Theoretical and Mathematical Physics, **187**(2): 718–729 (2016)

QUANTUM MECHANICAL MODEL IN GRAVITY THEORY

V. V. Losyakov*

We consider a model of a real massive scalar field defined as homogeneous on a d-dimensional sphere such that the sphere radius, time scale, and scalar field are related by the equations of the general theory of relativity. We quantize this system with three degrees of freedom, define the observables, and find dynamical mean values of observables in the regime where the scalar field mass is much less than the Planck mass.

Keywords: general relativity, quantization of gravity, quantum mechanics, quantization of gauge theories

DOI: 10.1134/S0040577916050081

1. Introduction

General relativity (the theory of gravity) and quantum theory were formulated almost simultaneously historically. While the first was recognized as the pinnacle of classical physics, the second demanded a major reconsideration of the foundations of classical theory.

The evolution of quantum theory despite the arising mathematical difficulties was astonishingly impressive: the positron prediction, calculation and experimental verification of radiative corrections, advances in condensed matter physics, atomic spectroscopy, etc. We here note that the mathematical formulation of quantum mechanics as a quantum theory with a finite number of degrees of freedom was established after von Neumann's works [1]. Under reasonable assumptions, it turned out that all representations of canonical commutation relations in quantum mechanics are unitarily equivalent. The situation in quantum field theory, the quantum theory with an infinite number of degrees of freedom, differs drastically. Emch wrote [2] that *the von Neumann uniqueness theorem does not admit a generalization to quantum field theory*: *it was incontrovertibly proved that nonequivalent representations not only existed but also seemed so numerous that they literally overwhelmed physicists*. Attempts to construct a mathematically rigorous formulation of a physically meaningful field theory containing interactions are still not crowned with success. As a result, physicists obtained an S-matrix formulation of quantum field theory, which allow computing the scattering cross sections of particles in the perturbation theory framework (see, e.g., [3]). The main achievement of this theory is that the calculated results are beautifully confirmed by experimental data. This primarily relates to the electroweak interaction.

The gravitational interaction is also weak on the elementary particle level, but the point is that it is *too* weak. In our opinion, from the standpoint of a reasonable person, for example, an experimentalist, taking quantum corrections caused by the gravitation field into account in scattering processes is the same

---

*Lebedev Physical Institute, RAS, Moscow, Russia; National Research University Higher School of Economics, Moscow, Russia, e-mail: losyakov@lpi.ru.

This research is supported by the Russian Foundation for Basic Research (Grant No. 14-01-90405).

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 187, No. 2, pp. 310–322, May, 2016. Original article submitted August 18, 2015.

as studying the Lamb shift before the experimental discovery of electromagnetic waves. In 1969, Dirac wrote that there is no experimental data for quantization of the gravitational field (p. 321 in Russian [4]). Very little has changed in that sense since then. Of course, this could not stop theoreticians, but they also encountered principal difficulties in this path: in contrast to the theory of electroweak (and also strong) interactions, the gravity theory turned out to be nonrenormalizable in the framework of the S-matrix approach.

Theoretical works on the quantization of gravity can be very conditionally divided into three directions:

1. a formal quantization of gravity as a gauge theory with first-class constraints outside the perturbation theory framework with all unsolved problems typical for any quantum field theory with interactions (see the monograph by Gitman and Tyutin [5]),
2. a generalization of the theory of gravity on the classical level aiming to ameliorate the ultraviolet behavior in the S-matrix approach (supergravity [6]), and
3. a quantization of gravity outside the framework of quantum field theory (the brightest theory in this direction, in our view, is string theory, initially formulated unrelated to gravity theory and decisively influencing modern theoretical and mathematical physics [7], [8]; Polyakov [9] wrote, "The unique feature of critical strings is that they describe gravitons").

Unfortunately or fortunately, physics is still physics: no experiment, no good theory.

But a small loophole exists (tested in constructing quantum theory). If there is no real experiment, sometimes a thought experiment is possible. Gravity must be amplified. From general considerations, we can assume that the role of quantum gravity is substantial at the beginning of our world or near a black hole (Hawking radiation). But it is interesting that in these thought experiments, the dependence on time is essential: the process is nonstationary. This does not in any way fit into the picture of the S-matrix approach, where the initial state is given in the infinitely remote past and the scattering result is observed in the infinitely remote future. And quantum mechanics is ideally suited to this.

Therefore, to obtain at least some kind of reliable information about the quantum properties of gravity, we attempt to consistently reduce the classical general relativity theory to a system with a finite number of degrees of freedom, quantize it, define the observables, and study the resulting quantum mechanical model with no regrets regarding unsolved quantum field theory problems.

