$O(S) = \{D^\alpha s \colon\ \alpha \in \mathbb{N}^n,\ s \in S\}.$$

Definition. A convergent differential series $f \in \mathcal{F}_a$ of the form $f = u_\alpha^i + g$ is said to be *solvable* for u_α^i if g it is independent of the orbital elements $O(u_\alpha^i)$. If the series $f \in \mathcal{F}_a$ is solvable for u_α^i, then the symbol $\operatorname{st} f$ denotes u_α^i. If S is a set of solvable series, then $\operatorname{st} S = \{\operatorname{st} f \colon f \in S\}$.

Definition. Let $S \subset \mathcal{F}_a$ be a local differential system consisting of solvable differential converging series. Let the ideal $\langle\!\langle S \rangle\!\rangle \neq \mathcal{F}_a$ have a normalized system of generators $\mathcal{B}$, and let the set of principal variables $\mathcal{L}$ of the system $\mathcal{B}$ satisfy the condition $\mathcal{L} = O(\operatorname{st} S)$. Then S is called a *passive system* of the ideal $\langle\!\langle S \rangle\!\rangle$.

A problem of verifying that a local differential system is passive arises. To solve it, we must introduce additional concepts, partially used in [12].

We recall that a preorder on a set is a reflexive, transitive relation and a strict preorder is an irreflexive, transitive relation [21].

Statement 4.3. *Let $\{H_\gamma\}_{\gamma\in\Gamma}$ be a partition of the set H, where Γ is a well-ordered set. Then a preorder $\preceq$ and a strict preorder $\prec$ can be defined on H by*

$$h_1 \preceq h_2 \iff \exists \gamma_1, \gamma_2 \colon\ h_1 \in H_{\gamma_1},\ h_2 \in H_{\gamma_2},\ \gamma_1 \leq \gamma_2, \tag{4.6}$$

$$h_1 \prec h_2 \iff \exists \gamma_1, \gamma_2 \colon\ h_1 \in H_{\gamma_1},\ h_2 \in H_{\gamma_2},\ \gamma_1 < \gamma_2. \tag{4.7}$$

Proof. The proof is obvious.

Let a semigroup G act on a set M (from the left), i.e., let there be a map $(g, m) \to gm$ of the set $G \times M$ to M satisfying

$$g_1(g_2 m) = (g_1 g_2)m, \quad m \in M, \quad g_1, g_2 \in G.$$

Definition. Let a semigroup G act on a set M, and let a partition $\{M_\gamma\}_{\gamma\in\Gamma}$ of the set M generate a strict preorder $\prec$ according to (4.7). The set M is called a *stratified G-set* if the following conditions are satisfied for all $g \in G$:

1. If $m_1 \prec m_2$, then $gm_1 \prec gm_2$ for all $m_1, m_2 \in M$.
2. For all $m \in M$, $m \prec gm$.

Let $x^0 \in \mathbb{K}^n$ and $\mathbb{K}[\langle x_1, \dots, x_n\rangle]_{x_0}$ be an algebra of germs of analytic functions at the point x^0 isomorphic to the corresponding algebra of convergent power series. We consider a point $a \in \mathbb{J}$ whose standard projection in $\mathbb{K}^n$ is equal to x^0; hence, the Cartesian coordinates of x^0 are part of the Cartesian coordinates of a. We let $\widehat{\mathcal{F}}_a$ denote the set $\mathcal{F}_a \setminus \mathbb{K}[\langle x_1, \dots, x_n\rangle]_{x_0}$.

Any partition $\{U_\gamma\}_{\gamma\in\Gamma}$ of the set U (see relations (3.7)) creates a partition of $\widehat{\mathcal{F}}_a$. Indeed, we consider the family of sets

$$Y_\gamma = X \cup \Bigg(\bigcup_{\Theta \le \gamma' \le \gamma} U_{\gamma'} \Bigg), \tag{4.8}$$

where X is defined in (3.7), $\Theta = \min\limits_{\gamma\in\Gamma} \gamma$. Obviously, we have the formulas

$$Y = \bigcup_{\gamma\in\Gamma} Y_\gamma, \qquad Y_{\gamma'} \subset Y_{\gamma''} \text{ for all } \gamma' < \gamma''.$$

We choose a point $a \in \mathbb{J}$ and consider the chain (by inclusion) of subalgebras of $\mathcal{F}_a$ given by

$$\mathcal{F}_a^\gamma = \{f \in \mathcal{F}_a\colon\ \operatorname{iv}(f) \subset Y_\gamma\}. \tag{4.9}$$

