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A Semianalytical Method for Constructing Nearly Equatorial Orbits of Hypothetical Satellites of Asteroids with an Almost Spheroidal Shape

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Abstract—The problem of the motion of a particle with a negligible mass (satellite) near the equatorial plane of a spheroidal body, in particular, an asteroid, is considered. To a first approximation, the motions can be separated into equatorial and latitudinal components for low inclinations of the satellite orbit. The equatorial central motion, when the force function depends only on the satellite's distance to the coordinate origin (the asteroid's center of mass), is constructed by the previously proposed semianalytical method. The construction of the latitudinal motion envisages the solution of a linearized system of second-order differential equations with periodic coefficients by numerically determining the monodromy matrix on the period of the equatorial motion and its temporal analytic continuation. The model problems of the perturbed motion of nearly equatorial hypothetical satellites of Ceres and Vesta are considered. The methodical accuracy has been estimated by a comparison with the numerical solution.

Keywords: spheroid potential, nearly equatorial orbits, hypothetical satellites of Ceres and Vesta

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INTRODUCTION AND POBLEM STATEMENT

The well-known classical problem of the motion of a test particle with a negligible mass in the attractive field of an arbitrary rigid body (or a system of bodies) in celestial mechanics is terminologically occasionally called the Fatou problem named after the French scientist who systematically studied it and revealed a number of dynamical properties (Fatou, 1931; Duboshin, 1964). In this paper we consider a special case of this problem where the attracting body is a homogeneous spheroid and the test particle moves near its equatorial plane.

The attractive force function of a homogeneous spheroid dependent on the satellite coordinates is expressed by well-known formulas (Duboshin, 1961; Kondrat'ev, 2007). In what follows, we will use a coordinate system with the origin at the center of mass of the attracting body O and the equatorial plane normal to its rotation axis Oz as the principal coordinate plane xOy. Due to the axial symmetry of the problem, the direction of the Ox axis is arbitrary, and it is natural and convenient to use the polar coordinates instead of the pair of rectangular coordinates x and y:

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x). \tag{1}$$

In this case, the attractive force function $U(\rho, z)$ does not depend on the polar angle (or longitude) θ and the equations of particle motion in chosen cylindrical coordinates ρ, θ, z

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 = \frac{\partial U}{\partial \rho},$$
$$\rho^2\frac{d^2\theta}{dt^2} + 2\rho\frac{d\rho}{dt}\frac{d\theta}{dt} = \frac{\partial U}{\partial \theta} = 0, \quad \frac{d^2 z}{dt^2} = \frac{\partial U}{\partial z} \tag{2}$$

admit two first integrals (Duboshin, 1964):

$$\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2(U + h), \quad \rho^2\frac{d\theta}{dt} = c. \tag{3}$$

The constants c and h are defined by the initial values of the coordinates and their derivatives with respect to time t at the initial $t_0 = 0$:

$$\rho_0, \quad \frac{d\rho_0}{dt} = \left.\frac{d\rho}{dt}\right|_{t_0=0}, \quad \theta_0, \quad \frac{d\theta_0}{dt} = \left.\frac{d\theta}{dt}\right|_{t_0=0},$$
$$z_0, \quad \frac{dz_0}{dt} = \left.\frac{dz}{dt}\right|_{t_0=0}.$$

The integrals (3) turn out to be useful in checking the accuracy of the numerical solution of Eqs. (2) with an arbitrary function $U(\rho,z)$.

With the integral c Eqs. (2) and (3) take the form

$$\frac{d^2\rho}{dt^2} = \frac{\partial U}{\partial \rho} + \frac{c^2}{\rho^3}, \quad \frac{d^2 z}{dt^2} = \frac{\partial U}{\partial z}, \tag{4}$$

$$\frac{1}{2}\left[\left(\frac{d\rho}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = U(\rho, z) - \frac{c^2}{2\rho^2} + h. \tag{5}$$

Once the explicit dependence of ρ on time t has been determined, the angle variable θ can be found by calculating the integral

