ISSN 0038-0946, Solar System Research, 2016, Vol. 50, No. 1, pp. 33–43. © Pleiades Publishing, Inc., 2016.
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# Qualitative Features of the Evolution of Some Polar Satellite Orbits

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Received April 29, 2015

**Abstract**—Two special cases of the problem of the secular perturbations in the orbital elements of a satellite with a negligible mass produced by the joint influence of the oblateness of the central planet and the attraction by its most massive (or main) satellites and the Sun are considered. These cases are among the integrable ones in the general nonintegrable evolution problem. The first case is realized when the plane of the satellite orbit and the rotation axis of the planet lie in its orbital plane. The second case is realized when the plane of the satellite orbit is orthogonal to the line of intersection between the equatorial and orbital planes of the planet. The corresponding particular solutions correspond to those polar satellite orbits for which the main qualitative features of the evolution of the eccentricity and pericenter argument are described here. Families of integral curves have been constructed in the phase plane of these elements for the satellite systems of Jupiter, Saturn, and Uranus.

*Keywords*: secular perturbations, polar satellite orbits

**DOI:** 10.1134/S0038094615060064

## INTRODUCTION AND FORMULATION OF THE PROBLEM

This paper is a continuation of our investigation of the integrable cases for the problem of the orbital evolution of a satellite with a negligible mass under the action of the joint gravitational perturbations from the oblateness of the central planet and the attraction by its main satellites and the Sun (Vashkov'yak et al., 2015). Some of the described cases may turn out to be useful in considering one interesting feature of the satellite systems of all giant planets. This feature is associated with the existence of zones of "avoidance" by small celestial bodies or zones separating the families of orbits of regular and irregular satellites in circumplanetary space. In these zones, the influence of the above perturbations is pairwise or, in aggregate, is comparable in magnitude.

The so-called secular equations provide a mathematical basis for the investigation. In the first approximation of the perturbation theory, these are the differential equations for the changes of orbital elements in which the full perturbing function $R$ is replaced by its secular part $W$. In the presumed absence of mutual orbital resonances, it is found by an independent averaging of $R$ over the mean planetocentric longitudes of the Sun, the main satellites, and the (test or hypothetical) satellite being investigated. The function $W$ depends on five planetocentric Keplerian orbital elements: the semimajor axis $a$, eccentricity $e$, inclination $i$, pericenter argument $\omega$, and ascending-node longitude $\Omega$. It is convenient to refer the angular elements in the problem under consideration to the plane of the planetary equator and the line of its intersection with the plane of the planet's heliocentric orbit (or the Sun's planetocentric orbit). For coherence, we will permit ourselves to repeat a number of formulas and designations from the above-mentioned paper, which we will call the original one for short.

The function $W$ consists of three terms corresponding to different named perturbing factors, while the two first integrals of the evolution problem are

$$a = \text{const},\ W = W_0 + W_1 + W_2 = \text{const}. \qquad (1)$$

Apart from the elements of the satellite orbit, we use the following notation:

$\mu_0, \mu_j, \mu'$ are the products of the gravitational constant by the masses of the planet, its $j$th main satellite, and the Sun, respectively;

$a_0, a_j, a'$ are, respectively, the mean equatorial radius of the planet and the semimajor axes of the orbit of its $j$th main satellite and the planetocentric orbit of the Sun;

$c_{20}$ is the coefficient at the second zonal harmonic of the planet's gravitational field;

$i_{\text{orb}}, \omega_{\text{orb}}$ are, respectively, the inclination and pericenter argument of the orbit of the test satellite referred to the planet's orbital plane.

In this paper (just as in the original one), the evolution problem is considered by taking into account the principal terms of the secular parts of the perturbing functions the expressions for which are given below.

The function

$$W_0 = \frac{3\mu_0 a_0^2 c_{20}}{4a^3}\left(1-e^2\right)^{-3/2}\left(\sin^2 i - 2/3\right) \tag{2}$$

describes the secular influence of the second zonal harmonic of the planet's gravitational field.

The function

$$W_1 = \frac{3\mu' a^2}{16a'^3}$$

$$\times\left\{\begin{aligned}&2e^2-\left(2+3e^2\right)\Big[\sin^2 i+\sin^2 I\\&\times\left(\cos 2i+\sin^2 i\sin^2\Omega\right)-\frac{1}{2}\sin 2I\sin 2i\cos\Omega\Big]\\&+5e^2\cos 2\omega\Big[\sin^2 i+\sin^2 I\Big(\cos 2i\\&+\left(\sin^2 i-2\right)\sin^2\Omega\Big)-\frac{1}{2}\sin 2I\sin 2i\cos\Omega\Big]\\&+5e^2\sin 2\omega\left[-\sin^2 I\cos i\sin 2\Omega+\sin 2I\sin i\sin\Omega\right]\end{aligned}\right\}, \tag{3}$$

describes the influence of the secular solar perturbations in the quadratic approximation in $a/a'$ or in the Hill approximation (the parameter $I$ is the angle between the equatorial plane of the planet and its orbital plane).

The function $W_2$ describes the secular perturbations from the planet's main satellites, which are assumed in our analysis to be noninteracting between themselves. This function is defined by the formula

$$W_2 = \frac{1}{2\pi}\int_0^{2\pi} V(E)(1-e\cos E)\,dE, \tag{4}$$

where $E$ is the eccentric anomaly of the test satellite, and $V$ in the dynamical interpretation is the force function of a system of a finite number $J$ of Gaussian rings with masses equal to the masses of the satellites. In our simplified model, we assume that the orbits of all main satellites lie in the planet's equatorial plane and have zero eccentricities. In this case, the function $V$ can be represented by a hypergeometric series,

$$\begin{aligned}V &= \sum_{j=1}^{J}\frac{\mu_j}{\sqrt{r^2+a_j^2}}F\left(\frac{1}{4},\frac{3}{4};1;\frac{4a_j^2\left(r^2-z^2\right)}{\left(r^2+a_j^2\right)^2}\right)\\&=\sum_{j=1}^{J}\frac{\mu_j}{\sqrt{r^2+a_j^2}}\sum_{n=0}^{\infty}B_n\left[\frac{4a_j^2\left(r^2-z^2\right)}{\left(r^2+a_j^2\right)^2}\right]^n,\end{aligned} \tag{5}$$

where

$$\begin{aligned}r &= a(1-e\cos E),\\ z &= a\sin i\left[(\cos E-e)\sin\omega+\left(1-e^2\right)^{1/2}\cos\omega\sin E\right],\end{aligned} \tag{6}$$

$B_n$ are the constant numerical coefficients defined by the general or recurrence formulas

$$\begin{aligned}B_n &= \frac{(4n)!}{2^{6n}(n!)^2(2n)!};\\ B_n &= \left(1-\frac{1}{n}+\frac{3}{16n^2}\right)B_{n-1},\quad n>0,\quad B_0=1.\end{aligned} \tag{7}$$

It can be shown that the function $W_2$ depends on $e$, $i$, and $\omega$, respectively, only via the combinations $e^2$, $\sin^2 i$, and $e^2\sin^2 i\cos 2\omega$. These properties are valid for any orbital eccentricity and inclination of the test satellite, while the expression for the function $W_2$ can be represented in general form as

$$\begin{aligned}W_2 &= \sum_{j=1}^{J}\frac{\mu_j}{\sqrt{a^2+a_j^2}}\\&\times\sum_k P_k^{(j)}\left(a,a_j\right)Q_k^{(j)}\left(e^2,\sin^2 i,e^2\sin^2 i\cos 2\omega\right),\end{aligned} \tag{8}$$

where $P_k^{(j)}, Q_k^{(j)}$ are some rational functions of their arguments.

