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SOME PROPERTIES OF THE DYNAMICS OF COLLAPSE IN MASSIVE AND MASSLESS RELATIVISTIC THEORIES OF GRAVITY

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We investigate the dynamics of collapse in massive and massless relativistic theories of gravity for different equations of state for matter numerically and analytically. This allows clarifying the character of the collapse dynamics in the massive relativistic theory of gravity; in particular, we establish the graviton-mass dependence of the time of reaching the turning point (i.e., the point of transition from contraction to expansion). For the massless relativistic theory of gravity, we clarify the relation between the known general relativity solution for cold dust and the corresponding solution in the relativistic theory of gravity. We show that the harmonic time is singular, including the case of a smooth distribution of matter corresponding to a compact object with a strongly diffused boundary, which means that the Oppenheimer–Snyder solution cannot be fully embedded into the Minkowski space. We in addition investigate the effect of a nonzero pressure on the collapse dynamics.

Keywords: gravity theory, general relativity, relativistic theory of gravity, collapse, graviton mass

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1. Introduction

It was already shown in the 1930s that in the general relativity (GR) framework, the evolution of a dust sphere leads to the formation of a black hole [1], [2]. In the relativistic theory of gravity (RTG) framework [3], [4], the existence of graviton mass changes the picture rather significantly. Of course, in the Newtonian and post-Newtonian domain, a graviton mass cannot significantly affect the nature of static solutions, but in the domain of strong fields, the existence of a nonzero mass becomes crucial. The effect of a graviton mass on static spherically symmetric solutions was generally studied in [5], where these solutions were shown to not have the properties of a black hole. The dynamics of a spherically symmetric configuration for cold dust were analyzed in [6], where it was shown that a ball of dust must pulsate and hence no black hole forms in the collapse of cold dust.

Here, we present a numerical problem of collapse for some specific spherically symmetric initial configurations. The initial configurations are given as follows. In models with a nonzero pressure, a static solution is constructed that satisfies one of the model equations of state (with a power-law dependence of pressure on energy). As a result, the static solution is a smooth distribution of matter, and the density of matter is everywhere nonzero from a formal standpoint. The computation shows that for all the used equations of state, the distribution of matter, first, decreases monotonically as the radius increases and, second, is "compact" in the sense that a characteristic radius can be specified inside which practically all

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matter is contained. The evolution of such a configuration is started by smoothly decreasing the pressure, which to a certain extent models nuclear fuel burning out. Hence, the initial conditions correspond to zero velocities and zero time derivatives of the metric. For models with zero pressure, we use a stationary configuration with a smooth compact distribution of the density of matter, zero velocity of matter, and zero time derivatives of the metric. Of course, the metric coefficients are then determined from the stationarity condition and the equations for the metric.

This problem setting, first, allows clarifying some details of the system dynamics. Second, it allows verifying the intuitively obvious statement that replacing the cold dust model with a more realistic model with a nonzero pressure does not result in qualitative changes. The existence of pressure must only somewhat decelerate (or even stop, if it is sufficiently large) the process of collapse. Third, in the framework of our approach, we can investigate how the dynamics of collapse is affected by the finite size of the transition domain between a nonzero density of matter and a nearly zero density of matter. In most works, "outer" (vacuum) and "inner" solutions are, by necessity, considered and matched on the boundary of the body, and the width of the transition domain is therefore set to zero. But experience from working with second-order differential equations suggests that the results in a model with zero-width boundaries can differ significantly from the results in a model with a transition domain of finite size. It suffices to recall the classic problem in quantum mechanics on the over-barrier reflection at a finite jump of the potential. Fourth, we analyze the massless limit of RTG, i.e., investigate the problem of collapse in massless RTG, and compare these results with the corresponding GR results. Finally, based on the analysis of the obtained numerical data, we obtain exact analytic results pertaining to the time of reaching the turning point in collapse and the behavior of the harmonic time in the collapse of cold dust in GR.

This paper is structured as follows. In Sec. 2, we analyze the massless theory case. We first investigate the properties of the cold-dust solution in the massless RTG, which follows by constructing harmonic coordinates for the Oppenheimer–Snyder solution and thus "embedding" that solution in the Minkowski space. We then investigate the effect of a nonzero pressure of matter on the dynamics of collapse in the massless RTG. In Sec. 3, we investigate the problem of collapse in the massive RTG and establish the dependence of the time of reaching the turning point on the graviton mass.