2. Notation and definitions

The metric tensor $g_{\mu\nu}$, $\mu,\nu = 0, 1, \ldots, d$, defines the invariant distance in the space–time (the metric tensor signature is $(-,+,\ldots,+)$):

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

The Christoffel symbols (connection) can be written as

$$\Gamma_{\mu,\nu\lambda} = \frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\nu\lambda}), \qquad \Gamma^\mu_{\nu\lambda} = g^{\mu\varrho}\Gamma_{\varrho,\nu\lambda},$$

where $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$. The Riemann curvature tensor, the Ricci tensor, and the scalar curvature have the forms

$$R^\mu_{\lambda\nu\varrho} = \partial_\nu \Gamma^\mu_{\lambda\varrho} - \partial_\varrho \Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\nu\sigma}\Gamma^\sigma_{\lambda\varrho} - \Gamma^\mu_{\varrho\sigma}\Gamma^\sigma_{\lambda\nu}, \qquad R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}.$$

For the Einstein action in the case of gravitation with a Λ-term ($\Lambda = m_\lambda^2$), we have the expression

$$S_{\mathrm{g}} = \frac{1}{4} m_{\mathrm{P}}^{d-1} \int d^{d+1}x\, \sqrt{-g}(R - 2m_\lambda^2), \tag{1}$$

where m_{P} is the Planck mass, $g = \det g_{\mu\nu}$, and $d^{d+1}x = dx_0\,dx_1\cdots dx_d$.

Below, we see that the simplest quantum mechanical model can be constructed if we add matter to it. We also choose the simplest matter, a real massive (mass m) scalar field ϕ. The corresponding action has the form

$$S_{\mathrm{m}} = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}(g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2), \tag{2}$$

and the equations of motion are

$$\begin{aligned} &m_{\mathrm{P}}^{d-1}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + m_\lambda^2 g_{\mu\nu}\right] = 2T_{\mu\nu},\\ &T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(g^{\lambda\sigma}\,\partial_\lambda\phi\,\partial_\sigma\phi + m^2\phi^2), \end{aligned} \tag{3}$$

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\sqrt{-g}g^{\mu\nu}\,\partial_\nu\phi - m^2\phi = 0. \tag{4}$$

3. Model

We consider the space–time with the metric

$$ds^2 = -e^2(t)\,dt^2 + a^2(t)\sum_{i=1}^{d} d\theta_i^2\prod_{k=1}^{i-1}\sin^2\theta_k, \tag{5}$$

where $0 \le \theta_i < \pi$, $i = 1,\dots,d-1$, $0 \le \theta_d < 2\pi$, d is the dimension of the space (not necessarily three), $-\infty < t < \infty$ is time, and $a(t) > 0$ and $e(t) > 0$ are the physical variables determining the considered space–time. The topology of such a space–time is $S^d \times \mathbb{R}^1$. It coincides with the topology of the de Sitter space–time. This metric is also known as the metric of a closed isotropic model [10].

The metric tensor $g_{\mu\nu}$ is diagonal:

$$\begin{aligned} &g_{00} = -e^2(t), \qquad g_{ik} = \delta_{ik}a^2(t)\prod_{m=1}^{i-1}\sin^2\theta_m \equiv \delta_{ik}a^2 g_i, \quad i,k = 1,\dots,d,\\ &g^{00} = -\frac{1}{e^2}, \qquad g^{ik} = \delta_{ik}\frac{1}{a^2 g_i}, \quad i,k = 1,\dots,d. \end{aligned}$$

The Ricci tensor can be written as

$$\begin{aligned} &R_{00} = -d\frac{\ddot a}{a} + d\frac{\dot e}{e}\frac{\dot a}{a}, \qquad R_{0i} = 0,\\ &R_{ik} = \delta_{ik}\frac{g_i}{e^2}\left(a\ddot a + (d-1)\dot a^2 + (d-1)^2 - \frac{\dot e}{e}a\dot a\right), \end{aligned} \tag{6}$$

and the scalar curvature is

$$R = \frac{d}{e^2a^2}\left[2a\ddot a + (d-1)\dot a^2 + (d-1)e^2 - 2\frac{\dot e}{e}a\dot a\right].$$

We also consider the model matter (the field is homogeneous on the surface of the sphere). Let

$$\phi(t,x) = \phi(t). \tag{7}$$

After substituting Ricci tensor (6) and scalar field constraint (7) in Einstein's equation (3) and scalar field equation (4), we obtain

$$\mu=\nu=0: \quad m_{\mathrm{P}}^{d-1}\left[\frac{d(d-1)}{2a^2}(\dot a^2+e^2)-m_\lambda^2 e^2\right]=\dot\phi^2+m^2e^2\phi^2,$$

$$\mu=i,\ \nu=0: \quad 0=0,$$

$$\mu=i,\ \nu=k: \quad m_{\mathrm{P}}^{d-1}\left[(d-1)\left(\frac{\ddot a}{a}+\frac{d-2}{2a^2}(\dot a^2+e^2)-\frac{\dot a}{a}\frac{\dot e}{e}\right)-m_\lambda^2 e^2\right]=-\dot\phi^2+m^2e^2\phi^2,$$

$$\ddot\phi+\left(d\frac{\dot a}{a}-\frac{\dot e}{e}\right)\dot\phi+m^2e^2\phi=0.$$