$$\theta = \theta_0 + c\int_{t_0}^{t} \frac{d\tau}{\rho^2(\tau)}. \tag{6}$$

The goal of this paper is to construct an approximate semianalytical solution of the system of Eqs. (2) for a limited change in ρ and small z_0 and $\frac{dz_0}{dt}$. The inclination of the satellite orbit i and the modulus of its latitude $\varphi = \arctan(z/\rho)$ are assumed to remain small in the time interval under consideration. The proposed solution can be used as the first (initial) approximation to describe the orbital motion of nearly equatorial satellites of celestial bodies whose geometric shapes are to a certain extent close to oblate spheroids, including the dwarf planet Ceres and the asteroid Vesta. Actually, the asteroids have a fairly complex structure of the gravitational field. In particular, the study performed during the Dawn mission begun in September 2007 has revealed from 10 to 25 gravitational field harmonics of Ceres and Vesta. It is described in detail in Konopliv et al. (2011), which also contains a review of the research on these celestial bodies performed with ground-based facilities. In addition, note that the model of a prolate ellipsoid of revolution (Ivashkin and Lan, 2018) and a triaxial ellipsoid (Guo and Ivashkin, 2018) is used in investigating the dynamics of prospective artificial satellites of some asteroids, in particular, Apophis.

THE ATTRACTIVE FORCE FUNCTION OF A HOMOGENEOUS SPHEROID AND ITS DERIVATIVES

The equation for the surface of a spheroid (ellipsoid of revolution) has the well-known canonical form

$$\frac{\rho^2}{a_1^2} + \frac{z^2}{a_3^2} = 1. \tag{7}$$

To express the force function of its attraction on an external point, we will use the formulas given in the monograph by Kondrat'ev (2007). In the case of an oblate spheroid, its semiaxes satisfy the conditions $a_1 = a_2 > a_3$ and the force function is

$$U(\rho, z, \lambda) = \frac{3\mu}{2\sqrt{a_1^2 - a_3^2}} \times \left\{ I(\lambda) - \frac{\rho^2}{2\left(a_1^2 - a_3^2\right)}\left[I(\lambda) - \frac{\sqrt{\left(a_1^2 - a_3^2\right)\left(a_3^2 + \lambda\right)}}{a_1^2 + \lambda}\right] - \frac{z^2}{a_1^2 - a_3^2}\left[\sqrt{\frac{a_1^2 - a_3^2}{a_3^2 + \lambda}} - I(\lambda)\right]\right\}. \tag{8}$$

Here, μ is the product of the gravitational constant and the mass of a homogeneous spheroid,

$$I(\lambda) = \arctan\left(\sqrt{\frac{a_1^2 - a_3^2}{a_3^2 + \lambda}}\right). \tag{9}$$

The parameter λ is the greatest positive root of the equation

$$\frac{\rho^2}{a_1^2 + \lambda} + \frac{z^2}{a_3^2 + \lambda} = 1, \tag{10}$$

or

$$\lambda^2 + \left(a_1^2 + a_3^2 - \rho^2 - z^2\right)\lambda + a_1^2 a_3^2 - a_3^2\rho^2 - a_1^2 z^2 = 0,$$

i.e.,

$$\lambda = \frac{1}{2}\left(\kappa - a_1^2 - a_3^2\right), \tag{11}$$

where

$$\kappa = \rho^2 + z^2 + \sqrt{\left(\rho^2 + z^2\right)^2 + 2\left(a_1^2 - a_3^2\right)\left(z^2 - \rho^2\right) + \left(a_1^2 - a_3^2\right)^2}. \tag{12}$$

With the introduction of simplifying notation,

$$s_1 = a_1^2 + \lambda = \frac{1}{2}\left(\kappa + b^2\right),$$
$$s_2 = a_3^2 + \lambda = \frac{1}{2}\left(\kappa - b^2\right), \tag{13}$$
$$b = \sqrt{a_1^2 - a_3^2}, \quad I(\lambda) = \arctan\left(\frac{b}{\sqrt{s_2}}\right)$$

expression (8) takes the form

$$U(\rho, z, \lambda) = \frac{3\mu}{2b}\left[\left(1 + \frac{2z^2 - \rho^2}{2b^2}\right)I(\lambda) + \frac{\rho^2\sqrt{s_2}}{2s_1 b} - \frac{z^2}{b\sqrt{s_2}}\right]. \tag{14}$$

Strictly speaking, the partial derivatives of the function U with respect to the coordinates must be calculated by differentiation with respect to the arguments ρ and z that enter into Eq. (14) both explicitly and via $\lambda(\rho, z)$. The first-order partial derivatives are defined by relatively simple formulas:

$$\frac{\partial U}{\partial \rho} = \frac{3\mu\rho}{2b^3}\left[\frac{b\sqrt{s_2}}{s_1} - I(\lambda)\right],$$
$$\frac{\partial U}{\partial z} = \frac{3\mu z}{b^3}\left[I(\lambda) - \frac{b}{\sqrt{s_2}}\right]. \tag{15}$$

It is easy to see that the sign of the derivative $\frac{\partial U}{\partial z}$ is opposite to the sign of z. As a corollary of the general property of motion in the Fatou problem (Fatou, 1931; Duboshin, 1964), this implies that the acceleration component $\frac{d^2 z}{dt^2}$ is constantly directed to the principal (equatorial) plane and the motion is stable with respect to small deviations of z from zero.