2. The case of a massless theory

In the case of a massless theory, we can use the known results obtained in GR. We need only establish which sectors of these solutions can be "embedded" in the Minkowski space. Of course, analytic results can be obtained only for cold dust. We analyze the case of a nonzero pressure numerically, but we can note immediately that a nonzero pressure must only decelerate the process of collapse somewhat, without inducing qualitative changes in the general picture. Direct numerical computation confirms this assertion.

We recall that for a noncomoving system in GR (for example, with the gauge chosen such that $g_{22} = -r^2$), the process of collapse requires infinite time. For a body with a boundary r_0 (i.e., in the case where the density of matter does not decrease smoothly as the radial variable r increases and instead vanishes at $r \geq r_0$),[1] this fact is easy to explain: by the Birkhoff theorem, the metric outside the body (for $r > r_0$) coincides with the Schwarzschild metric, and an element of the boundary of the body (the "outermost" grain of dust) therefore moves in the Schwarzschild metric and requires an infinite time to fall on the horizon. Inside the body, the metric coefficients are such that $g_{00}(r) < g_{00}(r_0)$, and both these quantities tend to zero sufficiently rapidly. Incidentally, just this causes the unbounded growth of the zero component T^{00} of the matter energy–momentum tensor also in the case of a finite density of matter $e(t, r)$, as was shown in [4], [7]. Despite the rapid decrease in the metric coefficient $g_{00}(r)$, it reaches zero in infinite time; in other words, the collapse proceeds infinitely long for a remote observer.

[1] This not altogether realistic case is quite useful for qualitative analysis.

We can therefore ask: does the picture of collapse in the massless RTG always coincide with the picture in a noncomoving coordinate system in GR? In other words, is infinite time always required for collapse in the massless RTG? To answer this question, it suffices to analyze the equation that allows passing from the Oppenheimer–Snyder solution to a similar solution in the massless RTG. We restrict ourself to the case where the collapse starts in a stationary configuration, i.e., the velocities are equal to zero at the initial time. We note that another case, the case with a power-law solution, was analyzed in [8] in the framework of a somewhat more conventional approach, where a body with a boundary is introduced and the outer (vacuum) and inner solutions are accordingly considered, being matched at the boundary. In the case considered here, the solution becomes [1], [2]

$$g_{ij} = \mathrm{diag}\bigl(1, -e^{\lambda(\tau,R)}, -r^2(\tau,R), -r^2(\tau,R)\sin^2\theta\bigr) \tag{1}$$

(here and hereafter, we use the notation in [9]), where

$$e^{\lambda(\tau,R)} = \frac{(\partial_R r(\tau,R))^2}{f(R)+1},$$

$$r(\tau,R) = \frac{F(R)}{-2f(r)}(1-\cos\eta), \qquad e(\tau,R) = \frac{\partial_R F(R)}{r^2(\tau,R)\,\partial_R r(\tau,R)},$$

and η is defined by the relation

$$\tau = \frac{F(R)}{2(-f)^{3/2}(\tau,R)}(\pi - \eta + \sin\eta). \tag{2}$$

It varies with time from π (at $\tau = 0$) to zero, and the functions $F(R)$ and $f(R)$ are arbitrary but must be selected such that the density of matter and the metric coefficients are finite at the initial time and have the correct sign and the needed asymptotic behavior at infinity. In specific numerical calculations, we choose them such that the matter distribution at the initial time satisfies the initial conditions described in the introduction, i.e., is a smooth monotonic function corresponding to a conventionally compact object with a fuzzy boundary. With our choice of the origin of time (in the notation in [9], this means choosing the function $\tau_0(R)$), we obtain a "stationary" initial condition: zero time derivatives of the metric coefficients and zero velocity of matter.

