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SOLUTION OF THE PROBLEM OF CHARGE MOTION IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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Using the method of first integrals, we find an exact solution for the relativistic motion of a charge in orthogonal and uniform electric and magnetic fields with respect to laboratory time and for any value of the dimensionless governing parameter equal to the ratio of the magnetic field strength to the electric field strength.

Keywords: equation of motion of a charged particle in relativistic mechanics, method of first integrals, uniform magnetic field, uniform electric field

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1. Introduction

As is known, during the acceleration of electrons, their radiative losses increase with their energy increase. For obtaining additional information about this effect, the picture of the behavior of charged particles in crossed uniform electric and magnetic fields under variation of the dimensionless governing parameter a, equal to the ratio of the magnetic field strength to the electric field strength, $a = |H|/|E|$, $0 < a < \infty$, can be interesting. The parameter a is called the governing parameter because its value should be changed in a certain way when building accelerators to decrease radiative losses and increase the stability of the charged particle beam.

A complete exact solution of the problem of electron motion in crossed uniform electric and magnetic fields with respect to the laboratory time is unknown to us. Solving this problem reduces to integrating an autonomous system of ordinary differential equations, and the solution has been found only for the governing parameter value unity in the nonrelativistic limit [1]. The laboratory time is a physical time because it is defined uniquely by the problem conditions up to translations and allows determining the velocity, momentum, and energy physically. There are many publications (see, e.g. [2]–[7] and the references therein) dedicated to a different problem statement in which a different evolution parameter, called proper time, is considered instead of a physical time.

Here, we present a complete exact solution of the problem posed in [1] of the motion of a charged particle in crossed uniform electric and magnetic fields with respect to physical time. Consequently, any informal motivation for using the proper time method disappears. In all subsequent editions of [1], there is no mention of the proper time method (although its definition is there), while the strategy for the complete solution in the first edition was not subsequently realized. The absence of references is easily understood because the proper time is related to a single particle and the trajectory of its motion. But the publications [2]–[7] indicated above are now undoubtedly interesting for explicating the physical meaning

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of proper time in a completely concrete physical situation. Because the exact solution that we give of a long-posed interesting physical problem is obtained by the method of first integrals with respect to physical time, it can be used to study the behavior of a charged particle beam not only in accelerators but also in plasma physics and also to study the energy spectrum of ultrahigh-energy cosmic rays.

2. Equations of motion and first integrals

We consider the motion of a charge e in uniform constant electric and magnetic fields. We choose the direction of the vector $\mathbf{E}$ along the y axis and the direction of the vector $\mathbf{H}$ along the z axis: $\mathbf{E} = (0, E, 0)$ and $\mathbf{H} = (0, 0, H)$. Because $(\mathbf{EH}) = 0$, the fields are crossed. We introduce the dimensionless governing parameter $a = H/E$, the characteristic time $T = mc/eE$, the corresponding frequency $\nu = 1/T$, and the dimensionless velocity components

$$\beta_x = \frac{v_x}{c}, \qquad \beta_y = \frac{v_y}{c}, \qquad \beta_z = \frac{v_z}{c}, \qquad \beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 < 1.$$

The relativistic equation of motion [1] for the charge e with the mass m

$$\frac{d\mathbf{v}}{dt} = \frac{e}{m}\sqrt{1 - \frac{v^2}{c^2}}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vH}] - \frac{1}{c^2}\mathbf{v}(\mathbf{vE})\right\}$$

can then be represented componentwise as a system of equations for β_x, β_y, and β_z:

$$\begin{aligned}
\frac{d\beta_x}{dt} &= \nu\sqrt{1-\beta^2}(a\beta_y - \beta_x\beta_y), \\
\frac{d\beta_y}{dt} &= \nu\sqrt{1-\beta^2}(1 - a\beta_x - \beta_y^2), \\
\frac{d\beta_z}{dt} &= \nu\sqrt{1-\beta^2}(-\beta_y\beta_z).
\end{aligned} \tag{1}$$

The first integrals of this system allow finding its solutions. To obtain them, we represent (1) as the system of relations

$$\frac{d\beta_x}{a\beta_y - \beta_x\beta_y} = \frac{d\beta_y}{1 - a\beta_x - \beta_y^2} = \frac{d\beta_z}{-\beta_y\beta_z} = \nu\sqrt{1-\beta^2}\,dt.$$

Multiplying the first equality by $2\beta_y$, we obtain

$$\frac{2\,d\beta_x}{a - \beta_x} = \frac{d(\beta_y^2)}{1 - a\beta_x - \beta_y^2}.$$