Here, as before, $\operatorname{iv}(f)$ means a set of variables that determine the series f. The chain $\{\mathcal{F}_a^\gamma\}_{\gamma\in\Gamma}$ then generates a partition $\{\Phi_a^\gamma\}_{\gamma\in\Gamma}$ of the set $\widehat{\mathcal{F}}_a$ on blocks

$$\begin{gathered}\Phi_a^\gamma = \mathcal{F}_a^\gamma \setminus \Bigg(\bigcup_{\gamma' < \gamma} \mathcal{F}_a^\gamma \Bigg), \quad \gamma > \Theta, \\ \Phi_a^\Theta = \mathcal{F}_a^\Theta \setminus \mathbb{K}[\langle x_1, \dots, x_n\rangle]_{x_0}.\end{gathered} \tag{4.10}$$

Moreover, subset chain (4.8) of the set Y generates a coordinate subspace chain

$$J_\gamma = \{z \in \mathbb{J}\colon\ y(z) = 0 \text{ for all } y \in Y \setminus Y_\gamma\} \tag{4.11}$$

of the space $\mathbb{J}$. Obviously, we have the formulas

$$\bigcup_{\gamma\in\Gamma} J_\gamma = \mathbb{J}, \qquad J_{\gamma'} \subset J_{\gamma''} \text{ for all } \gamma' < \gamma''.$$

On the sets U and $\widehat{\mathcal{F}}_a$, the action of the semigroup $\mathbb{N}^n_{-0} = \mathbb{N}^n \setminus \vec{0}$, where $\vec{0}$ is a tuple of zeros, is given by

$$\alpha u^i_\beta = u^i_{\alpha+\beta}, \qquad \alpha f = D^\alpha(f),$$

for any $\alpha \in \mathbb{N}^n_{-0}$. As follows from [11], $\widehat{\mathcal{F}}_a$ is a stratified $\mathbb{N}^n_{-0}$-set if U is a stratified $\mathbb{N}^n_{-0}$-set.

In what follows, we everywhere assume that $\widehat{\mathcal{F}}_a$ is a stratified $\mathbb{N}^n_{-0}$-set endowed with an appropriate preorder (4.6) and strict preorder (4.7).

Definition. A series $f \in \mathcal{F}_a$ of the form

$$f = u^i_\alpha + h, \quad h \prec u^i_\alpha, \tag{4.12}$$

is said to be *orderedly solvable* (with respect to u^i_α).

Definition. Let F be an arbitrary series in the algebra $\mathcal{F}_a$ and $f \in \mathcal{F}_a$ be an orderedly solvable series with respect to u^i_α. We say that the series F *reduces to a series* $r \in \mathcal{F}_a$ *with respect to* f if there exists an element $\delta \in \mathbb{N}^n$ for which $u^i_{\alpha+\delta} \in \operatorname{iv}(F)$ and if there is a series $q \in \mathcal{F}_a$ such that $F = qD^\delta f + r$, where $q \preceq F$, $r \preceq F$, and $u^i_{\alpha+\delta} \notin \operatorname{iv}(r)$. If the series F reduces to r with respect to f, we use the notation $F \underset{f}{\longrightarrow} r$.

The following statement was proved in [12].

Statement 4.4. *Let F be an arbitrary series in the algebra $\mathcal{F}_a$ and $u^i_\beta \in \operatorname{iv}(F)$. If the series $f \in \mathcal{F}_a$ is orderedly solvable with respect to u^i_α and there exists $\delta \in \mathbb{N}^n$ such that $u^i_\beta = u^i_{\alpha+\delta}$, then $F \underset{f}{\longrightarrow} r$.*

Definition. The subset S in $\mathcal{F}_a$ is said to be *weakly solvable* if each series $f \in S$ is orderedly solvable with respect to some u^i_α. An element u^i_α is called the *highest term* of the series f and is denoted by $\operatorname{lt} f$.

Definition. Let $F \in \mathcal{F}_a$ and $S = \{f_1, \dots, f_k\} \subset \mathcal{F}(a)$ be a weakly solvable subset. A series F *reduces to a series* $r \in \mathcal{F}(a)$ *with respect to* S if there is a finite sequence of one-step reductions of the form

$$F \underset{f_{i_1}}{\longrightarrow} r_1 \underset{f_{i_2}}{\longrightarrow} r_2 \underset{f_{i_3}}{\longrightarrow} \cdots \underset{f_{i_p}}{\longrightarrow} r, \tag{4.13}$$

where $f_{i_j} \in S$. This sequence of reductions is denoted by $F \underset{S}{\longrightarrow} r$ for short.