To construct a similar solution in the massless RTG, it suffices to solve the equations

$$\partial_\tau^2\tilde t(\tau,R) - \partial_R^2\tilde t(\tau,R)e^{-\lambda(\tau,R)} + \partial_\tau\tilde t(\tau,R)\biggl(\frac{\partial_\tau\lambda(\tau,R)}{2} + \frac{2\partial_\tau r(\tau,R)}{r(\tau,R)}\biggr) + {} + \partial_R\tilde t(\tau,R)\biggl(\frac{\partial_R\lambda(\tau,R)}{2} - \frac{2\partial_R r(\tau,R)}{r(\tau,R)e^{\lambda(\tau,R)}}\biggr) = 0, \tag{3}$$

$$\partial_\tau^2\tilde r(\tau,R) - \frac{\partial_\tau^2\tilde r(\tau,R)}{v(\tau,R)} + \partial_\tau\tilde r(\tau,R)\biggl(\frac{\partial_\tau\lambda(\tau,R)}{2} + \frac{2\partial_t r(\tau,R)}{r(\tau,R)}\biggr) + {} + \partial_R\tilde r(\tau,R)\biggl(\frac{\partial_r\lambda(\tau,R)}{2e^{\lambda(\tau,R)}} - \frac{2\partial_R r(\tau,R)}{r(\tau,R)e^{\lambda(\tau,R)}}\biggr) = -\frac{2\tilde r(\tau,R)}{r^2(\tau,R)} \tag{4}$$

for the new variables $\tilde t$ and $\tilde r$. In these variables, solution (1) is a solution of the RTG equations [3], [4], where the graviton mass μ is set to zero. We must note that in passing to the new coordinates, the diagonal form of the metric is violated, generally speaking. In the harmonic coordinates $\tilde t$, $\tilde r$, $\tilde\theta = \theta$, and $\tilde\varphi = \varphi$, we set the interval in the Minkowski space equal to

$$ds^2 = d\tilde t^{\,2} - d\tilde r^2 - \tilde r^2\,d\tilde\theta^2 - \tilde r^2\sin^2\tilde\theta\,d\tilde\varphi^2,$$

which means that of the metric in the Minkowski space and the effective metric g_{ij} are not diagonalized simultaneously.

We focus on Eq. (3) for the new time variable. If it turns out that time tends to infinity at the instant of collapse, then the RTG collapse requires an infinite time. In [10], [11], this question was already investigated via numerical analysis, but numerical proof of a singularity of one quantity or another can raise doubts. Moreover, in what follows, we also need the asymptotic form of the function $\tilde{t}(\tau, R)$ as $R \to 0$ at later stage of collapse, and we therefore obtain the corresponding analytic expression applicable with smooth functions such as $f(R)$ and $F(R)$, which ensure a finite energy density and finite metric coefficients at the initial time.

Directly solving Eq. (3) numerically for a rather broad choice of $f(R)$ and $F(R)$ shows that an unbounded growth of the time $\tilde{t}$ starts in a neighborhood of $R = 0$. Moreover, $\tilde{t}(\tau, R)$ and $\partial_\tau \tilde{t}(\tau, R)$ as functions of R decrease monotonically on the interval $[0, \infty)$. We therefore restrict ourself to considering a neighborhood of the point $R = 0$ and expand all quantities in solution (1) in a series in powers of R.

The finiteness requirement for the energy density and metric coefficients at the initial time lead to the expansions

$$f(R) = a_2 R^2 + a_3 R^3 + \ldots, \qquad F(R) = b_3 R^3 + b_4 R^4 + \ldots.$$

In this case, the energy density in a neighborhood of $R = 0$ becomes

$$e(\tau, R) = \frac{24 a_2^3}{b_3^3 (1 - \cos\eta)^3} R^0 + O(R^1).$$

For the parameter η, we use the expansion

$$\eta = c_0(\tau) R^0 + c_1(\tau) R^1 + \ldots.$$

Equation (2) allows finding the coefficients $c_i(\tau)$, in particular,

$$c_0(\tau) = \left[6 \left(\pi - \frac{2\tau a_2^{3/2}}{b_3} \right) \right]^{1/3}.$$

Finally, for the time $\tilde{t}$, we assume that

$$\tilde{t}(\tau, R) = d_0(\tau) r^0 + d_1(\tau) R^1 + \ldots.$$

Substituting all these series in (3) shows that $d_1(\tau) = 0$. Collecting the R^0 terms in (3), we next obtain

$$\partial_\tau^2 d_0(\tau) = \frac{4 a_2^{3/2} / b_3}{\pi - 2\tau a_2^{3/2} / b_3} \, \partial_\tau d_0(\tau).$$

Collapse corresponds to the denominator in the right-hand side of this equation vanishing. Hence, the time $\tilde{t}$ indeed grows without a bound under collapse, and this growth is in fact described by the asymptotic formula

$$d_0(\tau) \approx \mathrm{const} \cdot \left(\pi - \frac{2\tau k_2^{3/2}}{l_3} \right)^{-1}. \tag{5}$$

It is easy to verify that taking higher powers in the expansions in R into account does not change the obtained results.