It is hence immediately obvious that this reduction is consistent. Indeed, the considered model is characterized by the three variables $a(t)$, $e(t)$, and $\phi(t)$, which are obtained from solutions of three equations. Moreover, Einstein's equations are covariant under the general coordinate transformations

$$x^\mu \longrightarrow X^\mu = X^\mu(x),$$

$$g_{\mu\nu}(x)\longrightarrow G_{\mu\nu}(X)=\frac{\partial x^\lambda}{\partial X^\mu}\frac{\partial x^\sigma}{\partial X^\nu}g_{\lambda\sigma}(x), \qquad \phi(x)\longrightarrow\Phi(X)=\phi(x).$$

This leads us to the fact that in the total $(d+1)(d+2)/2+1$ equations of motion, there are $d+1$ fewer independent equations, which reflects the gauge nature of gravity theory. We can verify that this property is preserved in the reduced model. The third equation of motion follows from the first and last equations, and all the equations of motion are covariant under a time rescaling:

$$\begin{aligned}&t\longrightarrow T=f(t),\\ &a(t)\longrightarrow A(T)=a(t), \qquad e(t)\longrightarrow E(T)=\frac{e(t)}{\dot f(t)}, \qquad \phi(t)\longrightarrow\Phi(T)=\phi(t).\end{aligned} \tag{8}$$

The above equations of motion can be obtained from the principle of least action by substituting metric (5) in action (1).

The determinant of the metric tensor has the form

$$g\equiv\det g_{ik}=-e^2a^{2d}\prod_{m=1}^{d}\sin^{2(d-m)}\theta_m, \qquad \sqrt{-g}=ea^d\prod_{m=1}^{d}\sin^{d-m}\theta_m.$$

We define the element of the d-dimensional solid angle and the complete solid angle:

$$d\Omega_d=\prod_{m=1}^{d}d\theta_m\sin^{d-m}\theta_m, \qquad \Omega_d=\frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}.$$

Substituting, we obtain

$$S_{\mathrm{g}}=\frac{1}{4}m_{\mathrm{P}}^{d-1}\Omega_d\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}dt\,a^d\left[-d(d-1)\left(\frac{\dot a^2}{ea^2}+\frac{e}{a^2}\right)-2m_\lambda^2 e\right]+\frac{m_{\mathrm{P}}^{d-1}\Omega_d}{2e}\,\partial_0 a^d\Big|_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}.$$

Similarly, using (7), we find the matter action:

$$S_{\mathrm{m}}=\frac{1}{2}\Omega_d\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}dt\,a^d\left[\frac{\dot\phi^2}{e}-em^2\phi^2\right].$$

We bring the model variables to the dimensionless form. For this, we introduce the two dimensionless parameters

$$\mu = \sqrt{\frac{d(d-1)}{2}}\,\frac{m}{m_\lambda}, \qquad \zeta = \frac{d(d-1)}{2}\frac{m_{\mathrm{P}}^2}{m_\lambda^2}\left(\frac{\Omega_d m_\lambda^2}{m_{\mathrm{P}} m}\right)^{2/d}.$$

We measure time in units inverse to the scalar field mass, $t \longrightarrow t/m$, and define the new variables

$$q(t) = \left(\frac{\Omega_d m_\lambda^2}{m_{\mathrm{P}} m}\right)^{1/d} m_{\mathrm{P}} a(t), \qquad Q(t) = \frac{m}{m_\lambda} m_{\mathrm{P}}^{-(d-1)/2}\phi(t).$$

Hence, the complete action of the considered model has the form

$$S = \frac{1}{2}\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} dt\, q^d \left[\frac{1}{e}\left(\dot{Q}^2 - \mu^2\frac{\dot{q}^2}{q^2}\right) - e\left(Q^2 + 1 - \frac{\zeta}{q^2}\right)\right], \tag{9}$$

where $q(t)$, $Q(t)$, and $e(t)$ are model variables and μ and ζ are its parameters. The equations of motion corresponding to this action coincide with the previously obtained equations of motion. Moreover, this action is invariant under time reparameterization (8).

4. Generalized Hamiltonian form of the model

The Lagrangian of the model has the form

$$L(\mathcal{Q}, \dot{\mathcal{Q}}, e) = \frac{1}{2} q^d \left[\frac{1}{e}\left(\dot{Q}^2 - \mu^2\frac{\dot{q}^2}{q^2}\right) - e\left(Q^2 + 1 - \frac{\zeta}{q^2}\right)\right], \qquad \mathcal{Q} = (q, Q).$$

A feature of the theory is that the Lagrangian is independent of $\dot{e}$ and its Hessian is equal to zero. The generalized Hamiltonian approach to such theories was introduced by Dirac [11] and considered in detail by Gitman and Tyutin [5].