Definition. A series $f \in \mathcal{F}_a$ is said to be *irreducible* with respect to a weakly solvable set S if $\operatorname{iv}(f) \cap O(\operatorname{lt} S) = \varnothing$, where $\operatorname{lt} S = \{\operatorname{lt} f\colon\ f \in S\}$.

Definition. A series $r \in \mathcal{F}_a$ is called the *normal form of the series* $f \in \mathcal{F}_a$ *with respect to a weakly solvable subset* $S \subset \mathcal{F}_a$ if $f \underset{S}{\longrightarrow} r$ and r is a series irreducible to S. The normal form of a series f with respect to S is denoted by $\operatorname{NF}(f \downarrow S)$.

In the general case, the normal form of a series with respect to an arbitrary weakly solvable subset is not defined uniquely. For the normal form to be unique, it suffices for S to be a passive system [12].

We introduce a binary operation $\diamond$ on $\mathbb{N}^n$: for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, we set

$$\alpha \diamond \beta = (\mu_1, \dots, \mu_n), \qquad \mu_i = \max(\alpha_i, \beta_i) - \alpha_i.$$

Definition. Let f_1 and f_2 be two solvable series in $\mathcal{F}_a$ of the form

$$f_1 = u^i_\alpha + h_1, \qquad f_2 = u^i_\beta + h_2, \tag{4.14}$$

where $u^i_\alpha = \operatorname{lt} f_1$ and $u^i_\beta = \operatorname{lt} f_2$. Then the difference

$$D^{\alpha\diamond\beta} f_1 - D^{\beta\diamond\alpha} f_2 \tag{4.15}$$

is called the *τ-series of* f_1 and f_2 and is denoted by $\tau(f_1, f_2)$.

Definition. Let $S = \{f_1, \dots, f_k\}$ be a local differential system that is a weakly solvable subset of $\mathcal{F}_a$. We say that the system S *satisfies the compatibility conditions* if for each pair $f_i, f_j \in S$ of form (4.14), the corresponding τ-series (4.15) reduces to zero with respect to S.

The following statement and the theorem below answer the question about a series belonging to a differential ideal.

Statement 4.5. *Let $S = \{f_1, \dots, f_k\} \subset \mathcal{F}_a$ be a passive system and $\operatorname{lt} S = \operatorname{st} S$. A series $f \in \mathcal{F}_a$ belongs to the differential ideal $I = \langle\langle S \rangle\rangle$ if and only if f reduces to zero with respect to S.*

Proof. If $f \underset{S}{\to} 0$, then it follows from the definition of a reduction that

$$f = \sum a^i_\alpha D^\alpha f_i, \quad f_i \in S,$$

i.e., $f \in I$.

Conversely, let $f \in I$. As shown in [12], each nonzero element of the ideal I depends on at least one element in the orbit $O(\operatorname{lt} S)$. If $f \underset{S}{\to} 0$, where the series r cannot be reduced with respect to S, then r is independent of elements in $O(\operatorname{lt} S)$. Hence, $r = 0$.

If the system S is not passive, then Statement 4.5 is inapplicable in the general case because the result of a reduction is not defined uniquely.

Example. We consider the system $S = \{f_1 = u_{1,1} + u,\ f_2 = u_{0,2} - u\}$ and the series $f = u_{1,2} - u_{1,0}$. This system is not passive, because $D_2 f_1 - D_1 f_2 = u_{1,0} + u_{0,1}$ is independent of elements of the orbit $O(\operatorname{lt}(u_{0,2}), \operatorname{lt}(u_{1,2}))$. The series f can be represented in two ways: $f = D_2 f_1 - u_{1,0} - u_{0,1}$ and $f = D_1 f_2$, i.e., f reduces to $-u_{1,0} - u_{0,1}$ in the first case and to zero in the second case.

We note that if $f \in \mathcal{F}_b$, then $f \in \mathcal{F}_a$ for an infinite number of points a. Let $S = \{f_1, \dots, f_k\} \subset \mathcal{F}_b$ be a local differential system and $f_i(b) = 0$ for all $i \in \mathbb{N}_k$. We say that points $a, b \in \mathbb{J}$ are equivalent modulo S (and write $a \sim b \bmod S$) if $f_i \in \mathcal{F}_a$ and $f_i(a) = 0$ for all $f_i \in S$.