Below, when analyzing the nearly equatorial orbits and separating the motions into equatorial and latitudinal ones, we will also use the simplified expressions for the function U and its partial derivatives with respect to ρ and z as $z \to 0$ that directly follow from Eqs. (14) and (15):

$$U(\rho, 0) = \frac{3\mu}{2b}\left[\left(1 - \frac{\rho^2}{2b^2}\right)\arctan(\xi) + \frac{1}{2\xi}\right],$$
$$\left.\frac{\partial U}{\partial \rho}\right|_{z=0} = \frac{3\mu\rho}{2b^3}\left[\frac{b^2}{\xi\rho^2} - \arctan(\xi)\right], \tag{16}$$
$$\left.\frac{\partial U}{\partial z}\right|_{z\to 0} = \frac{3\mu z}{b^3}[\arctan(\xi) - \xi],$$

where $\xi = \frac{b}{\sqrt{\rho^2 - b^2}}$.

SEPARATION OF THE MOTIONS AND AN APPROXIMATE CONSTRUCTION OF THE SOLUTION

Equatorial Motion

Following the method of successive approximations, to construct the solution at small z/ρ, we will neglect the square and the cube of this ratio in the first and second equations (4), respectively. Let us first consider the equatorial motion, when $z_0 = 0, \frac{dz_0}{dt} = 0$. In this approximation the change in ρ defined by the first integral (5) occurs in a central field and is found by inverting the quadrature

$$t = \pm\frac{1}{\sqrt{2}}\int_{\rho_0}^{\rho}\frac{d\rho}{\sqrt{f(\rho)}}, \tag{17}$$

where

$$f(\rho) = U(\rho, 0) - \frac{c^2}{2\rho^2} + h, \tag{18}$$

for the real motion the following condition should be satisfied:

$$h \geq \frac{c^2}{2\rho^2} - U(\rho, 0). \tag{19}$$

In what follows, without introducing any significant restrictions, we will assume that

$$\rho_0 = \rho_{\min}, \quad \frac{d\rho_0}{dt} = 0, \quad \theta_0 = 0, \quad \frac{d\theta_0}{dt} = \frac{c}{\rho_0^2} \tag{20}$$

at $t = 0$.

To find the dependence $\rho(t)$ in the case of its limited (librational) change $(0 < \rho_{\min} < \rho < \rho_{\max})$, we apply the previously developed approximate semianalytical method (Vashkov'yak, 2018) briefly described below.

The constants c and h are uniquely defined by the following formulas by specifying the extreme values $\rho_{\min}, \rho_{\max}$:

$$c = \rho_{\min}\rho_{\max}\sqrt{\frac{2[U(\rho_{\min}, 0) - U(\rho_{\max}, 0)]}{\rho_{\max}^2 - \rho_{\min}^2}},$$
$$h = \frac{\rho_{\min}^2 U(\rho_{\min}, 0) - \rho_{\max}^2 U(\rho_{\max}, 0)}{\rho_{\max}^2 - \rho_{\min}^2}. \tag{21}$$

Assuming that $\rho_{\min}$ and $\rho_{\max}$ are simple zeros of the function $f(\rho)$, it can be represented as

$$f(\rho) = g(\rho)(\rho - \rho_{\min})(\rho_{\max} - \rho), \tag{22}$$

with the last two multipliers, obviously, reflecting the main qualitative property of a restricted motion. In this case, the function $g(\rho)$ is calculated from the well-known expression (18) and the specified extreme values $\rho_{\min}$ and $\rho_{\max}$ as

$$g(\rho) = \frac{f(\rho)}{(\rho - \rho_{\min})(\rho_{\max} - \rho)}. \tag{23}$$

Next, to bring the integral (17) to an elliptic form, we approximate the function $g(\rho)$ by a quadratic polynomial

$$P(\rho) = p_1\rho^2 + p_2\rho + p_3$$
$$= p_1(\rho - \rho_1)(\rho - \rho_2) \approx g(\rho). \tag{24}$$

Thus, at fixed constants c and h the construction of an analytical solution requires precalculating the coefficients of the approximating polynomial. Next, we find the dependence $\rho(t)$ after the substitution of $Q(\rho) = P(\rho)(\rho - \rho_{\min})(\rho_{\max} - \rho)$ for $f(\rho)$ in the quadrature (17) using its inversion and then also determine $\theta(t)$ by calculating the integral (6). These dependences containing elliptic functions and integrals are described in Vashkov'yak (2018), while here we will present only their general form, when the discriminant of the quadratic equation $P(\rho) = 0$ is negative and the roots ρ_1 and ρ_2 are complex conjugate ones (precisely this case is realized in the specific numerical examples considered below).

The time dependence of the distance is defined by the formulas

$$\rho(t) = \frac{\alpha + \beta \mathrm{cn}\nu t}{\gamma + \delta \mathrm{cn}\nu t}, \quad k^2 = \frac{(\rho_4 - \rho_3)^2 - \delta^2}{4pq}, \quad \nu = \sqrt{2p_1 pq}, \tag{25}$$

where cn$\nu t$is the Jacobi elliptic cosine with modulus k and the constants α, β, γ, δ, p, q depend on the coefficients of the approximating polynomial $P(\rho)$ and the two real (specified) roots $\rho_{\min}$, $\rho_{\max}$ of the polynomial $Q(\rho)$.

The dependence $\rho(t)$ is a periodic function of time with a period

$$T_\rho = \frac{4}{\nu}\mathbf{K}(k) \tag{26}$$

where $\mathbf{K}(k)$ is a complete elliptic integral of the first kind with modulus k.

The time dependence of the polar angle θ is determined by finding the integral in Eq. (6). Substituting the dependence $\rho(\tau)$ into its integrand, according to Eqs. (25), gives

$$\theta(t) = \frac{c}{\nu}\left(\frac{\delta}{\beta}\right)^2 \left[\nu t + 2sI_1(t) + s^2 I_2(t)\right], \tag{27}$$

where

$$I_l(t) = \int_0^{\nu t} \frac{dw}{(b + \mathrm{cn}w)^l}, \quad (l = 1,2), \quad b = \frac{\alpha}{\beta}, \quad s = \frac{\gamma}{\delta} - b.$$

The integrals I_l are expressed via trigonometric functions and incomplete elliptic integrals of the first, second, and third kinds.

Latitudinal Motion

When $\frac{dz_0}{dt}$ is nonzero, there will be deviations related to the change in the z coordinate or latitude $\varphi = \arctan(z/\rho)$ in the satellite motion. By analogy with an unperturbed Keplerian motion, we will specify the initial conditions for the latitudinal motion at the orbital pericenter by the formulas

$$z_0 = 0, \quad \frac{dz_0}{dt} = V_0 \sin i_0, \quad V_0 = \sqrt{\frac{\mu(1+e)}{a(1-e)}}, \quad a = \frac{1}{2}(\rho_{\max} + \rho_{\min}), \quad e = \frac{1}{2a}(\rho_{\max} - \rho_{\min}), \tag{28}$$

where the initial orbital inclination i_0 will be assumed to be a sufficiently small quantity of the order of a few degrees and $|\varphi| \le i_0$.

At small z the change in this coordinate is defined by a linear second-order differential equation:

$$\frac{d^2 z}{dt^2} + R(t)z = 0. \tag{29}$$

Here, according to the last formula (16) for the derivative $\left.\frac{\partial U}{\partial z}\right|_{z\to 0}$, the function

$$R(t) = \frac{3\mu}{b^3}[\xi\text{-}\arctan(\xi)] \ge 0. \tag{30}$$

It originally depends on the distance ρ via ξ and is actually a function of time, because, to a first approximation, $\rho(t)$ is defined by the periodic dependence (25), so that

$$R(t + T_\rho) = R(t), \tag{31}$$

while the period of this function is given by Eq. (26).

The differential equation (29), known as Hill's equation, in a vector form is

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}, \tag{32}$$

where

$$\mathbf{x} = \left(z, \frac{dz}{dt}\right)^{\mathrm{T}}, \quad A(t + T_\rho) = A(t) = \begin{pmatrix} 0 & 1 \\ -R(t) & 0 \end{pmatrix}, \tag{33}$$

and the superscript "T" denotes transposition.

According to the Lyapunov–Floquet theory (see, e.g., Yakubovich and Strazhinskii, 1972, Ch. VII; Malkin, 1966, Ch. V), the differential equation (32) with periodic coefficients can be solved by calculating the fundamental matrix (matrizant) $X(t)$ satisfying the equation

$$\frac{dX}{dt} = A(t)X \tag{34}$$

under the initial condition $X(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E_2$ and the monodromy matrix $M = X(T_\rho)$. In this case, the vector $\mathbf{x}(t)$ is found from the formula

$$\mathbf{x}(t) = X(t)\mathbf{x}_0, \tag{35}$$

where $\mathbf{x}_0 = \left(z_0, \frac{dz_0}{dt}\right)^{\mathrm{T}}$ is the vector of initial data. The matrizant $X(t)$ for an arbitrary time $t > T_\rho$ is defined by the formula

$$X(t > T_\rho) = X(0 \le t \le T_\rho)M^n, \quad n = \mathrm{Entier}(t/T_\rho). \tag{36}$$

Thus, to construct the latitudinal motion requires finding the matrizant $X(t)$ only in a limited time interval $[0 \le t \le T_\rho]$, which can be done without any significant computational burden by a numerical method with a check of the equality $\det X(t) \equiv 1$.

Table 1. Parameters of the asteroids and the satellite orbits

Parameter	Ceres	Vesta
μ, km^3 s^{-2}	62.6	17.8
R_H, km	223500	125500
$D_1 = 2a_1$, km	964.4	572.6
$D_2 = 2a_2$, km	964.2	557.2
$D_3 = 2a_3$, km	891.8	446.4
T_{rot}, h	9.074	5.342
i, deg	3	3
ρ_{min}, km	800	400
ρ_{max}, km	1583.64	700.85

The properties of the latitudinal motion depend significantly on the eigenvalues σ of the monodromy matrix M (multipliers) satisfying the equation

$$\det(M - \sigma E_2) = 0 \quad (37)$$

and on the characteristic exponents $\alpha = \frac{1}{T_\rho}\ln\sigma$. The general solution of Eq. (32) can be represented as a product of T_ρ-periodic functions by $\exp(\alpha t)$ and σ are defined by very simple formulas:

$$\sigma_{1,2} = B \pm \sqrt{B^2 - 1}, \quad (38)$$

where $B = \frac{1}{2}\mathrm{Tr}M$ is half the trace of the monodromy matrix.

If $B > 1$, then the multipliers and characteristic exponents are positive. In this case, the latitudinal motion is described by oscillations of the period T_ρ with an exponentially growing amplitude, so that z is outside the range of applicability of the linear approximation (29).

If $B < 1$, then the complex conjugate multipliers lie on the unit circumference $|\sigma_1| = |\sigma_2| = 1$ and the characteristic exponents are purely imaginary. The latitudinal motion is restricted and is described by a superposition of oscillations with limited amplitudes and periods T_ρ and $T_\alpha = 2\pi/\alpha$.

METHODICAL EXAMPLES OF SOLVING MODEL PROBLEMS

In this section we compare the results obtained by two fundamentally different methods. The initial motion parameters are defined by Eqs. (20) and (28), and this complete set of initial data serves to calculate the restricted motion near the equatorial plane of an asteroid both by the semianalytical method and by numerically integrating Eqs. (2) with a check of the constancy of c and h along the solution. As examples, we consider the motion of nearly equatorial hypothetical satellites of the asteroids Ceres and Vesta that are assumed to be oblate spheroids. Table 1 gives their physical characteristics: the gravitational parameters μ, the radii of Hill's spheres R_H, the geometrical sizes $D_1 \approx D_2 > D_3$, the axial rotation periods T_{rot}, the initial inclinations of the satellite orbits, and their apsidal distances.

In the adopted spheroid model it is assumed that $D_1 = D_2$ ($a_1 = a_2$) for both asteroids and their axial rotations occur around the smaller semiaxes a_3.

The minimum distances of the satellites of Ceres and Vesta are assumed to be specified and the pericenter heights are about 318 and 114 km, respectively. The maximum distances are taken to be such that they correspond to the Keplerian orbits of satellites with revo-