If we define the generalized momenta $\mathcal{P} = (p, P)$ as

$$\mathcal{P} = \frac{\partial L}{\partial \dot{\mathcal{Q}}}, \qquad p = -\mu^2 \frac{q^{d-2}}{e}\dot{q}, \qquad P = \frac{q^d}{e}\dot{Q},$$

then we can obtain the equations of motion of the model by varying the action in the generalized Hamiltonian form

$$S = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} dt\,[\mathcal{P}\dot{\mathcal{Q}} - \lambda\varphi(\mathcal{P}, \mathcal{Q})], \tag{10}$$

and in our case, there is the constraint

$$\varphi(\mathcal{P}, \mathcal{Q}) = \frac{1}{2q^d}\left(P^2 - \frac{1}{\mu^2}q^2 p^2\right) + \frac{1}{2}q^d\left(Q^2 + 1 - \frac{\zeta}{q^2}\right)$$

with the Lagrange multiplier $\lambda = e$ and the canonical Hamiltonian

$$\mathcal{H}(\mathcal{P}, \mathcal{Q}, e) = \mathcal{P}\dot{\mathcal{Q}} - L(\mathcal{Q}, \dot{\mathcal{Q}}, e)|_{\dot{\mathcal{Q}} = f(\mathcal{P}, \mathcal{Q}, e)} = \lambda\varphi(\mathcal{P}, \mathcal{Q}),$$

proportional to the constraint and vanishing on it. It can be shown that this type of action in the generalized Hamiltonian dynamics fixes the considered model as a gauge theory with a primary first-class constraint.

The Hamiltonian equations of motion have the forms

$$\frac{\delta S}{\delta q} = \dot{p} + e\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial q} = 0, \qquad \frac{\delta S}{\delta p} = \dot{q} - e\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial p} = 0,$$

$$\frac{\delta S}{\delta Q} = \dot{P} + e\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial Q} = 0, \qquad \frac{\delta S}{\delta P} = \dot{Q} - e\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial P} = 0, \qquad \varphi(\mathcal{P},\mathcal{Q}) = 0.$$

From the condition for preserving the constraint with time without using the vanishing of the corresponding variational derivatives in the equations of motion, we can obtain

$$\frac{\delta S}{\delta q} = \mu^2\frac{q^{d-2}}{p}\left[\frac{\delta S}{\delta p}\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial q} + \frac{\delta S}{\delta Q}\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial P} + \frac{\delta S}{\delta P}\frac{\partial\varphi(\mathcal{P},\mathcal{Q})}{\partial Q}\right].$$

Hence, there is no need at all to consider the equation $\delta S/\delta q = 0$. The number of independent equations of motion is one less than the number of determinable variables. For the solution of the classical equations of motion to be unique, we must fix one of the variables (or a function of it) by imposing a gauge condition that completely eliminates the arbitrariness. This must be done consistently. To choose the gauge class, we propose the gauge[1]

$$q = e^t. \tag{11}$$

At the same time, we have the previously imposed conditions $q > 0$, $-\infty < t < \infty$. But that is not the point. In this case, the equation $\delta S/\delta p = 0$ *uniquely* determines the Lagrange multiplier

$$e = -\mu^2\frac{e^{(d-1)t}}{p}, \tag{12}$$

and from the constraint equation $\varphi(\mathcal{P},\mathcal{Q}) = 0$, we find the momentum conjugate to the coordinate q,

$$p = -\mu e^{-t}\sqrt{P^2 + e^{2dt}(Q^2 + 1 - \zeta e^{-2t})}. \tag{13}$$

The sign of the square root in the constraint solution is chosen to satisfy the condition $e > 0$ (see (12)).

As a result, there are two equations $\delta S/\delta Q = 0$ and $\delta S/\delta P = 0$, which become

$$\dot{P} = -\mu e^{2dt}\frac{Q}{\sqrt{P^2 + e^{2dt}(Q^2 + 1 - \zeta e^{-2t})}}, \qquad \dot{Q} = \mu\frac{P}{\sqrt{P^2 + e^{2dt}(Q^2 + 1 - \zeta e^{-2t})}}. \tag{14}$$

These equations of motion are Hamiltonian. The classical dynamics (not generalized and simply Hamiltonian) occur in the phase space $-\infty < P, Q < \infty$ without any constraints and are determined by the physical Hamiltonian

$$H_{\text{phys}} = \mu\sqrt{P^2 + e^{2dt}(Q^2 + 1 - \zeta e^{-2t})}. \tag{15}$$

We note that according to the "canonical" procedure [5], gauge condition (11) and the primary first-class constraint $\varphi(\mathcal{P},\mathcal{Q}) = 0$ form a secondary second-class constraint. Further, we should perform the canonical transformation

$$P, p; Q, q \longrightarrow P, p; Q, q - e^t.$$

Such a time-dependent canonical transformation implies that the canonical Hamiltonian, vanishing on the equations of motion, maps to H_{phys}, and the Dirac brackets for the variables P and Q become ordinary Poisson brackets.

[1] We could also choose other gauge classes, $Q = f(t)$ for example. In that case, the physical variables would be q and p, and it would already be another model.

5. Quantizing the model

The canonical variables P and Q are independent and defined on the entire phase plane. Quantization means that we define two self-adjoint operators $\widehat{P}$ and $\widehat{Q}$ (it is possible in this situation) and impose canonical commutation relations on them,

$$\widehat{Q}\widehat{P} - \widehat{P}\widehat{Q} \equiv [\widehat{Q}, \widehat{P}] = i.$$

We construct the Hamiltonian $\widehat{H}_{\text{phys}}$. For this, we use the spectral theorem and its corollaries (see [12]).

Let $\hat{h}$ be a self-adjoint operator acting on a Hilbert space $\mathcal{H}$. Let $\hat{h}$ have a pure point spectrum λ_N, i.e., the eigenvectors $|\phi_N\rangle$ of this operator form a basis of $\mathcal{H}$. The operator $\hat{h}$ can then be represented in the form of a linear combination of projectors,

$$\hat{h} = \sum_N \lambda_N |\phi_N\rangle\langle\phi_N|,$$

and its domain consists of those vectors $|\phi\rangle \in \mathcal{H}$ for which

$$\sum_N \lambda_N^2 |\langle\phi_N|\phi\rangle|^2 < \infty.$$

If there is a classical observable $f(h)$, then the corresponding observable $\hat{f}(h)$ in quantum mechanics is defined as

$$\hat{f}(h) = \sum_N f(\lambda_N)|\phi_N\rangle\langle\phi_N|,$$

and if the function $f(\lambda)$ is real-valued, then the operator $\hat{f}(h)$ is self-adjoint.

In the considered case,

$$H_{\text{phys}} = f(h) = \mu\sqrt{h}, \qquad \hat{h} = \widehat{P}^2 + e^{2dt}(\widehat{Q}^2 + 1 - \zeta e^{-2t}).$$

The self-adjoint operator $\hat{h}$, up to the coefficient $1/2$, is the Hamiltonian of the harmonic oscillator with a unit weight and a frequency equal to $\omega(t) = e^{dt}$. Its eigenstates $|\phi_N(t)\rangle$ and eigenvalues $\lambda_N(t)$ can be found from the equation $\hat{h}|\phi_N(t)\rangle = \lambda_N(t)|\phi_N(t)\rangle$ (time plays the role of a parameter in this problem) and are well known.

We introduce the creation and annihilation operators

$$\hat{a}^+(t) = \frac{1}{\sqrt{2}}(e^{-dt/2}\widehat{P} + ie^{dt/2}\widehat{Q}), \qquad \hat{a}(t) = \frac{1}{\sqrt{2}}(e^{-dt/2}\widehat{P} - ie^{dt/2}\widehat{Q}),$$

and then

$$|\phi_N(t)\rangle = \frac{1}{\sqrt{N!}}(\hat{a}^+(t))^N|\phi_0(t)\rangle, \qquad \lambda_N(t) = e^{dt}(2N+1) + e^{2dt}(1 - \zeta e^{-2t}), \quad N = 0, 1, \ldots,$$

where $|\phi_0(t)\rangle$ is the solution of the equation $\hat{a}(t)|\phi_0(t)\rangle = 0$.

We can now define the physical Hamiltonian

$$\widehat{H}_{\text{phys}} = \mu \sum_{N=0}^{\infty} \sqrt{e^{dt}(2N+1) + e^{2dt}(1 - \zeta e^{-2t})}\,|\phi_N(t)\rangle\langle\phi_N(t)|. \tag{16}$$

For the physical Hamiltonian to be self-adjoint, it is necessary and sufficient that the condition

$$1+e^{dt}(1-\zeta e^{-2t})\geq 0$$

be satisfied. If we now require that the considered problem be well defined for any times ($-\infty<t<\infty$), then it is necessary to impose the condition

$$\zeta\leq d\frac{(d-2)^{2/d-1}}{2^{2/d}},\quad d>2, \tag{17}$$

on the problem parameters.[2]

A given model state $|\Phi_0\rangle$ in the domain of $\widehat{H}_{\rm phys}(t_0)$ evolves according to the Schrödinger equation[3]

$$i\frac{d}{dt}|\Phi(t)\rangle=\widehat{H}_{\rm phys}|\Phi(t)\rangle,\qquad |\Phi(t_0)\rangle=|\Phi_0\rangle.$$

If we use the expansion

$$|\Phi(t)\rangle=\sum_{N=0}^{\infty}c_N(t)|\phi_N(t)\rangle, \tag{18}$$

then we write a system of differential recurrence relations for the expansion coefficients $c_N(t)$:

$$\begin{aligned}\dot c_N(t)+i\mu\sqrt{\lambda_N(t)}c_N(t)=\frac{1}{4}d(&\sqrt{(N+2)(N+1)}c_{N+2}(t)-\\ &-\sqrt{N(N-1)}c_{N-2}(t)),\quad N=0,1,\ldots.\end{aligned}$$

6. Model observables

Which model properties are interesting? First of all, both the geometry of the space–time obtained in the process of the evolution of the initial state and the state of matter in this space–time are interesting. Therefore, to begin, we restrict ourself to such observables: the metric of the model space–time and the stress–energy tensor of the scalar field. To find a quantum observable, we must define the classical observable.

The metric in the model is determined by the scale factor $a(t)$ and the metric tensor component $g_{00}=-e^2$:

$$a=\frac{\mu}{m\sqrt{\zeta}}e^t,\qquad g_{00}=-\mu^2\frac{e^{2dt}}{P^2+e^{2dt}(Q^2+1-\zeta e^{-2t})}.$$

The stress–energy tensor of the scalar field is given by (see (3))

$$T_0^0=-\frac{me^{-dt}}{2\Omega_d}(e^{-dt}P^2+e^{dt}Q^2),\qquad T_k^i=\delta_k^i\frac{me^{-dt}}{2\Omega_d}(e^{-dt}P^2-e^{dt}Q^2),\qquad T_0^i=0,$$

and the quantum mechanical observables are

$$\hat g_{00}=-\mu^2e^{dt}\sum_{N=0}^{\infty}|\phi_N(t)\rangle\frac{1}{2N+1+e^{dt}(1-\zeta e^{-2t})}\langle\phi_N(t)|, \tag{19}$$

$$\widehat{T}_0^0=-\frac{me^{-dt}}{2\Omega_d}(\hat a^+(t)\hat a(t)+\hat a(t)\hat a^+(t)),\qquad \widehat{T}_k^i=\delta_k^i\frac{me^{-dt}}{2\Omega_d}((\hat a^+(t))^2+(\hat a(t))^2). \tag{20}$$

[2] The condition that e is finite transforms this relation into a strict inequality.

[3] The one-parameter family of the Hamiltonian $\widehat{H}_{\rm phys}(t)$ has a common domain.

In the selected gauge class, the scale factor is a parameter. Moreover, because the scale factor is invariant under reparameterization transformations (8), the time t should be expressed in terms of the invariant scale factor a after all the calculations.

Knowing the expansion coefficients $c_N(t)$ for the state $|\Phi(t)\rangle$ (see (18)) allows easily calculating the mean values of the observables in this state:

$$g_{00}(t) \equiv \langle\Phi(t)|\hat{g}_{00}|\Phi(t)\rangle = -\mu^2 e^{dt} \sum_{N=0}^{\infty} \frac{\bar{c}_N(t)c_N(t)}{2N+1+e^{dt}(1-\zeta e^{-2t})},$$

$$T_0^0(t) \equiv \langle\Phi(t)|\widehat{T}_0^0|\Phi(t)\rangle = -\frac{me^{-dt}}{2\Omega_d} \sum_{N=0}^{\infty} (2N+1)\bar{c}_N(t)c_N(t),$$

$$T(t)\delta_k^i \equiv \langle\Phi(t)|\widehat{T}_k^i|\Phi(t)\rangle = \delta_k^i \frac{me^{-dt}}{2\Omega_d} \sum_{N=2}^{\infty} \sqrt{N(N-1)}(\bar{c}_N(t)c_{N-2}(t) + \bar{c}_{N-2}(t)c_N(t)).$$

7. Calculations in the regime $m \ll m_{\mathrm{P}}$

From relation (17), we can obtain the restriction

$$\mu < \frac{1}{[(d-1)\Omega_d]^{1/(d-2)}} \sqrt{\frac{d}{d-2}\left(\frac{m}{m_{\mathrm{P}}}\right)^{(d-1)/(d-2)}}, \quad d > 2,$$

and we can therefore assume that $\mu \to 0$ if $m \ll m_{\mathrm{P}}$. And because the physical Hamiltonian $\widehat{H}_{\mathrm{phys}}$ is proportional to that small parameter, a given initial state $|\Phi_0\rangle$ remains almost unchanged in some sufficiently large[4] interval after the initial instant t_0.

Let the initial state be the vacuum state. Then we have

$$|\Phi(t)\rangle = |\Phi_0\rangle = |\phi_0(t_0)\rangle, \qquad \hat{a}(t_0)|\phi_0(t_0)\rangle = 0. \tag{21}$$

This state should now be expanded in the basis $|\phi_N(t)\rangle$ to determine the coefficients $c_N(t)$ and calculate the mean values of the observables as functions of the scale factor a.

Omitting the intermediate calculations, we present the answers. The expansion coefficients are

$$c_{2N} = \frac{(-1)^N\sqrt{(2N)!}}{2^N N!\sqrt{\cosh(d(t-t_0)/2)}} \tanh^N \frac{d(t-t_0)}{2}, \qquad c_{2N+1}(t) = 0, \quad N = 0, 1, \ldots.$$

The strength–energy tensor (see Fig. 1) is

$$T_0^0(a) = -\frac{m}{4\Omega_d a_0^d}\left[1 + \left(\frac{a_0}{a}\right)^{2d}\right], \qquad T(a) = -\frac{m}{4\Omega_d a_0^d}\left[1 - \left(\frac{a_0}{a}\right)^{2d}\right], \tag{22}$$

and the quantum component of the metric tensor[5] (see Fig. 2) is

$$\begin{gathered} g_{00}(a) = -\frac{d(d-1)}{4} m m_{\mathrm{P}}^{d-1} a_0^{d/2} \frac{a^{3d/2}}{a^d + a_0^d} \int_0^1 \frac{du\, u^{\beta(a)-1}}{\sqrt{1-uz^2(a)}}, \\ \beta(a) = \frac{1}{4}\left[1 + \frac{1}{2m}\Omega_d m_{\mathrm{P}}^{d-1} a^{d-2}(2m_\lambda^2 a^2 - d(d-1))\right], \qquad z(a) = \frac{a^d - a_0^d}{a^d + a_0^d}, \end{gathered} \tag{23}$$

[4]If the initial state is not strongly excited, then this approximation works roughly when the dimensionless scale factor $0 < q \ll (d/\mu)^{1/d} \gg 1$.