In the original paper, we used a simplification that, for moderate eccentricity and sine of inclination, allowed us to restrict ourselves to the partial sum of the series in powers of these small parameters in Eq. (8). Therefore, two of the seven integrable cases specified in this paper (IV and VI) could not be legitimately considered, because $\sin i = 1$ for both of them.

The goal of this paper is to derive an analytical expression for the function $W_2$ for $i = 90°$ and moderate eccentricities and to perform a qualitative analysis of the above two integrable cases, i.e., the system of Lagrange differential equations in elements with the full averaged perturbing function $W$, based on this expression.

## EXPRESSION FOR THE FUNCTION $W$ FOR POLAR ORBITS

To derive an analytical expression for the function $W_2$, it is necessary to perform integration in Eq. (4) by setting $\sin i = 1$ in Eq. (6) and by taking into account, just as in the original paper, only the terms of order $e^2$ and $e^4$. Note that its expression for arbitrary $i$, which is also valid, in particular, for $i = 90°$, was derived by Vashkovjak (1976), but only to within $e^2$ inclusive.

Making the necessary standard but somewhat cumbersome transformations leads to the expression

$$\begin{aligned}W_2 &= \sum_{j=1}^{J}\frac{\mu_j}{\sqrt{a^2+a_j^2}}\Big(P_1^{(j)}e^2+P_2^{(j)}e^4\\&+P_3^{(j)}e^4\cos 2\omega+P_4^{(j)}e^4\cos 4\omega+P_5^{(j)}e^2\cos 2\omega\Big).\end{aligned} \tag{9}$$

Series in powers of the "unified" parameter

$$\zeta_j = \left(\frac{2aa_j}{a^2 + a_j^2}\right)^2, \quad (10)$$

which remains less than unity for any relation between the semimajor axes $a$ and $a_j$, is used to represent the functions $P_k^{(j)}$. This parameter was introduced to derive a single analytical expression for the secular part of the perturbing function in Vashkov'yak et al. (2013). Such a representation itself requires introducing some auxiliary functions acting as the Laplace coefficients in classical expansions. For the polar orbits considered here, we use the functions

$$\begin{aligned} C_j^{(m)} &= \delta_{m0} + \sum_{n=1}^{\infty} \frac{n^m}{n+1} G_n \zeta_j^n, \\ D_j^{(m)} &= \frac{1}{2}\delta_{m0} + \sum_{n=1}^{\infty} \frac{n^m}{(n+1)(n+2)} G_n \zeta_j^n, \end{aligned} \quad (11)$$

where $\delta_{m0} = \begin{cases} 1, & m = 0, \\ 0, & m \neq 0, \end{cases}$ and the constant numerical coefficients $G_n$ are defined by the general or recurrence formulas

$$\begin{aligned} G_n &= \frac{(2n-1)!!}{(2n)!!} B_n = \frac{(4n)!}{2^{8n}(n!)^4}; \\ G_n &= G_{n-1}\left(1 - \frac{1}{2n}\right)\left(1 - \frac{1}{4n}\right)\left(1 - \frac{3}{4n}\right), \quad G_0 = 1. \end{aligned} \quad (12)$$

The general expression for the functions $P_k^{(j)}$ is

$$\begin{aligned} P_k^{(j)}(a, a_j) &= \sum_{m=0}^{M_k} g_k^{(m)} p_{k,m}(\alpha_j) C_j^{(m)} \quad (k \neq 4); \\ P_k^{(j)}(a, a_j) &= \sum_{m=0}^{M_k} g_k^{(m)} p_{k,m}(\alpha_j) D_j^{(m)} \quad (k = 4). \end{aligned} \quad (13)$$

The functions $p_{k,m}(\alpha_j)$ are polynomials of degree $S_k$,

$$p_{k,m}(\alpha_j) = \sum_{s=0}^{S_k} p_{k,m}^{(s)} \alpha_j^s \quad (14)$$

for the dimensionless quantity (a function of the semimajor axes $a$ and $a_j$)

$$\alpha_j = \frac{a^2}{a_j^2 + a^2}, \quad (15)$$

The quantities $M_k$, $S_k$, and $p_{k,m}^{(s)}$ are integers, while $g_k^{(m)}$ can also be fractions. In order not to overload the text, the entire tabular material is given in the Appendix.

In the original paper, we introduced several constant parameters,

$$\begin{aligned} \alpha_0 &= -\frac{3}{16}\left(\frac{a_0}{a}\right)^2 c_{20}, \quad \alpha_{k>0} = \frac{a}{\mu_0} S_k, \\ S_k &= \sum_{j=1}^{J} \frac{\mu_j}{\sqrt{a^2 + a_j^2}} P_k^{(j)}(a, a_j), \quad (1 \le k \le 5), \end{aligned} \quad (16)$$

$$\beta = \frac{3}{16}\frac{\mu'}{\mu_0}\left(\frac{a}{a'}\right)^3, \quad \gamma_k = \alpha_k/\beta, \quad (0 \le k \le 5) \quad (17)$$

and a new independent variable,

$$\tau = \beta n(t - t_0), \quad (18)$$

where $n = \sqrt{\mu_0/a^3}$ is the mean motion of the test satellite, $t_0$ and $t$ are the initial and current instants of time, respectively. In what follows, we will use these parameters and the normalized perturbing function

$$W_N = \frac{W}{\beta n^2 a^2} = \frac{Wa}{\beta \mu_0}. \quad (19)$$

After the preliminary transformations performed to find the function $W_2$ and the substitution of $\sin i = 1$ into Eqs. (2) and (3), the full normalized perturbing function for polar orbits takes the form

$$\begin{aligned} W_N &= -\gamma_0\left(2e^2 + \frac{5}{2}e^4\right) + \gamma_1 e^2 + \gamma_2 e^4 + \gamma_3 e^4 \cos 2\omega \\ &+ \gamma_4 e^4 \cos 4\omega + \gamma_5 e^2 \cos 2\omega + 5e^2 \cos 2\omega \\ &\times \left[1 - \sin^2 I\left(1 + \sin^2\Omega\right)\right] + 5e^2 \sin 2\omega \sin 2I \sin\Omega \\ &+ 2e^2 - \left(2 + 3e^2\right)\left(1 - \sin^2 I \cos^2\Omega\right) = \text{const}. \end{aligned} \quad (20)$$

Here, the constant in the part proportional to $\gamma_0$ and the terms of order $e^6$ or higher were discarded. In the next section, we will separately consider two cases of different mutually orthogonal orientations of polar orbits with $\cos i = 0$:

(1) $\sin\Omega = 0$ at $\cos I = 0$ (the equatorial plane of the planet is orthogonal to the plane of its heliocentric orbit, and the satellite moves in this plane, $\sin i_{\text{orb}} = 0$).

