ISSN 0038-0946, Solar System Research, 2015, Vol. 49, No. 4, pp. 247–262. © Pleiades Publishing, Inc., 2015.
Original Russian Text © M.A. Vashkov'yak, S.N. Vashkov'yak, N.V. Emel'yanov, 2015, published in Astronomicheskii Vestnik, 2015, Vol. 49, No. 4, pp. 283–299.

On the Evolution of Satellite Orbits under the Action of the Planet's Oblateness and Attraction by Its Massive Satellites and the Sun

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Received January 26, 2015

Abstract—The problem of the joint influence of the oblateness of a central planet and attraction by its most massive (or main) satellites and the Sun on the orbital evolution of a satellite with a negligible mass is considered. For an arbitrary angle between the equatorial plane of the planet and the plane of its heliocentric orbit, the evolution equations have been derived in the planeto-equatorial elements of the satellite orbit. Integrable cases of the evolution problem are described. The influence of Uranus's main satellites on the orbital evolution of its real and hypothetical satellites has been revealed through numerical calculations and analytical estimations.

Keywords: averaged perturbing function, orbital evolution, Uranian satellites

DOI: 10.1134/S0038094615040085

INTRODUCTION AND FORMULATION OF THE PROBLEM

This paper is a natural continuation of the research on the evolution of orbits under the action of gravitational perturbations performed by Lidov (1961) and Kozai (1962), who revealed the main features of the orbital evolution of satellites and asteroids under the influence of secular perturbations from an external attracting mass point. These, in particular, include the effect of a dramatic increase in the orbital eccentricity at a constant semimajor axis with a simultaneous decrease in the pericenter distance, up to the fall of the satellite to the surface of a planet with a finite radius. This effect, called the Lidov–Kozai mechanism (and resonance), arises when the satellite (or asteroid) orbit is inclined to the plane where the perturbing point moves by an angle close to 90°. