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Gravitational Instability in the Dust Layer of a Protoplanetary Disk: Interaction of Solid Particles with Turbulent Gas in the Layer

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Abstract—Gravitational instability of the dust layer formed after the aggregates of dust particles settle toward the midplane of a protoplanetary disk under turbulence is considered. A linearized system of hydrodynamic equations for perturbations of dust (monodisperse) and gas phases in the incompressible gas approximation is solved. Turbulent diffusion and the velocity dispersion of solid particles and the perturbation of gas azimuthal velocity in the layer upon the transfer of angular momentum from the dust phase due to gas drag are taken into account. Such an interaction of the particles and the gas establishes upper and lower bounds on the perturbation wavelength that renders the instability possible. The dispersion equation for the layer in the case when the ratio of surface densities of the dust phase and the gas in the layer is well above unity is obtained and solved. An approximate gravitational instability criterion, which takes the size-dependent stopping time of a particle (aggregate) in the gas into account, is derived. The following parameters of the layer instability are calculated: the wavelength range of its subsistence and the dependence of the perturbation growth rate on the perturbation wavelength in the circumsolar disk at a radial distance of 1 and 10 AU. It is demonstrated that at a distance of 1 AU, the gas–dust disk should be enriched with solids by a factor of 5–10 relative to the initial abundance as well as the particle aggregates should grow to the sizes higher than about 0.3 m in order for the instability to emerge in the layer in the available turbulence models. Such high disk enrichment and aggregate growth is not needed at a distance of 10 AU. The conditions under which this gravitational instability in the layer may be examined with no allowance made for the transfer of angular momentum from the gas in the layer to the gas in a protoplanetary disk outside the layer are discussed.

Keywords: planet formation, protoplanetary disk, gravitational instability, planetesimals
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INTRODUCTION

The process of formation of self-gravitating planetesimals (≥1 km in size) in a protoplanetary disk remains understudied. At least two obstacles to their formation are known. The first one is the inefficiency of growth (by sticking) of dust particle aggregates with boulder sizes ranging from 0.1 to 10 m (and above). The second one consists in that such bodies have the maximum velocity of radial drift to the disk center and thus may fall to the Sun (or the central star) from a distance of 1 AU in just 100 years (bodies with sizes in the decimeter to meter range) or several thousand years (10-meter bodies) (Weidenschilling, 1977). The presence of these barriers motivates the search for other options for the formation of self-gravitating planetesimals. It is likely that the studies into gravitational instability (GI) in the dust layer, which forms as particles settle toward the midplane of a gas–dust protoplanetary disk and, upon reaching critical density $\rho_{cr} \approx 2\rho^* = 3M_*/2\pi r^3$, where M_* is the mass of the central star and r is the distance to it, segregates into ring dust condensations (Safronov, 1969), were chronologically the first to be initiated within this research trend. At distance r = 1 AU, relation $\rho_{cr} \sim 10^2\rho_g$, where ρ_g is the gas density, holds, while $\rho_{cr} \sim 10\rho_g$, at 10 AU, and the spatial density of the dust phase before dust deposition at any distance r is $\rho_p \sim 10^{-2}\rho_g$. The term “dust layer” or “particle layer” is misleading, since the typical size of “particles” in it may reach 10 cm^{-1} m (Safronov, 1991; Cuzzi et al., 1993). They grow this large by coalescing in collisions while settling to the midplane (or within the layer). Therefore, the dust phase in the layer may be denoted as a solid coarse phase. Note that the volume fraction of the dust phase in the layer is very small (even at the critical density): $s_p = \rho_{cr}/\rho_s \sim 10^{-7}$–$10^{-4}$ at distances ranging from 1 to 10 AU, where ρ_s is the solid particle density.

Gas turbulence hinders the process of thinning and densifying of the dust layer to the state in which GI is possible. Turbulence remains localized near the midlplane in the region of the dust layer even after global turbulence in the disk subsides (Weidenschilling, 1984). It is generated by shear flow induced by the difference between the velocities of gas rotation in the dust layer and in the remaining (upper- and lower-lying) parts of the disk. This difference is attributed to the fact that gas in the dust layer, the mass of which dominated by dust, is accelerated by solid particles through viscous drag to an almost Keplerian velocity, while gas outside the layer flows with a lower velocity supported by the radial pressure gradient (Goldreich and Ward, 1973). Considerable shear stresses are produced owing to a large vertical gradient of orbital velocity of gas. These stresses induce turbulence, which establishes turbulent diffusion of particles, prevents their further sedimentation, and stabilizes the layer thickness (Weidenschilling, 1984; Cuzzi et al., 1993).

Shear stresses not only induce turbulence in the layer, but also establish the transfer of angular momentum from the dust layer to the surrounding (upper- and lower-lying) gas in the disk. Consequently, the particle orbital radii are reduced, leading to radial compression of the layer. As a result, the layer density may reach the value needed for GI (Youdin and Chiang, 2004; Makalkin and Ziglina, 2004). At the same time, it was noted that the efficiency of turbulence may be too low to prevent its decay and the rise of GI in a thin sublayer in the midplane of the dust layer (Safronov, 1991), especially in the peripheral disk regions (Makalkin and Ziglina, 2004).

Certain studies where the influence of solid particles on turbulence in the disk and in the dust layer was taken into account analytically (Kolesnichenko, 2000; Kolesnichenko and Marov, 2006) and (for a single parameter set) numerically (Dobrovolskis, 1999) have been published. The particles suppress turbulence considerably and reduce the turbulent viscosity and the thickness of the dust layer.

Coradini et al. (1981) were the first to analyze GI in the dust layer of a protoplanetary disk with the interaction between dust (monodisperse) and gas phases taken into account. This interaction was represented by the viscous drag force. The system of hydrodynamic (motion and continuity) equations for density and velocity perturbations of each phase was solved numerically. The key difference between these results and the earlier single-phase models (Safronov, 1969; Goldreich and Ward, 1973) consists in a considerable increase in the time of GI development due to drag between the dust particles and gas. The results of analytical evaluation of the effect of gas drag on GI in the dust layer (Safronov, 1991) agreed with the numerical data provided by (Coradini et al., 1981) and demonstrated a significant increase in the characteristic time of GI development. It was also determined that the density corresponding to the onset of instability is similar to the one found in the models that disregard the gas (at $\varepsilon = \rho_p/\rho_g \geqslant 10^2$). Although the estimate obtained by Safronov (1991) included only the radial perturbations of dust movement (chaotic particle velocities were neglected), it revealed a mostly accurate GI pattern, since the specific angular momentum of matter in the dust layer was conserved with a sufficient accuracy. Marov et al. (2008) presented a model of GI in the dust layer, where a system of equations for perturbations of radial and azimuthal velocities and surface densities of gas and dust phases in the layer was solved analytically. The dispersion equation of the fifth order in complex growth rate was obtained. Its solution for small particles with their stopping time in gas being much shorter than the orbital Keplerian period ($t_s \ll 2\pi/\Omega$) provided a formula for the growth rate, which yielded a characteristic time of GI development close to that obtained with the simplified estimate (Safronov, 1991). The maximum wavelength at which GI may emerge in the presence of gas drag (Coradini et al., 1981; Marov et al., 2008) is close to the wavelength derived in the model with no gas (Safronov, 1969; Goldreich and Ward, 1973). The point is that the maximum wavelength is related to the condition of conservation of the total angular momentum, which is governed by dust dominating the dust layer. The minimum wavelength needed for GI is defined differently: it is affected considerably by turbulent particle diffusion. All these results were obtained in models with different velocities of gas and dust phases (two-fluid model). A number of studies relying on the one-fluid model with equal velocities of gas and dust have also been published. However, the development of GI in this model requires an unnaturally large layer density (Coradini et al., 1981; Safronov, 1991), which corresponds to a layer thickness of 1–100 m that is unrealistically small if even the slightest turbulence is present.

A term representing turbulent diffusion of solid particles in the radial direction was introduced into the continuity equation for the dust layer in (Goodman and Pindor, 2000; Youdin, 2011; Shariff and Cuzzi, 2011). In earlier calculations, only the vertical turbulent particle diffusion, which defines the layer thickness, was taken into account (Cuzzi et al., 1993). In common with vertical particle diffusion, the radial one inhibits GI, since it prevents densification and induces the dispersion of forming annular condensations in the layer if the wavelength of density perturbations is insufficiently long. As a result, both the minimum perturbation wavelength needed for GI and the wavelength with the highest rate of perturbation growth increase considerably. This is essential to estimating the characteristic sizes and masses of forming dust clusters (precursors of planetesimals). However, the gas flow along circular orbits was assumed to be unperturbed in these studies. Therefore, the equations for gas perturbations were lacking, and azimuthal perturbations of gas velocity, which govern the angular momentum transfer, were neglected. Thus, the angular momentum in unit volume of the disk medium enriched with dust was not conserved in the indicated studies, which led to excess transfer of the angular momentum from the dust phase to gas. As a result, the upper bound on the wavelength enabling the growth of density perturbations in the dust phase was removed without a reasonable basis, and the perturbation growth rate was distorted. The results of this analysis of GI imply that it may emerge at dust continuum densities being 2–3 orders of magnitude lower than ρ_{cr} and that the perturbation wavelengths are bounded from the above only by the radial disk size.

In addition to turbulent diffusion, the perturbations of gas velocity and density were taken into account in the study of GI in (Takahashi and Inutsuka, 2014). The corresponding equations were introduced into the system, which was then analyzed and solved numerically for the peripheral protoplanetary disk region (r = 100 AU) that is located at a great distance from the region of planet formation. At 100 AU, the gas–dust disk as a whole (gas included) is close to the GI state: GI parameter Q (Toomre, 1964) was close to 3 in (Takahashi and Inutsuka, 2014) even in the case of 10-fold disk depletion with gas, while $Q \sim 10$ at a distance of 10 AU (instability emerges at $Q \lesssim 1$). The solution was obtained only for small particles with Stokes number St = 0.01. Parameter $\mathrm{St} \equiv t_s\Omega$ is the dimensionless particle stopping time in gas and serves as the measure of dynamic influence of gas drag on the particle movement.

The solution obtained in (Takahashi and Inutsuka, 2014) yields the maximum growth rate at a wavelength shorter than the disk thickness and, at the same time, relies on the two-dimensional (thickness-averaged) model. Therefore, the validity of these results is debatable. Most significantly, this model does not take the sedimentation of particles into account and is thus not suitable for the analysis of GI in the region of planet formation.

It was proposed in recent years that streaming instability, which may emerge before GI and take its place, is more probable in the process of formation of planetesimals (Johansen et al., 2014). Streaming instability is caused by the fact that dust particles, which move faster along an orbit than the gas, accelerate this gas and decelerate themselves due to gas drag. Since the angular momentum is conserved, particles drift radially inward, and gas flows outward. The linear perturbation of these two opposite radial flows has a single growing mode corresponding to the dust phase compaction (Youdin and Goodman, 2005). Numerical modeling demonstrates that the dust phase density may exceed the gas density in the midplane by a factor of 100–300 in the case of large dust aggregates with Stokes number St ~ 0.3 in a disk with mass fraction of solids Z = 0.02 (higher than the protosolar value Z = 0.015) and with mutual collisions taken into account (Johansen et al., 2012). The density then exceeds the critical density for GI, and the gravitational collapse of particle clumps ensues. However, the theory of streaming instability is not without its problems. Specifically, the equations of hydrodynamics of a two-phase medium do not include turbulent viscosity (the gas flow equation) and turbulent diffusion (continuity equations of both phases), although even a slight turbulence may stir particles and prevent them from clumping (Bai and Stone, 2010). This should be applicable not only to turbulence generated by external sources, but also to turbulence induced by the streaming instability itself. The results of our calculations reported below also speak in favor of this: they demonstrate that GI in a layer with turbulence in it being induced by the Kelvin–Helmholtz instability is suppressed markedly by turbulent diffusion of particles in the radial direction. In addition, it is still not clear why a hundred-fold increase in the dust phase density in condensations (from $\rho_p/\rho_g \approx 10$ to 10^3) is triggered by just a slight change (by a factor of 1.5–2) with mass fraction of solids Z, which increases from 0.02 to 0.03 (Bai and Stone, 2010) or from Z = 0.01 to Z = 0.02 (Johansen et al., 2014). Lastly, the mass fraction of solids is Z = 0.005 in the inner ice-free region of a protoplanetary disk prior to the onset of gas loss from the disk (Lodders, 2003); therefore, the gas abundance needs to be reduced by a factor of 4–5 in order for streaming instability to become efficient here. At the same time, if a sufficient amount of solid matter is accumulated in the inner (along r) region of the dust layer, GI is possible (Youdin and Chiang, 2004; Makalkin and Ziglina, 2004). Thus, it is still early to say that the problem of gravitational instability of the dust phase in the disk as a mechanism of formation of planetesimals has lost its relevance.

In the present study, GI in the dust layer of a protoplanetary disk is analyzed in the context of linear dynamics, which characterizes radial perturbations of parameters of solid (monodisperse) and gas phases of an axially symmetric thickness-averaged layer on the assumption that the perturbation wavelength is much shorter than the disk radius. The incompressible gas approximation is substantiated and applied; as a result, the variables to be determined are the perturbations of radial velocity and surface density of the dust phase and the perturbations of azimuthal velocities of both phases. The equations contain terms representing the influence of gas drag on solid bodies (dust particles aggregates), turbulent gas viscosity, turbulent diffusion of bodies, and root-mean-square (rms) velocities of bodies in turbulent gas. The transfer of angular momentum from the dust phase to the gas in the layer is taken into account, while the transfer of angular momentum to the remaining gas lying above and below the layer in the disk is neglected. This approximation is justified if GI in the layer develops faster than the transfer of its angular momentum to the remaining gas. Parameter perturbations are related to the perturbation of the self-gravity of the layer, which is calculated with the layer thickness taken into account. The thickness itself also depends on self-gravity, solar gravity, and vertical turbulent particle diffusion.

The formulation of the problem is discussed in the next section. The incompressible gas approximation is also substantiated there, and the basic equations are given. The fourth-degree dispersion equation is then derived, and it is demonstrated that this equation may be simplified and reduced to a cubic one, which is solved in the present study, in the dust layer at the characteristic distances of planet formation, where $\varepsilon = \rho_p/\rho_g \gg 1$. It is also shown that the cubic equation may be reduced to a quadratic one in the case of sufficiently small particles. The solution of this quadratic equation is used to derive the modified GI criterion incorporating the particle Stokes number. The results of calculations (critical values of surface density of the dust phase at different values of parameter St, the wavelength range where GI is possible, and the perturbation development rate for various wavelengths and particle parameters) are presented in the subsequent section. The characteristic equation is solved numerically with the parameters of the circumsolar protoplanetary disk for two radial distances in the region of planet formation: r = 1 and 10 AU. The results (specifically, the surface density of the dust phase, the sizes of dust aggregates, and the turbulent viscosity enabling GI at various radial distances) are then discussed. The conditions under which the transfer of angular momentum from gas in the layer to gas outside the layer may be neglected are estimated. Conclusions regarding the emergence of GI at various radial distances are made. The prospects for further study of GI in a protoplanetary disk are considered.

DUST LAYER PARAMETERS

Let us consider the GI condition in a layer where its mass is dominated by the dust phase. It is thus called the dust layer, although it would be more accurate to call it the dust–gas layer, since gas, which dominates the layer volume, influences the dynamics of solid particles. These particles (aggregates of dust particles) form such a layer by settling toward the disk midplane along with the particle growth in collisions and/or turbulence decay in a protoplanetary disk. The layer thickness is several orders of magnitude smaller than the protoplanetary disk thickness defined by the gas.

When the sound speed in the gas remains constant throughout the disk thickness, the vertical profile of gas density is Gaussian: $\rho_g(z) \propto \exp(-z^2/2h_g^2)$. The gas density scale height of the protoplanetary disk is

$$h_g = c_s/\Omega. \tag{1}$$

Here, c_s is the isothermal sound speed, Ω is the Keplerian angular velocity defined by the gravity of the central star, and $c_s^2 = R_g T/\bar{\mu}$, where T is temperature and R_g is the gas constant. The mean molar mass $\bar{\mu}$ is 2.34 for the gas with a protosolar composition (Lodders, 2003). The following relation is valid in the case of a Gaussian vertical density profile:

$$\Sigma_g = \sqrt{2\pi}\, h_g \rho_{g,0}, \tag{2}$$

where Σ_g is the gas surface density in the disk, and $\rho_{g,0}$ is the gas density in the disk midplane (all parameters are functions of radial coordinate r).

The vertical density profile of the dust phase in the dust layer is defined by the turbulent diffusion coefficient for particles, which remains constant in the vertical direction (Dubrulle et al., 1995) when particles are sufficiently small (St ≤ 1). If this is the case, the vertical density profile of the dust phase is, in common with that of gas, Gaussian: $\rho_p(z) = \rho_{p,0}\exp(-z^2/2h_p^2)$. Therefore, relation $\Sigma_p = \sqrt{2\pi} h_p \rho_{p,0} = H_p \rho_{p,0}$. holds true. Here, Σ_p is the surface density of the dust phase in the layer, $\rho_{p,0}$ is the spatial density of the dust phase (distributed density of solids) in the disk midplane, h_p is the density scale height of the dust layer, and H_p is its effective thickness (homogeneous layer thickness). Since $h_p \ll h_g$, the gas density throughout the entire layer thickness is almost equal to the density in the midplane; therefore, gas surface density $\Sigma_{g,l}$ in the layer may be defined as $\Sigma_{g,l} = H_p \rho_{g,0}$. Using the last two equalities, one may derive a relation for parameter ε, which characterizes the degree of enrichment of the layer with dust matter:

$$\varepsilon \equiv \Sigma_p/\Sigma_{g,l} = \rho_{p,0}/\rho_{g,0}. \tag{3}$$

The $h_p = h_g$ equality is satisfied prior to the onset of sedimentation of solid particles. In addition, the initial surface density of dust matter $\Sigma_{p,i}$ is related to the surface density of gas as $\Sigma_{p,i}/\Sigma_g = Z$, where Z is the mass fraction of the solid (dust) phase in the disk. The value of Z varies from ≈0.005 at radial distance r = 1 AU, where ice had evaporated, to ≈0.015 at r = 10 AU (Lodders, 2003). Following sedimentation and large-scale radial transfer of dust matter, its surface density in the layer Σ_p deviates from the initial value: $\Sigma_p/\Sigma_{p,i} = \beta$. The sedimentation of dust should yield $\beta < 1$, since small dust particles are not involved in this process and in the layer formation; they remain distributed throughout the entire disk thickness even under a very weak turbulence. At the same time, the radial drift of particles in the layer may be accompanied by radial compaction of the layer with an increase in its surface density Σ_p (Youdin and Chiang, 2004; Makalkin and Ziglina, 2004), which gives $\beta > 1$. Owing to the fact that the value of β is uncertain, ratio $\Sigma_p/\Sigma_g = \beta Z$ also remains ill-defined; therefore, parameter ε is calculated using the specified values of Σ_g and Σ_p; height h_g, which depends on temperature in accordance with (1); and height h_p, which is defined by Eqs. (11) and (18) given below. Relying on Eq. (2), a similar relation for dust matter below Eq. (2), and Eq. (3), one obtains the following expression for ε, which is needed to calculate GI parameters in the layer:

$$\varepsilon = \beta Z \frac{h_g}{h_p} = \frac{\Sigma_p}{\Sigma_g}\frac{h_g}{h_p}. \quad (4)$$

The surface densities in (4) are set parameters, height h_g is determined from (1) for a given temperature T, and height h_p is defined below.

The scale height h_p of the dust layer with a Gaussian vertical density distribution was obtained with particle turbulent diffusion, gas drag (Dubrulle et al., 1995), and self-gravity of the dust layer (Youdin, 2011) taken into consideration:

$$h_p = \sqrt{\frac{D_z}{t_s\Omega^2\psi}} = \frac{l_E}{\sqrt{\mathrm{St}\psi}}. \quad (5)$$

It is taken into account here that the turbulent diffusivity for solid particles in the vertical direction, D_z, is almost equal to the gas turbulent diffusion coefficient (Dubrulle et al., 1995; Youdin and Lithwick, 2007), which matches its kinematic turbulent viscosity ν; thus, $D_z = \nu$. Diffusivity D_z differs from the particle turbulent diffusivity in the radial direction (Youdin and Lithwick, 2007). Parameter ψ in Eq. (5) represents additional compression of the dust layer in the vertical direction due to an increase in the vertical component of the gravitational force caused by self-gravity of the layer (Youdin, 2011). Length scale $l_E = \sqrt{\nu/\Omega}$, related to viscosity and rotation, is discussed below. Particle Stokes number $\mathrm{St} \equiv t_s\Omega$. is also found at the right-hand side of relation (5).

Relation (5) and the Stokes number include characteristic particle stopping time t_s in gas. In the case of small particles with their radius a satisfying the inequality $a \le (9/4)l_g$ (l_g is the mean free path of molecules), time t_s depends on the molecule mean thermal velocity close to c_s (Epstein flow regime). Time t_s for larger particles ($a > (9/4)l_g$) depends on the mean velocity of particles relative to gas, and the drag force is defined by the Stokes formula (Stokes flow regime). Time t_s for both regimes may be expressed in terms of gas density ρ_g (Cuzzi et al., 1993; Dubrulle et al., 1995) or gas surface density Σ_g in the disk (Youdin, 2011). In order to do that, one should use the above formulas that relate the following quantities: $\rho_g \to \Sigma_g \to H_g \to h_g = c_s/\Omega$. As a result, the following is obtained for t_s:

$$t_s = t_{\mathrm{Ep}} = \frac{\rho_s a}{\rho_g c_s} = \frac{\sqrt{2\pi}\rho_s a}{\Sigma_g\Omega} \quad \text{for} \quad a \le (9/4)l_g$$
$$t_s = t_{\mathrm{St}} = t_{\mathrm{Ep}}\frac{4a}{9l_g} \quad \text{for} \quad a > (9/4)l_g. \quad (6)$$

Here, ρ_s is the density of material of a solid particle (body) that is a highly porous, loose aggregate of dust particles. The mean free path of molecules l_g may be written as

$$l_g = \frac{\sqrt{2\pi}m_p\bar{\mu}c_s}{Q_\mu\Sigma_g\Omega}, \quad (7)$$

where m_p is the proton mass, and $Q_\mu \approx Q_{H_2} \approx 3 \times 10^{-15}$ cm^2 is the mean collision cross section of molecules. In the circumsolar disk, $l_g \sim 1$ cm at distance $r = 1$ AU, and $l_g \sim 1$ m at distance $r = 10$ AU. Therefore, decimeter-sized bodies should be in the Stokes deceleration regime at 1 AU and in the Epstein regime at 10 AU.

The initial formula for stopping time in the Stokes flow regime (t_{St} in (6)) is as follows:

$$t_s = \frac{8}{3}\frac{\rho_s}{\rho_g}\frac{a}{C_D|\mathbf{V} - \mathbf{U}|}.$$

Drag coefficient C_D depends on Reynolds number $\mathrm{Re}_p = 2a|\mathbf{V} - \mathbf{U}|/\nu_g < 1$, where $\nu_g = \frac{1}{2}c_s l_g$, is the kinematic molecular viscosity of gas, and $\mathbf{V}$ and $\mathbf{U}$ are the particle and gas velocities. At $\mathrm{Re}_p < 1$, coefficient $C_D = 24\,\mathrm{Re}_p^{-1}$, which yields the second formula in (6). According to our calculations, the condition $\mathrm{Re}_p < 1$ is satisfied within the dust layer in a wide range of protoplanetary disk parameters for particle sizes $a \lesssim 0.8$–1 m at distance $r = 1$ AU and $a \lesssim 6$–10 m at distance 10 AU. The particle size ranges depend mainly on density ρ_s of the material of solid bodies and gas density ρ_g.

Dimensionless parameter ψ in Eq. (5) (Youdin, 2011) is given by

$$\psi = 1 + \frac{2\pi G\Sigma_p}{\Omega^2 h_p}. \quad (8)$$

The ratio of vertical accelerations of the forces of the layer self-gravity and the gravity of the central star is found at the right-hand side of Eq. (8). Self-gravity has already been taken into account in earlier studies of GI in the dust layer (Safronov, 1969), although the dependence of the layer thickness on gas drag and vertical turbulent particle diffusion, which is present in Eq. (5), was neglected.

In the case when turbulence extends to a considerable part of the protoplanetary disk thickness, the kinematic turbulent viscosity ν in (5) may be written using α-parameterization (Shakura and Sunyaev, 1973): $\nu = \alpha c_s h_g$, where $\alpha < 1$. The mean velocity of the largest turbulent eddies is $V_g = \sqrt{\alpha} c_s$, and their characteristic turnover time is $t_t = 1/\Omega$ (Shakura et al., 1978; Dubrulle et al., 1995; Youdin, 2011). The following relation between the above-indicated turbulence parameters is then obtained with (1) taken into account: $\nu = V_g^2/\Omega$.

However, the effect of the dust phase (mass and heat transfer) on the coefficient of turbulent viscosity should also be taken into account, and the following correction factor thus needs to be introduced into the formula for ν (Kolesnichenko and Marov, 2006):

$$\theta = \sqrt{1 - (\mathrm{Ri} + \mathrm{K})/\mathrm{Sc}}, \tag{9}$$

where Ri is the Richardson number, K is the Kolmogorov number, and Sc is the Schmidt number, which is written as $\mathrm{Sc} = \mathrm{Sc}_z = \nu/D_z \approx 1$ (Youdin and Lithwick, 2007) for the dust layer and shear flow $U = U(z)$. Using the expressions for Ri and K (see (Kolesnichenko and Marov, 2006), formulas (201) and (202)) and the parameter values for a protoplanetary disk and the dust layer found in the present study, we have obtained an approximate estimate $\theta \approx 0.7$. A close value $\theta \approx 0.65$ is provided by the numerical model of (Dobrovolskis et al., 1999), as one can see from comparing turbulent viscosities on Figs. 5 and 6 found in the indicated paper.

The following expression for the coefficient of turbulent viscosity is obtained within the α-model:

$$\nu = V_g^2/\Omega = \alpha\theta h_g^2 \Omega, \tag{10}$$

where

$$V_g = \sqrt{\alpha\theta} c_s \tag{11}$$

is the rms turbulent velocity of gas (the largest eddies) in the dust layer. Coefficient θ is introduced into (10) and (11) in order to include additional attenuation of turbulence by particles. Inserting expression (10) for the kinematic viscosity into (5) with (8) taken into account, we obtain

$$h_p = \sqrt{l_G^2 + \frac{\alpha\theta}{\mathrm{St}} h_g^2} - l_G, \tag{12}$$

where

$$l_G = \pi G \Sigma_p / \Omega^2 \tag{13}$$

is the gravitational length scale that is 4π times smaller than the maximum perturbation wavelength at which GI in the dust layer is possible (Safronov, 1991). Relation (12) for h_p (without θ) was obtained in (Youdin, 2011). It was also demonstrated there that this relation holds true both at $\mathrm{St} \ll 1$ (small particles) and at $\mathrm{St} > 1$.

Even if global turbulence characterized by the α-model of viscosity decays to a very low level of $\alpha < 10^{-6}$–10^{-7}, "local" turbulence generated by shear stresses between the gas in the layer and the gas above and below the layer persists. The dust layer resides in this turbulent layer, which is similar to an Ekman boundary layer. Ekman length scale $l_E = \sqrt{\nu/\Omega}$ for the shear turbulent layer in the disk is (Goldreich and Ward, 1973)

$$l_E = \frac{\Delta V_g}{\Omega \mathrm{Re}^*}. \tag{14}$$

Turbulent layer thickness h_t may not be smaller than the Ekman one ($h_t \geq l_E \geq h_p$), but may be larger than homogeneous layer height h_p at large particle sizes (St ~ 1) (Cuzzi et al., 1993; Dobrovolskis et al., 1999). We assume that the turbulent layer thickness may be larger than the Ekman one at small particle sizes, when turbulence is associated with the Kelvin–Helmholtz instability (Youdin and Chiang, 2004). It may then be assumed that the thicknesses of the turbulent layer and the dust layer are equal.

Re* in Eq. (14) is the critical Reynolds number, the value of which for protoplanetary disks has not been determined reliably. It was assumed in (Goldreich and Ward, 1973) that Re* = 500, Cuzzi et al. (1993) varied Re* within the range of 45–180, and Re* = 20 was adopted in (Dobrovolskis et al., 1999). The value of Re* in our GI modeling is varied from 20 to 200.

Parameter ΔV_g in (14) is the difference between azimuthal gas velocities within the dust layer and outside of it. Since this velocity within the layer is close to the Keplerian circular velocity, it was assumed in the majority of studies that these velocities are equal. However, the difference may be as large as 10% if the ratio of densities of the dust phase and the gas in the layer is $\varepsilon = 10$ (ε is defined by formula (3)) and $\mathrm{St} \leq 1$. This follows from the relation derived by (Makalkin and Ziglina, 2004) with the use of formulas for regular velocities of solid particles relative to gas (Nakagawa et al., 1986):

$$\Delta V_g = \frac{\varepsilon}{(\varepsilon + 1)} \frac{\Delta V}{(1 + C^2)}, \quad \text{where} \quad C = \frac{\mathrm{St}}{\varepsilon + 1}. \tag{15}$$

Parameter ΔV in (15) is the difference between the Keplerian circular velocity and the gas orbital velocity outside the layer:

$$\Delta V = -\frac{1}{2\rho_{g,0}\Omega} \frac{\partial P}{\partial r}, \tag{16}$$

where P is the gas pressure. A relation equivalent to (15) was also derived in (Youdin and Chiang, 2004), but parameter C was left out (i.e., the relation held true for small particles). At $\varepsilon \gg 1$ and $\mathrm{St} \leq 1$, velocity difference ΔV_g is as large as 40–50 m/s at distance $r = 1$ AU and may be somewhat lower at $r = 10$ AU (depending on temperature distribution $T(r)$). In the case of turbulence in an Ekman-like layer, the kinematic viscosity is defined, with (14) taken into account, by the following equation:

$$\nu = \frac{\theta \Delta V_g^2}{\Omega \, \mathrm{Re}^{*2}} = \theta l_E^2 \Omega = \frac{V_g^2}{\Omega}, \qquad (17)$$

where V_g, just as in (11), is the rms turbulent gas velocity in the largest eddies:

$$V_g = \sqrt{\theta} \Delta V_g / \mathrm{Re}^*. \qquad (18)$$

Equations (17) and (18) differ from the relations found in (Goldreich and Ward, 1973; Dobrovolskis et al., 1999) only in the additional factor θ, which characterizes the attenuation of turbulence by solid particles. Inserting viscosity ν from Eq. (17) into Eq. (5) with regard to (8), we obtain the following expression for the scale height of the dust layer in the case of local shear turbulence with gas drag, attenuation of turbulence by particles, and self-gravity of the layer taken into account:

$$h_p = \sqrt{l_G^2 + \frac{\theta}{\mathrm{St}} l_E^2} - l_G, \qquad (19)$$

where l_G is defined by (13). In common with the layer scale height h_p, the semithickness of the homogeneous layer, which is equal to $\sqrt{\pi/2} h_p$, may be both lower and higher (depending on the sizes of particles governing gas drag at St ~ 1) than the Ekman layer thickness, while turbulent layer thickness h_t may not be smaller than the Ekman one l_E. It follows that the turbulent viscosity, the rms turbulent gas velocity, and the scale height of the dust layer, which are defined by relations (10)–(12) and (17)–(19) that include parameter θ, may be overstated at $\mathrm{St} \geq 1$, (when $h_t \geq l_E \geq h_p$,), since the attenuation of turbulence at such a ratio of thicknesses should not occur in the upper part of the turbulent layer where the particle abundance is reduced greatly.

The particle turbulent diffusivity in radial direction, D_r, is also important for the analysis of GI in the layer. Youdin and Lithwick (2007) have demonstrated that D_r differs from the corresponding diffusivity in the vertical direction (D_z) found in Eq. (5). This difference becomes significant at large dimensionless stopping times $\mathrm{St} \gg 1$ (i.e., at large particle sizes). The following approximate relation (Youdin, 2011) containing Schmidt number Sc may be written for coefficient D_r, which is hereinafter referred to as D:

$$\nu/D = \mathrm{Sc} = 1 + \mathrm{St}^2/4 \quad \text{or} \quad D = \nu/(1 + \mathrm{St}^2/4). \qquad (20)$$

The rms velocity of solid particles in the turbulent layer is defined approximately (Youdin, 2011; Völk et al., 1980) as

$$V_p = V_g / \sqrt{1 + \mathrm{St}}, \qquad (21)$$

where rms gas velocity V_g is defined by equalities (11) or (18) depending on the turbulence type.

GRAVITATIONAL INSTABILITY IN THE DUST LAYER: PROBLEM FORMULATION AND BASIC EQUATIONS

Turbulence stabilizes the dust layer thickness; as a result, the mean particle velocity in the vertical direction is zero. Therefore, scale height h_p and parameter ε, which characterizes the dust phase density in the layer and is tied to h_p by relation (3), may be taken to be constant in the analysis of GI.

The initial system of equations includes the equations of motion and continuity for solid and gas phases in the protoplanetary disk conditions. This system was considered in a number of studies on hydrodynamics of protoplanetary and other astrophysical disks and GI in them, specifically, in (Coradini et al., 1981; Gor'kavyi and Fridman, 1994; Goodman and Pindor, 2000; Kolesnichenko and Marov, 2006; Takahashi and Inutsuka, 2014).

We consider this system of equations in the cylindrical coordinate system for the dust layer in a protoplanetary disk under the assumption of axial symmetry ($\partial/\partial\varphi = 0$). The system in this approximation includes six equations: radial and azimuthal components of the equations of motion of the solid phase and gas and continuity equations for both phases. The surface densities of dust and gas phases in the dust layer (Σ_p and $\Sigma_{g,l}$) and two velocity components for each of the two phases are the unknown functions of the radial coordinate and time. The circular Keplerian motion of gas and dust is regarded as the unperturbed one.

Let us examine radial perturbations of an axisymmetric dust layer under the following assumptions: the perturbation wavelength is (1) much smaller than the current radial coordinate value ($\lambda \ll r$), and (2) larger than the dust layer thickness, while the characteristic perturbation growth time is (3) much shorter than the time of layer thickness variation and (4) longer than turbulence timescale Ω^{-1}. The first restriction allows one to use the local approximation and neglect radial variations of unperturbed parameters. The second one provides an opportunity to consider the layer in the thin disk approximation and use the density integrated over z (i.e., surface density) and radial and azimuthal velocities averaged over z (Goodman and Pindor, 2000). The third restriction allows one to ignore the reduction in layer scale height h_p and the associated increase in the density ratio of dust and gas phases ε in the process of development of GI in the layer. The fourth restriction enables the use of averaged parameters of turbulent motion: rms turbulent velocity and turbulent diffusivity.

Let us solve the system of equations for the perturbations of surface densities and radial and azimuthal components of velocity of gas and dust phases in the linear approximation under the assumption of gas incompressibility. The latter assumption simplifies the solution considerably. However, its applicability should be substantiated.

In the case of a nonstationary flow of gas, it may be considered incompressible if the following two conditions are satisfied (Landau and Lifshitz, 1986): (1) the gas flow speed is low compared to sound speed c_s; (2) characteristic time τ and characteristic distance l of the gas velocity variation satisfy inequality $\tau \gg l/c_s$. The first condition is satisfied in a protoplanetary disk due to the fact that the velocities of turbulent gas flows are much lower than the sound speed. Let us check whether the second condition is satisfied.

The gas surface density perturbation in a protoplanetary disk may be written as a monochromatic wave

$$\sigma_g = \hat{\sigma}_g e^{nt+ikr}, \tag{22}$$

where complex parameter n is related to complex frequency ω as $n = -i\omega$.

If the real part of n is positive, it has the meaning of the perturbation growth rate. The $\tau \gg l/c_s$ inequality for perturbations (22) takes the form

$$|n|^{-1} \gg k^{-1}/c_s. \tag{23}$$

It is sometimes preferable to use dimensionless parameter $\gamma = n/\Omega$. The following condition of validity of incompressible gas approximation is derived from (23) with Eq. (1) and the definition of wave number $k = 2\pi/\lambda$ taken into account:

$$|\gamma| = |n|/\Omega \ll h_g k = 2\pi h_g/\lambda. \tag{24}$$

According to (Safronov, 1991), GI of the dust layer in gas is established at wavelength $\lambda \leq 50 h_p$, while the calculations in the present study (performed for different sets of parameters) yield the following wavelength of the earliest onset of GI with the higher growth rate: $\lambda \sim (20–60)h_p$. Let us recall that, in contrast to (Safronov, 1991), we take turbulent particle diffusion into account. The following is derived from (24) with the use of these estimates: $|\gamma| \ll 0.1 h_g/h_p$. In view of relation (4), the condition under which the gas may be considered incompressible takes the form

$$|\gamma| \ll 0.1\varepsilon \Sigma_g/\Sigma_p. \tag{25}$$

Densities $\rho_p \sim \rho^*$, were obtained in our calculations for the GI onset. These values agree with the results presented in (Coradini et al., 1981; Safronov, 1991) and correspond to $\varepsilon > 100$ at a distance of 1 AU and $\varepsilon > 20$ at 10 AU. The minimum ratio used in these calculations was $\Sigma_g/\Sigma_p \sim 5$ at 1 AU and ~25 at 10 AU. Thus, the minimum value at the right-hand side of inequality (25) was ~50. As for the left-hand side of (25), it was determined in our calculations that $|\gamma|$ is of the same order of magnitude as the Stokes number for particles in the layer (i.e., $|\gamma|/\mathrm{St} \sim 1$). This relation is obtained due to the fact that $|\mathrm{Im}[\gamma]| = |\mathrm{Re}[\omega]|/\Omega \sim \mathrm{St}$; in addition, $|\mathrm{Re}[\gamma]| \lesssim |\mathrm{Im}[\gamma]|$. The maximum value of St used in calculations was two. Therefore, inequality (25) and the condition of gas incompressibility were well satisfied in our calculations. According to our estimates, the assumption of gas incompressibility is a fine approximation for the analysis of GI in the entire region of planet formation, although it may be violated at $r \gtrsim 100$ AU. Inequality (25) is not satisfied under the parameter values used for GI analysis at a distance of 100 AU in (Takahashi and Inutsuka, 2014); therefore, all six hydrodynamic equations of the system should be solved, and the authors of the indicated study did just that.

Let us decompose all functions into unperturbed and perturbed parts in order to proceed to the system of equations for perturbations. In contrast to the complete function and its unperturbed part, the perturbed part is denoted by lower-case characters.

Let us proceed to the system of equations for the surface density and velocity perturbations in the incompressible gas approximation. Since the gas surface density perturbation is zero in this approximation, it follows from the continuity equation for gas that the perturbation of the radial gas velocity component is also zero. As a result, the continuity equation for gas and the equation for the radial component of perturbed gas flow in the radial direction are excluded from the system, while three equations for the dust phase and one for gas (the equation for the azimuthal gas velocity perturbation) remain. The perturbation of surface density of the dust phase σ_p and perturbations for two particle velocity components (v_r and v_φ) and azimuthal gas velocity u_φ are the unknown parameters. The system of equations for perturbations is as follows:

$$\frac{\partial v_r}{\partial t} - 2\Omega v_\varphi = -\frac{V_p^2}{\Sigma_p}\frac{\partial \sigma_p}{\partial r} - \frac{\partial \Phi_1}{\partial r} - \frac{v_r}{t_s}, \tag{26}$$

$$\frac{\partial v_\varphi}{\partial t} + \frac{1}{2}\Omega v_r = \frac{u_\varphi - v_\varphi}{t_s}, \tag{27}$$

$$\frac{\partial u_\varphi}{\partial t} = \frac{\varepsilon(v_\varphi - u_\varphi)}{t_s} + \nu\frac{\partial^2 u_\varphi}{\partial r^2}, \tag{28}$$

$$\frac{\partial \sigma_p}{\partial t} + \Sigma_p\frac{\partial v_r}{\partial r} = D\frac{\partial^2 \sigma_p}{\partial r^2}. \tag{29}$$

Φ_1 in (26) is the perturbed gravitational potential of an infinitely thin layer satisfying the Poisson equation

$$\Delta\Phi_1 = 4\pi G \sigma_p \delta(z).$$

The equation of motion of the dust phase includes gas drag, which equals (for unit mass) the difference between gas and dust velocities divided by particle stopping time (6): $(\mathbf{u}-\mathbf{v})/t_s$. The same force (acting in the opposite direction) is found in the equation of motion of the gas. Since $u_r = 0$ in an incompressible gas, only v_r. is present in Eq. (26). The use of expression (6) for time t_s implies that the dust phase is monodisperse.

Since the gas flow in the layer is turbulent, turbulent diffusion of solid particles in the radial direction should be included into the continuity equation (Monin and Yaglom, 1965; Goodman and Pindor, 2000; Kolesnichenko and Marov, 2006).

Equation (26) is also written with consideration for the radial gradient of pressure of solid particles with rms velocitiy V_p, which is defined by turbulent gas velocity in accordance with Eq. (21). The term with V_p^2 is written in the form that was used in the studies into GI in protoplanetary disks (Coradini et al., 1981; Youdin, 2011; Takahashi and Inutsuka, 2014).

Equation (28) includes the term with turbulent gas viscosity ν, which characterizes turbulent stress and the transfer of angular momentum in turbulent gas in a layer with Keplerian rotation (Lynden-Bell and Pringle, 1974).

The system of Eqs. (26)–(29) differs from a similar one in (Coradini et al., 1981) in that turbulent diffusion and viscosity (the terms with D and ν) are taken into account; it also features more equations than the system in (Goodman and Pindor, 2000; Youdin, 2011), which lacks Eq. (28) that characterizes quantitatively the transfer of angular momentum from the solid phase to gas. At the same time, the incompressible gas approximation, which is valid at $\varepsilon \gg 1$, provides an opportunity to examine GI (specifically, study it analytically) in the region of planet formation in more detail.

Assuming that small perturbations of all parameters have the form of a monochromatic wave (22), we derive the system for amplitudes (Fourier coefficients) of all perturbations. The following is obtained for the amplitude of perturbation of the gravitational force:

$$\hat{F} = -\partial\hat{\Phi}_1/\partial r = 2\pi i G\hat{\sigma}_p(1+kh_p)^{-1}. \quad (30)$$

Here, an additional reducing factor $(1+kh_p)^{-1}$ (Genkin and Safronov, 1975; Shu, 1984) was introduced into the perturbed gravitational force (Gor'kavyi and Fridman, 1994) in order to take the nonzero layer thickness into account. The value of h_p is defined by Eqs. (12) or (19) depending on the type of turbulence in the layer. The thin disk approximation implies that $kh_p \le 1$. As a result, the following system of equations for amplitudes is obtained:

$$n\hat{v}_r - 2\Omega\hat{v}_\varphi + \frac{ikV_p^2\hat{\sigma}_p}{\Sigma_p} - \frac{2\pi i G\hat{\sigma}_p}{1+kh_p} + \frac{\hat{v}_r}{t_s} = 0, \quad (31)$$

$$n\hat{v}_\varphi + \frac{1}{2}\Omega\hat{v}_r - \frac{\hat{u}_\varphi - \hat{v}_\varphi}{t_s} = 0, \quad (32)$$

$$n\hat{u}_\varphi - \frac{\varepsilon(\hat{v}_\varphi - \hat{u}_\varphi)}{t_s} + k^2\nu\hat{u}_\varphi = 0, \quad (33)$$

$$(n + k^2D)\hat{\sigma}_p + ik\Sigma_p\hat{v}_r = 0. \quad (34)$$

DERIVATION AND ANALYSIS OF THE DISPERSION EQUATION

Let us first examine system (31)–(34) with no restrictions on n, t_s, and ε. It follows from Eqs. (32)–(33) that

$$v_\varphi = -\frac{A\Omega}{2n}v_r, \quad (35)$$

where

$$A = \frac{\varepsilon + t_s n + t_s\nu k^2}{1 + \varepsilon + t_s n + t_s\nu k^2 + \nu k^2 n^{-1}}. \quad (36)$$

Inserting Eqs. (34) and (35) into Eq. (31), we obtain the dispersion relation

$$\left(n + t_s^{-1}\right) + A\Omega^2 n^{-1} + \left[V_p^2k^2 - 2\pi G\Sigma_p k(1+kh_p)^{-1}\right]\left(n + Dk^2\right)^{-1} = 0. \quad (37)$$

Taking the $A(n)$ dependence in relation (36) into account, one may write Eq. (37) as an equation of the fourth order in n.

If inequality $t_s|n| \ll \varepsilon$, is satisfied, term $t_s n$ in the numerator and the denominator of expression (36) for parameter A may be omitted; the dispersion equation of the third order is then obtained. It was already noted that the results of our calculations revealed that the imaginary part of n is larger than the real part and $|\mathrm{Im}[n]|/\Omega \equiv |\mathrm{Im}[\gamma]| \sim \mathrm{St}$, where St is the particle Stokes number, which is the dimensionless stopping time. Therefore, inequality $t_s|n| \ll \varepsilon$ (or $\mathrm{St}|\gamma| \ll \varepsilon$) may be satisfied for sufficiently large particle aggregates with $\mathrm{St} \sim 1$, if $\varepsilon \gg 1$. In the case of smaller particles (at $\mathrm{St} \ll 1$), inequality $t_s|n| \ll \varepsilon$ is also satisfied at smaller values of $\varepsilon \gtrsim 1$.

The results reported in (Coradini et al., 1981; Marov et al., 2008) and in the present study demonstrate that inequality $\varepsilon \gg 1$ is satisfied under GI in the entire region of planet formation. This is confirmed by a simple estimate for the circumsolar disk: $\varepsilon \sim \rho^*/\rho_g \sim 10^2(r/1\,\mathrm{AU})^{-3}/(r/1\,\mathrm{AU})^{-(2-2.5)}$.

Thus, inequality $t_s|n| \ll \varepsilon$ is satisfied at $r < 40$ AU even in the case of large particle aggregates ($\mathrm{St} \sim 1$). Therefore, term $t_s n$ may be removed from expression (36), and this expression the takes the form

$$A = \frac{\varepsilon + t_s \nu k^2}{1 + \varepsilon + t_s \nu k^2 + n^{-1}\nu k^2}. \quad (38)$$

Inserting expression (38) into Eq. (37), we obtain the following cubic dispersion equation for complex parameter n:

$$a_0 n^3 + a_1 n^2 + a_2 n + a_3 = 0. \quad (39)$$

The coefficients of this equation are as follows:

$$a_0 = 1, \quad (40)$$

$$a_1 = t_s^{-1} + Dk^2 + B, \quad (41)$$

$$a_2 = \Omega^2\left(\frac{\varepsilon}{1 + \varepsilon + t_s \nu k^2} + t_s B\right) + \frac{Dk^2}{t_s} + V_p^2 k^2 - \frac{2\pi G \Sigma_p k}{1 + kh_p} + B(t_s^{-1} + Dk^2), \quad (42)$$

$$a_3 = \Omega^2 Dk^2\left(\frac{\varepsilon}{1 + \varepsilon + \nu k^2 t_s} + t_s B\right) + B\left(\frac{Dk^2}{t_s} + V_p^2 k^2 - \frac{2\pi G \Sigma_p k}{1 + kh_p}\right), \quad (43)$$

where letter B denotes the parameter with the dimension of frequency:

$$B = \frac{\nu k^2}{1 + \varepsilon + t_s \nu k^2}. \quad (44)$$

The results obtained by solving Eq. (39) with coefficients (40)–(43) numerically are detailed below in the relevant section.

Let us use the Routh–Hurwitz theorem for equations with real coefficients to derive an analytical criterion for GI in the dust layer. It follows from this theorem that the number of roots of Eq. (39) with a positive real part equals the number of sign changes in the sequence T_0, T_1, T_1T_2, a_3, where $T_0 = a_0$, $T_1 = a_1$, and $T_2 = a_1a_2 - a_0a_3$. According to Eqs. (40) and (41), quantities T_0 and T_1 are positive. Equation (39) was solved for a protoplanetary disk at a distance of 1 and 10 AU with independent variables t_s, Σ_p, and ν varied in wide ranges, and either two complex conjugate roots (with a negative real part or a positive one) or two real positive roots (the third root was negative) were always obtained. In the case of complex conjugate roots, the ones with a positive real part corresponded to instability. According to the theorem mentioned above, such a result at positive T_0 and T_1 is possible only if $T_2 < 0$, $a_3 > 0$.

Let us make certain simplifying assumptions (resting on the parameter values in the region of planet formation) in order to derive an analytical GI criterion based on the last two inequalities. According to the numerical results, $\nu k^2 \lessdot \Omega$ (at $\alpha \le 10^{-5}$); therefore, $B \sim \nu k^2/\varepsilon \lessdot \Omega/\varepsilon$. Consequently, the following inequalities should be satisfied at $\varepsilon \gg 1$:

$$Bt_s \sim \mathrm{St}/\varepsilon \ll 1, \quad B/(Dk^2) \sim \varepsilon^{-1} \ll 1. \quad (45)$$

The second inequality in (45) was obtained with the fact that $\nu \sim D$ at $\mathrm{St} \lessdot 1$ is taken into account (this follows from (21)). The minimum value of $\varepsilon \approx 19$ at the maximum St = 2 (at 10 AU) was determined in our calculations; therefore, inequalities (44) are satisfied. It then follows that the terms with parameters B and ν, which are small relative to the other terms, may be neglected in the derivation of the expression for T_2 with the use of relations (40)–(43). As a result, the expression obtained for T_2 is the same as the one derived if $\nu = 0$ and, consequently, $B = 0$ are set in Eq. (39) with coefficients (40)–(43). It can be seen from (43) that inequality $a_3 > 0$ is satisfied in this case. Therefore, as follows from the Routh–Hurwitz theorem, inequality $T_2 < 0$ is the gravitational instability condition. Considering that $\varepsilon \gg 1$, this inequality may be written as

$$T_2 = (t_s^{-1} + Dk^2)\left(\frac{Dk^2}{t_s} + V_p^2 k^2 - \frac{2\pi G \Sigma_p k}{1 + kh_p} + \Omega^2\right) - \Omega^2 Dk^2 < 0. \quad (46)$$

SIMPLIFIED GRAVITATIONAL INSTABILITY CONDITION IN THE LAYER

Expression (46) of the fourth order in k is still too complex to derive an analytical GI criterion. If inequality $|n| \ll t_s^{-1} + Dk^2$, which holds true at small values of the particle Stokes number ($\mathrm{St} \ll 1$), is satisfied, Eq. (39) is reduced to the following quadratic equation:

$$a_1 n^2 + a_2 n + a_3 = 0, \quad (47)$$

where coefficients a_1, a_2, and a_3 are defined by Eqs. (41)–(43). The satisfaction of the $a_2 < 0$ inequality is then necessary and sufficient for the unstable solutions to exist, since $a_1 > 0$, and inequality $a_3 > 0$ also holds true if inequality $\varepsilon \gg 1$ and conditions (45) are satisfied. The GI criterion then takes the following form:

$$\left(\frac{D}{t_s} + V_p^2\right)k^2 - \frac{2\pi G \Sigma_p k}{1 + kh_p} + \Omega^2 < 0. \quad (48)$$

The upper bound on the wavelength at GI, which results from disk rotation, follows from (48). It is necessary that inequality

$$\frac{2\pi G \Sigma_p k}{1 + kh_p} - \Omega^2 > 0,$$

be satisfied. It is equivalent to the condition that was obtained earlier (Safronov, 1987) and placed a lower bound on the critical density for GI onset:

$$\lambda < 2\pi h_{\rm p}\left(\frac{\rho_{\rm p,0}}{\rho^*/3} - 1\right). \qquad (49)$$

Another prerequisite for instability also follows from (48):

$$2\pi G\Sigma_{\rm p} f > (Dt_{\rm s}^{-1} + V_{\rm p}^2)k, \qquad (50)$$

where f is the coefficient representing the reduction in self-gravity of the disk due to the inclusion of its thickness:

$$f = (1 + kh_{\rm p})^{-1}. \qquad (51)$$

In our calculations performed in a fairly wide range of parameter variation, f varied within the relatively narrow interval of 0.76–0.83.

At $\mathrm{St} \ll 1$ (i.e., $t_{\rm s} \ll \Omega^{-1}$), the $D \approx V_{\rm p}^2/\Omega$, relation is derived from relation $\nu = V_{\rm g}^2/\Omega$ (Eqs. (10) and (17)) and Eqs. (20) and (21). It then follows that

$$Dt_{\rm s}^{-1} + V_{\rm p}^2 \approx D/t_{\rm s} \approx V_{\rm p}^2/\mathrm{St} \approx V_{\rm g}^2/\mathrm{St}. \qquad (52)$$

In view of relations (52), the GI condition expressed by inequality (50) yields a lower bound on wavelength λ enabling the growth of perturbations:

$$\lambda > \lambda_{\min} \approx \frac{D}{G\Sigma_{\rm p} f t_{\rm s}} \approx \frac{V_{\rm g}^2}{G\Sigma_{\rm p} f \mathrm{St}}. \qquad (53)$$

At such values of λ that satisfy inequality (53), the diffuse spreading of the annular condensation of the layer is slower that its gravitational contraction. It follows from (49) and (53) that if surface density $\Sigma_{\rm p}$ of solid matter in the layer is not sufficiently high, GI is not established at any wavelength.

It follows from the comparison of (46) and (48) that if inequality (48) is satisfied, inequality (46) is also satisfied, while the reverse is not true (i.e., the region of parameters satisfying the GI condition for the third-order equation is wider than that for the second-order one). This is one of the reasons why the cubic equation yielded critical (for GI onset) surface densities $\Sigma_{\rm p,cr}$ that were considerably (by a factor of up to 1.5) smaller than the ones obtained by solving the quadratic equation.

Assuming the constancy of the parameter f defined by Eq. (51), inequality (48) may be written as $y(k) < 0$, where $y(k)$ is the quadratic polynomial in wave number k with a positive coefficient $(Dt_{\rm s}^{-1} + V_{\rm p}^2)$ at the highest degree. This new inequality is satisfied when the discrete of quadratic equation $y(k) = 0$ is positive; i.e.,

$$\pi^2 G^2 \Sigma_{\rm p}^2 f^2 - \left(Dt_{\rm s}^{-1} + V_{\rm p}^2\right)\Omega^2 > 0. \qquad (54)$$

We then rewrite the GI criterion using the equivalent of the Toomre stability parameter (Toomre, 1964) for disk $Q < 1$ with sound speed $c_{\rm s}$ and gas surface density $\Sigma_{\rm g}$ replaced by rms velocity $V_{\rm p}$ and surface density $\Sigma_{\rm p}$ of solid particles:

$$Q_{\rm p} = \frac{V_{\rm p}\Omega}{\pi G\Sigma_{\rm p}}. \qquad (55)$$

In view of relations (52) obtained with $\mathrm{St} \ll 1$, GI condition (54) may be written as

$$Q_{\rm p}/f < \sqrt{\mathrm{St}}. \qquad (56)$$

All parameters in relations (48)–(56) (except for gravitational constant G) depend on radial coordinate r. It can be seen from relation (52) that particle diffusion, which, as indicated by inequality (56), inhibits GI in the layer, is of paramount importance for the GI condition in the dust layer containing small particles with $\mathrm{St} \ll 1$. At a given turbulence in the layer (i.e., at given ν and $V_{\rm g}$, which are tied to D and $V_{\rm p}$ by relations (20) and (21)), the instability condition, as follows from (55) and (56), is defined just by parameter product $\Sigma_{\rm p}^2\mathrm{St}$. It follows from (6) that the GI condition depends (through Stokes number $\mathrm{St} = t_{\rm s}\Omega$) both on the size and density (a and $\rho_{\rm s}$) of a solid particle (aggregate) and on the temperature and density (or surface density) of the gas.

Although this analysis of GI was performed based on simplified formulas obtained by solving the quadratic dispersion equation at $\mathrm{St} \ll 1$, it still provides a fairly accurate description of the onset of GI in the St ~ 1 case. This is demonstrated in the next section, where the results of solving a more accurate cubic dispersion relation (39) are reported.

NUMERICAL RESULTS AND DISCUSSION

The analysis of GI was performed for the conditions established in the circumsolar protoplanetary disk $\Omega = \sqrt{GM_{\rm S}/r^3} = \Omega_1 (r/1\,\mathrm{AU})^{-1.5}$, where $M_{\rm S}$ is the solar mass. Lower index 1 denotes the values of parameters at $r = r_1 = 1\,\mathrm{AU}$.

The following distribution was adopted for the surface gas density:

$$\Sigma_{\rm g} = \Sigma_1 (r/1\ \mathrm{AU})^{-p}. \qquad (57)$$

The basic values of Σ_1 and p were $\Sigma_1 = 2000$ g/cm^2 and $p = 1$. The temperature distribution was as follows:

$$T = T_1 (r/1\,\mathrm{AU})^{-q}, \qquad (58)$$

where $T_1 = 300$ K, and index q varied within the range of 0.5–0.8. The value of $q = 0.78$ corresponds to temperature $T = 50$ K at 10 AU, which agrees with the geochemical constraints and theoretical models (Makalkin and Dorofeeva, 2009).

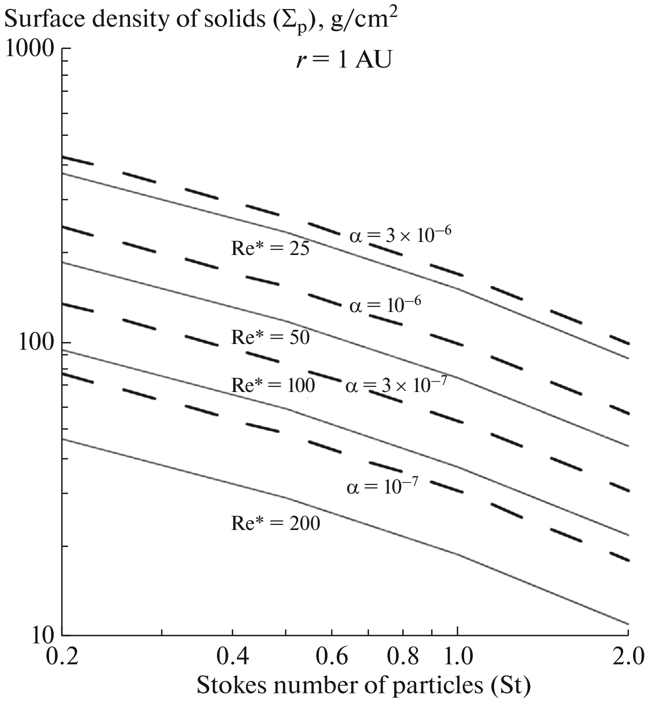

Fig. 1. Relation between the critical values of parameters Σ_p and St of the dust layer at distance $r = 1$ AU. A pair of Σ_p and St values corresponding to the onset of GI in the layer is assigned to each point of curves. Dashed curves correspond to different values of parameter α in the turbulent viscosity defined by Eq. (10). Solid curves correspond to the turbulent viscosity, which is expressed in terms of the critical Reynolds number Re* (see Eq. (17)), at different values of Re*.

The radial dependence of the other parameters is determined in consideration of the $c_s^2 = R_g T/\bar{\mu}$ equality at $\bar{\mu} = 2.34$. Specifically, the value of ΔV determined by formula (16) with (57) and (58) taken into account is

$$\Delta V = \frac{(p + q/2 + 1.5)}{2} \frac{c_{s,1}^2}{V_{K,1}} \left(\frac{r}{1\,\mathrm{AU}}\right)^{0.5-q},$$

where $V_{K,1} = \Omega_1 r_1$. When the adopted parameter values are used, ΔV_1 varies from 40 to 47 m/s. Parameter θ defined by Eq. (9) was set to be equal to 0.7.

All GI calculations were performed by solving dispersion equation (39) with coefficients (40)–(43) for radial distances $r = 1$ and 10 AU.

Surface density Σ_p of the dust phase and the Stokes parameter for a solid particle (aggregate of solid particles) were used as the input variable parameters. As was explained above, these parameters are governed by the little studied processes of transfer and growth of solid particles and may vary widely.

Let us take a look at the results of calculation of critical surface density $\Sigma_{p,cr}$, at which a positive real part emerges in two complex conjugate roots of Eq. (39) solved with respect to n. Complex parameter n is used to characterize perturbations (22) of the surface density and velocities. With $\mathrm{Re}[n] > 0$, parameter $\mathrm{Re}[n]$ has the meaning of the perturbation growth rate. With $\mathrm{Re}[n] < 0$, the real part of complex conjugate roots is negative (i.e., GI is lacking).

Figures 1 and 2 show the relations between the Stokes number and the surface density of particles (St and Σ_p) corresponding to the onset of GI. Lower values of St at a set Σ_p (and lower values of Σ_p at a set St) correspond to a stable layer. Thus, the set of parameters St and Σ_p assumes a critical value at the given parameters of the protoplanetary disk and turbulence within it. The demonstrated results agree well with approximate criterion (56), where product $\Sigma_p^2 \mathrm{St} \approx$ const at the given parameters of turbulence in the disk at set radial distance r. The critical pairs of values Σ_p and St in both figures are given for two mechanisms of generation of turbulence: the "global" one, which is characterized using the α-parameterization of turbulent viscosity under turbulence generated in a considerable part of the disk, and a more local one with turbulence limited to the equatorial part of the disk with shear flow generated by the dust layer.

Since the value of the turbulent viscosity found in Eqs. (10) and (17) is fairly uncertain in both turbulence models, parameters α and Re* were varied by a factor of 100 and 8, respectively, in calculations presented in Figs. 1 and 2. The eight-fold variation of Re* leads to a 64-fold turbulent viscosity variation (see Eq. (17)). These parameters were varied so that GI at the distances of 1 and 10 AU was established at such parameter values that, in our view, remained within reasonable bounds. At $\alpha = 10^{-5}$ and $r = 1$ AU, critical value $\Sigma_p \approx 310$ g/cm² was obtained at St = 1, while even larger values corresponded to St < 1. Such Σ_p values appear improbable, although they should not be excluded. The range of variation of parameter θ, which represents the effect of the dust phase on turbulence, is, according to our estimates, narrower than that of α and Re^{*2}. Since θ appears in Eqs. (10) and (17) for the viscosity coefficient in combinations $\alpha\theta$ or θ/Re^{*2}, Figs. 1 and 2 provide an opportunity to estimate the effect of variation of θ on GI.

It can be seen from Fig. 1 that Σ_p should be considerably higher than $\Sigma_{p,1} \approx 10$ g/cm² (the value corresponding to the standard model of a "minimum-mass" protoplanetary disk) in order for GI to emerge at 1 AU. The sole exception is the case when $\alpha \le 10^{-7}$ or $\mathrm{Re}^* \ge 100$, with St ≥ 1. Thus, our results agree with those obtained in (Cuzzi et al., 1993; Dobrovolskis et al., 1999) by numerical modeling with a very low value of $\Sigma_{p,l} \approx 8$ g/cm^2. The maximum value St = 2 demonstrated in the figures and covered by our calculations still satisfies condition $\mathrm{Re}[n] < \Omega$, when the GI development time is longer than the turnover time of the largest turbulent eddy. At higher values of St, inequality $|n| > \Omega$ holds, and the use of averaged characteristics of turbulent velocity and the turbulent diffusivity becomes unfounded. At the same time, surface density Σ_p of solid matter may become fairly large as solid particles accumulate in the process of their radial drift in the dust layer with the accompanying radial contraction and compaction of the layer (Youdin and Chiang, 2004; Makalkin and Ziglina, 2004). However, a ten-fold enrichment of the inner disk region with solid matter produces the problem of "eliminating" it so that only ~10% of its mass goes to form terrestrial planets later.

The results also indicate that GI becomes possible when dust aggregates reach a certain size. It can be seen from Fig. 1 that if particles within the layer had St ≥ 1 at a distance of 1 AU, which yields dust aggregate radii $a \gtrsim 30$ cm at matter density $\rho_s = 2$ g/cm^3, GI may emerge only at a relatively low level of turbulence, which corresponds to $\alpha \le 10^{-6}$ or $\mathrm{Re}^* \ge 50$, and at moderate dust matter accumulation ($\Sigma_p \le 100$ g/cm^2) owing to radial transfer. Since the Stokes flow regime is established at 1 AU, it follows from Eqs. (6) that the estimate depends only slightly on the surface density of gas in the disk (Σ_g).

At distance r = 10 AU (Fig. 2), the GI conditions are less stringent. Instability emerges at a relatively low surface density of the solid phase $\Sigma_p \sim 3$–10 g/cm^2, which corresponds to a just (1–3)-fold enrichment with respect to protosolar abundance $\Sigma_{p,i} = Z\Sigma_g$, where $Z \approx 0.015$, and Σ_g is defined by relation (57). GI is possible even under fairly strong turbulence with parameter $\alpha = 10^{-5}$, but only at a sufficiently large Stokes number St = 0.3–1, which corresponds to dust aggregate radii of 0.4–1.5 m at density $\rho_s \sim 0.5$–0.7 g/cm^3. Instability may also emerge without any enrichment with solid matter if $\alpha = 3 \times 10^{-6}$ or Re* = 25 (at the St values given above) or if $\alpha = 10^{-6}$ or Re* = 50 (at St > 0.1); in other words, it is feasible in the case of decimeter-sized bodies. Lastly, GI is possible even in a layer of centimeter-sized particles with St $\gtrsim$ 0.01 in the case of turbulent viscosity with $\alpha = 10^{-7}$ (without enrichment with dust) or $\alpha = 10^{-6}$ or Re* = 50 (with three-fold enrichment with dust material).

In addition to affecting turbulent diffusivity and rms velocity of particles, D and V_p, turbulent gas viscosity ν appears, owing to differential Keplerian rotation of the dust layer, in perturbation equation (28) as a coefficient in the right term for viscous drag. However, our calculations showed that the influence of this term on the critical values of parameters Σ_p and St was very weak (less than 1%) in the majority of examined cases; the exception was provided by cases with the maximum considered values of turbulent viscosity ($\alpha = 10^{-5}$) and parameter St at 10 AU. The critical value of Σ_p was reduced by 3% at St = 1 and by 7% at St = 2.

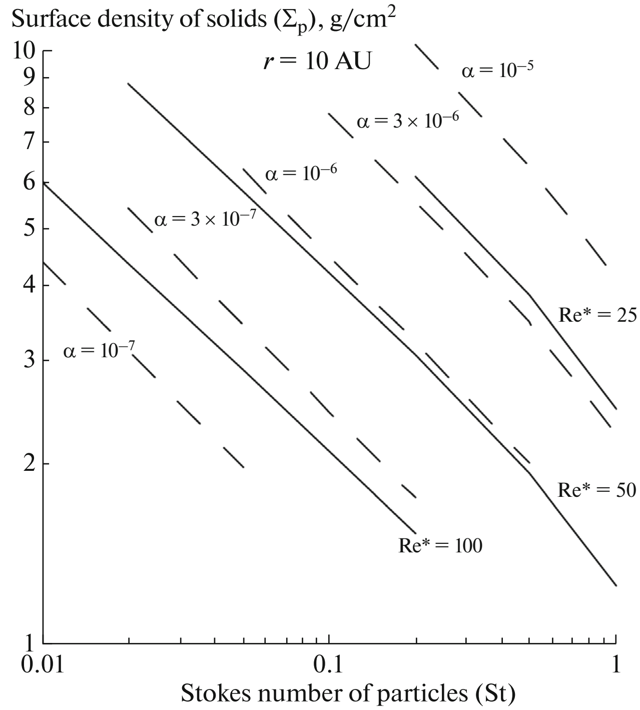

Fig. 2. Relation between the critical values of parameters Σ_p and St of the dust layer at distance r = 10 AU. See Fig. 1 for notation.

We have also compared the critical values of Σ_p and St from solution of cubic dispersion equation (39) with their values obtained from solution of quadratic equation (47). It follows from the above analysis that the difference grows with particle Stokes number. At St = 1, the difference is ≈20%; at St = 2, it is as large as 41%.

Spatial density $\rho_{p,0}$ of the dust phase in the layer at the onset of GI under the limiting values of parameters Σ_p and St was typically close in our calculations to the critical density for GI $\rho_{cr} \approx 2\rho^*$, which was obtained earlier by Safronov (1969). At distance r = 1 AU and St = 1, density $\rho_{p,0} \approx 2.2\rho^*$, at St = 2, it is reduced to $\rho_{p,0} \approx 1.6\rho^*$, at St = 0.2, $\rho_{p,0}$ reaches $3\rho^*$ regardless of the turbulent viscosity parameters (α or Re*). At 10 AU, the minimum ratio $\rho_{p,0}/\rho^*$ is approximately 20% higher than that at 1 AU, although the maximum ratio is below three even at St = 0.01. The value of parameter ε defined by relation (3) is proportional to $\rho_{p,0}$ at a constant gas density ρ_g. Parameter ε was varied in our calculations, the results of which are presented in Figs. 1 and 2, from 290 to 510 at distance r = 1 AU

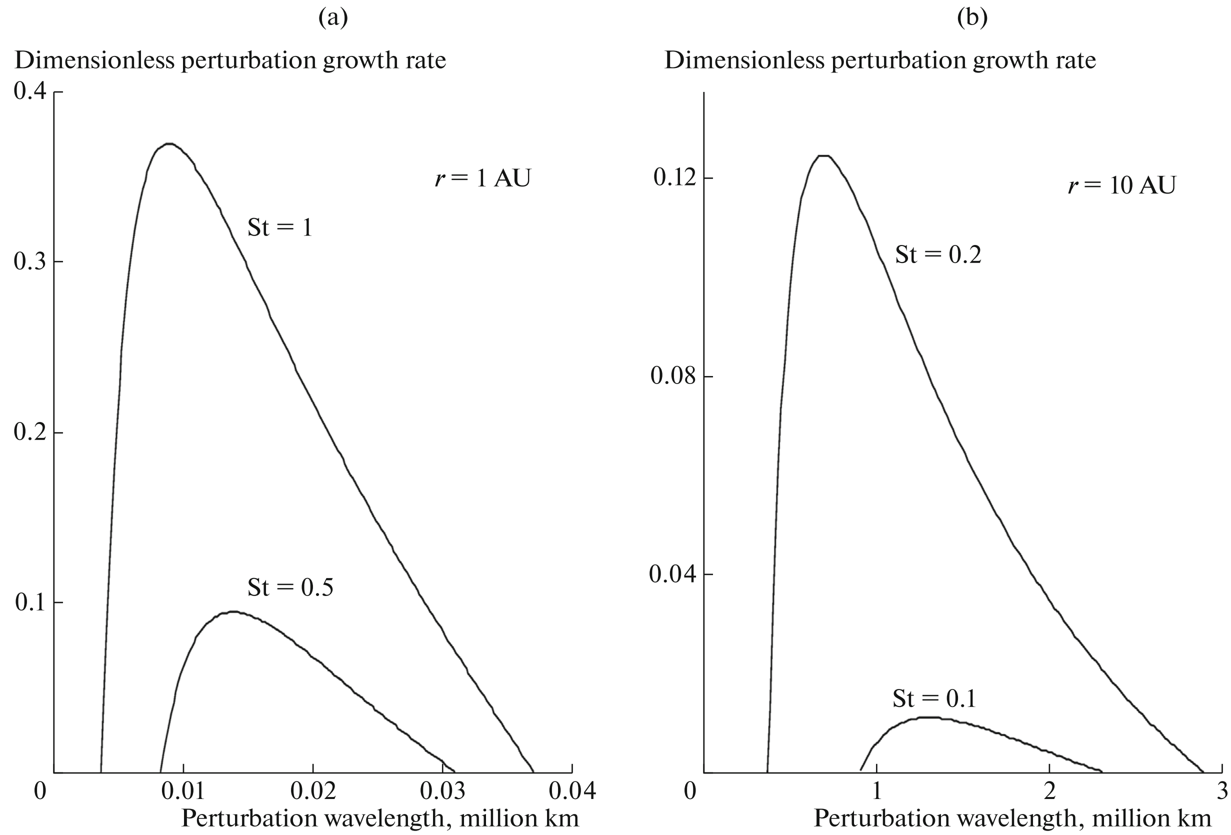

Fig. 3. Dependence of the dimensionless growth rate of surface density and velocity perturbations of the dust phase ($\Gamma \equiv \mathrm{Re}[\gamma] = \mathrm{Re}[n]/\Omega$) on perturbation wavelength ($\lambda = 2\pi/k$,) at radial distance $r = 1$ AU (a) and 10 AU (b). The results of calculations for two particle Stokes numbers (St) with a twofold difference between them are shown in each panel. The values of surface density of solids Σ_p, temperature T, and turbulence parameter α used in calculations are given in the text.

and from 19 to 51 at $r = 10$ AU. Thus, condition $\mathrm{St}|\gamma| \ll \varepsilon$, which is a requirement for dispersion relation (39) to be applicable, was fulfilled in all cases. Parameter ψ, which characterizes the contribution of the self-gravity of the disk to its thickness, also, according to Eqs. (5) and (8), grows with the variation of ratio $\rho_{p,0}/\rho^*$; as a result, ψ varies from ≈6 to 10, thus illustrating the importance of the self-gravity of the layer in the process of its thinning, which facilitates GI. At St ≥ 0.05, scale height h_p of the dust layer becomes smaller than Ekman length scale l_E, yielding a ratio of $h_p/l_E \approx 0.3$ at St = 1. The value of h_p varies from ≈400 to 4000 km at 1 AU and from 3×10^4 to 2×10^5 km at 10 AU.

Our calculations showed that the GI growth factor and the wavelength interval in which GI may develop both get considerably bigger if surface density Σ_p of the dust phase deviates even slightly from its critical value and when Stokes number St of particles increases. Figure 3 shows dimensionless perturbation growth rate $\Gamma = \mathrm{Re}[\gamma]$ at distances $r = 1$ and 10 AU as a function of perturbation wavelength λ at two values of St with a twofold difference between them. Calculations were performed for the following parameter values at radial distances of 1 and 10 AU, respectively: surface density of the dust phase $\Sigma_p = 60$ and 5 g/cm^2, temperature $T = 300$ and 50 K, and turbulent viscosity parameter $\alpha = 10^{-7}$ and 10^{-6}.

Figure 3 demonstrates strong dependences $\Gamma_{max}(\mathrm{St})$ and $\Delta\lambda(\mathrm{St})$, where $\Delta\lambda = \lambda_{max} - \lambda_{min}$ is the difference between the maximum and the minimum wavelengths at which perturbations may grow. It follows from Fig. 3 that $\Gamma_{max} \propto \mathrm{St}^a$, where $a \sim 2$–3, and $\Delta\lambda \propto \mathrm{St}^b$, where $b \sim 0.5$–0.8. Dependence $\Gamma(\lambda)$, shown in Fig. 3a is plotted for two St values at $\Sigma_p = 60$ g/cm^2; at St = 0.5, critical value $\Sigma_{p,cr} = 48.6$ g/cm^2, while $\Sigma_{p,cr} = 31.0$ g/cm^2 corresponds to St = 1. Dependences $\Gamma(\lambda)$ in Fig. 3b are plotted for $\Sigma_p = 5$ g/cm^2; at St = 0.1, critical value $\Sigma_{p,cr} = 4.50$ g/cm^2, while $\Sigma_{p,cr} = 3.22$ g/cm^2 corresponds to St = 0.2. It follows that at $\Sigma_p/\Sigma_{p,cr} \approx 1.5$–2, dimensionless growth factor Γ increases from zero to $\Gamma \approx \mathrm{St}/2$. The case of an even larger ratio of surface densities ($\Sigma_p/\Sigma_{p,cr} > 2$) is not discussed here, since we assume it to be unlikely due to the fact that GI should develop earlier.

It follows from the comparison of Figs. 3a and 3b that the wavelengths at which GI may be established increase by a factor of approximately 100 (from ~10^{-2} million km to ~1 million km) as radial distance r is varied by a factor of 10 (from 1 to 10 AU). This difference in λ values at $r = 1$ and 10 AU also corresponds to a two orders of magnitude difference in the dust layer thicknesses and scale heights h_p. Our calculations and the estimates presented in (Safronov, 1991) indicate that the latter value satisfies relation $k_{\Gamma\max} h_p = 2\pi h_p / \lambda_{\Gamma\max} \approx 0.2–0.3$. Here, $\lambda_{\Gamma\max}$ is the wavelength corresponding to the maximum perturbation growth rate. It can be seen from Fig. 3 that the values of $\lambda_{\Gamma\max}$ are approximately two times higher than $\lambda_{\min}$ (the wavelength at which GI is possible).

At St $\ll 1$, $\lambda_{\min}$ is defined by relation (53). In view of formula (11), which is applicable under the α-parameterization of turbulent viscosity, it follows from (53) that $\lambda_{\min} \propto \alpha T/(\Sigma_p \mathrm{St})$. The results of calculations for $\lambda_{\min}$ (and for $\lambda_{\Gamma\max}$) presented in Fig. 3 satisfy the indicated relation even at St $\lesssim 1$. If the parameter values given above are used, this relation yields an approximately two orders of magnitude difference in the indicated wavelengths at $r = 1$ and 10 AU.

INTERACTION OF THE DUST LAYER WITH THE DISK

In the present study, gravitational instability in the dust layer was examined without considering its interaction with gas outside the layer. Let us show that this approximation is justified (i.e., determine when the transfer of angular momentum from the dust layer to the surrounding disk gas may be neglected in the analysis of GI in the layer).

The equation for variation of the angular momentum of a layer rotating with Keplerian circular velocity V_K and affected by drag, which is produced by gas rotating with a velocity reduced by $\Delta V_g \approx \Delta V$, (see Eq. (16)), may be written as $\mathrm{d}(j\delta m)/\mathrm{d}t = 2S2\pi r^2\delta r$, where S is the shear stress acting on each of the two surfaces of the layer, $\delta m \approx \Sigma_p 2\pi r\delta r$, and $j = V_K r$. Characteristic time $t_l(r)$ of momentum transfer from the layer to the disk is then obtained:

$$t_l \sim \Sigma_p \Delta V / 2|S|. \qquad (59)$$

Approximate equality $S \approx -\rho_g \nu \Delta V/h$ holds true for S, where h is the thickness of the shear layer. Defining viscosity through parameter α using Eq. (10), we obtain $h = h_p$. In the case of viscosity in the shear layer (17), we get $h \geq l_E$, where l_E is given by Eq. (14). Since inequality $h_p < l_E$ is satisfied at St ~ 1, we may write $h = \kappa h_p$, where $\kappa \geq 1$.

As a result, we obtain from relation (59) under the α-parameterization of viscosity that $t_l \sim (\beta^2 Z^2/\alpha\theta\varepsilon)\Omega^{-1}$. It follows that the denser the dust layer and the higher the turbulent viscosity are, the faster is the angular momentum transfer; the lower the gas fraction in a protoplanetary disk or the greater the layer enrichment with dust matter (characterized by parameter β) are, the slower is the transfer. The angular momentum transfer from the particle layer to the disk may be neglected only if the characteristic perturbation development time is significantly shorter than the characteristic time of momentum transfer:

$$\mathrm{Re}[n]^{-1} \ll t_l \text{ or } \Gamma \equiv \mathrm{Re}[\gamma] \gg \frac{\alpha\theta\varepsilon}{\beta^2 Z^2}.$$

The values needed for layer GI were determined in our calculations. At $r = 1$ AU, they are as follows: $\varepsilon \geq 145$, $Z \approx 0.005$, and $\beta \approx 1-10$; at a distance of 10 AU, $\varepsilon \geq 19$, $Z \approx 0.015$, and $\beta \approx 1-3$. Using these parameters and $\theta = 0.7$, $\kappa = 1$, we obtain

$$\alpha \ll 3 \times 10^{-7}\,\beta^2\Gamma \text{ at 1 AU}$$
$$\text{and } \alpha \ll 2 \times 10^{-5}\,\beta^2\Gamma \text{ at 10 AU}.$$

It can be seen that in the case of global turbulence, the model of GI in the layer with no angular momentum transfer to the disk is applicable at a much higher turbulent viscosity at $r = 10$ AU than at 1 AU. At the same time, under moderate enrichment with dust material, $\beta \approx 5$, and Re[γ] ~ 0.1–0.3, the conditions for the development of GI "isolated" from the rest of the disk are improved at 1 AU.

If global turbulence is lacking (or negligibly weak), shear turbulence in the Ekman layer occurs, and from relation (59) one can obtain $t_l = (\varepsilon/\theta\kappa)\Omega^{-1}$. This implies that $\Gamma \gg \theta\kappa/\varepsilon$. Parameter κ at St ~ 1 is as large as $\kappa \sim 3$. As a result, the following condition is obtained:

$$\Gamma \gg 10^{-2} \text{ at 1 AU and } \Gamma \gg 10^{-1} \text{ at 10 AU}.$$

Thus, the opposite is true for shear turbulence in the Ekman layer: the constructed model with no angular momentum transfer to the disk is applicable at 1 AU better than at 10 AU.

It may then be concluded that under certain conditions stated above, GI may develop in the dust layer and induce the formation of annular condensations before the angular momentum is transferred to gas outside the layer. In the case of global α-turbulence, this scenario becomes more likely as radial distance is increased and may come true at $r = 10$ AU under moderate enrichment with dust matter. When global turbulence is very weak, the mechanism of local turbulence, which is induced by shear stresses between gas within the layer and outside of it, is in action. If this is the case, the angular momentum transfer from the layer to the disk proceeds slower at shorter radial distances; as a result, GI in the layer may emerge at $r = 1$ AU before the development of interaction with the surrounding gas.

Naturally, this interaction of the layer with the surrounding gas should be taken into account in more detailed formulations of the problem of GI in the layer. The preliminary solution of the problem of GI in the layer with the transfer of angular momentum from the layer to the surrounding disk gas taken into account demonstrates that the inclusion of this transfer leads to the development of a longer-wavelength and slower-growing GI mode. In a first approximation, it is superimposed on the shorter-wavelength and higher-amplitude mode obtained in the present study, but does not remove it.

CONCLUSIONS

The problem of gravitational instability of the dust layer in the midplane of a protoplanetary disk with turbulence was analyzed. The layer forms as dust particle aggregates settle toward the midplane of a protoplanetary disk in the process of growth of particles, which stick in mutual collisions, and/or in the process of decay of global turbulence in the disk. We have continued earlier research into GI in the region of planet formation and the estimation of its significance in the process of planet formation.

The linearized system of hydrodynamic equations for perturbations of the dust (monodisperse) and gas phases was solved. The incompressible gas approximation was used. The system was solved in the thin-disk approximation (i.e., the perturbations of parameters averaged over the dust layer thickness were considered). The perturbations of surface density and radial and azimuthal velocities of the solid phase and the azimuthal velocity of gas were the unknown variables. The equations included turbulent diffusion, the root-mean-square velocity of solid particles in turbulent gas, and the term with turbulent gas viscosity, which characterizes shear stress in a disk with differential Keplerian rotation. The present study differs from the earlier analytical investigations of GI in the layer in that we took into account the perturbation of azimuthal gas velocity in the layer induced by the transfer of angular momentum from the dust phase. The conservation of the total angular momentum of the dust and gas phases in the dust layer places an upper bound on the perturbation wavelength at which instability is possible. Turbulent particle diffusion defines the lower bound on the wavelength of such perturbations.

The system of equations for perturbations was solved, and the cubic dispersion equation, which is valid for the dust layer in the case when the ratio of surface densities of the dust phase and gas in the layer is much higher than unity, was then derived and solved. The dispersion relation was also reduced to a quadratic equation in the case of small dust particle aggregates with their characteristic stopping time in gas being considerably shorter than the orbital period. As a result, the approximate GI criterion, which takes the deceleration of dust particles by gas into account, was obtained. It is similar to the known criterion derived by Toomre (parameter $Q < 1$), but the sound speed and the gas surface density in it are replaced by the root-mean-square particle velocity and the surface dust density. In addition, the new criterion differs from the common one in that it features particle Stokes number St (the dimensionless size-dependent stopping time of an individual particle (aggregate) in gas). The relation between surface layer density Σ_p and the particle Stokes number corresponding to the onset of instability was obtained. The approximate form of this relation is $\Sigma_p^2 \mathrm{St} = A$, where parameter A depends on the radial distance and the turbulence intensity (turbulent velocity V_g). The numerical solution of the third-order dispersion equation (see Figs. 1 and 2) confirmed that this relation is approximately satisfied in the case of small particle sizes ($\mathrm{St} \ll 1$) and revealed a moderate deviation from this relation at St ~ 1.

The parameters corresponding to the onset of gravitational instability, the wavelength interval in which this instability emerges, and the dependence of the perturbation growth rate on the perturbation wavelength were determined by solving the dispersion equation. Calculations were performed for two radial distances in the circumsolar disk: 1 and 10 AU. An instability development time of 10–10^2 Keplerian periods and characteristic unstable wavelengths of ~10 layer thicknesses follow from the obtained values of the perturbation growth rate.

It was demonstrated that GI at a distance of 10 AU may emerge at the protosolar abundance of solids, $Z \approx 0.015$, or their slight enrichment at a turbulence level corresponding to parameter $\alpha \le 10^{-5}$ and decimeter particle sizes. At a distance of 1 AU, a 5–10-fold enrichment with solids relative to the initial (protosolar) abundance $Z \approx 0.005$ is required in order for the instability to emerge in the turbulent dust layer. However, this conclusion remains valid only if the assumed level of turbulence, which is generated by shear stresses in the process of interaction of gas in the layer and above the layer, is sufficiently high. If the assumed critical Reynolds number $\mathrm{Re}^* \approx 20$ is replaced by a value that is ten times higher, no enrichment with solid matter is needed. The values of $\mathrm{Re}^* \geqslant 200$ were used in certain studies. This problem likely requires further examination.

In view of the obtained results, the conclusions made in certain studies regarding the inefficiency of GI in the formation of planetesimals seem hasty and insufficiently substantiated.

In the present study, GI in the layer was analyzed with no consideration for the transfer of angular momentum from gas in the layer to gas in a protoplanetary disk outside the layer. However, our estimate of the conditions under which this transfer may be neglected demonstrates that, under certain conditions, GI in the dust layer may develop and lead to the formation of annular condensations before the angular momentum is transferred to gas outside the layer.

Naturally, the more complete formulation of the problem of GI in the dust layer with the transfer of angular momentum from the layer to the surrounding gas should also be examined. The preliminary solution of the problem in this formulation demonstrates that the inclusion of this transfer leads to the development of a long-wavelength and slower-growing GI mode. In a rough approximation, it is superimposed on the shorter-wavelength and higher-amplitude mode obtained in the present study, but does not remove it.

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