(2) $\cos\Omega = 0$ at an arbitrary angle $I$ (the plane of the satellite orbit is orthogonal to the line of their intersection).

In the original paper, these cases are numbered IV and VI, respectively. This numbering of integrable cases was initially introduced by Lidov and Yarskaya (1974), who considered the evolution problem by taking into account the oblateness of the central planet and the attraction by the Sun, i.e., without allowance for the attraction by its massive satellites or at $\gamma_k = 0$ $(1 \le k \le 5)$.

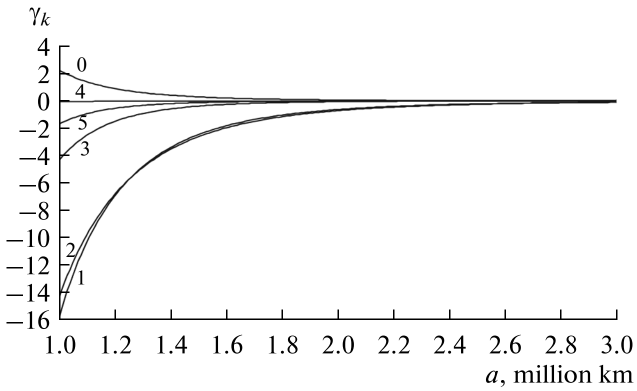


**Fig. 1.** Dependences of $\gamma_k$ on semimajor axis.

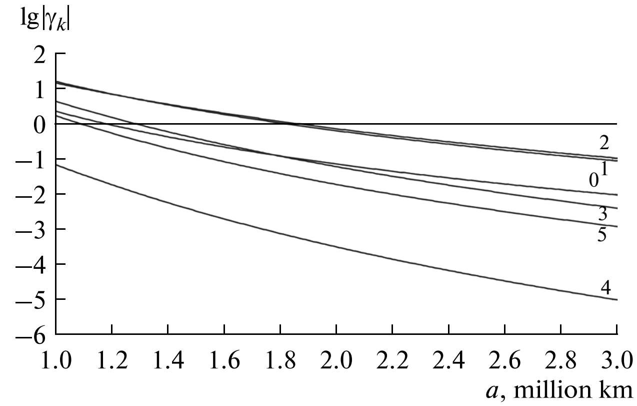


**Fig. 2.** Dependences of $\log|\gamma_k|$ on semimajor axis.

The evolution system of the problem is simplified, because the right-hand sides of the two equations become zero for polar orbits at $\cos i = 0$ in both cases:

$$\frac{di}{d\tau} = \frac{d\Omega}{d\tau} = 0. \tag{21}$$

The simplified system is reduced to two differential equations,

$$\frac{de}{d\tau} = -\frac{\sqrt{1-e^2}}{e}\frac{\partial W_N}{\partial \omega}, \quad \frac{d\omega}{d\tau} = \frac{\sqrt{1-e^2}}{e}\frac{\partial W_N}{\partial e} \tag{22}$$

with the first integral (20), whose presence allows us to perform its qualitative analysis and, in principle, to obtain the necessary quadratures.

Since the problem is a multiparameter one, its investigation for arbitrary $I$ and $\gamma_k$ ($0 \le k \le 5$) seems difficult and not too justified. Therefore, just as in our original paper, below we will consider mainly the satellite system of Uranus with its specific physical parameters. As has already been pointed out in the Introduction, studying the region of circumplanetary space where the influence of the above perturbations on the satellite is pairwise or, in aggregate, is comparable in magnitude is of greatest interest. Figure 1 shows the dependences of coefficients $\gamma_k$ on the semimajor axis of Uranus's satellite orbit. The numbers near the lines correspond to six values of the index $k$. No real satellites are observed in the presented range of semimajor axes; therefore, we can only talk about hypothetical satellites. All curves $\gamma_k(a)$, except $\gamma_0(a)$, go to minus infinity when $a$ tends to $a_5 \approx 0.6$ million km, the semimajor axis of Uranus's most distant main satellite.

For greater detail, the same dependences are presented in Fig. 2 on a logarithmic scale. They give an idea of the orders of the perturbations from the oblateness and massive satellites compared to the solar ones, the horizontal straight line corresponding to zero value of the logarithm. The perturbing influence of Uranus's massive satellites is seen to exceed considerably the influence of its oblateness in this range of semimajor axes.

Since the terms of order $e^6$ or higher are disregarded in Eq. (9), the need for estimating whether the derived expression for the function $W_2$ is suitable for high eccentricities inevitably arises. Such an estimate was made through comparative calculations of this function by two methods: the proposed analytical one ($W_a$) and by numerically finding the quadrature (4) ($W_c$), which can be performed in principle for arbitrary $e < 1$. Figure 3 presents the results of such a comparison in the same range of semimajor axes. The numbers above the lines correspond to two boundary values of $\omega$ in degrees. The solid and dashed lines correspond to the analytical and numerical methods, respectively. For the comparison to be proper, the terms independent of $e$ were discarded in the function $W_c$ (just as in $W_a$). For our comparison, we chose the eccentricity $e = 0.4$, which differs markedly from zero. In this case, the discarded terms of the perturbing function of order $e^6$ are approximately 0.0041 in magnitude. It seems to us that an accuracy of even ~0.01 is quite acceptable for the investigation of the evolution problem, while the qualitative features of the evolution

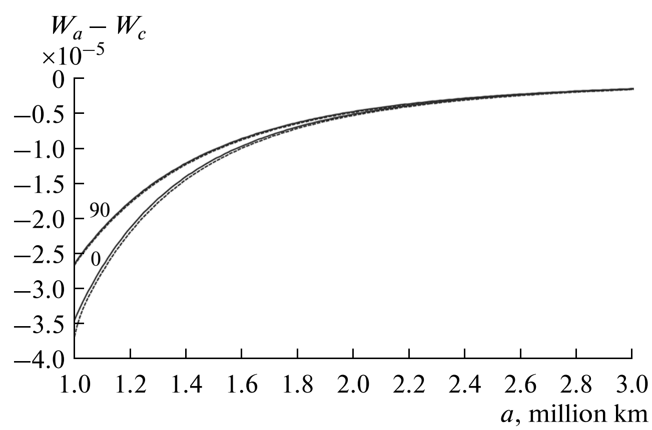


**Fig. 3.** Dependences of the functions $W$ derived by different methods on semimajor axis.

**Table 1.** Absolute error of determining the function $W$

| $e$ | $e^6$ | $\max_{a,\omega}\lvert W_a - W_c\rvert$ |
|---|---|---|
| 0.05 | $1.6 \times 10^{-8}$ | $4.9 \times 10^{-11}$ |
| 0.1 | $1.0 \times 10^{-6}$ | $5.5 \times 10^{-11}$ |
| 0.2 | $6.4 \times 10^{-5}$ | $1.2 \times 10^{-9}$ |
| 0.3 | $7.3 \times 10^{-4}$ | $1.5 \times 10^{-8}$ |
| 0.4 | $4.1 \times 10^{-3}$ | $2.3 \times 10^{-7}$ |

of the satellite orbit will also be adequate at slightly higher $e \sim 0.5–0.7$.