We multiply the numerator and the denominator of the first fraction in the obtained equality by β_x, and by the property of equal fractions

$$\frac{a}{b} = \frac{c}{d} = \frac{c + \mu a}{d + \mu b},$$

which holds for any $\mu \neq 0$, we obtain (choosing $\mu = 1$)

$$\frac{2\,d\beta_x}{a - \beta_x} = \frac{d(\beta_x^2 + \beta_y^2)}{1 - \beta_x^2 - \beta_y^2}.$$

Further, using the equalities

$$d(\beta_x^2 + \beta_y^2) = -d(1 - \beta_x^2 - \beta_y^2), \qquad \frac{2\,d\beta_x}{a - \beta_x} = -\frac{d(\beta_x - a)^2}{(\beta_x - a)^2},$$

we obtain the easily integrated equation

$$\frac{d(\beta_x - a)^2}{(\beta_x - a)^2} = \frac{d(1 - \beta_x^2 - \beta_y^2)}{1 - \beta_x^2 - \beta_y^2}.$$

The equation

$$\frac{d\beta_x}{a\beta_y - \beta_x\beta_y} = \frac{d\beta_z}{-\beta_y\beta_z}$$

can also be easily integrated.

Therefore, first integrals of system (1) are the functions

$$\frac{1 - \beta_x^2 - \beta_y^2}{(\beta_x - a)^2}, \qquad \frac{\beta_z}{\beta_x - a},$$

and we hence equate them to the constants of motion A^2 and B and obtain

$$1 - \beta_x^2 - \beta_y^2 = A^2(\beta_x - a)^2, \tag{2}$$

$$\beta_z = B(\beta_x - a). \tag{3}$$

We write Eq. (2) in the form of an equation for an ellipse

$$\frac{\big(\beta_x - aA^2/(1 + A^2)\big)^2}{F^2} + \frac{\beta_y^2}{G^2} = 1,$$

where

$$F = \frac{ap}{1 + A^2}, \qquad G = \frac{ap}{\sqrt{1 + A^2}}, \qquad p = \frac{\sqrt{1 + (1 - a^2)A^2}}{a}. \tag{4}$$

This implies that Eq. (2) admits the parametric representation defined by basic trigonometric functions

$$\beta_x - \frac{aA^2}{1 + A^2} = F\sin\varphi, \qquad \beta_y = G\cos\varphi, \tag{5}$$

and in accordance with Eq. (3), β_z can also be expressed in terms of $\sin\varphi$.

The problem of integrating Eqs. (1) thus reduces to integrating the equation satisfied by φ. We find this equation. Equalities (2) and (3) imply that

$$\sqrt{1 - \beta^2} = \sqrt{A^2 - B^2}\,|\beta_x - a|. \tag{6}$$

Further, from Eqs. (5), we find that

$$\frac{d\beta_x}{dt} = F\cos\varphi\frac{d\varphi}{dt} = \frac{F}{G}\beta_y\frac{d\varphi}{dt}.$$

Substituting this equality in the first equation in system (1), after eliminating β_y and taking relations (6) into account, we obtain

$$\frac{d\varphi}{dt} = \varepsilon\nu\sqrt{1 - \gamma^2}(1 + A^2)(a - \beta_x)^2,$$

where $\sqrt{A^2 - B^2} = \sqrt{1 + A^2}\sqrt{1 - \gamma^2}$ and $\varepsilon = 1$ if $a - \beta_x > 0$ and $\varepsilon = -1$ if $a - \beta_x < 0$. Because

$$a - \beta_x = \frac{a}{1 + A^2}(1 - p\sin\varphi)$$

in accordance with (5), the sought equation is written as

$$\frac{d\varphi}{dt} = \frac{\varepsilon\nu a^2\sqrt{1-\gamma^2}}{1+A^2}(1-p\sin\varphi)^2. \tag{7}$$

Using Eq. (7), we find the equations for x, y, and z as functions of φ. Because

$$\beta_x = \frac{1}{c}\frac{dx}{dt} = \frac{1}{c}\frac{dx}{d\varphi}\frac{d\varphi}{dt} = \frac{a}{1+A^2}(A^2+1+p\sin\varphi-1)$$

in accordance with relations (4) and (5), we have the equations for x and analogously for y and z:

$$\frac{dx}{d\varphi} = \frac{\varepsilon c}{\nu a\sqrt{1-\gamma^2}}\left(\frac{1+A^2}{(1-p\sin\varphi)^2} - \frac{1}{1-p\sin\varphi}\right), \tag{8}$$

$$\frac{dy}{d\varphi} = \frac{\varepsilon c\sqrt{1+A^2}}{\nu a\sqrt{1-\gamma^2}}\frac{p\cos\varphi}{(1-p\sin\varphi)^2}, \tag{9}$$

$$\frac{dz}{d\varphi} = \frac{\varepsilon cB}{\nu a\sqrt{1-\gamma^2}}\frac{1}{1-p\sin\varphi}. \tag{10}$$