Theorem 4.1. *Let $S = \{f_1, \dots, f_k\} \subset \mathcal{F}_b$ be a local differential system that is a weakly solvable subset of $\mathcal{F}_b$ satisfying the compatibility conditions and $f_i(b) = 0$ for all $f_i \in S$. Then S is a passive system, there exist a point $a \in \mathbb{J}$ equivalent to b modulo S and a parallelepiped W containing this point such that the set $Z_W(f)$ of type (4.1) is an analytic manifold in W with the system of parameters $Y \setminus \operatorname{lt} S$.*

Proof. The passivity of the system S was proved in [11]. It was also shown that there exists a point $a \in \mathbb{J}$ (where $a \sim b \bmod S$) such that $f(a) = 0$ for all series f in the orbit $O(S)$. Moreover, it was proved that there exists a normalized system $\mathcal{B}$ of generators of the ideal $\langle\langle S \rangle\rangle$, whose set of principal variables coincides with $\operatorname{lt} S$.

Without loss of generality, we can assume that S is a normalized set (or, in the terms in [22], is orthonormalized) because according to the results in [11], [12], S is a passive system and there exists a canonical set S' such that $\langle\langle S \rangle\rangle = \langle\langle S' \rangle\rangle$. It then follows from Statement 4.2 that the germs $\mathbb{Z}_a(S)$ and $\mathbb{Z}_a(S')$ coincide. It therefore suffices to prove that there is a parallelepiped W containing the point a such that the analytic set $Z_W(S')$ is a manifold in the parallelepiped W.

According to Statement 2.2, we must show that the set of analytic functions in $\widetilde{\mathcal{B}}$ corresponding to the normalized system of generators $\mathcal{B}$ of the ideal $\langle\langle S' \rangle\rangle$ is given in some parallelepiped $V \subset \mathbb{J}$. As follows from Statement 3.1, there exists a parallelepiped $U(a_\tau, \rho)$ in which all functions in $\widetilde{O(S')}$ are analytic. It therefore suffices to show that the set of analytic functions in $\widetilde{\mathcal{B}}$ is also defined in $U(a_\tau, \rho)$. For this, we recall how a normalized system of generators was constructed in [11].

As previously noted, the partition of U in (3.7) generates chain (4.9) of subalgebras $\mathcal{F}_a^\gamma$, chain (4.11) of subspaces J_γ, and partition (4.10), $\{\Phi_a^\gamma\}_{\gamma\in\Gamma}$. We let a_γ denote the natural projection of the point a on the subspace J_γ. We introduce the notation

$$\gamma_0 = \min\{\gamma \in \Gamma\colon\ O(S) \cap \mathcal{F}_a^\gamma \neq \varnothing\}, \qquad \Pi_\gamma = U(a_\tau, \rho) \cap J_\gamma,$$

$$O_\gamma = O(S) \cap \mathcal{F}_a^\gamma, \qquad C_\gamma = O(S) \cap \Phi_a^\gamma,$$

where $\mathcal{F}_a^\gamma$ and Φ_a^γ are respectively given by (4.9) and (4.10). Obviously, for any $\gamma_\star \geq \gamma$, we have

$$O_{\gamma_\star} = C_{\gamma_\star} \cup \Bigg(\bigcup_{\gamma_0 \leq \gamma < \gamma_\star} C_\gamma \Bigg). \tag{4.16}$$

We let $\langle O_\gamma \rangle_{a_\gamma}$ denote the algebra ideal $\mathcal{F}_a^\gamma$ generated by the set O_γ.

Using the transfinite induction principle, we show that

1. for any $\gamma \geq \gamma_0$, there exists a normalized system of generators B_γ of the ideals $\langle O_\gamma \rangle_{a_\gamma}$ of the algebra $\mathcal{F}_a^\gamma$ such that $L_\gamma = \operatorname{lt} O_\gamma$, where L_γ is the set of principal variables of the system B_γ, and
2. all functions in $\widetilde{B}_\gamma$ are analytic in Π_γ.

At $\gamma = \gamma_0$, we have the equalities

$$O_{\gamma_0} = S \cap \mathcal{F}_a^{\gamma_0} = S \cap \Phi_a^{\gamma_0} = C_{\gamma_0}.$$

It is easy to see (also see [11]) that the set O_{γ_0} is a normalized system of generators of the ideal $\langle O_{\gamma_0} \rangle_{a_{\gamma_0}}$. Therefore, according to Statement 2.2, the analytic set

$$Z_{\gamma_0} = \{z \in \Pi_{\gamma_0}\colon\ \tilde{f}(z) = 0 \text{ for all } f \in \tilde{O}_{\gamma_0}\}$$

is a manifold in Π_{γ_0}. Obviously, $L_{\gamma_0} = \operatorname{lt} O_{\gamma_0}$.

We now suppose that for all values of γ satisfying the inequalities $\gamma_0 \leq \gamma < \gamma_\star$, there exists a normalized system of generators B_γ of the ideal $\langle O_\gamma \rangle_{a_\gamma}$ such that $L_\gamma = \operatorname{lt} O_\gamma$ and all functions in $\widetilde{B}_\gamma$ are analytic in Π_γ. We show that these properties also hold for $\gamma = \gamma_\star$.

According to equality (4.16) and the induction hypothesis, the set

$$G_{\gamma_\star} = C_{\gamma_\star} \cup \Bigg(\bigcup_{\gamma_0 \leq \gamma < \gamma_\star} B_\gamma \Bigg)$$

is a system of generators of the ideal $\langle O_{\gamma_\star} \rangle_{a_{\gamma_\star}}$. A normalized system of generators $B_{\gamma_\star}$ was constructed in [11] using a set $G_{\gamma_\star}$. Our case where S' is a canonical set has its own specific features. Namely, any series $f \in C_{\gamma_\star}$ has the form $f = u_\alpha^i + g$, where $g \prec u_\alpha^i$, while at the same time, because of derivation formulas (3.8), the series $g \in \mathcal{F}_a^\gamma$ for $\gamma < \gamma_\star$ is a polynomial in the principal variables of the normalized system of generators B_γ. According to relation (2.5), the coefficients of this polynomial are power series converging in Π_γ and depending only on the parametric variables of the system B_γ. If we express the principal variables in g in terms of the series in B_γ, then we obtain series $f_\star \in B_{\gamma_\star}$ that converge in $\Pi_{\gamma_\star}$.

As follows from the preceding considerations, the set $B = \bigcup_{\gamma_0 \leq \gamma} B_\gamma$ is a normalized system of generators of the ideal $\langle\!\langle S \rangle\!\rangle$ and the corresponding set $\widetilde{B}$ consists of functions analytic in the parallelepiped $U(a_\tau, \rho)$. It remains to refer to Statement 2.2 to complete the proof.

Remark. We can standardly define tangent bundles and vector fields on manifolds and also sheaves of germs of analytic functions [14], [23].

Example. We consider the Dubreil-Jacotin nonlinear equation [24]

$$\Delta\psi + \frac{\rho'}{2\rho}[(\nabla\psi)^2 + y] + F = 0, \tag{4.17}$$

where ψ is a current function depending on x and y, ρ is the fluid density depending on ψ, Δ is the Laplace operator, $(\nabla\psi)^2 = \psi_x^2 + \psi_y^2$, and F is some function of ψ. This equation describes stationary planar flows of a stratified fluid in a gravitational field. In our notation (Cartesian coordinates on $\mathbb{J}$), this equation is given by the series

$$u_{(2,0)} + u_{(0,2)} + \frac{\rho'}{2\rho}[u_{(1,0)}^2 + u_{(0,1)}^2 + x_2] + F,$$

where ρ and F are analytic functions of $u_{(0,0)}$. In what follows, we use the standard notation and, for example, write u_x and u_y instead of $u_{(1,0)}$ and $u_{(0,1)}$.

We find the functions ρ and F for which Eq. (4.17) allows a generalized separation of variables, i.e., solutions of the form $\psi = G(\alpha(x) + \beta(y))$, where G is a certain function and α and β are respectively functions of x and y. This type of solutions of the nonlinear Laplace equation was described in [20].

The problem can be reformulated as the problem of investigating the compatibility of the system formed by Eq. (4.17) and the equation

$$\phi(\psi)_{xy} = 0, \tag{4.18}$$

where ϕ is the inverse function of G.

We introduce a new function $w = \phi(\psi)$, and Eqs. (4.17) and (4.18) then become

$$\Delta w + (\nabla w)^2 r + yf + g = 0, \tag{4.19}$$

$$w_{xy} = 0. \tag{4.20}$$

Here, r, f, and g are some functions of w. We briefly describe the study of the compatibility of system (4.19), (4.20) using the methods proposed above. Total differential operators (3.8) are denoted by D_x and D_y. The calculations are voluminous, and we therefore used the computer algebra system Maple [10].