Therefore, in the RTG with a zero graviton mass, the evolution of a fuzzy-boundary body consisting of cold dust (in the spherically symmetric case) is to some extent similar to the evolution of that body in

GR from the standpoint of a remote observer. The harmonic time grows without a bound under collapse, and the process of collapse hence requires infinite time.

As regards a more realistic case with a nonzero pressure, we must limit ourself to numerical analysis. As the initial condition, we choose a static configuration corresponding to a given equation of state of matter. We restrict ourself to power-law dependences

$$p(r) = Ce^{\alpha}(r) \tag{6}$$

with different C and α. Because the static configuration interests us not for itself but as the initial condition for a dynamic problem, it is convenient to parameterize the metric coefficients as

$$\begin{aligned} g_{00}(r) = u(r) = \frac{v(r)K^4(r)}{L^4(r)}, \qquad -g_{22}(r) = w(r) = K^2(r), \\ -g_{11}(r) = v(r) = \frac{2K(r)K'(r)}{r} - \frac{K^2(r)L'(r)}{rL(r)} \end{aligned} \tag{7}$$

(the equations $D_i\tilde{g}^{ij} = 0$ are then satisfied automatically). For the functions $K(r)$, $L(r)$, and $p(r)$, we then obtain the equations (omitting the argument r of all functions for brevity)

$$\begin{aligned} K'' &= \frac{1}{2KLr}\bigg[(K^4L' - 2K^3K'L)(e - \mu^2 - p) + 2K^2L'\left(\frac{\mu^2r^2}{2} - 1\right) + \\ &\quad + 4KK'\left(-L\frac{\mu^2r^2}{2} + L + L'r\right) - 6K'^2Lr\bigg], \\ L'' &= \frac{1}{2K^4K'Lr}\bigg[2K^6K'LL'(e + \mu^2 + 3p) - 2K^7(L')^2\left(\frac{\mu^2}{2} + p\right) + \\ &\quad + K^5\left(-4e(K')^2L^2 - 4(K')^2L^2p + LL'\frac{\mu^2r}{2} + 2(L')^2\left(\frac{\mu^2r^2}{2} - 1\right)\right) - \\ &\quad - 2K^4K'\left(L^2\frac{\mu^2r}{2} + LL'(\mu^2r^2 - 3) - 5(L')^2r\right) - \\ &\quad - 2K^3(K')^2L(2L + 11L'r) + 12K^2(K')^3L^2r - KL^5L'\frac{\mu^2}{2r} + 2K'L^6\frac{\mu^2}{2r}\bigg], \\ p' &= \frac{1}{4K^3K'Lr}\bigg[(e + p)\bigg((2K^6L' - 4K^5K'L)\left(\frac{\mu^2}{2} + p\right) - K^4L\frac{\mu^2r}{2} + \\ &\quad + 2K^3(2K'L - KL')\left(\frac{\mu^2r^2}{2} - 1\right) + 2K^2(K')^2Lr + L^5\frac{\mu^2r}{2}\bigg)\bigg], \end{aligned} \tag{8}$$

and the graviton mass μ must be set to zero in this case. This system of equations is to be solved with the boundary conditions $K(0) = 0$, $L(0) = 0$, $p(\infty) = 0$, $K'(\infty) = 1$, and $L'(\infty) = 1$. Of course, the role of infinity in a numerical solution is played by a sufficiently large r, and exact boundary conditions are replaced with expansions in powers of $1/r$.

It is of some interest that if we do not especially select the solution mass m and the constants C and α, then the solution in the domain where the matter density can be considered vanishing, generally speaking, does not coincide with the simplest form of the static solution in the massless RTG in the vacuum

$$u = \frac{\tilde{r} - m}{\tilde{r} + m}, \qquad v = \frac{\tilde{r} + m}{\tilde{r} - m}, \qquad w = (\tilde{r} + m)^2.$$