The problem of integrating these equations, which determine the trajectory of the motion, thus reduces to finding the integrals

$$I_1(p,\varphi) = \int\frac{d\varphi}{1-p\sin\varphi}, \qquad I_2(p,\varphi) = \int\frac{d\varphi}{(1-p\sin\varphi)^2},$$

which, as is easily seen, are algebraically dependent,

$$I_1(p,\varphi) + (p^2-1)I_2(p,\varphi) = \frac{p\cos\varphi}{1-p\sin\varphi}. \tag{11}$$

With $p=1$ (with $a=1$), this formula is inapplicable, we must use the equations

$$I_1(1,\varphi) = \frac{1+\sin\varphi}{\cos\varphi}, \qquad I_2(1,\varphi) = \frac{2}{3}\frac{1}{\cos\varphi}\frac{1}{1-\sin\varphi} + \frac{1}{3}\tan\varphi. \tag{12}$$

We present the formulas for the first integral for other values of a. By definition

$$1-p^2 = \frac{(a^2-1)(1+A^2)}{a^2},$$

and we therefore distinguish two cases. If $a>1$, then $1-p^2>0$, and we obtain

$$I_1(p,\varphi) = \frac{2}{\sqrt{1-p^2}}\arctan\frac{\sin\varphi - p\cos\varphi - p}{\sqrt{1-p^2}(1+\cos\varphi)}. \tag{13}$$

If $a<1$, then $p^2-1>0$, and we obtain

$$I_1(p,\varphi) = \frac{1}{\sqrt{p^2-1}}\log\left(1 - \frac{2\sqrt{p^2-1}(1+\cos\varphi)}{\sin\varphi + (\sqrt{p^2-1}-p)(1+\cos\varphi)}\right). \tag{14}$$

3. Motion trajectories

Different values of the governing parameter a imply different trajectories. In accordance with this, we consider three cases.

Let the governing parameter be equal to unity. From relations (7)–(10) and (12), we then obtain

$$\frac{2}{3}\frac{1}{\cos\varphi}\frac{1}{1-\sin\varphi}+\frac{1}{3}\tan\varphi=\frac{\nu\sqrt{1-\gamma^2}}{1+A^2}t+C_0 \tag{15}$$

and

$$x=\frac{c}{\nu\sqrt{1-\gamma^2}}\left((1+A^2)\left(\frac{2}{3}\frac{1}{\cos\varphi}\frac{1}{1-\sin\varphi}+\frac{1}{3}\tan\varphi\right)-\frac{1+\sin\varphi}{\cos\varphi}\right)+C_1, \tag{16}$$

$$y=\frac{c\sqrt{1+A^2}}{\nu\sqrt{1-\gamma^2}}\frac{1}{1-\sin\varphi}+C_2, \tag{17}$$

$$z=-\frac{Bc}{\nu\sqrt{1-\gamma^2}}\frac{1+\sin\varphi}{\cos\varphi}+C_3, \tag{18}$$

where C_0, C_1, C_2, and C_3 are integration constants. It follows from (15) and (16) that

$$x=ct-\frac{c}{\nu\sqrt{1-\gamma^2}}\frac{1+\sin\varphi}{\cos\varphi}+C.$$

As already noted above, the case just now considered for $a=1$ was also investigated in [1]. The solution presented there is related to the parameter p_y. Because

$$p_y=\frac{mc\beta_y}{\sqrt{1-\beta^2}},$$

the connection between the two parameterizations is established using formulas (5) and (6). We present the result:

$$p_y=\frac{mc}{\sqrt{1-\gamma^2}}\frac{1+\sin\varphi}{\cos\varphi}.$$

If the governing parameter is greater than unity, then $\varepsilon=1$, and in accordance with (7), we have

$$I_2(p,\varphi)=\frac{\nu a^2\sqrt{1-\gamma^2}}{1+A^2}t,$$

which in accordance with (11) and (13) yields

$$\frac{p\cos\varphi}{1-p\sin\varphi}-\frac{2}{\sqrt{1-p^2}}\arctan\frac{\sin\varphi-p\cos\varphi-p}{\sqrt{1-p^2}(1+\cos\varphi)}=\varepsilon\nu(1-a^2)\sqrt{1-\gamma^2}\,t+C_0. \tag{19}$$

Similarly, it follows from (8), (11), and (13) that

$$x=\frac{c}{\nu a(1-a^2)\sqrt{1-\gamma^2}}\left\{\frac{a^2p\cos\varphi}{1-p\sin\varphi}-\frac{2}{\sqrt{1-p^2}}\arctan\frac{\sin\varphi-p\cos\varphi-p}{\sqrt{1-p^2}(1+\cos\varphi)}\right\}+C_1.$$

