ISSN 0038-0946, Solar System Research, 2017, Vol. 51, No. 4, pp. 315–326. © Pleiades Publishing, Inc., 2017.
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Constructive-Analytical Solution of the Problem of the Secular Evolution of Polar Satellite Orbits

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Received January 10, 2017

Abstract—The well-known twice-averaged Hill problem is considered by taking into account the oblateness of the central body. This problem has several integrable cases that have been studied qualitatively by many scientists, beginning with M.L. Lidov and Y. Kozai. However, no rigorous analytical solution can be obtained in these cases due to the complexity of the integrals. This paper is devoted to studying the case where the equatorial plane of the central body coincides with the plane of its orbital motion relative to the perturbing body, while the satellite itself moves in a polar orbit. A more detailed qualitative study is performed, and an approximate constructive-analytical solution of the evolution system in the form of explicit time dependences of the eccentricity and pericenter argument of the satellite orbit is proposed. The methodical accuracy for the polar orbits of lunar satellites has been estimated by comparison with the numerical solution of the system.

Keywords: polar satellite orbits, secular perturbations
DOI: 10.1134/S0038094617040074

INTRODUCTION

In this paper we consider the problem of the joint influence of the oblateness of a central body and the attraction from a remote perturbing point moving in a nearly circular orbit on the secular evolution of the orbit of a satellite with a negligible mass. The full perturbing function from these two factors in the twice-averaged problem statement is replaced by its secular part. Being generally nonintegrable, this problem, nevertheless, has several integrable cases and particular solutions that can serve as models in various astronomical applications. A detailed classification and a qualitative study of these cases can be found in Lidov and Yarskaya (1974). Its results are also presented briefly in Lidov (1974). Below we will use the corresponding numbering of these cases (from I to VII) introduced in the above papers.

The integrable case I, where only the attraction from an external point is taken into account in the twice-averaged problem, was studied in the fundamental papers by Lidov (1961) and Kozai (1962). The results of these works, along with numerous and multipronged astronomical manifestations of the Lidov-Kozai effect, are described in detail in the monograph by Shevchenko (2017).

Lidov (1963a, 1963b) used one integrable case V, where the orbital plane of the perturbing point is orthogonal to the equatorial plane of the planet. It was invoked to explain the existence of Uranus's main satellites whose equatorial orbits (in the hypothetical absence of its oblateness) must have inevitably cross the planet's surface in the course of their evolution due to the secular solar perturbations.

Another integrable case where the equatorial plane of the central body coincides with the plane of its orbital motion relative to the perturbing point (the so-called coplanar case III) was applied in analyzing the evolution of the orbits of artificial lunar satellites perturbed by the Earth's attraction (Kozai, 1963), in particular, the satellites moving in a polar orbit (Uphoff, 1976). In these papers, just as in Kudielka (1997), particular attention is given to the so-called "frozen" (equilibrium or stationary) orbits with constant elements of the satellite orbit. The stability of such orbits was studied by Vashkov'yak (1998).

Previously, the circular satellite orbits (case VII) have also been studied. Sekiguchi (1961) and Allan and Cook (1964) described the evolution of the orbital plane, while Vashkov'yak (1974) also analyzed the stability of orbits with respect to their eccentricity.

Some integrable cases are also applied in designing long-lived orbits of planetary satellites (Prokhorenko, 2010).

Of course, the integrable cases of the twice-averaged Hill problem under consideration including the oblateness of the central planet can serve only as a first approximation and as a qualitative guide in solving a particular real celestial-mechanics problem. The well-known methods of the perturbation theory use more complete physical models of orbital evolution. For example, Oesterwinter (1970) and Brumberg et al. (1971), apart from allowance for many additional perturbations in the motion of lunar satellites, in contrast to the integrable case III, naturally, also assume the equatorial and orbital planes of the Moon to be noncoplanar.

This paper is devoted to studying the evolution of polar satellite orbits as a particular solution of the integrable case III. No rigorous analytical solution can be obtained even in known special functions due to the complexity of the quadratures. Apart from a more detailed qualitative study and the calculation of some quantitative characteristics, here we propose an approximate method for constructing a constructive-analytical solution of the evolution system of two differential equations describing the change in the eccentricity and pericenter argument of a satellite orbit. The method envisages an approximation of the irrational part of the integrand that leads to elliptic integrals whose inversion allows a solution in elliptic functions to be obtained. To estimate the model and methodical accuracies, we use a comparison with the numerical solution of a more complete (nonintegrable) evolution system in which the relatively small orbital inclination of the perturbing point to the equatorial plane of the central body is taken into account. Our comparative calculations were performed for several polar orbits of lunar satellites perturbed by the attraction from the Earth and, to a lesser extent, the Sun.

FORMULATION OF THE PROBLEM

The secular evolution of a satellite orbit in the integrable case III under consideration is described by a system of equations in Keplerian elements that can be derived from the equations in Lidov and Yarskaya (1974). Using the eccentricity $e = \sqrt{1-\varepsilon}$, instead of the element ε introduced in this paper, the parameter $\gamma = -5\beta/2$ instead of β, and the independent variable $\tau = n/20$ instead of n, this system takes the form (Vashkov'yak, 1996)

$$\frac{de}{d\tau} = 10e\sin^2 i\sqrt{1-e^2}\sin 2\omega,$$

$$\frac{di}{d\tau} = -\frac{10e^2\sin i\cos i}{\sqrt{1-e^2}}\sin 2\omega,$$

$$\frac{d\omega}{d\tau} = \frac{2}{\sqrt{1-e^2}} \times\left[e^2 - 1 + 5\cos^2 i + 5\left(\sin^2 i - e^2\right)\cos 2\omega\right] + 4\gamma\left(1-e^2\right)^{-2}\left(5\cos^2 i - 1\right), \tag{1}$$

$$\frac{d\Omega}{d\tau} = 2\cos i \times\left[\frac{1}{\sqrt{1-e^2}}\left(5e^2\cos 2\omega - 3e^2 - 2\right) - 4\gamma\left(1-e^2\right)^{-2}\right].$$

Here, i, ω, Ω are the standard designations for the Keplerian angular equatorial elements, while the constant parameter γ and the independent variable τ are defined by the formulas

$$\gamma = \frac{\alpha}{2\beta}, \alpha = -\frac{3}{8}\left(\frac{a_0}{a}\right)^2 c_{20}, \qquad \beta = \frac{3a^3}{16\mu}\sum_{\mathrm{j}=1}^{N}\frac{\mu_{\mathrm{j}}}{a_{\mathrm{j}}^3}, \tau = \beta n(t-t_0). \tag{2}$$

We introduced the parameters β used in this paper previously, while the formula for it is written when the problem is naturally generalized to an arbitrary number N of perturbing points revolving in equatorial orbits relative to the central one and under an additional insignificant assumption about the eccentricities of these orbits ($e_j = 0$). So, μ, μ_{j} are the products of the gravitational constant by the masses of the planet and the jth perturbing point respectively, a_0, a_j, a are the mean equatorial radius of the planet and the semimajor axes of the orbits of the jth perturbing point and the satellite, respectively (naturally, in the averaged model of the problem the semimajor axis a is constant). The remaining quantities are: c_{20} is the coefficient at the second zonal harmonic of the planet's gravitational field, $n = \sqrt{\mu/a^3}$ is the mean motion of the satellite, t_0 and t are the initial and current times, respectively.

The first integrals of the evolution system (1) are (Lidov and Yarskaya, 1974)

$$\left(1-e^2\right)\cos^2 i = c_1, \qquad e^2\left(\frac{2}{5} - \sin^2 i\sin^2\omega\right) + \frac{2}{5}\gamma\left(1-e^2\right)^{-3/2}\left(\cos^2 i - \frac{1}{3}\right) = c_2, \tag{3}$$

while the longitude of the ascending node Ω that does not appear on the right-hand sides of Eqs. (1), once the dependences $e(\tau)$, $i(\tau)$, and $\omega(\tau)$ have been found, is defined by the quadrature

$$\Omega(\tau) = \Omega_0 + 2\int_0^\tau \cos i(\varsigma)\left\{\begin{aligned}&\left[1-e^2(\varsigma)\right]^{-1/2}\\ &\times\left[5e^2(\varsigma)\cos 2\omega(\varsigma) - 3e^2(\varsigma) - 2\right]\\ &-4\gamma\left[1-e^2(\varsigma)\right]^{-2}\end{aligned}\right\}d\varsigma. \tag{4}$$

Using the first integrals allows us to eliminate the variables i and ω from the first equation in (1) and to

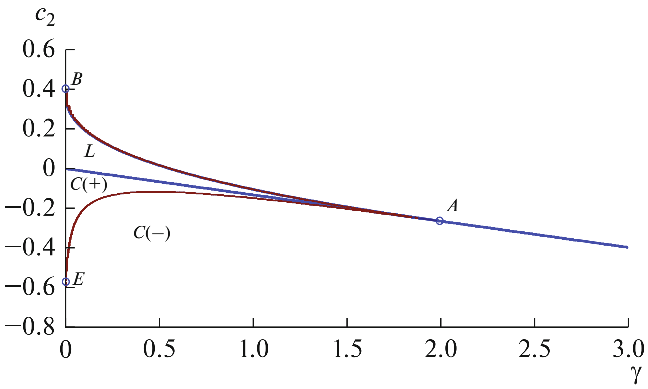

Fig. 1. Libration and circulation regions of the pericenter argument in the plane of problem parameters.

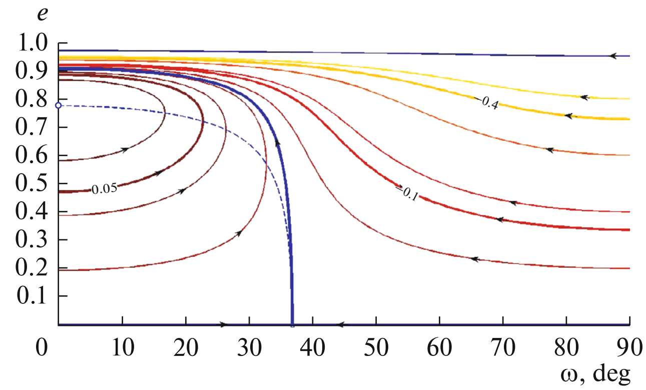

Fig. 2. Family of phase trajectories in the (ω, *e*) plane for $\gamma = 0.2$.

consider only one equation for the eccentricity at various admissible values of the constants c_1 and c_2. Once its solution has been found, the time dependences of the inclination i and pericenter argument ω are determined from Eqs. (3); by symmetry, it will suffice to obtain the dependence $\omega(t)$ only for $0 \le \omega \le 90°$.

Substituting $z = e^2$ reduces the problem to inverting the quadrature

$$\tau = \frac{\operatorname{sign}(\sin 2\omega_0)}{8\sqrt{6}} \int_{z_0}^{z} \frac{d\varsigma}{\sqrt{f(\varsigma, \gamma, c_1, c_2)}}, \quad (5)$$

where

$$f(z, \gamma, c_1, c_2) = f_1(z, \gamma, c_1, c_2)\, f_2(z, \gamma, c_1, c_2),$$
$$f_1(z, \gamma, c_1, c_2) = -z^2 + \left[1 - \frac{5}{3}(c_1 + c_2)\right] z + \frac{5}{3} c_2 + \frac{2}{9}\gamma(1 - z - 3c_1)(1 - z)^{-3/2}, \quad (6)$$
$$f_2(z, \gamma, c_1, c_2) = z - \frac{5}{2} c_2 - \frac{1}{3}\gamma(1 - z - 3c_1)(1 - z)^{-5/2},$$

with the functions f, f_1, f_2, apart from their arguments, also depending on three constants: the parameter of the problem γ and the integral constants c_1, c_2.

The goal of this paper is to investigate the evolution of polar orbits with inclination $i = 90°$ ($c_1 = 0$), including the construction of an approximate analytical solution for the evolution system based on a more comprehensive qualitative analysis.

Remark. At $c_1 \ne 0$ and $\gamma = 0$ the twice-averaged Hill problem (without allowance for the negligible influence of the planet's oblateness) was qualitatively studied in the already mentioned papers (Lidov, 1961; Kozai, 1962), while its general solution was obtained by Vashkov'yak (1999) and Kinoshita and Nakai (1999, 2007). In this case, the function $f(z)$ is a cubic polynomial, while the time dependences of all four elements are described by elliptic functions and integrals. In particular, at $c_1 \approx 0$ ($i \approx 90°$) the Lidov-Kozai effect manifests itself, when an evolving satellite orbit inevitably (in a finite time) crosses the surface of a central planet with a finite radius irrespective of its initial elements. In the general case, the crossing conditions are studied in detail in Prokhorenko (2007).

QUALITATIVE PECULIARITIES OF THE EVOLUTION OF POLAR SATELLITE ORBITS ($i = 90°$, $c_1 = 0$)

This particular solution was first described in Lidov and Yarskaya (1974) and Lidov (1974), where simplified evolution equations and an expression for the first integral were given, the conditions for the existence of singular points were revealed, their types were described, and the qualitative behavior of the integral curves was presented. To avoid misunderstandings, it should be noted that the inaccuracy, which, unfortunately, turned out to be uncorrected after proofreading, of course, was known to the authors of these papers. More specifically, fragments *b* and *c* in Fig. 2 from these papers must be interchanged (but with the retention of the captions!).

The equations defining the equilibrium and extreme values of the variable $\varepsilon = 1 - e^2 = 1 - z$, which is uniquely related to the eccentricity, are given in Prokhorenko (2010, p. 184, Table 4 (1)) as a supplement to the described study.

For coherency, in this and next sections we will repeat several formulas from the above-mentioned papers derived independently in a slightly different notation that we have already used previously and will describe additional details of the qualitative peculiarities of the orbital evolution.

At $c_1 = 0$ the evolution system (1) admits the particular solution $i = i_0 = 90°$, $\Omega = \Omega_0 = \text{const}$ and takes a simpler form:

$$\frac{dz}{d\tau} = 20z\sqrt{1-z}\sin 2\omega,$$
$$\frac{d\omega}{d\tau} = 4\left[\sqrt{1-z}\left(2-5\sin^2\omega\right)-\gamma(1-z)^{-2}\right]. \quad (7)$$

The second of the integrals in Eqs. (3) is also simplified and takes the form

$$z\left(\frac{2}{5}-\sin^2\omega\right)-\frac{2}{15}\gamma(1-z)^{-3/2} = c_2, \quad (8)$$

while in Eqs. (6)

$$f_1(z,\gamma,c_1=0,c_2) = (1-z)\,g_1(z,\gamma,c_2),$$
$$g_1(z,\gamma,c_2) = z+\frac{5}{3}c_2+\frac{2}{9}\gamma(1-z)^{-3/2},$$
$$f_2(z,\gamma,c_1=0,c_2) = g_2(z,\gamma,c_2)$$
$$= z-\frac{5}{2}c_2-\frac{1}{3}\gamma(1-z)^{-3/2}. \quad (9)$$

The constant c_2 is defined by the initial values $z_0 = e_0^2$, ω_0, and, of course, the parameter γ:

$$c_2(\gamma,e_0,\omega_0)$$
$$= e_0^2\left(\frac{2}{5}-\sin^2\omega_0\right)-\frac{2}{15}\gamma\left(1-e_0^2\right)^{-3/2}. \quad (10)$$

In this paper we will consider only the case of positive values of the parameter γ ($c_{20} < 0$), which corresponds to most of the known, actually existing astronomical objects.

In the (ω, e) or (ω, z) plane the evolution of polar orbits is described by a family of phase trajectories corresponding to different values of the integral constant c_2. For $0 \le \gamma < 2$ there exist singular (equilibrium) values

$$\omega^* = 0; \quad e^* = \sqrt{1-(\gamma/2)^{2/5}}. \quad (11)$$

The integral constant

$$c_2^*(\gamma) = c_2(\gamma, e_0 = e^*, \omega_0 = 0)$$
$$= \frac{2}{15}\left[3-5\left(\frac{\gamma}{2}\right)^{2/5}\right], \quad (12)$$

which is marginally possible at a fixed value of the parameter $\gamma < 2$, corresponds to the equilibrium value e^*. The bifurcation value of $\gamma = 2$ $\left[c_2^*(2) = -4/15\right]$ corresponds to the emergence (disappearance) of singular points. No singular points exist for $\gamma > 2$. The family of closed phase trajectories bounded by the limiting trajectory (separatrix) that separates the libration and circulation regions of the pericenter argument ω is located in the vicinity of the stationary points (11). In the phase plane the separatrix is defined by the equation

$$\omega(e,\gamma) = \arcsin\sqrt{\frac{2}{5}\left[1-\frac{\gamma}{3e^2}\left[\left(1-e^2\right)^{-3/2}-1\right]\right]}, \quad (13)$$

which is derived from the integral (8) at

$$c_2 = c_2^{(s)}(\gamma) = -2\gamma/15. \quad (14)$$

At $\sin\omega = 0$ and a fixed value of the parameter $\gamma < 2$ the maximum "separatrix" value $z^{(s)} = e^{(s)2}$ is found as a root of the equation

$$z^{(s)}+\frac{1}{3}\gamma\left[1-\left(1-z^{(s)}\right)^{-3/2}\right] = 0. \quad (15)$$

At $e = 0$ the corresponding "separatrix" value

$$\omega^{(s)}(0,\gamma) = \arcsin\sqrt{(2-\gamma)/5} \quad (16)$$

is also found. When $\gamma = 2$, $c_2^{(s)}(2) = c_2^*(2) = -\frac{4}{15}, \omega^{(s)} = 0$.

The condition $d\omega/d\tau = 0$ gives an equation of the curve of vertical tangents to all libration trajectories of the family:

$$2-5\sin^2\omega-\gamma(1-z)^{-5/2} = 0. \quad (17)$$

Eliminating $\sin^2\omega$ from (17) and (8), we will obtain an equation to find $z^{(l)}$:

$$z^{(l)}-3c_2\left(1-z^{(l)}\right)^{5/2}\Big/\gamma-2/5 = 0. \quad (18)$$

After the numerical solution of this equation, substituting $z^{(l)}$ into (17) gives $\omega^{(l)}$ corresponding to the maximum value of the pericenter argument.

In the plane of parameters (γ, c_2) the open set of possible values for the constant c_2 is defined by the inequalities

$$-\infty < c_2 < c_2^*(\gamma), \gamma \le 2;$$
$$-\infty < c_2 < c_2^{(s)}(\gamma), \gamma > 2.$$

In Fig. 1 the corresponding upper boundary is indicated by the thick lines joining at point A with coordinates (2, – 4/15). At $\gamma < 2$ the thin straight line $c_2^{(s)}(\gamma)$ separates region L, where the pericenter argument ω librates, from regions $C(+)$ and $C(-)$, where ω circulates. The thin curve separating regions $C(+)$ and $C(-)$ corresponds to zero value of the discriminant D of some quadratic equation, which will be described in one of the succeeding sections.

Note that at $\gamma = 0$ points B and E and straight line BE correspond to the boundary points and the base ($c_1 = 0$) of the curvilinear triangle constructed in the

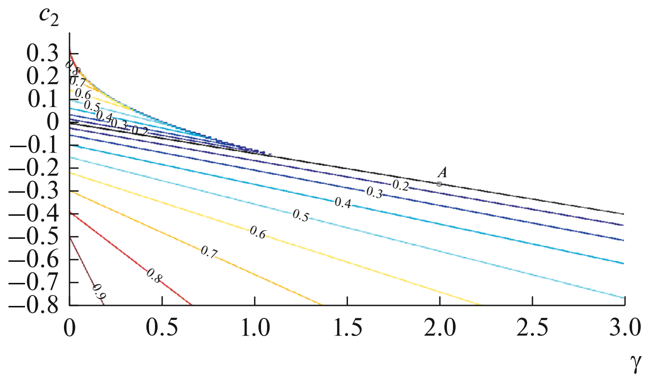

Fig. 3. Isolines $e_{\min}(\gamma, c_2)$ = const.

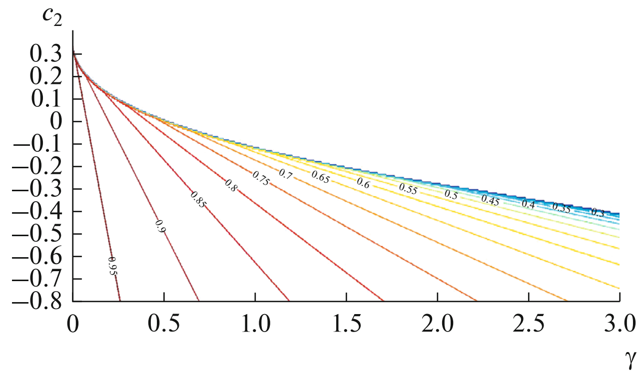

Fig. 4. Isolines $e_{\max}(\gamma, c_2)$ = const.

plane of parameters (c_2, c_1) in Lidov (1961, 1963a, 1962). Curve AB corresponds to a stable equilibrium point (11), while the straight line going away from point A into the region $\gamma > 2$, extending the "separatrix" straight line (14), corresponds to trajectories with $e = 0$.

To illustrate what has been said above, we will give an example of the family of phase trajectories. As a rule, the tools for constructing the isolines of a function of two variables (along with other standard numerical procedures used in this paper) are contained in any mathematical software system, in particular, in MatLab.

Figure 2 shows the family of phase trajectories for $\gamma = 0.2$ and various admissible values of the integral constant c_2. Three of these trajectories indicated by the thick lines on which $c_2 = 0.05$, –0.1, and –0.4 are plotted correspond to regions L, $C(+)$, and $C(-)$, respectively, while the direction of the arrows corresponds to an increase of the independent variable τ and the physical time t. The thick line also indicates the separatrix corresponding to the limiting solution with an infinite period and a maximum eccentricity $e^{(s)} \approx 0.91$. The dashed line indicates the curve of vertical tangents (17) passing through the two singular points of this family marked by the circles.

THE RANGES OF ECCENTRICITY VARIATIONS AND CHARACTERISTIC PERIODS OF THE EVOLUTION OF POLAR ORBITS

The ranges of eccentricity variations for both libration and circulation regions can be estimated by solving some equations. They follow from the integral c_2 at $\omega = 0$ or 90° and are different in form for different intervals of parameters γ and c_2.

At $0 < \gamma < 2$ and $c_2 < c_2^{(s)}(\gamma)$ the extreme values of the variable $z = e^2$ for the circulation region of ω are defined by the equations

$$\begin{aligned} g_1(z_{\min}, \gamma, c_2) &= z_{\min} + \frac{2}{9}\gamma(1 - z_{\min})^{-3/2} \\ &+ \frac{5}{3}c_2 = 0, \omega = 90^\circ, \\ g_2(z_{\max}, \gamma, c_2) &= z_{\max} - \frac{1}{3}\gamma(1 - z_{\max})^{-3/2} \\ &- \frac{5}{2}c_2 = 0, \omega = 0, \end{aligned} \tag{19}$$

each having one positive root in the interval ($0 < z < 1$). In the same range of γ but at $c_2 > c_2^{(s)}(\gamma)$ for the libration region of ω the extreme values of z are defined by two roots of one equation coincident in form with the second of Eqs. (19). These roots are, respectively, in the intervals $(0 < z_{\min} < z^*)$ and $(z^* < z_{\max} < z^{(s)})$. At $\gamma > 2$ the pericenter argument can only circulate, while the extreme values of z are defined by the same equations (19).

Figures 3 and 4 present the results of our numerical solution of Eqs. (19). The isolines of minimum and maximum eccentricities, respectively, are shown in the plane of parameters (γ, c_2). The thick straight line in Fig. 3 corresponds to the dependence (14). The extreme values of the eccentricity, in particular, for the trajectory marked in Fig. 2 by the numerical value of $c_2 = -0.1$, can be approximately determined from these figures. The values of $e_{\min} \approx 0.34$ and $e_{\max} \approx 0.92$, which roughly correspond to the extreme values of e in Fig. 2, are found by graphical interpolation.

The numerical solution of Eq. (8) and the subsequent substitution into (17) make it possible to construct the isolines $\omega^{(l)}(\gamma, c_2)$ = const. This procedure gives a family similar to $e_{\min}(\gamma, c_2)$ = const shown in the narrow region L (Fig. 3). All isolines of $\omega^{(l)}$, starting at point A, end at the boundary of the region at $\gamma = 0$, while the constant values themselves are bounded by

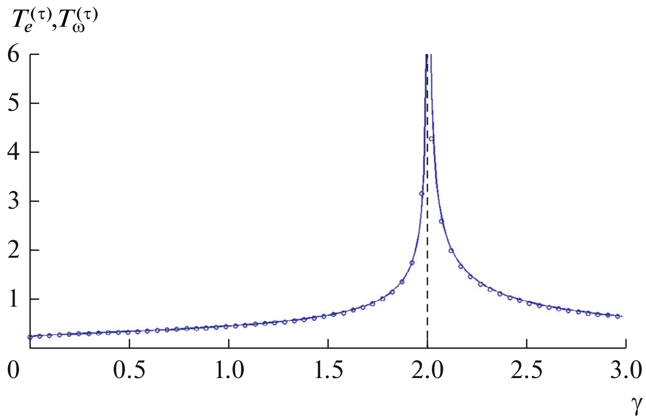

Fig. 5. Dimensionless libration, $T_e^{(\tau)}$ $(\gamma < 2)$, and circulation, $T_\omega^{(\tau)}$ $(\gamma > 2)$, periods versus γ.

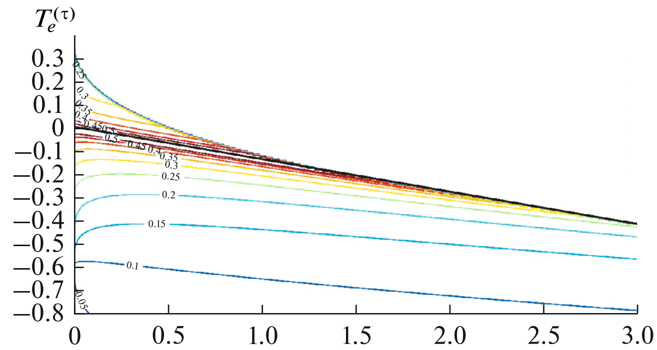

Fig. 6. Isolines $T_e^{(\tau)}(\gamma, c_2) = \text{const}$.

the range $0 \le \omega^{(l)} \le \arcsin\sqrt{2/5}$. No graphical image of this family is given here due to its low informativeness.

The dynamical evolution characteristics, in particular, the libration and circulation periods of the pericenter argument, are also of interest, along with the estimate of the ranges of eccentricity variations. Implying the possibility of numerically finding a definite integral, for example, by the Gaussian method, let us denote

$$J(\gamma, c_2) = \int_{z_{\min}}^{z_{\max}} \frac{dz}{\sqrt{f(z, \gamma, c_2)}}, \tag{20}$$

where the function $f(z)$ is defined by Eqs. (6) at $c_1 = 0$. Then, according to (5), the dimensionless periods of the eccentricity and pericenter argument variations (with τ) can be written as

$$T_e^{(\tau)}(\gamma, c_2) = \frac{J(\gamma, c_2)}{4\sqrt{6}},$$
$$T_\omega^{(\tau)}(\gamma, c_2) = \begin{cases} T_e^{(\tau)}(\gamma, c_2), c_2 > c_2^{(s)} \ (\text{libration of } \omega) \\ 2T_e^{(\tau)}(\gamma, c_2), c_2 < c_2^{(s)} \left(\text{circulation of } \omega \text{ from } 0 \text{ to } 360^\circ\right). \end{cases} \tag{21}$$

The libration and circulation periods of ω near a singular point and zero value of z can be found analytically.

In the vicinity of a stable equilibrium point the period of small e and ω oscillations can be derived from the solution of Eqs. (7) linearized in $e - e^*$ and ω. Its dependence on the parameter γ is defined by the formula

$$T_e^{(\tau)}(\gamma) = \frac{\pi}{10\sqrt{2}}\left[1 - \left(\frac{\gamma}{2}\right)^{2/5}\right]^{-1/2}. \tag{22}$$

The circulation period of the pericenter argument can be derived from the second equation in (7) for $z \to 0$ or

$$\frac{d\omega}{d\tau} = -4\left(\gamma - 2 + 5\sin^2\omega\right). \tag{23}$$

Calculating the modulus of the definite integral $\tau = \frac{1}{4}\int_{\omega_0}^{\omega} \frac{d\vartheta}{2 - \gamma - 5\sin^2\vartheta}$ within the range from 0 to 360° gives the following formula at $\gamma > 2$:

$$T_\omega^{(\tau)}(\gamma) = \frac{\pi}{2\sqrt{(\gamma - 2)(\gamma + 3)}}. \tag{24}$$

The solid curves in Fig. 5 indicate the analytical dependences of the dimensionless libration, $T_e^{(\tau)}$ $(\gamma < 2$, $c_2 \approx c_2^*)$, and circulation, $T_\omega^{(\tau)}$ $(\gamma > 2, c_2 \approx c_2^{(s)})$, periods on γ, the dashed vertical asymptote corresponds to $\gamma = 2$, and the circles mark the values obtained numerically. Figure 6 shows the isolines $T_e^{(t)}(g, c_2) = \text{const}$ in the plane of parameters (γ, c_2).

The following relations serve to pass to the physical time t according to (2):

$$t = \frac{\tau}{n\beta}, T_{e,\omega}^{(t)} = \frac{T_{e,\omega}^{(\tau)}}{n\beta}. \tag{25}$$

Remark. Linearizing Eqs. (7) for $z \to 0$ and $\omega \to \omega^{(s)}$ gives the following expression for the roots $\lambda_{1,2}$ of the characteristic equation:

$$\lambda_{1,2} = \pm 8\sqrt{(2 - \gamma)(3 + \gamma)}, \tag{26}$$

Table 1. Extreme values of the eccentricity and the period of its variations for $\gamma = 0.2$ and the parameter c_2 belonging to different regions

Region	c_2	$e_{\min}$	e^*	$e_{\max}$	$e^{(s)}$	$T_e^{(\tau)}$
L	0.05	0.4714	0.7758	0.8846	0.9067	0.3518
$C(+)$	–0.1	0.3368	–	0.9194	–	0.3390
$C(-)$	–0.4	0.7274	–	0.9447	–	0.1561

which corresponds to the saddle type of this singular point at $\gamma < 2$.

The isolines displayed in Figs. 3, 4, and 6, along with Eq. (25), make it possible to roughly estimate the main characteristics of an evolving orbit for specific values of the parameters γ and c_2: the extreme values of the eccentricity and the period of its variations. Above, in Table 1, these characteristics obtained by numerically solving the corresponding equations are given for $\gamma = 0.2$ and three values of c_2 corresponding to three different regions in Fig. 1.

The values of $e_{\min}$, $e_{\max}$, and $T_e^{(\tau)}$ can be approximately found by graphical interpolation using Figs. 3, 4, and 6, respectively. In addition, the extreme and singular values of the eccentricity can also be obtained from Fig. 2. For the constant $c_2 = 0.05$ (region L) the numerical solution gives $\omega^{(l)} \approx 22°.7$ for the libration amplitude of the pericenter argument and the corresponding value of $e^{(l)} \approx 0.72$. In the figure these are approximate coordinates of the point of intersection between the dashed and thick libration curves.

In the next sections we will describe a method for finding an approximate solution of the evolution system (7) using the results of its qualitative study and the possibility of a rigorous calculation of the main quantitative characteristics of the evolution of orbital elements.