[5]Let $F(\alpha, \beta; \gamma \mid z)$ be the Gauss hypergeometric function. Then

$$\int_0^1 \frac{du\, u^{\beta-1}}{\sqrt{1-z^2 u}} = \frac{1}{\beta} F\left(\frac{1}{2}, \beta; \beta+1 \,\middle|\, z^2\right).$$

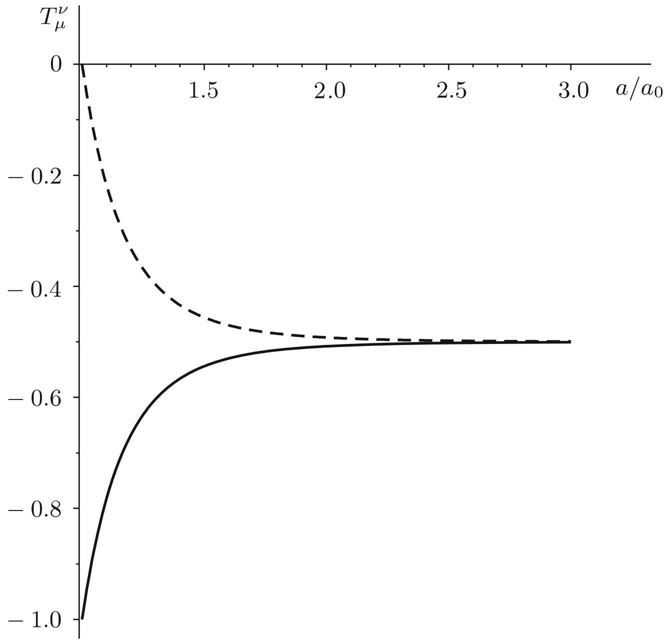

Fig. 1. Plots of T_0^0 (continuous curve) and T (dashed curve) depending on the scale factor: the unit of measurement of the components T_μ^ν is $m/(2a_0^d\Omega_d)$ and $d = 3$.

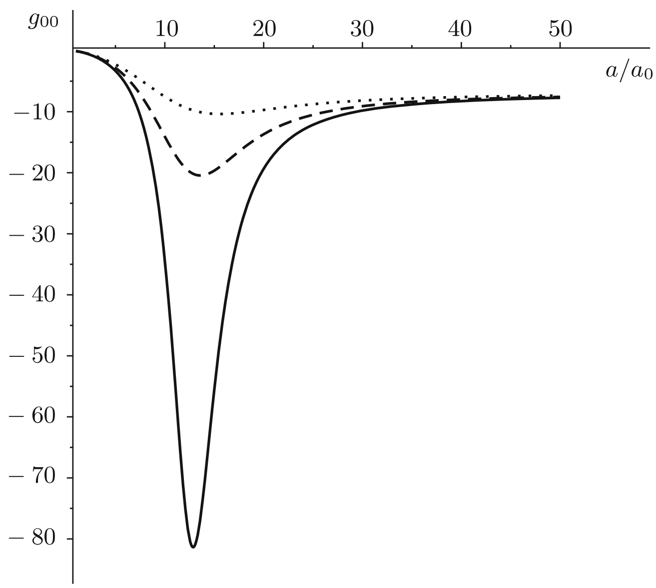

Fig. 2. Plots of g_{00} depending on the scale factor (in the units μ^2): the continuous curve is $\zeta = 1.8$, the dashed curve is $\zeta = 1.5$, and the dotted curve is $\zeta = 1$, $d = 3$, and $q_0 = 1/16$.

where a_0 is the initial scale factor.

We do not analyze the obtained formulas in detail. Our purpose here is to construct a quantum mechanical model in the framework of general relativity, and these formulas merely illustrate its capabilities. We only note the asymptotic behavior of the solutions.