Table 1 lists the maximum (in $a$ and $\omega$) absolute values of the differences obtained by the analytical and numerical methods, giving numerical estimates for the absolute error of finding the function $W$ for various eccentricities. Based on the data from this table and Fig. 3, the maximum in $a$ and $\omega$ of the relative error in $W$ for $e = 0.4$ can be estimated to be ~0.05.

## QUALITATIVE ANALYSIS OF CASE IV

At $\cos I = 0$, $\cos i = 0$, and $\sin\Omega = 0$, the function $W_N$ (and the first integral) is

$$W_N = A(\omega)e^4 + B(\omega)e^2 = c, \tag{23}$$

where

$$\begin{gathered} A(\omega) = -\frac{5}{2}\gamma_0 + \gamma_2 + \gamma_3\cos 2\omega + \gamma_4\cos 4\omega, \\ B(\omega) = 2(1-\gamma_0) + \gamma_1 + \gamma_5\cos 2\omega. \end{gathered} \tag{24}$$

From integral (23) it is easy to find the dependence of $e$ on $\omega$ and the constant of the integral $c$ defined by the initial values $e_0$ and $\omega_0$ at $\tau = 0$,

$$c = A(\omega_0)e_0^4 + B(\omega_0)e_0^2. \tag{25}$$

Of course, apart from $\omega$, the functions $A$ and $B$ depend on the semimajor axis $a$ via $\gamma_k$. The changes in the eccentricity and pericenter argument are described by the equations

$$\frac{de}{d\tau} = 2\sqrt{1-e^2}\,g_1(e,\omega), \quad \frac{d\omega}{d\tau} = 2\sqrt{1-e^2}\,g_2(e,\omega), \tag{26}$$

where

$$\begin{gathered} g_1(e,\omega) = e\left[\gamma_5 + e^2(\gamma_3 + 4\gamma_4\cos 2\omega)\right]\sin 2\omega, \\ g_2(e,\omega) = B(\omega) + 2A(\omega)e^2. \end{gathered} \tag{27}$$

The existence of the first integral (23), in principle, allows the solution of the problem to be obtained in quadratures. This requires finding the dependence $\omega(e)$ from it and substituting it into the first of Eqs. (26). Thereafter, a nonlinear first-order differential equation is obtained for the variable $x = e^2$. However, here we perform only a qualitative analysis of the evolution system.

Stationary solutions of system (26) exist only at $\sin 2\omega^* = 0$ or $\omega^* = \pm\pi/2, 0, \pi$, because it follows from Fig. 1 that $\gamma_k < 0$ for all $1 \le k \le 5$ and $\gamma_3 \pm 4\gamma_4 < 0$. The stationary values of the eccentricity in the $(\omega, e)$ phase plane are defined by the formula

$$(e^*)^2 = -\frac{B(\omega^*)}{2A(\omega^*)}. \tag{28}$$

In contrast to the case of $\gamma_k = 0$ $(1 \le k \le 5)$, there exist relatively small ranges of semimajor axes in which $(e^*)^2 > 0$ and $0 < e^* < 1$ both for $\omega^* = \pm\pi/2$ and for $\omega^* = 0, \pi$. Figure 4 shows the dependences of the stationary eccentricities on the semimajor axis in the range of their existence (the numbers near the lines correspond to two values of $\omega^*$ in degrees). Four special values of the semimajor axis corresponding to the change in the structures of the families of phase trajectories in the $(\omega, e)$ plane are marked on the horizontal axis. It can be shown that the singular point is a saddle at $\omega^* = 0, \pi(\cos 2\omega^* = 1)$ and a center at $\omega^* = \pm\pi/2(\cos 2\omega^* = -1)$. The following values of the constant of the integral correspond to the saddle points:

$$c_s = 0 \text{ and } c_s = -\frac{B(0)^2}{2A(0)}, \tag{29}$$

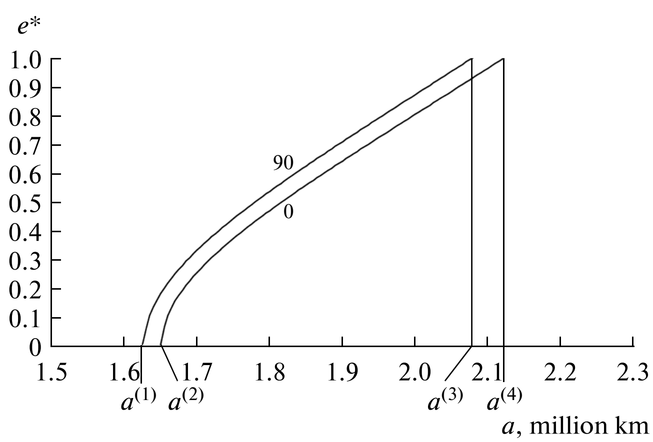


**Fig. 4.** Dependences of the stationary eccentricities on semimajor axis.

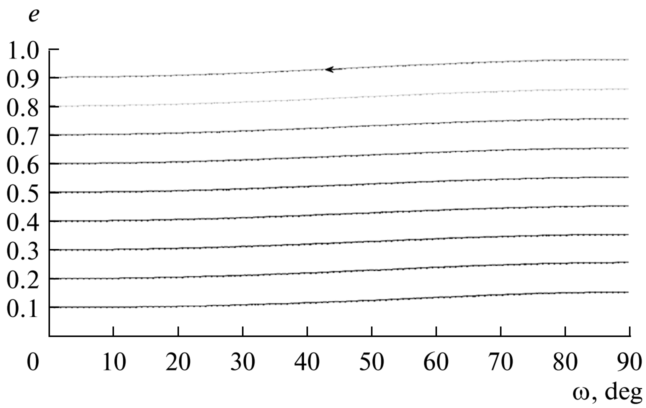


**Fig. 5.** Family of phase trajectories for case IV in the (ω, $e$) plane for $a$ = 1.62 million km.

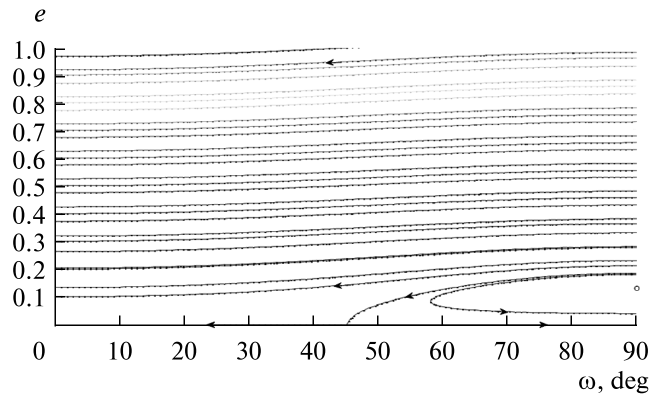


**Fig. 6.** Same as Fig. 5 for $a$ = 1.64 million km.

while the equations for one or two branches of the separatrices are obtained from (23) at $c = c_s$:

$$e^2 = \frac{-B(\omega) \pm \sqrt{B(\omega)^2 + 4A(\omega)c_s}}{2A(\omega)}. \tag{30}$$

At $a < a^{(1)} \approx 1.625$ million km, no singular points exist in the (ω, $e$) plane, while the orbital evolution is reduced to a monotonic (circulational) decrease in the pericenter argument at insignificant eccentricity oscillations (Fig. 5).