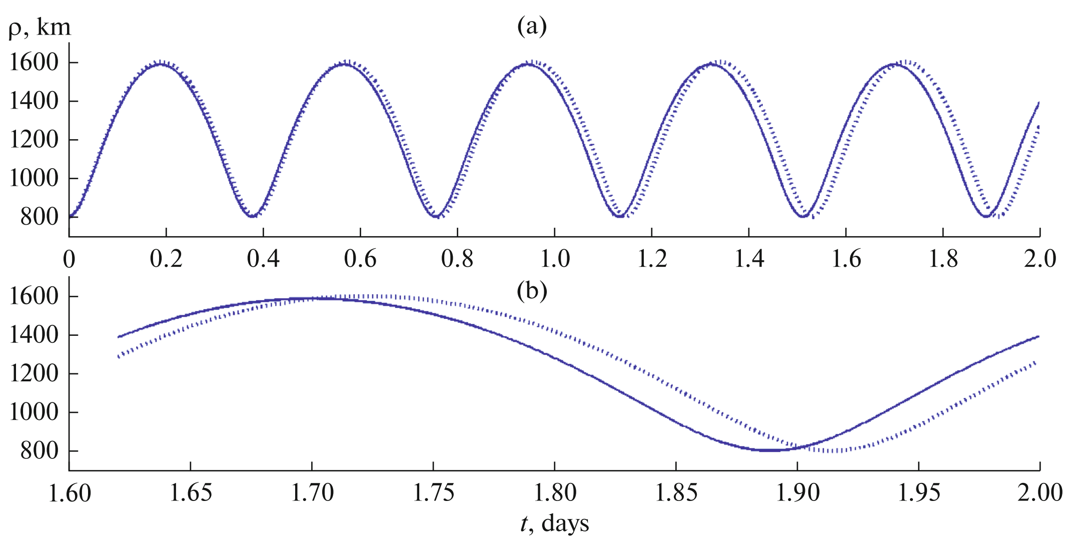

Fig. 1. Time dependences of the distance ρ for Ceres' satellite (the solid curves are the semianalytical solution, the dotted curves are the numerical solution).

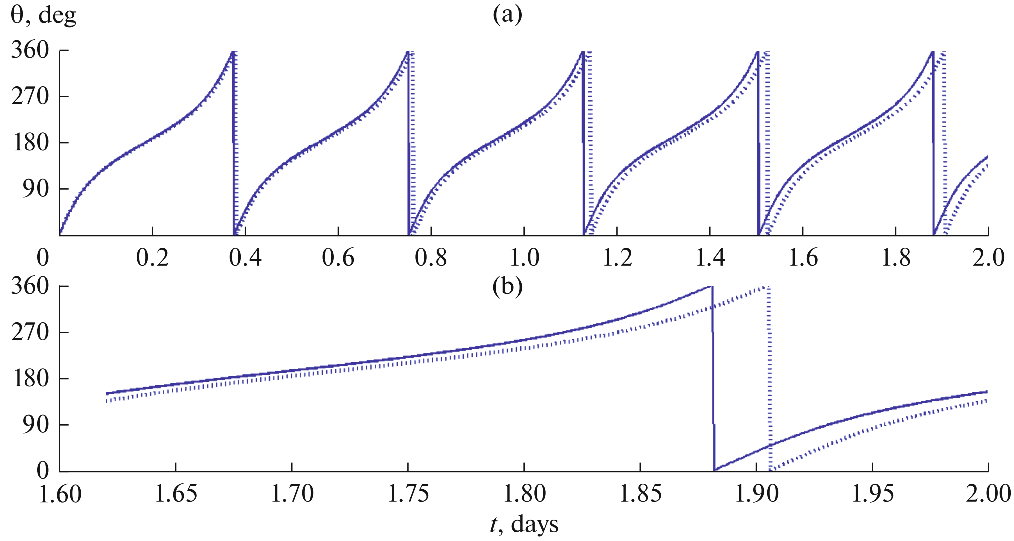

Fig. 2. Same as Fig. 1 for the polar angle θ.

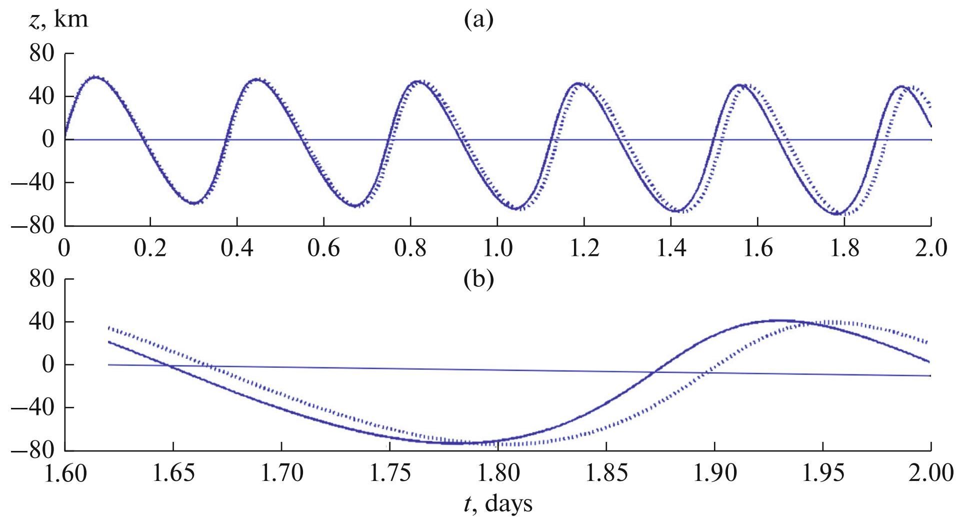

Fig. 3. Same as Fig. 1 for the *z* coordinate.

lution periods equal to T_{rot}, i.e., synchronous to the satellites of the asteroids. The initial low inclination of the satellite orbits is taken to be 3°. The computational time interval is two days or approximately from 5–9 orbital periods of the satellites.