APPROXIMATION OF THE IRRATIONAL PART OF THE INTEGRAND AND INVERSION OF THE INTEGRAL

The dependence of τ on z is defined by Eqs. (5) and (6), where the expressions for the functions f_1 and f_2 take the simpler form (9) at $c_1 = 0$. In this case, one conceivable way to find the integral (5) could be an expansion of the irrational terms of these functions into power series of z. However, for the integral (5) to be reduced to an elliptic (or at least ultra-elliptic) one, the maximum degree of such expansions must be low (3 or 4). In this way, a constraint in eccentricity (or in z) on the correctness of finding the integral and on the applicability of its inversion would be introduced. This will be illustrated below with a numerical example.

In this paper we propose a different approximate method of solving the problem. For the motion to be real, the function f in the integrand of (5) must be nonnegative, and, as our qualitative analysis of system (7) shows, the variable z changes in the limited range $z_{\min} \le z \le z_{\max}$.

Let us represent the function f as

$$f(z,\gamma,c_2) = g(z,\gamma,c_2)(z - z_{\min})(z_{\max} - z), \quad (27)$$

and the following obvious equalities serve as both a definition and a way of calculating the function $g(z,\gamma,c_2)$:

$$g(z,\gamma,c_2) = (1-z)\frac{g_1(z,\gamma,c_2)\,g_2(z,\gamma,c_2)}{(z - z_{\min})(z_{\max} - z)} = \frac{f(z,\gamma,c_2)}{(z - z_{\min})(z_{\max} - z)}. \quad (28)$$

The expressions for the function g at special values of its argument can be derived from this equality, where g_1 and g_2 are defined by Eqs. (9):

Table 2. Location of the roots of the polynomial $P(z)$ in different sets of the (γ, c_2) plane

Set	Roots of polynomial $P(z)$	Discriminant	Moduli of elliptic functions
Line BA $c_2(\gamma) = c_2^*(\gamma)$	$z_2 < z_1 < 0 < z_3 = z = z_4$	$D > 0$	$k = 0$
Region L	$z_2 < z_1 < 0 < z_3 < z < z_4$	$D > 0$	$0 < k < 1$
Line $c_2(\gamma) = c_2^{(s)}(\gamma)$ In region $0 < \gamma < 2$	$z_2 < z_1 = 0 = z_3 < z < z_4$	$D > 0$	$k = 1$
Region $C(+)$	$z_2 < z_1 < 0 < z_3 < z < z_4$	$D > 0$	$0 < k < 1$
Line EA	$z_2 = z_1 < 0 < z_3 < z < z_4$	$D = 0$	$k = \sigma = 0$
Region $C(-)$	z_1, z_2 complex-conjugate, $0 < z_3 < z < z_4$	$D < 0$	$0 < \sigma < 1$
Line $c_2 = c_2^{(s)}(\gamma)$ In region $\gamma > 2$	z_1, z_2 complex-conjugate, $0 < z_3 = z = z_4$	$D < 0$	$\sigma = 0$

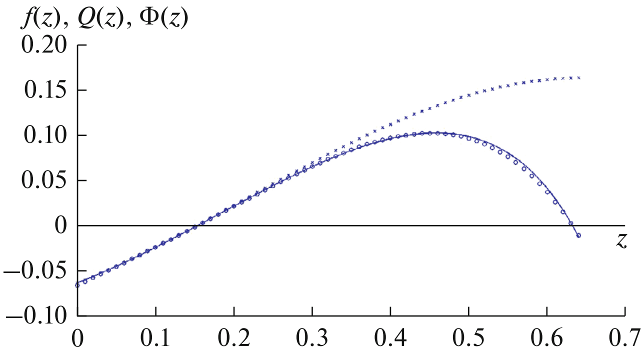

Fig. 7. The z dependences of the functions $f(z)$ (solid curve), $Q(z)$ (circles), and $\Phi(z)$ (crosses).

$$g(0,\gamma,c_2) = -\frac{g_1(0,\gamma,c_2)\,g_2(0,\gamma,c_2)}{z_{\min}z_{\max}} = \frac{2}{3z_{\min}z_{\max}}\left(\frac{1}{3}\gamma + \frac{5}{2}c_2\right)^2, \tag{29}$$

$$\lim_{z\to z_{\min}} g(z,\gamma,c_2) = \frac{5}{2}\left[\frac{\gamma}{3}(1-z_{\min})^{-5/2}+1\right]\frac{z_{\min}(1-z_{\min})}{z_{\max}-z_{\min}}, \tag{30}$$

$$\lim_{z\to z_{\max}} g(z,\gamma,c_2) = \frac{5}{3}\left[\frac{\gamma}{2}(1-z_{\max})^{-5/2}-1\right]\frac{z_{\max}(1-z_{\max})}{z_{\max}-z_{\min}}. \tag{31}$$

Below we will use a methodical technique that simplifies the function g (and, consequently, also f) to such an extent that the integral (5) at $c_1 = 0$ can be reduced to an elliptic form. In the product (27) the extreme values of the variable z are known from the numerical solution of the corresponding equations (19). The product of two multipliers $(z - z_{\min})(z_{\max} - z)$, which mainly determines the behavior of the function f in the range $0 \le z < 1$, below will be arbitrarily called the "kernel" for short. Leaving only the arguments of the functions in the subsequent formulas for their simplification, we will note that the function $g(z)$ is obtained by dividing the original function $f(z)$ by the kernel according to Eq. (28). To reduce the integral (5) at $c_1 = 0$ to an elliptic form, we will approximate the function $g(z)$ by a quadratic polynomial:

$$P(z) = p_1z^2 + p_2z + p_3 \approx g(z), \tag{32}$$

where the coefficients p_1, p_2, p_3 are determined by the least-squares technique (such a procedure in the MatLab environment also belongs to the class of standard ones). The roots of the polynomial $P(z)$ are defined by very simple formulas:

$$z_1 = \frac{1}{2p_1}\left(-p_2 + \sqrt{D}\right),$$
$$z_2 = \frac{1}{2p_1}\left(-p_2 - \sqrt{D}\right), D = p_2^2 - 4p_1p_3. \tag{33}$$

Turning to Fig. 1, we will point out that the thin curve, which was previously mentioned when describing it, corresponds to zero values of the discriminant D, with $D > 0$ in regions L, $C(+)$ and $D < 0$ in region $C(-)$. The quartic polynomial

$$Q(z) = P(z)(z - z_{\min})(z_{\max} - z) \approx f(z) \tag{34}$$

will be an approximating one for the function $f(z)$ and will be represented as

$$Q(z) = p_1(z - z_1)(z - z_2)(z - z_3)(z_4 - z), \tag{35}$$

where $z_3 = z_{\min}, z_4 = z_{\max}$.

Since, by its physical meaning, the variable $z = e^2$ satisfies the inequalities $0 \le z < 1$, it is natural to construct the approximation on the same set of values. However, specific calculations show that due to the singularity of the function $f(z)$ at $z = 1$, for the approximation to be successful, its upper boundary must be moved leftward to some positive value z_a that will thus play the role of a methodical constant. In our subsequent calculations we take $z_a = 0.5$. In addition, note that the coefficients p_1, p_2, and p_3 are positive in all of our test calculations.

The solid line and the circles in Fig. 7 indicate the dependence $f(z)$ and its approximation $Q(z)$, respectively. The crosses indicate the corresponding dependence $\Phi(z)$ obtained by expanding the irrational terms in the product of the functions $g_1(z)$ and $g_2(z)$ to within z^3 inclusive. The correctness of these expansions is limited by $z \approx 0.3$. This illustration is given for the orbit of a lunar satellite with the following parameters: $a =$ 3500 km ($\gamma \approx 0.8121$), $e_0 = 0.4$, $i_0 = 90°$, $\omega_0 = 80°$ ($c_2 \approx$ -0.2318), while the perturbing bodies are the Earth and the Sun.

Thus, at fixed parameters of the problem γ and c_2 the sought-for analytical solution requires performing a number of preliminary calculations of both the extreme values of the eccentricity and the coefficients of the approximating polynomial for its construction. Thereafter, a general solution of the problem at $c_1 = 0$ in the form of time dependences of the elements e and ω is found by inverting the quadrature (5). This inversion is performed differently for different ranges of parameters γ and c_2. Below we provide the formulas to find $e(\tau)$, $\omega(\tau)$, and some intermediate quantities derived using the reference book by Gradshteyn and Ryzhik (1962).

The Solution for Regions L and C(+)

In the case of $D > 0$ under consideration, all four roots of the polynomial $P(z)$ are real and, together with the variable z, satisfy the inequalities

$$z_2 < z_1 < z_3 < z < z_4, \tag{36}$$

while the extreme values of z are defined by the obvious formulas

$$z_3 = e^2_{\min}, z_4 = e^2_{\max}. \tag{37}$$

The dependence of the eccentricity on τ is defined by the formulas

$$\begin{gathered} e(\tau) = \sqrt{\frac{z_1(z_4 - z_3)\operatorname{sn}^2 u - z_3(z_4 - z_1)}{(z_4 - z_3)\operatorname{sn}^2 u - z_4 + z_1}}, \\ k^2 = \frac{(z_4 - z_3)(z_1 - z_2)}{(z_4 - z_1)(z_3 - z_2)}, \\ u = 4\operatorname{sign}(\sin 2\omega_0)\sqrt{6p_1(z_4 - z_1)(z_3 - z_2)}\tau \\ + F(\varphi_0, k^2), \\ \sin^2\varphi_0 = \frac{(z_4 - z_1)(z_0 - z_3)}{(z_4 - z_3)(z_0 - z_1)}, z_0 = e_0^2, \end{gathered} \tag{38}$$

where sn$u$ and $F(\varphi_0, k^2)$ are, respectively, the Jacobi elliptic sine and an incomplete elliptical integral of the first kind with modulus k.

In the range of the variable z the modulus k remains less than one. It becomes zero at $z_3 = z_4$, i.e., on line BA in Fig. 1, on which $c_2(\gamma) = c_2^*(\gamma)$, corresponding to a stable singular point in the (ω, e) plane, and at $z_1 = z_2$, when $D = 0$, i.e., on line EA. The value of $k = 1$ is reached at $z_1 = z_3 = 0$, i.e., on the straight line $c_2(\gamma) = c_2^{(s)}(\gamma)$, separating regions L and $C(+)$ that corresponds to the separatrix in the (ω, e) plane.

The dependence $\omega(\tau)$ is found using the integral (8) and the general formula following from it

$$\omega(\tau) = \operatorname{Arcsin}\sqrt{\frac{2}{5} - \frac{1}{e^2(\tau)}\left\{\frac{2}{15}\gamma\left[1 - e^2(\tau)\right]^{-3/2} + c_2\right\}}. \tag{39}$$

The Solution for Region C*(–)*

In the case of $D < 0$ under consideration, the polynomial $P(z)$ has two complex-conjugate roots

$$\begin{gathered} z_1 = \frac{1}{2p_1}\left(-p_2 + i\sqrt{-D}\right), \\ z_2 = \frac{1}{2p_1}\left(-p_2 - i\sqrt{-D}\right), i = \sqrt{-1} \end{gathered} \tag{40}$$

(obviously, here it is impossible to confuse the imaginary unit with the orbital inclination i) and two real roots (37), with $z_3 < z < z_4$. The dependence of the eccentricity on τ is defined by the formulas

$$\begin{gathered} e(\tau) = \sqrt{\frac{pz_3 + qz_4 + (pz_3 - qz_4)\operatorname{cn}u}{p + q + (p - q)\operatorname{cn}u}}, \\ \sigma^2 = \frac{(z_4 - z_3)^2 - (p - q)^2}{4pq}, \\ u = 8\operatorname{sign}(\sin 2\omega_0)\sqrt{6p_1pq}\tau + F(\varphi_0, \sigma^2), \\ \tan^2\frac{\varphi_0}{2} = \frac{p(z_0 - z_3)}{q(z_4 - z_0)}, \\ p = \sqrt{(m_1 - z_4)^2 + m_2^2}, q = \sqrt{(m_1 - z_3)^2 + m_2^2}, \\ m_1 = -\frac{p_2}{2p_1}, m_2 = \frac{\sqrt{-D}}{2p_1}, \end{gathered} \tag{41}$$

where cn$u$ and $F(\varphi_0, \sigma^2)$ are, respectively, the Jacobi elliptic cosine and an incomplete elliptic integral of the first kind with modulus σ.

In the range of the variable z the modulus σ remains less than one. It becomes zero either at $p = q$, i.e., at $D = 0$ (curve EA in Fig. 1) or at $z_3 = z_4$, i.e., on the straight line $c_2(\gamma) = c_2^{(s)}(\gamma)$, going to the right downward from point A. In this case, there are no singular points in the (ω, e) plane.

The dependence $\omega(\tau)$, as in case 5.1, is found using Eq. (39).

Above we provide a Table 2 with the inequalities defining the type and location of the roots of the polynomial $P(z)$. In addition, it gives the signs of the discriminant as well as the ranges and limiting values of the moduli of the elliptic functions entering into the solution.

Note that in the rows corresponding to regions L and $C(+)$ the inequalities of each column coincide, while at the bifurcation point A (2, –4/15) itself all four roots z_1, z_2, z_3, z_4 become zero.

COMPARISON OF THE PROPOSED SOLUTION WITH THE NUMERICAL ONE

In this section, as a numerical example, we consider the evolution of several orbits of lunar satellites under the secular perturbations caused by the attraction from the Earth and the Sun, which, however, exerts an influence smaller by two orders of magnitude. In the proposed constructive-analytical solution the equatorial and orbital planes of the Moon are assumed to coincide between themselves and to lie in the ecliptic plane. When obtaining a comparative solution, we numerically integrate an evolution system including the mean inclination i of a "nonevolving" lunar orbit to the ecliptic that is more rigorous than system (1). This system in equatorial Keplerian elements was taken from Vashkov'yak et al. (2015), in which we should adopt $\gamma_0 = \gamma, \gamma_{k\neq 0} = 0, I = 5^\circ.15$.

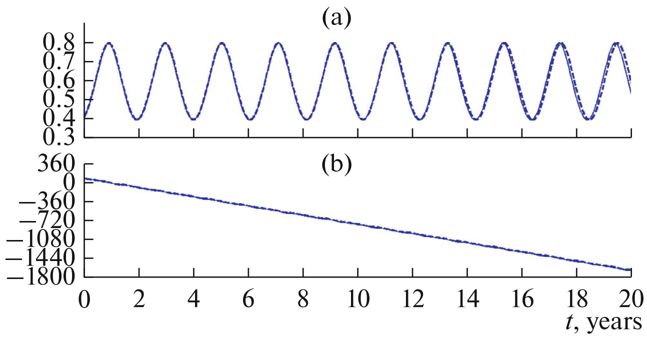

Fig. 8. The time dependences of the elements for $\omega_0 = 80°$: (a) the eccentricity and (b) the pericenter argument.

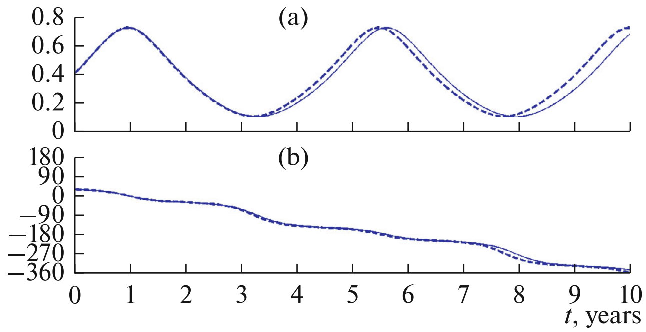

Fig. 9. The same as Fig. 8 on a ten-year time interval and for $\omega_0 = 30°$.

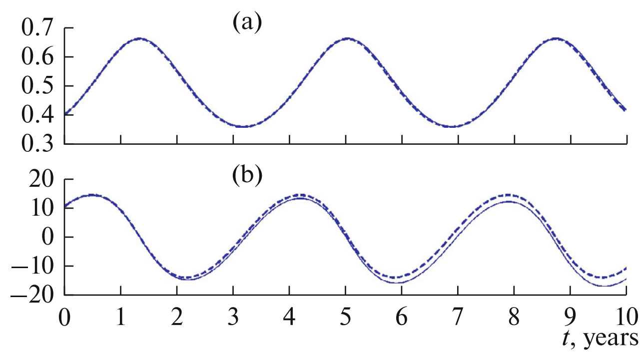

Fig. 10. The same as Fig. 9 for $\omega_0 = 10°$.

As in the previous section, the semimajor axis of the satellite orbit was chosen to be 3500 km, corresponding to $\gamma \approx 0.8121$ and the satellite's orbital period of about 5 h 10 min. As the remaining initial elements we took $e_0 = 0.4$, $i_0 = 90°$, and several different values of ω_0.

The constructed approximate solution of the evolution problem at various values of the parameters γ and c_2 showed quite satisfactory agreement with the results of our numerical integration of the rigorous system. Below we demonstrate the results of our comparative calculations performed by the above two methods. **These examples are given only with a methodical goal** and are associated with fictitious satellite orbits, because the minimum pericenter distance becomes less than the lunar radius during their evolution.

Figures 8a and 8b shows the results of our calculations with $\omega_0 = 80°$, the integral constant is $c_2 \approx -0.2318$, ($D < 0$, $\sigma^2 \approx 0.004$), corresponding to region $C(-)$.

The solid and dashed curves indicate the time dependences of the elements derived by numerically integrating the evolution system and by the proposed method, respectively. The values of ω on fragments (b) of these figures are given in degrees. Our comparison was made on a twenty-year time interval, which is tens of thousands of orbital periods of the satellite and definitely exceeds the orbital periods of the perturbing bodies as well as the characteristic periods of the eccentricity variations.

Figures 9a and 9b show the results of our calculations with $\omega_0 = 30°$, the integral constant is $c_2 \approx -0.1166$, ($D > 0$, $k^2 \approx 0.75$), corresponding to region $C(+)$.

Figures 10a and 10b show the results of our calculations with $\omega_0 = 10°$, the integral constant is $c_2 \approx -0.0815$, ($D > 0$, $k^2 \approx 0.43$), corresponding to region L.

The above examples of applying the proposed method of solving the model problem allow its error relative to the nonintegrable problem to be estimated (in a real situation $I \neq 0$), ~10%. Note that in our test calculations, when specifying $I = 0$ on the right-hand sides of the evolution system, the results agree much better. More specifically, the solid and dashed curves on the time intervals under consideration in the scales of the figures turn out to be virtually indistinguishable visually, which, incidentally, suggests that the derived formulas are correct. Therefore, replacing the function $f(z)$ by the approximating polynomial $P(z)$ seems quite proper and provides an acceptable methodical error of the solution for the model evolution problem.

ON THE STABILITY OF POLAR ORBITS WITH RESPECT TO THE INCLINATION

Obviously, the revealed peculiarities of the evolution of polar orbits, its quantitative characteristics, and the constructive-analytical solution of the evolution problem itself are valid only when the inclination i of the satellite orbit on the time interval under consideration remains close to 90°. Although in the integrable special case of the model problem ($I = 0$) this condition is fulfilled, $i(t) = i_0 = 90°$, in general, the actual deviation of the plane of motion of the perturbing body from the equatorial one can change i appreciably. However, one might expect at a small value of the parameter

$$s = \sin I \qquad (42)$$

the satellite orbit to remain nevertheless nearly polar. This assertion is confirmed by considering the evolution system used in the comparative calculations of the previous section. When the small quantities $\sim s^2$ are discarded, linearizing this system in variable

$$\delta = i - \pi/2 \qquad (43)$$

leads to the following equations:

$$\frac{de}{d\tau} = 10e\sqrt{1-e^2} \times [\sin 2\omega - 2s(\sin\Omega\cos 2\omega - \delta\cos\Omega\sin 2\omega)], \tag{44}$$

$$\frac{d\delta}{d\tau} = \frac{1}{\sqrt{1-e^2}}\{-10se^2\cos\Omega\sin 2\omega + 2\delta \times \left[5e^2\sin 2\omega - s\sin\Omega\left(2 + 3e^2 + 5e^2\cos 2\omega\right)\right]\}, \tag{45}$$

$$\frac{d\omega}{d\tau} = \frac{1}{\sqrt{1-e^2}} \times \left\{\begin{array}{l} -4\gamma\left(1-e^2\right)^{-3/2} + 2\left(1-e^2\right)(5\cos 2\omega - 1) \\ +20s\left(1-e^2\right)\sin\Omega\sin 2\omega - \\ -2s\delta\cos\Omega\left[8 - 3e^2 + 5\left(e^2 - 2\right)\cos 2\omega\right] \end{array}\right\}, \tag{46}$$

$$\frac{d\Omega}{d\tau} = \frac{1}{\sqrt{1-e^2}} \times \left\{\begin{array}{l} 2s\left(5e^2\cos 2\omega - 3e^2 - 2\right)\cos\Omega \\ -\,2\delta\left[\begin{array}{l} -4\gamma\left(1-e^2\right)^{-3/2} + 5e^2\cos 2\omega - 3e^2 - 2 \\ +\,5se^2\sin\Omega\sin 2\omega \end{array}\right] \end{array}\right\}. \tag{47}$$

At small s and δ finding the time dependences of the variables δ and Ω in the first approximation is reduced to solving two first-order differential equations (one of them is linear in δ). Their right-hand sides contain both $\sin\Omega$, $\cos\Omega$ and $e(\tau)$, $\omega(\tau)$ – the periodic functions of an independent variable known from the solution of the evolution system at $s = \delta = 0$. The dependences $\delta(\tau)$ and $\Omega(\tau)$ are easiest to find in the cases of a constant eccentricity, when $e = 0$ at any ω or when $e = e^*$ at $\omega = 0$.

If $e = 0$, then the approximate dependences

$$\delta(\tau) = \delta_0\exp\left\{\frac{4s}{\Omega_0'}[\cos\Omega(\tau) - \cos\Omega_0]\right\},$$
$$\Omega(\tau) = \Omega_0 + \Omega_0'\tau, \tag{48}$$
$$\Omega_0' = 4[(1 + 2\gamma)\delta_0 - s\cos\Omega_0],$$

where δ_0 and Ω_0 are the initial values of the variables at $\tau = 0$, follow from Eqs. (45) and (47).

If $e = e^*$ and $\omega = 0$, then it follows from the same equations that

$$\delta(\tau) = \delta_0\exp\left\{\frac{s}{5\delta_0 - s\cos\Omega_0} \times \left[5\left(\frac{\gamma}{2}\right)^{2/5} - 4\right][\cos\Omega(\tau) - \cos\Omega_0]\right\},$$
$$\Omega(\tau) = \Omega_0 + \Omega_0'\tau, \tag{49}$$
$$\Omega_0' = 4\left(\frac{\gamma}{2}\right)^{1/5}(5\delta_0 - s\cos\Omega_0).$$

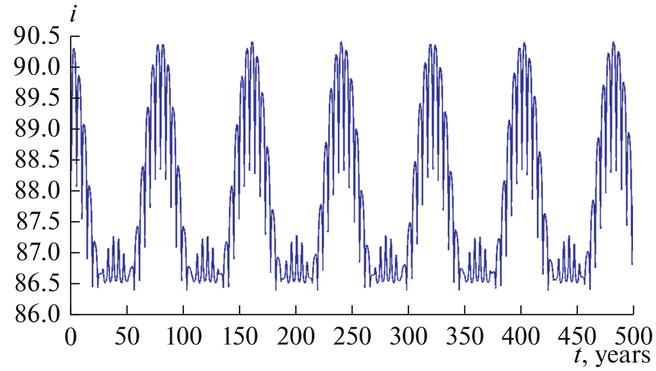

Fig. 11. The time dependence of the inclination for $\omega_0 = 30°$.

Thus, in both special cases, $\Omega(\tau)$ changes linearly with time, while $\delta(\tau)$ is a bounded function. Of course, to be more convincing, the stability of polar orbits with respect to the inclination is required to be studied in the case of arbitrary eccentricities different from the constant values 0 and e^*.

For $I = 5°.15$ our numerical integration of the full evolution system on an interval of 500 years showed that for the three variants of initial conditions adopted in the previous section the inclination of the satellite orbit differs from 90° by no more than ~4°. Interestingly, for the range of initial values of the pericenter argument $20° < \omega_0 < 50°$ the inclination can exceed 90° on relatively short intervals. These short, so-called flips, the transitions of the orbit from prograde to retrograde and conversely, can be seen in Fig. 11 ($\omega_0 = 30°$).

Here the shorter-period ($T_e \approx 4.5$ years) oscillations associated with the eccentricity variations (Fig. 8) are superimposed on the long-period inclination variations associated with the motion of the ascending node ($T_\Omega \approx 85$ years).

CONCLUSIONS

In this paper we proposed a method for constructing an approximate analytical solution of the integrable problem of the secular (long-period) evolution of a class of polar satellite orbits. The physical model includes the main perturbing factors from the oblateness of the central planet and from a remote attracting point, while the corresponding evolution system was obtained by averaging the full perturbing function over all fast variables. Using the well-known first integral of the system, we described its qualitative peculiarities and determined the main quantitative evolution characteristics.

The solution of the evolution system proposed here is basically a constructive-analytical one, because the predetermined extreme values of the eccentricity of the evolving orbit should be used for its construction. These values are found numerically as solutions of the rigorous equations. The subsequently performed reduction of quadrature (5) at $c_1 = 0$ to an elliptic form is based on the representation of the function $f(z)$ as a product $(z - z_{\min})(z_{\max} - z)$, the so-called "kernel" and the function $g(z)$, followed by its approximation by a quadratic polynomial. In a given interval of the approximation its discriminant defines the type of roots for the approximating polynomial and the different form of the analytical time dependences of the eccentricity, while the pericenter argument is found from the known first integral of the evolution system.

Using several polar orbits of lunar satellites as an example, we showed satisfactory agreement between the results of our calculations performed by the proposed method and by numerically integrating the more rigorous evolution system in which the deviation of the orbital planes of the perturbing bodies (the Earth and the Sun) from the plane of the lunar equator was taken into account. The relative stability of the inclination of the orbits under consideration on long time intervals determines the correctness of using the qualitative peculiarities of the evolution and its quantitative characteristics as well as the legitimacy of applying our analytical solution to some satellite problems. Applying the proposed constructive-analytical approach to the general coplanar case of the evolution problem under consideration at $c_1 \neq 0$ could become the development of this work.

REFERENCES

Allan, R.R. and Cook, G.E., The long-period motion of the plane of a distant circular orbit, *Proc. R. Soc. London, Ser. A*, 1964, vol. 280, no. 1380, pp. 97–109.

Brumberg, V.A., Evdokimova, L.S., and Kochina, N.G., Analytical methods for the orbits of artificial satellites of the Moon, *Celest. Mech.*, 1971, vol. 3, pp. 197–221.

Gradshtein, I.S. and Ryzhik, I.M., *Tablitsy integralov, sum, ryadov i proizvedenii (Tables of Integrals, Sums, Series, and Products)*, Moscow: Fizmatgiz, 1962.

Kinoshita, H. and Nakai, H., Analytical solution of the Kozai resonance and its application, *Celest. Mech. Dyn. Astron.*, 1999, vol. 75, pp. 125–147.

Kinoshita, H. and Nakai, H., General solution of the Kozai mechanism, *Celest. Mech. Dyn. Astron.*, 2007, vol. 98, pp. 67–74.

Kozai, Y., Secular perturbations of asteroids with high inclination and eccentricity, *Astron. J.*, 1962, vol. 67, pp. 591–598.

Kozai, Y., Motion of Lunar orbiter, *Publ. Astron Soc. Jpn.*, 1963, vol. 15, no. 3, pp. 301–312.

Kudielka, V.W., Equilibria bifurcations of satellite orbits, in *The Dynamical Behavior of our Planetary System*, Dvorak, R. and Henrard, J., Eds., Dordrecht: Kluwer, 1997, pp. 243–255.

Lidov, M.L., Evolution of the orbits of artificial satellites of planets affected by gravitational perturbations of external objects, *Iskusstv. Sputniki Zemli*, 1961, no. 8, pp. 5–45.

Lidov, M.L., The evolution of orbits of artificial satellites of planets under the action of gravitational perturbations of external bodies, *Planet. Space Sci.*, 1962, vol. 9, pp. 719–759).

Lidov, M.L., *The approximate analysis of the evolution of orbits of artificial satellites, Problemy dvizheniya iskusstvennykh nebesnykh tel (The Motion of Artificial Celestial Objects)*, Moscow: Akad. Nauk SSSR, 1963a, pp. 119–134.

Lidov, M.L., On the approximated analysis of the orbit evolution of artificial satellites, *International Union for Theoretical and Applied Mechanics (IUTAM) Symp. "Dynamics of Satellites," Paris, May 28–30, 1962*, Roy, M., Ed., Berlin: Springer-Verlag, 1963b, pp. 168–179.

Lidov, M.L., Integrable cases of satellite problem with the third body and the oblate planet, *Proc. 62nd International Astronomical Union Symp. "The Stability of the Solar System and Small Stellar Systems,"* Kozai, Y., Ed., Boston: D. Reidel, 1974, pp. 117–124.

Lidov, M.L. and Yarskaya, M.V., Integrable cases in the problem of the evolution of a satellite orbit under the joint effect of an outside body and of the noncentrality of the planetary field, *Cosmic Res.*, 1974, vol. 12, no. 2, pp. 139–152.

Prokhorenko, V.I., The conditions for the intersection of the satellite's orbit with the surface of a central body of finite radius in a double-averaged restricted problem of the three bodies, Tr. Mat. Inst. im. *V.A. Steklova*, 2007, vol. 259, pp. 156–173.

Prokhorenko, V.I., Planetocentric gravitational sphere of the dominant influence of perturbations from planet oblateness over perturbations from external bodies, *Cosmic Res.*, 2010. vol. 48, no 2, pp. 174–187.

Qesterwinter, C., The motion of a lunar satellite, *Celest. Mech.*, 1970, vol. 1, pp. 368–436.

Sekiguchi, N., On the libration of the orbital plane of a stationary artificial satellite with a circular orbit, *Publ. Astron. Soc. Jpn.*, 1961, vol. 13, no. 2, pp. 207–211.

Shevchenko, I.I., *The Lidov-Kozai Effect—Applications in Exoplanet Research and Dynamical Astronomy*, New York: Springer-Verlag, 2017.

Uphoff, C., Stabilizing influence of Earth perturbations on Polar Lunar Orbiters, *AIAA Pap.*, 1976, vol. 843, pp. 1–6.

Vashkov'yak, M.A., Stability of circular satellite orbits for combined action of perturbations from an external body and from the noncentrality of the planetary gravitational field, *Cosmic Res.*, 1974, vol. 12, no. 6, pp. 757–770.

Vashkov'yak, M.A., On the special particular solutions of a double-averaged Hill's problem with allowance for flattering of the central planet, *Astron. Lett.*, 1996, vol. 22, no. 2, pp. 207–216.

Vashkov'yak, M.A., On the stability of stationary solutions of the double-averaged Hill's problem with an oblate central planet, *Astron. Lett.*, 1998, vol. 24, no. 5, pp. 682–691.

Vashkov'yak, M.A., Evolution of the orbits of distant satellites of Uranus, *Astron. Lett.*, 1999, vol. 25, no. 7, pp. 476–481.

Vashkov'yak, M.A., Vashkov'yak, S.N., Emel'yanov, N.V., On the evolution of satellite orbits under the action of the planet's oblateness and attraction by its massive satellites and the Sun, *Sol. Syst. Res.*, 2015, vol. 49, no. 4, pp. 283–299.

Translated by V. Astakhov