At the first bifurcation value of $a = a^{(1)}$, a pair of singular points arises in the lower right corner of the displayed rectangle. In view of the double symmetry of the phase trajectories and the possibility of their natural extension, only the range $0 \le \omega \le 90°$ is shown. The value $a^{(1)}$ is the solution of the equation $B(\pm\pi/2) = 0$ or

$$2 - 2\gamma_0(a) + \gamma_1(a) - \gamma_5(a) = 0. \tag{31}$$

At a $a^{(1)} < a < a^{(2)} \approx 1.652$ million km, two singular points (one of them is marked by the circle) exist in the phase plane, and a libration region of the pericenter argument appears (Fig. 6).

The value $\omega_s$ corresponding to the intersection of the separatrix with the $e = 0$ axis is determined from the formula $B(\omega_s) = 0$ or

$$\cos 2\omega_s = \frac{2(\gamma_0 - 1) - \gamma_1}{\gamma_5}. \tag{32}$$

The value of $e$ corresponding to the intersection of the separatrix with the $\omega = 90°$ axis is determined from the formula

$$e_s^2 = -\frac{B(\pm\pi/2)}{A(\pm\pi/2)}. \tag{33}$$

At the second bifurcation value of $a = a^{(2)}$, the separatrix at $e = 0$ "enters" the lower left corner. The value $a^{(2)}$ is the solution of the equation $B(0) = 0$ or

$$2\gamma_0(a) - 2 - \gamma_1(a) - \gamma_5(a) = 0. \tag{34}$$

At a $a^{(2)} < a < a^{(3)} \approx 2.08$ million km, two singular points and a libration region of the pericenter argument also exist in the phase plane. However, in contrast to Fig. 6, the second branch appears in the separatrix, while the saddle point is located on the $\omega = 0$ axis (Fig. 7).

The maximum (in eccentricity) width of the libration zone is determined using Eq. (30), in which we should set $\omega = \pm\pi/2$.

Apart from the phase trajectories and singular points, the special horizontal straight line corresponding to the so-called critical eccentricity is marked by the larger crosses. For the external variant of the problem, it is determined by the condition for equality between the pericenter distance of the orbit of the test satellite and either the orbital radius of one of the planet's massive satellites ($1 \le j \le J$) or its own radius ($j = 0$). For polar orbits, such a condition is met only at $\omega = 0$ and π. At $e > \overline{e}_j$, the investigation of the evolution problem is only of formal interest, because the regularity of the orbit of the test satellite breaks down due to the possibility of its intersection either with the orbits of massive satellites or with the planet's surface.

The "cross" straight line (not shown in Figs. 5 and 6) in Fig. 7 corresponds to $j = 5$, i.e., to the contact at the pericenter of the orbit of the test satellite and the orbit of Oberon, the most distant main satellite of Uranus. Clearly, the straight lines corresponding to the critical eccentricities for $0 \le j \le 4$ are also implicitly present in the (ω, $e$) phase plane. For a fixed semimajor axis, they are located above the "cross" straight line, but they are not shown in the figure.

As the semimajor axis increases, both singular points move into the region of high eccentricities (Fig. 8). As the semimajor axis increases further, qualitative changes in the structures of the families of phase trajectories occur in the region $\overline{e}_j < e \approx 1$. Therefore, we give $a = a^{(4)} \approx 2.12$ million km (which formally follows from the model) and note that only circulational trajectories with a monotonic increase in the pericenter argument exist in the phase plane at $a > a^{(4)}$. In addition, since our analysis is limited to moderate

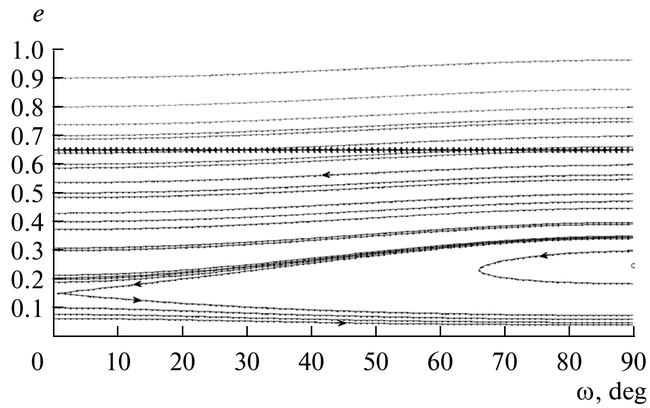


**Fig. 7.** Same as Fig. 5 for $a = 1.67$ million km.

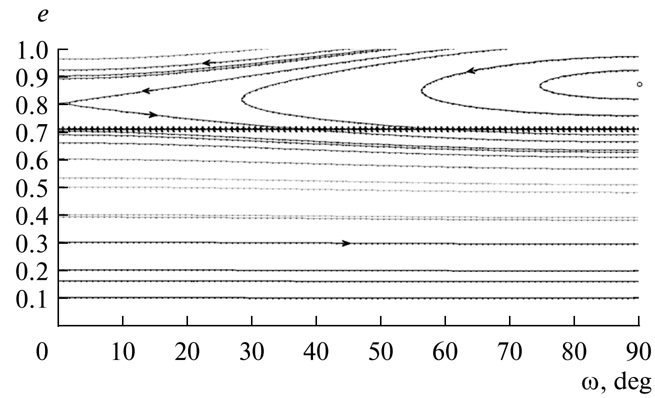


**Fig. 8.** Same as Fig. 5 for $a = 2.00$ million km.

eccentricities, we will end our analysis of the integrable case IV here.

It is interesting only to note that the revealed structures of the families of phase trajectories presented in Figs. 6–8 (the libration zones of the pericenter argument) owe their existence exclusively to the perturbing influence of the massive satellites. At $\gamma_k = 0$ for $1 \le k \le 5$, i.e., in their absence, the orbital eccentricity of the test satellite remains constant, while the pericenter argument decreases or increases monotonically with a constant rate dependent on the semimajor axis (Lidov and Yarskaya, 1974).