First, we define the energy density $\varepsilon = -T_0^0$ and pressure $p\delta_i^k = T_i^k$. The equation of state of matter at the initial instant $\varepsilon = \varepsilon(a_0), p(a_0) = 0$ then reaches its asymptotic value ($a \gg a_0$)

$$\varepsilon = -p = \frac{1}{2}\varepsilon(a_0). \tag{24}$$

Second, we consider a classical analogue of the solutions. At the initial instant, we define the state of the field with an energy minimum (an analogue of the vacuum state) as $\phi(t_0) = \dot{\phi}(t_0) = 0$. The solution of the classical equations of motion in the proper time τ [10] is then well known. It is the de Sitter metric, and the scalar field remains unexcited:

$$\phi(\tau) = 0, \qquad e(\tau) = 1, \qquad a(\tau) = a(\tau_0)\cosh\kappa_{\mathrm{dS}}(\tau - \tau_0), \qquad \kappa_{\mathrm{dS}}^2 = \frac{2m_\lambda^2}{d(d-1)}.$$

The asymptotic form of this solution as $\tau \to \infty$ is

$$a(\tau) \xrightarrow{\tau\to\infty} e^{\kappa_{\mathrm{dS}}\tau}.$$

In the considered model, we can easily find the asymptotic mean value g_{00}:

$$g_{00}(a) \xrightarrow{a\to\infty} -\frac{d(d-1)}{2}\sqrt{\pi\Omega_d}(m_{\mathrm{P}}a_0)^{(d-1)/2}\frac{m}{m_\lambda}\sqrt{\frac{m}{m_{\mathrm{P}}}}.$$

If we now pass to the proper time in this metric, then the law of growth of the scale factor is also exponential (de Sitter metric), but the exponent is very different:

$$a(\tau) \xrightarrow{\tau\to\infty} e^{\sqrt{\kappa_{\mathrm{dS}}\kappa_{\mathrm{Q}}}\tau}, \qquad \kappa_{\mathrm{Q}}^2 = \frac{4\varepsilon(a_0)}{\pi d(d-1)m_{\mathrm{P}}^{d-1}}, \qquad \varepsilon(a_0) = \frac{m}{2\Omega_d a_0^d}, \tag{25}$$

where $\varepsilon(a_0)$ is the energy density of the zero-point oscillation at the initial instant. Moreover, this growth indicator depends on the initial scale factor a_0, the value of which is not among the initial parameters of the Lagrangian in the classical theory. It is determined by the initial quantum state, which is the vacuum state only at the initial instant (a feature of a nonstationary problem). What then "swings" the scale factor differently in quantum theory? The answer is quite obvious: the zero-point oscillations, which are responsible for matter in the considered model. We also note that the "seed" Λ term (defined by κ_{dS}^2) interacts with the quantum κ_{Q}^2 not by the rule of the arithmetic mean (i.e., addition) but by the rule of the geometric mean (multiplication with taking the square root), which indicates a significant mutual influence of gravity and matter in this model.

Acknowledgments. B. L. Voronov says that if a participant "takes away" one idea from a seminar, then it was a good seminar, and if more, then it was outstanding. In that sense, any conversation with Igor Viktorovich Tyutin is outstanding. That is why I am heartily grateful to Igor Viktorovich for an opportunity to communicate with him (including the discussions about this paper) and to wish him well on his birthday.

The author thanks S. Apenko, A. Semenov, and E. Akhmedov for the discussions and P. Arseev, C. Zybin, A. Marshakov, and the participants in their seminars for the chances to talk and the remarks. Special thanks are due to E. Goncharov for the useful remark and to A. Smirnov for the refining explanation.

REFERENCES

1. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton Univ. Press, Princeton, N. J. (1996).
2. G. G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory* (Interscience Monogr. Texts Phys. Astron., Vol. 26), Interscience, New York (1972).
3. A. A. Slavnov and L. D. Faddeev, *Introduction to the Quantum Theory of Gauge Fields* [in Russian], Nauka, Moscow (1988); English transl.: L. D. Faddeev and A. A. Slavnov *Gauge Fields*: *Introduction to Quantum Theory*, Addison-Wesley, Redwood City, Calif. (1991).
4. P. A. M. Dirac, *Collected Scientific Works* [in Russian], Vol. 4, *Gravitation and Cosmology*: *Memoirs and Thoughts* (*Lectures, Scientific Papers* 1937–1984), Fizmatlit, Moscow (2005).
5. D. M. Gitman and I. V. Tyutin, *Canonical Quantization of Fields with Constraints* [in Russian], Nauka, Moscow (1986); English transl.: *Quantization of Fields with Constraints*, Springer, Berlin (1990).
6. D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, *Phys. Rev. D*, **13**, 3214–3218 (1976).
7. A. Yu. Morozov, *Sov. Phys. Usp.*, **35**, 671–714 (1992).
8. A. V. Marshakov, *Phys. Usp.*, **45**, 915–954 (2002).
9. A. M. Polyakov, *Gauge Fields and Strings* (Contemp. Concepts Phys., Vol. 3), Harwood Academic, Chur (1987).
10. L. Landau and E. Lifshitz, *Course of Theoretical Physics* [in Russian], Vol. 2, *The Classical Theory of Fields*, Nauka, Moscow (1988); English transl. prev. ed., Pergamon, New York (1971).
11. P. A. M. Dirac, *Lectures on the Quantum Mechanics*, Acad. Press, New York (1967).
12. F. A. Berezin and M. A. Shubin, *The Schrödinger Equation* [in Russian], Moscow Univ. Press, Moscow (1983); English transl. (Math. Its Appl. Soviet Ser., Vol. 66), Kluwer Academic, Dordrecht (1991).