First, for each of the examples we approximated the function $g(\rho)$ defined by Eq. (23) by a quadratic polynomial $P(\rho)$ in the segment $\rho_{min} = \rho_3 \le \rho \le \rho_4 = \rho_{max}$ by the least-squares method and calculated its discriminant D and roots ρ_1, ρ_2.

The comparative results of our calculations of the time dependences of the distance, polar angle, and z coordinate for Ceres' satellite are shown in Figs. 1–3.

Panels (a) and (b) of these figures present a time interval of two days and a considerably shorter interval corresponding to the last revolution of the satellite. Panels (b) provide a clear quantitative estimate of the methodical error in the proposed approximate solution due to the approximation of the function $g(\rho)$ (23) by a quadratic polynomial $P(\rho)$ (24). A minor and seemingly secular change in the mean and extreme values of z can be noticed in Fig. 3a. Actually, this is a long-period variation with a period T_α that is about 22.5 days here and modulates a short period variation of z with a period $T_\rho \approx T_{rot} = 9.074$ h.

Similar dependences for Vesta's satellite are shown in Figs. 4–6.

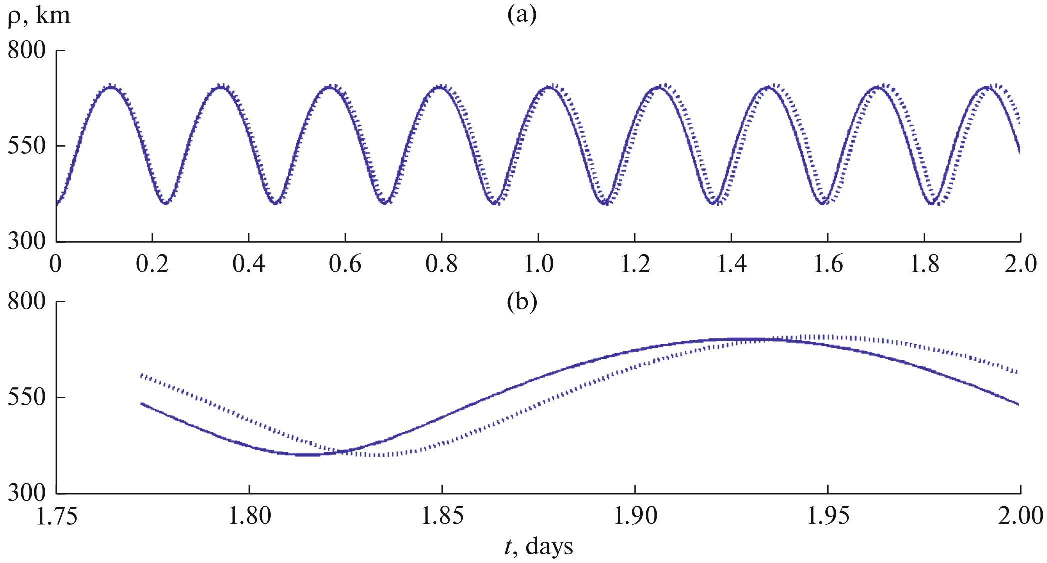

Fig. 4. Same as Fig. 1 for Vesta's satellite.