Comparing this equality with (19), we obtain the relation

$$x=act-\frac{c}{\nu a\sqrt{1-\gamma^2}}\frac{2}{\sqrt{1-p^2}}\arctan\frac{\sin\varphi-p\cos\varphi-p}{\sqrt{1-p^2}(1+\cos\varphi)}. \tag{20}$$

For the coordinates y and z, we obtain

$$y = \frac{c\sqrt{1+A^2}}{a\nu\sqrt{1-\gamma^2}}\frac{1}{1-p\sin\varphi} + C_2, \tag{21}$$

$$z = -\frac{Bc}{\nu a\sqrt{1-\gamma^2}}\left\{\frac{2}{\sqrt{1-p^2}}\arctan\frac{\sin\varphi - p\cos\varphi - p}{\sqrt{1-p^2}(1+\cos\varphi)}\right\} + C_3. \tag{22}$$

Finally, we consider the case where the governing parameter is less than unity and $a-\beta_x$ is consequently not sign-definite. Because all necessary explanations have previously been made, we present only the additional necessary relations: we have

$$\frac{p\cos\varphi}{1-p\sin\varphi} - \frac{1}{\sqrt{p^2-1}}\log\left(1 - \frac{2\sqrt{p^2-1}(1+\cos\varphi)}{\sin\varphi + (\sqrt{p^2-1}-p)(1+\cos\varphi)}\right) = \varepsilon\nu(1-a^2)\sqrt{1-\gamma^2}\,t + C_0,$$

whence we obtain

$$x = \frac{\varepsilon c}{\nu a(1-a^2)\sqrt{1-\gamma^2}}\left\{\frac{a^2 p\cos\varphi}{1-p\sin\varphi} - \right.$$
$$\left. - \frac{1}{\sqrt{p^2-1}}\log\left(1 - \frac{2\sqrt{p^2-1}(1+\cos\varphi)}{\sin\varphi + (\sqrt{p^2-1}-p)(1+\cos\varphi)}\right)\right\} + C_1$$

and in another form

$$x = act - \frac{\varepsilon c}{\nu a\sqrt{1-\gamma^2}}\left\{\frac{1}{\sqrt{p^2-1}}\log\left(1 - \frac{2\sqrt{p^2-1}(1+\cos\varphi)}{\sin\varphi + (\sqrt{p^2-1}-p)(1+\cos\varphi)}\right)\right\},$$

$$y = \frac{\varepsilon c\sqrt{1+A^2}}{a\nu\sqrt{1-\gamma^2}}\frac{1}{1-p\sin\varphi} + C_2,$$

$$z = -\frac{\varepsilon Bc}{\nu a\sqrt{1-\gamma^2}}\left\{\frac{1}{\sqrt{p^2-1}}\log\left(1 - \frac{2\sqrt{p^2-1}(1+\cos\varphi)}{\sin\varphi + (\sqrt{p^2-1}-p)(1+\cos\varphi)}\right)\right\} + C_3.$$

We have thus established all the necessary relations.

When the initial data are given, the integrals of motion are expressed in terms of them. Let $\beta_x(t_0) = \beta_x^0$, and similarly for the other velocity components. By continuity, we then obtain

$$A^2 = \frac{1-(\beta_x^0)^2-(\beta_y^0)^2}{(a-\beta_x^0)^2}, \qquad B = \frac{\beta_z^0}{\beta_x^0 - a}.$$

Consequently, the problem can be simplified if we impose the condition $\beta_z^0 = 0$, because then $B = 0$ and the motion occurs in the plane (x, y).

In conclusion, we note that because the connection between classical and quantum mechanics is well known [8], it would be interesting to give an exact solution of the problem of charged particle motion in crossed electric and magnetic fields at the quantum level.

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* [in Russian], Vol. 2, *The Classical Theory of Fields*, Nauka, Moscow (1967); English transl., Pergamon, Oxford (1971).

2. G. A. Schott, *Electromagnetic Radiation and the Mechanical Reaction, Arising from It, Being an Adams Prize Essay in the University of Cambridge*, Cambridge Univ. Press, Cambridge (1912).
3. S. A. Boguslavsky, *Selected Papers on Physics* [in Russian], Fizmatgiz, Moscow (1961).
4. V. V. Batygin and I. N. Toptygin, *Collection of Problems in Electrodynamics and the Special Theory of Relativity* [in Russian], Lan', St. Petersburg (2010).
5. C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill International, New York (1980).
6. Y. Friedman and M. D. Semon, *Phys. Rev. E*, **72**, 026603 (2005).
7. S. A. Chin, *J. Math. Phys.*, **50**, 012904 (2009); arXiv:0809.0859v1 [math-ph] (2008).
8. D. I. Blokhintsev, *Fundamentals of Quantum Mechanics* [in Russian], Vyshaya Shkola, Moscow (1963); English transl.: *Quantum Mechanics*, D. Reidel, Dordrecht (1964).