This form in turn corresponds to the simplest solution of Eq. (4) in the static case in the vacuum: $\tilde{r}_1(r) = r - m$. In the numerical solutions that we obtained, the function $u(r)$ in the domain of large r behaves such that the coefficient at $1/r^3$, as a rule, does not coincide with $-2m^3$, which means that the coefficient of the second (logarithmic) solution of Eq. (4),

$$\tilde{r}_2(r) = 2m + (r - m)\log\left(1 - \frac{2m}{r}\right),$$

is nonzero. This was already noted in [12], [13], where it was noted that in considering the harmonicity conditions [14], matching with the inner solution does not allow setting the coefficient of the logarithmic solution of Eq. (4) to zero. This result is in fact obvious because as $r \to 0$, the solution is determined by three arbitrary constants ($K(0) = L(0) = 0$), and as $r \to \infty$, we therefore need both arbitrary constants: the coefficient of $1/r$, which is the corresponding mass of the solution, and the coefficient of $1/r^3$, which determines the coefficient of the logarithmic term.

We realize the time variation of the static solution thus constructed by modifying the matter equation of state (6)—by decreasing the constant C (which, as it were, models nuclear fuel burning out). For the purposes of computation, it is more convenient to vary C not in jumps but smoothly,

$$C(t) = C_1 + (C_0 - C_1)e^{-\beta t^2},$$

where β is a constant, C_0 is a constant value corresponding to the constructed static solution, and C_1 is the final value of the constant (the case $C_1 = 0$ corresponds to cold dust).

In the dynamical problem, we parameterize the metric coefficient exactly the same as in the static problem: by formulas (7). We recall that we use the static solution as the initial condition for the dynamical problem, and hence

$$u(0,r) \equiv u(r), \quad v(0,r) \equiv v(r), \quad w(0,r) \equiv w(r), \quad K(0,r) \equiv K(r), \quad L(0,r) \equiv L(r).$$

The system dynamics are determined by the dynamics of the metric and of matter. The reason we chose parameterization of (7) by the functions $K(t,r)$ and $L(t,r)$ is quite simple: the dynamics of the metric then reduce to an equation for $K(t,r)$, while the equation $D_i\tilde{g}_{i0} = 0$ gives

$$\partial_0 \frac{w^2(t,r)v(t,r)}{u(t,r)} = \partial_0 L^4(t,r) = 0,$$

and $L(t,r)$ is hence independent of time and coincides with its initial value, i.e., with $L(r)$ taken from the static solution: $L(t,r) = L(0,r) \equiv L(r)$.

For the function $K(t,r)$, we use the RTG equations [3], [4] or, more precisely, the equation for R_{11} to obtain the equation

$$\begin{aligned}\ddot{K} = \frac{1}{4KL^5r(KL' - 2K'L)}\Bigg[&-K^7\left(LL'r\left(2es^2+\frac{\mu^2}{2}\right)-4(K')^2L^2\mu^2 + 2(L')^2\left(\frac{\mu^2}{2r^2} - 1\right)\right)+ \\ &+ 2K^6K'\left(L^2r\left(2e\eta^2 + \frac{\mu^2}{2}\right) + L\left(L'\left(\frac{2\mu^2}{r^2} - 5\right) + L''r\right) - 5(L')^2r\right)+ \\ &+ 2K^9(L')^2\frac{\mu^2}{2} - 8K^8K'LL'\frac{\mu^2}{2} + 2K^5K'L\big(15K'L'r - 2L\big(K'(\mu^2r^2 - 3) + K''r\big)\big)- \\ &- 24K^4K'^3L^2r + K^3L^5L'\frac{\mu^2r}{2} - 2K^2K'L^6\frac{\mu^2r}{2} + \\ &+ 2K\dot{K}L^5r(5\dot{K}L' - 2\dot{K}'L) - 16\dot{K}^2K'L^6r\Bigg]\end{aligned} \tag{9}$$

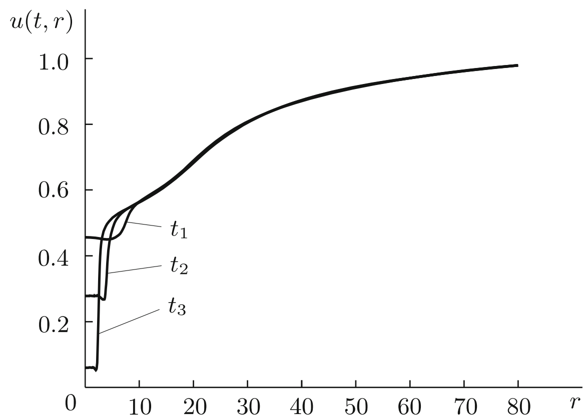

Fig. 1. Evolution of the metric coefficient $u(t,r)$ at sufficiently large times $t_3 > t_2 > t_1$.