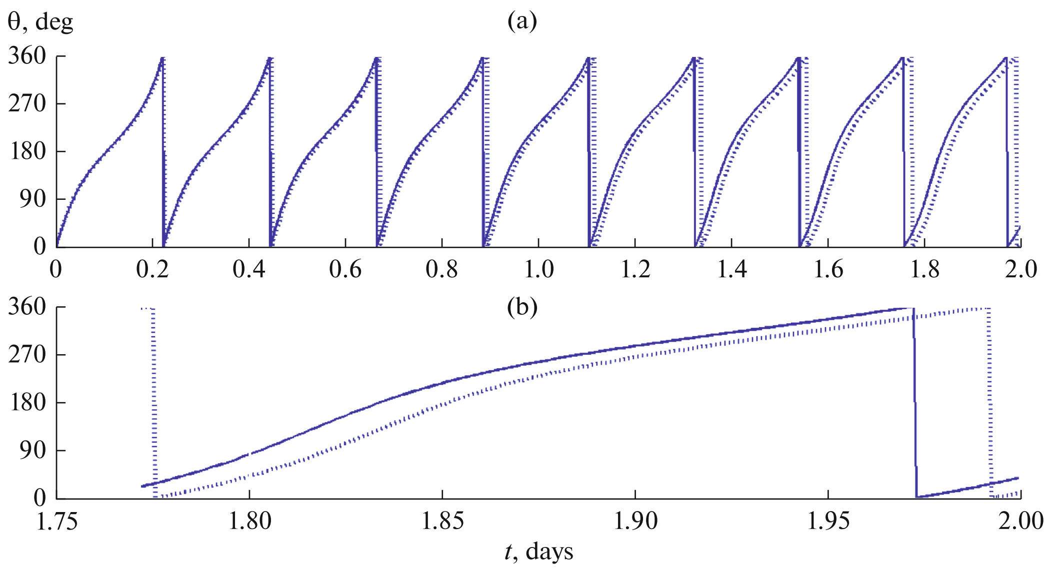

Fig. 5. Same as Fig. 2 for Vesta's satellite.

A long-period variation with a period T_α that is about 2.6 days here and modulates a short-period variation of z with a period $T_\rho \approx T_{rot} = 5.034$ h can be seen in Fig. 6a more clearly than in Fig. 3a.

CONCLUSIONS

We proposed a semianalytical method and described its potential for finding an approximate solution of the problem of the motion of a satellite near the equatorial plane of a homogeneous oblate spheroid in the form of time dependences of the coordinates. In general, when constructing the equatorial motion of the satellite for the adopted dynamical and geometrical characteristics of the spheroid, using the method envisages either predetermining the extreme (apsidal) distances by numerically solving the rigorous equations or specifying these parameters a priori. The analytical time dependences of the distance and polar angle are derived using the previously developed technique for constructing the motion in a central gravitational field. To solve the equation describing the latitudinal motion requires its numerical integration in a relatively short time interval, the period of the distance change in the equatorial motion. This solution is con-

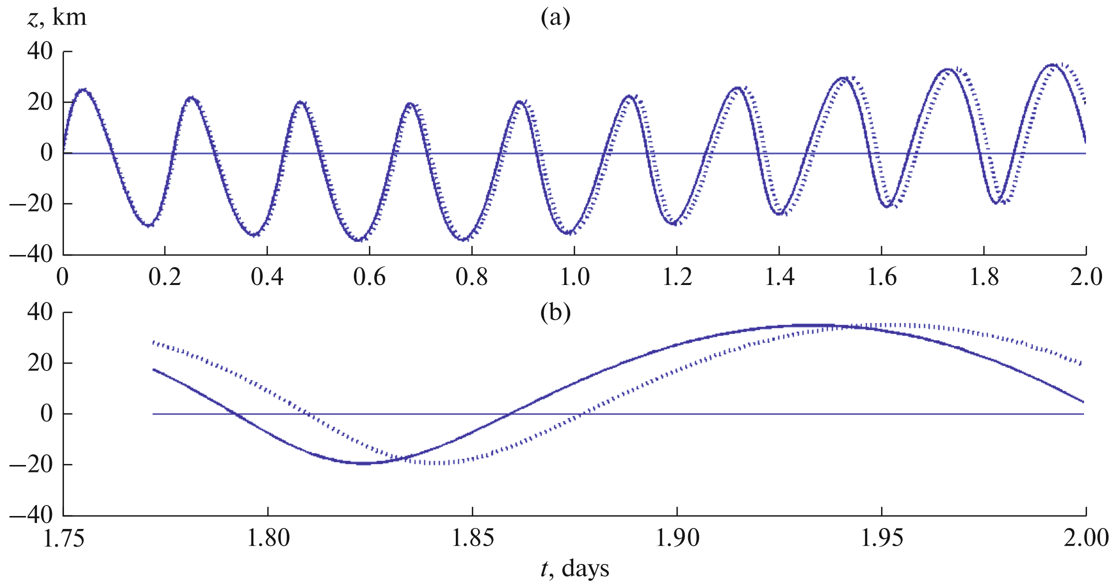

Fig. 6. Same as Fig. 3 for Vesta's satellite.

tinued for an arbitrary time using the monodromy matrix.

Using the orbits of hypothetical satellites of Ceres and Vesta as special examples, we showed qualitative and approximately quantitative agreement between the results of our calculations performed by the proposed method and the method for numerically integrating the equations of motion in the attractive field of a homogeneous oblate spheroid.

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