## QUALITATIVE ANALYSIS OF CASE VI

At $\cos i = 0$, $\cos\Omega = 0$, and an arbitrary angle $I$, the function $W_N$ (and the first integral) is

$$W_N = A(\omega)e^4 + D(\omega)e^2 = c, \tag{35}$$

where $A(\omega)$ is defined by the first of Eqs. (24),

$$\begin{aligned} D(\omega) &= -1 - 2\gamma_0 + \gamma_1 + \gamma_5\cos 2\omega \\ &+ 5[\cos 2\omega\cos 2I + \sin 2\omega\sin 2I\,\mathrm{sign}(\sin\Omega)]. \end{aligned} \tag{36}$$

From integral (35) it is easy to find the dependence of $e$ on $\omega$ and the constant of the integral $c$ defined by the initial values $e_0$ and $\omega_0$ at $\tau = 0$,

$$c = A(\omega_0)e_0^4 + D(\omega_0)e_0^2. \tag{37}$$

Apart from $\omega$, the function $D$ (just as $A$ and $B$) depends on the semimajor axis $a$ via $\gamma_k$. The changes in the eccentricity and pericenter argument are described by the equations

$$\frac{de}{d\tau} = 2\sqrt{1-e^2}f_1(e,\omega), \quad \frac{d\omega}{d\tau} = 2\sqrt{1-e^2}f_2(e,\omega), \tag{38}$$

where

$$\begin{aligned} &f_1(e,\omega) \\ &= e\begin{Bmatrix}\left[\gamma_5 + 5\cos 2I + e^2(\gamma_3 + 4\gamma_4\cos 2\omega)\right] \\ \times \sin 2\omega - 5\cos 2\omega\sin 2I\,\mathrm{sign}(\sin\Omega)\end{Bmatrix}, \\ &f_2(e,\omega) = D(\omega) + 2A(\omega)e^2. \end{aligned} \tag{39}$$

Just as in case IV, the existence of the first integral (35), in principle, allows the solution of the problem to be obtained in quadratures. This requires finding the dependence $\omega(e)$ from it and substituting it into the first of Eqs. (38). Thereafter, a nonlinear first-order differential equation is obtained for the variable $x = e^2$.

To find stationary solutions of system (38), it is necessary to solve the equations

$$f_1(e,\omega) = 0, \quad f_2(e,\omega) = 0. \tag{40}$$

Let us first consider the two simplest model cases where $\sin 2I = 0$:

(a) $I = 90°(\cos 2I = -1)$ and (b) $I = 0$ $(\cos 2I = 1)$.

Whereas case (a) is fairly close to the actual orientation of Uranus's equatorial and orbital planes, case (b) seems hypothetical. Nevertheless, the formal solution of Eqs. (40) in both these cases is defined by simple formulas:

$$\begin{aligned} &\sin 2\omega^* = 0, \\ &e^* = \sqrt{\frac{-1 - 2\gamma_0 + \gamma_1 + (\gamma_5 + 5\cos 2I)\cos 2\omega^*}{5\gamma_0 - 2(\gamma_2 + \gamma_4 + \gamma_3\cos 2\omega^*)}}. \end{aligned} \tag{41}$$

It follows from these formulas and our calculations that only the solution $\omega^* = \pm\pi/2$ in the narrow range of semimajor axes

$$a^{(1)}(I = 90°) \approx 1.40 \text{ million km}$$

$$\le a \le 1.80 \text{ million km} \approx a^{(2)}(I = 90°),$$

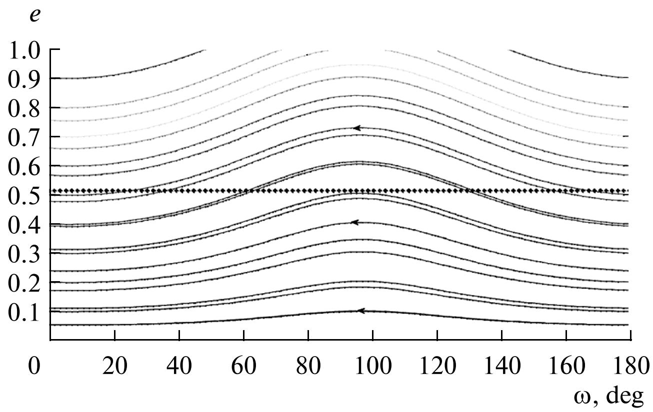


**Fig. 9.** Family of phase trajectories for case VI in the (ω, $e$) plane for $a = 1.2$ million km (Uranus's system).

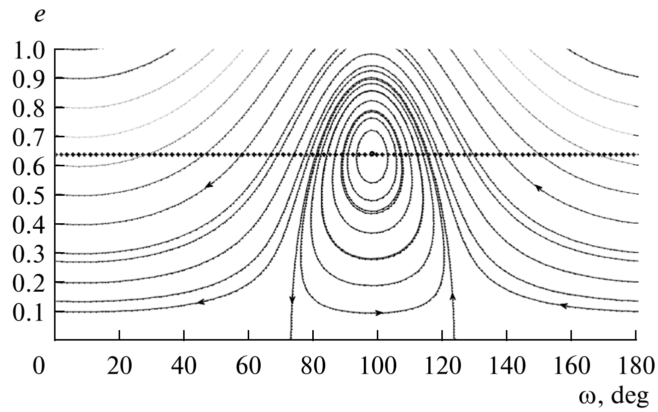


**Fig. 10.** Same as Fig. 9 for $a = 1.6$ million km.

exists in case (a) and only the solution $\omega^* = 0, \pi$ in the range

$$a^{(1)}(I = 0) \approx 1.43 \text{ million km}$$
$$\leq a \leq 1.84 \text{ million km} \approx a^{(2)}(I = 0).$$

exists in case (b).

For an arbitrary angle $I$, the dependence $\omega(e)$ can be approximately found by taking into account the simplifying assumption of $\gamma_4 = 0$. In particular, it is justified for Uranus's system of main satellites, because, as can be seen from Fig. 1, the inequality $|\gamma_4| \ll |\gamma_k|$ ($k =$ 0, 1, 2, 3, 5) is valid for this system. At $\gamma_4 = 0$, from the first of Eqs. (40) we will obtain the following approximate expression:

$$\tan 2\omega = \frac{5\sin 2I \operatorname{sign}(\sin\Omega)}{5\cos 2I + \gamma_5 + \gamma_3 e^2}. \tag{42}$$

Substituting $\sin 2\omega$ and $\cos 2\omega$ into the second of Eqs. (40), we will obtain a quartic equation for $e^2$. Neglecting the terms of order $e^6$ and $e^8$, in accordance with the adopted accuracy of representing the perturbing function $W$, we will obtain a biquadratic equation for the eccentricity

$$b_2 e^4 + b_1 e^2 + b_0 = 0, \tag{43}$$

where

$$\begin{gathered} b_0 = d_1\left(d_2^2 - d_1\right), \quad b_1 = 2d_1d_2d_4 + 2d_3\left(d_2^2 - 3d_1\right)\gamma_3, \\ b_2 = d_1d_4^2 + 4d_2d_3d_4\gamma_3 + \left(d_2^2 - 4d_1 - 9d_3^2\right)\gamma_3^2, \\ d_1 = 25 + (10\cos 2I + \gamma_5)\gamma_5, \quad d_2 = 1 + 2\gamma_0 - \gamma_1, \\ d_3 = 5\cos 2I + \gamma_5, \quad d_4 = 5\gamma_0 - 2\gamma_2. \end{gathered} \tag{44}$$