where, as previously, we omit the arguments t and r of all functions for brevity, differentiations with respect to time and the radius are respectively denoted by a dot and a prime, and the graviton mass μ must be set to zero. As previously, the matter density is here denoted by $e(t,r)$, and the first component of its velocity dx^1/ds is denoted by $\eta(t,r)$. The matter dynamics can be obtained from the conservation law, i.e., from the relation $\nabla_i T^{ij} = 0$. Both nontrivial expressions (with $j = 0$ and $j = 1$) have the structure

$$A_1\,\partial_0 e(t,r) + A_2\,\partial_0 \eta(t,r) + A_3 = 0$$

(the expressions for A_i are too cumbersome to be given here). This allows finding $\partial_0 e(t,r)$ and $\partial_0 \eta(t,r)$ separately. We obtain the initial conditions for $e(0,r)$ from the static solution: $e(0,r) = e(r)$ and $\eta(0,r) = 0$.

The numerical results confirm the original conjecture: the collapse of cold dust (the case $C_1 = 0$) proceeds much faster than the collapse of matter with the residual pressure proportional to C_1 (of course, if C_1 is sufficiently small for the collapse to occur at all for a given mass of the solution), and the overall behavior of metric coefficients coincides with the behavior of the metric coefficients for the Oppenheimer–Snyder solution in the reference frame of a remote observer.

In Fig. 1, we give a characteristic picture of the evolution of the metric coefficient $g_{00} = u(t,r)$ at sufficiently large times. In our case, the notion of the "boundary of the body" is inapplicable (matter is distributed continuously), but a conventional boundary domain where matter is concentrated roughly coincides with the domain of sharp growth of the function $u(t,r)$. We note that in accordance with the Birkhoff theorem (and its RTG counterpart [15], [16]) in the domain that can be conventionally called the vacuum (with a sufficiently low density of matter), the metric does not evolve. We can see from Fig. 1 that the velocity of motion of the "boundary of the body" decreases, while the decrease in $u(t,0)$ does not decelerate. These two observations completely describe the subsequent evolution of the solution: the boundary stops, and $u(t,0)$ continues to decrease, which fully agrees with the above analysis of the behavior of the time $\tilde{t}(\tau,R)$ for the Oppenheimer–Snyder solution.

3. The case of a nonvanishing graviton mass

In the case of the massive RTG, we do not have any explicit analytic solution, and we must therefore turn to direct numerical computation for the preliminary analysis of the system dynamics. In the preceding section, we already gave all the necessary relations for seeking a static solution of (8) and for posing dynamical problem (9) in the case of a nonzero graviton mass μ.

Unfortunately, it was necessary to compute with totally unrealistic graviton masses. The known estimate of the graviton rest mass is $m_{\mathrm{g}} < 3 \cdot 10^{-66}\,\mathrm{g}$ (whence $\mu < 8 \cdot 10^{-29}\,\mathrm{cm}^{-1}$). Therefore, the solution

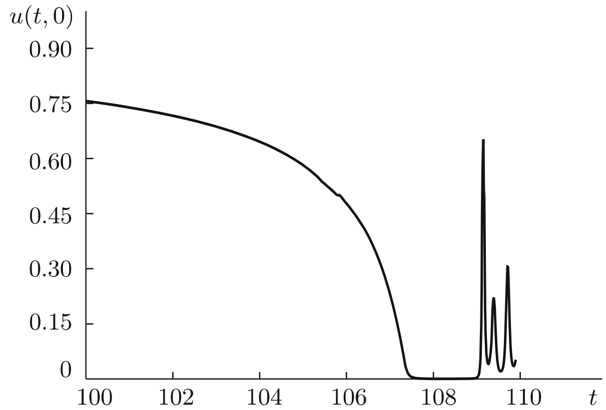

Fig. 2. Dependence of the metric coefficient $u(t, 0)$ at the origin on time before and after the turning point $t_0 = 108.8$.

mass must differ from the graviton rest mass by at least 100 orders of magnitude. This leads to technical complications in seeking a static solution: to investigate the effect of mass on the solution, the computation must be done with at least sixty-digit arithmetic, which dramatically slows the computation. This leads to principal difficulties in investigating the dynamics of collapse. As is shown below, the effect of a nonzero mass on the dynamics of collapse is manifested only after a time proportional to $1/\mu$, and numerical analysis over a time span proportional to 10^{30} is impossible in principle. We therefore investigate a model problem where the graviton mass is less than the solution mass by only a few orders of magnitude.