Applying the operator $D_x D_y$ to (4.19) and reducing this expression with respect to (4.20), i.e., replacing all the derivatives w_{xy}, w_{xxy}, and w_{xxxy} with zero, we obtain a second-order equation. Reducing the last equation with respect to (4.19), we obtain a polynomial E_1 in w_x and w_y. It is easy to show that all the coefficients of the polynomial vanish only if Eq. (4.19) has the form

$$\Delta w + (\nabla w)^2 + ay + bw = 0, \quad a, b \in \mathbb{R}.$$

The generalized method of separation of variables for this equation can be found in [20], where the corresponding pictures of the streamlines are also presented. We note that the differential ideal generated by the last equation and by Eq. (4.19) has a normalized system of generators. We can regard $D_x^n D_y^m w_{xx}$ and $D_x^n D_y^m w_{xy}$ for $n, m \in \mathbb{N}$ as the principal variables of this system of generators.

If the polynomial E_1 is nonzero, then acting on it by D_x and reducing this expression with respect to (4.19) and (4.20), we obtain a relation. If we act on the polynomial E_1 by D_y and reduce the resulting expression with respect to (4.19) and (4.20), then we obtain a new relation. Using (4.19) and (4.20), we can eliminate all the variables w_{xy}, w_{xx}, and w_{yy} from these two relations and obtain a new polynomial E_2 in w_x and w_y.

We now assume that the polynomials E_1 and E_2 differ only by a factor. This gives seven ordinary differential equations for the three functions f, g, and r in the left-hand side of Eq. (4.19). From these seven equations, we can obtain three first-order differential equations, which we solve and obtain

$$r = \frac{1}{-2w + c_0}, \qquad g = \frac{c_1}{-2w + c_0}, \qquad f = \frac{c_2}{-2w + c_0},$$

where $c_0, c_1, c_2 \in \mathbb{R}$. Therefore, Eq. (4.19) can be written as

$$\Delta w + (\nabla w)^2 \frac{1}{-2w + c_0} + y \frac{c_2}{-2w + c_0} + \frac{c_1}{-2w + c_0} = 0. \tag{4.21}$$

It is easy to see that the common solution of this equation and Eq. (4.20) is the polynomial

$$w = b_1 y + a_2 x^2 + a_1 x + a_0, \tag{4.22}$$

where $b_1, a_1 \in \mathbb{R}$, $a_2 = c_2/4b_1$, and $a_0 = (c_1 + 2c_0 a_2 + a_1^2 + b_1^2)/4a_2$. If we introduce new variables

$$y' = y + \frac{c_1}{c_2}, \qquad w' = w - \frac{c_0}{2},$$

then Eq. (4.21) becomes

$$\Delta w' + \frac{(\nabla w')^2}{-2w'} + \frac{c_2 y'}{-2w'} = 0.$$

By replacing $w' = \psi^2 - c_2/4$, we obtain an equation for the stream function in the Dubreil-Jacotin form:

$$\Delta \psi + \frac{c_2}{c_2 \psi - 4\psi^3} [(\nabla \psi)^2 + y] = 0. \tag{4.23}$$

The solution of this equation found above has the form $\psi = (w - c_0/2 + c_2/4)^{1/2}$, where the function w is given by (4.22). Some other Dubreil-Jacotin equations that allow a generalized separation of variables were presented in [25].

A group classification of the Dubreil-Jacotin equation was shown in [19]. It follows from the results of this classification that Eq. (4.23) allows a two-dimensional symmetry algebra and solution (4.22) that we found is not invariant.

5. Conclusion

We have considered the algebra $\mathcal{F}_a$ of convergent power series. Instead of it, the ring of germs of smooth (infinitely differentiable) real-valued functions can be studied with the majority of our results remaining applicable. In particular, our observations are based on the Weierstrass division theorem, but the statement on the uniqueness of the remainder also occurs in the smooth case if there is a normalized system of generators for the corresponding ideal. This is related to the fact that the Weierstrass theorem in this case follows from the implicit function theorem [26].

Extending this approach to difference algebras and equations is undoubtedly interesting. Unfortunately, the study of difference equations cannot be limited to local algebras. Nevertheless, symmetries of the difference equations introduced by Dorodnitsyn [27] are currently being studied actively.

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