The solution of Eq. (43) gives stationary values of the eccentricity ($0 \leq e^* \leq 1$) for various angles $I$ that exist only in a fairly narrow range of semimajor axes, $a^{(1)}(I) \leq a \leq a^{(2)}(I)$. For $I = 97.86°$ corresponding to the inclination of Uranus's equator to its orbit, $a^{(1)} \approx$ 1.4 million km and $a^{(2)} \approx 1.8$ million km. In this range of semimajor axes, naturally, there also exists a libration zone of the pericenter argument in the (ω, $e$) phase plane, with the separatrix defined by Eq. (35) at $c = 0$ serving as its boundary:

$$e_s^2 = -\frac{D(\omega)}{A(\omega)}. \tag{45}$$

In contrast to case IV, the succeeding figures for case VI show the structure of the families of π-periodic and naturally extended phase trajectories in the wider range $0 \leq \omega \leq 180°$. Just as in case IV, the trajectories located above the critical "cross" straight line are only of formal interest.

At $a < a^{(1)}$, no singular points exist in the (ω, $e$) plane. The orbital evolution is reduced to a monotonic (circulational) decrease in the pericenter argument at restricted eccentricity oscillations (Fig. 9). It can be seen that at $e = 0$ and $\omega \approx I$, there is a tendency for the formation of a singular point that occurs at the bifurcation value $a^{(1)}$.

At $a^{(1)} < a < a^{(2)}$, there exists center- and saddle-type singular points in the (ω, $e$) plane. A situation where $e^* = \bar{e} < e_s$ is realized for the intermediate value of $a = 1.6$ million km (Fig. 10). All of the phase points moving along the family's trajectories cross the straight line of the critical eccentricity in a finite time. In the evolution problem under consideration, this serves as a qualitative analog of the well-known Lidov–Kozai effect in the twice-averaged Hill problem (Lidov, 1961; Kozai, 1962).

We do not provide here the obvious structure of the family for $e^* < e_s < \bar{e}$. At $a > a^{(2)}$, all of the phase points moving along the family's trajectories also cross the straight line of the critical eccentricity in a finite time, but no center-type singular point exists in the (ω, $e$) plane (Fig. 11).

The integrable case VI considered in this section can also reflect the qualitative features of the orbital evolution of a test satellite in Jupiter's system. To obtain results similar to those obtained in Uranus's system, it is necessary to specify the parameters $\mu_j$, $a_j$,

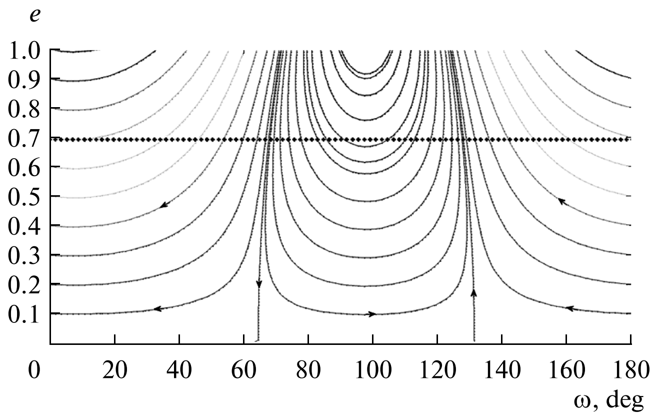


**Fig. 11.** Same as Fig. 9 for $a$ = 1.9 million km.

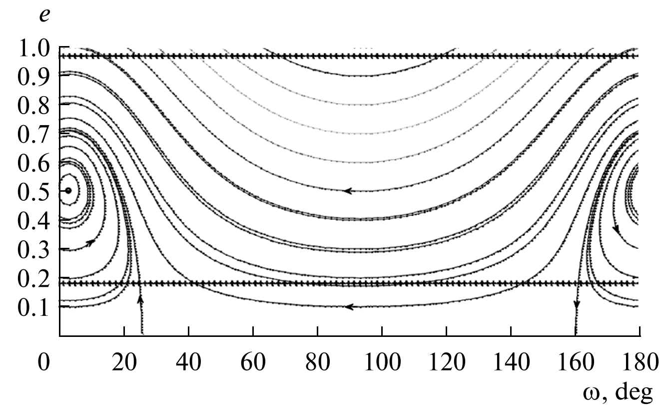


**Fig. 12.** Same as Fig. 10 for $a$ = 2.3 million km (Jupiter's system).

$a_0$, $c_{20}$, $I = 3.13°$, $J = 4$ corresponding to Jupiter and its main (Galilean) satellites. Our calculations showed that the qualitative behavior of the dependences of coefficients $\gamma_k$ on the semimajor axis is retained, with $|\gamma_4| \ll |\gamma_k|$ ($k$ = 0, 1, 2, 3, 5). The families of phase trajectories are modified in approximately the same way as those for Uranus's system. The difference lies in the numerical values of the parameters, in particular, a $a^{(1)} \approx 2.09$ million km and $a^{(2)} \approx\approx 2.69$ million km. For $a < a^{(1)}$, the families are qualitatively similar to those shown in Fig. 9, with the "generation" of a singular point also occurring at $e = 0$ and $\omega \approx I$. At $a^{(1)} < a < a^{(2)}$, the stationary value $e^*$ changes from 0 to 1. Figure 12, which is qualitatively similar to Fig. 10, shows the families of phase trajectories for $a = 2.3$ million km. It presents not one but two critical cross straight lines, $e = \overline{e}_4$ (the contact with Callisto's orbit) and $e = \overline{e}_0$ (the contact with Jupiter's "surface"). Note that the massive satellites of Jupiter (just as those of Uranus and, subsequently, Saturn) are numbered in order of increasing semimajor axes of their orbits.

All of what has been said about Jupiter's system also fully applies to Saturn's system. The parameters $\mu_j$, $a_j$, $a_0$, $c_{20}$, $I = 25.33°$, $J = 7$ corresponding to Saturn and its main satellites are specified. The families of phase trajectories are modified in approximately the same way as those for Uranus's system. The difference lies in the numerical values of the parameters, in particular, $a^{(1)} \approx 2.4$ million km and $a^{(2)} \approx 3.1$ million km. For $a < a^{(1)}$, the families are qualitatively similar to those shown in Fig. 9, with the "generation" of a singular point also occurring at $e = 0$ and $\omega \approx I$. At $a^{(1)} < a < a^{(2)}$, the stationary value $e^*$ changes from 0 to 1. Figure 13, which is also qualitatively similar to Fig. 10, shows the families of phase trajectories for $a = 2.5$ million km.