For static configurations, a nonzero graviton mass leads to some increase in $u(0)$ (the larger μ is, the closer $u(0)$ is to unity). As regards the dynamics of collapse, the picture at the early stage is the same as in the massless theory, and at sufficiently large times, the value of $u(t, 0)$ reaches the turning point and starts increasing. The smaller μ is, the later the turning point in time occurs. The subsequent evolution of the solution is given by oscillations of $u(t, 0)$ (see Fig. 2), attended by not very regular oscillations (waves) in the distribution of matter.

We emphasize that the oscillations of $u(t, 0)$ are not harmonic, just as the matter density oscillations are not given by uniform compression–rarefaction cycles (see Fig. 3). The typical initial radius of the domain occupied by matter is not reached in the course of these oscillations; in other words, in the course of oscillations, the matter density at large distances from the center turns out to be much smaller than its initial value.

The obtained numerical results are, of course, only an indication about the nature of the solution dynamics, and a direct analytic proof must be given that for $\mu \neq 0$, the dynamics of $u(t, 0)$ contains a turning point, while this turning point is absent at $\mu = 0$. As a by-product, we establish how the time of reaching the turning point depends on the graviton mass. Because we are interested in a neighborhood of the point $r = 0$, we again use expansions in power series in r.

For the functions $K(t, r)$ and $L(r)$, taking the boundary conditions at the origin into account, we write

$$K(t, r) = K_1(t)r + K_2(t)r^2 + K_3(t)r^3 + \ldots, \qquad L(r) = L_1 r + L_2 r^2 + L_3 r^3 + \ldots.$$

Substituting these expansions in (8) and (9) gives $K_2(t) = 0$ and $L_2 = 0$. Similarly, we introduce the expansions

$$e(t, r) = e_0(t)r^0 + e_1(t)r^1 + e_2(t)r^2 + \ldots, \qquad \eta(r) = \eta_1(t)r + \eta_2(t)r^2 + \eta_3(t)r^3 + \ldots.$$

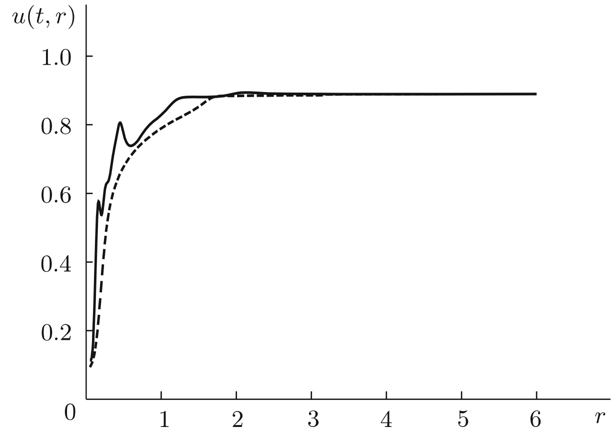

Fig. 3. Metric coefficients $u(t_1, r)$ and $u(t_2, r)$ as functions of r for a time $t_1 < t_0$ (dashed line) and for a time $t_2 > t_0$ (solid line): t_0 is the turning point.

Substituting all these expansions in (9) and segregating the coefficient of r^0, we obtain the equation (as usual, we omit the argument t of $K_1(t)$ and $K_3(t)$ for brevity)

$$\ddot{K}_1 = -\frac{K_1^7\mu^2}{4L_1^4} - \frac{5K_1^5L_3}{L_1^5} + \frac{K_1^5\mu^2}{8L_1^4} + \frac{10K_1^4K_3}{L_1^4} + \frac{5\dot{K}_1^2}{2K_1} + \frac{K_1\mu^2}{8}.$$

The collapse corresponds to the function $K_1(t)$ tending to zero, and we can therefore keep only the terms linear in $K_1(t)$ in this expression, neglecting the terms with $K_1^4(t)$, $K_1^5(t)$, and $K_1^7(t)$. We then obtain

$$\ddot{K}_1 = \frac{5\dot{K}_1^2}{2K_1} + \frac{K_1\mu^2}{8}. \tag{10}$$

Interestingly, the solution of this equation, which is applicable only in a small neighborhood of the origin and at a late stage of collapse, coincides with the solution of the equation occurring in a cosmological RTG model [17], obtained under the assumption that the pressure is zero and the maximum value of the scale factor $R_{\max}$ of the universe is sufficiently large.