Apart from the two "cross" straight lines reflecting the critical eccentricity for the external variant of the problem, the triangles in Fig. 13 indicate one more straight line reflecting the critical eccentricity, but for the internal variant of the problem, when the orbit of the test satellite at its apocenter touches the orbit of the massive one (in our case, Iapetus, $j = 7$). The latter figure is also characteristic in that not all of the phase trajectories intersect the straight lines of the critical eccentricity ($j$ = 6 and 0), and regular trajectories with $e < \overline{e}_7$ also exist in a small neighborhood of the singular point.

Note that the abscissas of the equilibrium points in Figs. 10, 12, and 13 are defined by the equality $\omega^* = I$, which corresponds to the condition $\omega^*_{\text{orb}} = 0$.

In addition, we will point out that at $\gamma_k = 0$ for $1 \le k \le 5$, i.e., in the absence of massive satellites, the ranges of existence of singular points are shifted. The corresponding values $a^{(1)}$ and $a^{(2)}$ are reduced approximately by 0.4–0.7 million km. However, in contrast to case IV, the structures of the families of phase trajectories shown in Figs. 9–13 are qualitatively retained (Lidov and Yarskaya, 1974). To be precise, note that Figs. 2b and 2c in the above paper should be interchanged (while retaining the captions to them).

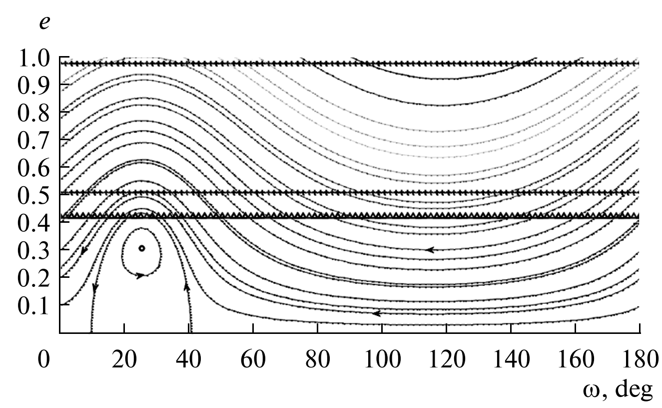


**Fig. 13.** Same as Fig. 10 for $a$ = 2.5 million km (Saturn's system).

**Table 2.** Values of $M_k$, $S_k$, $g_k^{(m)}$, and $p_{k,m}^{(s)}$

| $k$ | $M_k$ | $S_k$ | $m$ | $g_k^{(m)}$ | $p_{k,m}^{(0)}$ | $p_{k,m}^{(1)}$ | $p_{k,m}^{(2)}$ | $p_{k,m}^{(3)}$ | $p_{k,m}^{(4)}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 2 | 0 | 3/4 | 0 | −1 | 1 | | |
| | | | 1 | 1/4 | 2 | −19 | 19 | | |
| | | | 2 | 1/2 | 3 | −1 | 1 | | |
| | | | 3 | 1 | 1 | −4 | 4 | | |
| 2 | 5 | 4 | 0 | 15/64 | 0 | 0 | 3 | −10 | 7 |
| | | | 1 | 1/64 | 4 | −4 | 393 | −1190 | 809 |
| | | | 2 | 1/16 | 0 | −11 | 309 | −804 | 520 |
| | | | 3 | 1/16 | −5 | −42 | 462 | −992 | 600 |
| | | | 4 | 1 | 0 | −4 | 21 | −36 | 20 |
| | | | 5 | 1/4 | 1 | −8 | 24 | −32 | 16 |
| 3 | 5 | 4 | 1 | 1/16 | 4 | 2 | 33 | −70 | 35 |
| | | | 2 | 1/12 | −4 | −1 | 177 | −352 | 176 |
| | | | 3 | 1/12 | −7 | −46 | 390 | −688 | 344 |
| | | | 4 | 1/3 | 1 | −20 | 84 | −128 | 64 |
| | | | 5 | 1/3 | 1 | −8 | 24 | −32 | 16 |
| 4 | 6 | 4 | 1 | 1/64 | −40 | 120 | −183 | 130 | −35 |
| | | | 2 | 1/192 | −188 | 1564 | −2775 | 2122 | −599 |
| | | | 3 | 1/16 | 2 | 19 | −149 | 164 | −56 |
| | | | 4 | 1/48 | 43 | −314 | 654 | −416 | 88 |
| | | | 5 | 1/2 | 1 | −8 | 22 | −24 | 8 |
| | | | 6 | 1/12 | 1 | −8 | 24 | −32 | 16 |
| 5 | 3 | 2 | 1 | 1/4 | 6 | −7 | 3 | | |
| | | | 2 | 1/2 | 5 | −16 | 8 | | |
| | | | 3 | 1 | 1 | −4 | 4 | | |

## CONCLUSIONS

This paper supplements our investigation of the integrable cases for the general problem of the evolution of a satellite orbit under the joint influence of three different perturbing factors: the noncentrality of the planet's gravitational field and the attraction by its main satellites and the Sun. For two special integrable cases, we performed a qualitative analysis of the secular equations that, to a first approximation, define the evolution of the eccentricity and pericenter argument of polar orbits. The investigation was carried out for various inclinations of the orbital plane of a remote perturbing point to the planet's equatorial plane, in particular, for the orthogonal orientation of these planes (case IV) that roughly corresponds to the position of Uranus's equator relative to its heliocentric orbit ($I = 97°.86$). For hypothetical and relatively distant polar satellite orbits that, in principle, could be located in the orbital plane of Uranus, the perturbing influence of its massive satellites leads (along with the circulational one) to the possibility of a librational variation in the pericenter argument. However, this can be realized only in a fairly narrow range of semimajor axes, approximately from 1.6 to 2.1 million km. The polar orbits of Uranus's hypothetical satellites, the orthogonal lines of intersection between both principal planes (case IV), could in principle exist only near $a$ smaller than or approximately equal to 1.6 million km. For the same integrable case, we investigated the behavior of the polar orbits of Jupiter's and Saturn's hypothetical satellites. The integrable cases of the evolution problem considered here cannot be applied to the satellite system of Neptune due to the peculiarity of the orbits of its main satellites: either a significant eccentricity or a significant inclination.

Our analysis establishes the threshold values of the constant semimajor axes for satellite orbits starting from which singular points appear in the phase (pericenter argument—eccentricity) plane, i.e., the stability of circular orbits with respect to the eccentricity is lost. As the semimajor axes increase further, the families of phase trajectories acquire a peculiar feature associated with the manifestation of the Lidov—Kozai effect. The attainment of a critical eccentricity for various "effective" radii of contact of the orbits leads to their intersections. Under these conditions, the probability of the collision of hypothetical satellites with real (massive) satellites or with the "surfaces" of the giant planets increases sharply.

The revealed qualitative features of the evolution can help in explaining the existence of "empty" zones separating the families of orbits of regular and irregular satellites in circumplanetary space.

## APPENDIX

Table 2 gives the numerical values of all the quantities needed to calculate the function $W_2$ from Eqs. (9)–(15).

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*Translated by V. Astakhov*