A kind of "homogeneous isotropic universe" thus emerges in a small neighborhood of the origin at a late stage of collapse. In reality, to estimate the time of reaching the turning point, it suffices to note that the equation for the scaling factor of the universe $R(\tau)$ in a neighborhood of the lower turning point $R_{\min}$, obtained in [18], coincides with (10) if it is also considered only in a neighborhood of the turning point.

For $\mu \neq 0$, Eq. (10) is solved by

$$K_1(t) = \text{const} \cdot \left(\cos\frac{\sqrt{3}\mu(t-t_0)}{4}\right)^{-2/3}, \tag{11}$$

and for $\mu = 0$, we have the solution

$$K_1(t) = \text{const} \cdot (t-t_0)^{-2/3}. \tag{12}$$

The answer in the case of zero mass, of course, coincides exactly with the expression that can be easily obtained from asymptotic form (5) for the time $\tilde{t}(\tau, R)$ in the preceding section.

It follows from Eq. (10) that for $\mu t \ll 1$, we can neglect the second term in the right-hand side of (10), and the system evolution is described by relation (12). As times of the order of $1/\mu$ are reached, the mass term starts playing a significant role, the turning point is reached, and the function $K_1(t)$ starts increasing, with its behavior in the neighborhood of the turning point (minimum) described by relation (11). The numerical simulation results showed that $K_1(t)$ continues to grow after the turning point and linear approximation (10) loses applicability after some time (see Fig. 2). During growth, $K_1(t)$ reaches values comparable to the initial value $K_1(0)$ and then starts decreasing again.

We have thus shown that for any arbitrarily small graviton mass, unlike in the massless theory, collapse terminates, and the characteristic time from the collapse start to the turning point is proportional to $1/\mu$.

4. Conclusions

The results of numerically analyzing the system dynamics presented here, on one hand, confirm the intuitively obvious assumption that the existence of a nonzero pressure (for a rather broad class of model equations of state of matter) decelerates the process of collapse but does not lead to qualitative changes in the system dynamics. Nevertheless, we note that we worked in the sector of relatively small pressures such that neither the collapse stops nor critical phenomena [19] occur. On the other hand, our computations confirm the status of the exact analytic solution of the GR equations for collapsing dust in the framework of the massless RTG. The solution of the massless RTG equations, as expected, exactly coincides with the GR solution from the standpoint of a remote observer (or, equivalently, in the "noncomoving" coordinate system). Specifically, the quantity g_{00} decreases (exponentially fast) without bound in a domain where the density of matter is relatively large (this leads to an unbounded growth of T^{00} at a finite density $e(t,r)$ of matter, as was shown previously [4], [7]), but reaches zero only in infinite time.

We proved that the harmonic time is singular for any Oppenheimer–Snyder solution with a smooth compact distribution of matter, and collapse in the Minkowski space hence requires infinite time. Finally, we investigated the dynamics of collapse in the massive RTG numerically and analytically. In analyzing the system dynamics in a neighborhood of the turning point, we showed that the time of reaching the turning point is of the order of $1/\mu$, where μ is the graviton mass. With the available estimates of the graviton rest mass $m_{\mathrm{g}} < 3 \cdot 10^{-66}\,\mathrm{g}$, this time is proportional to $4 \cdot 10^{17}\,\mathrm{s} = 12 \cdot 10^{9}\,\mathrm{yr}$, i.e., comparable to the age of the universe. As already noted above, the dynamics of collapse in a neighborhood of the turning point coincides with the dynamics of a Friedmann universe [17], [18], and this coincidence is hence unsurprising.

In our numerical analysis of the dynamics of collapse, we had to consider a quite unrealistic model where the solution mass and the graviton mass differ not by the desired 60 orders of magnitude but by only a few (3 to 5) orders. Yet the results fully confirm the conclusion in [6]; we only note that the pulsations of a dust ball are irregular and the initial size of the ball is not restored in the course of pulsations.

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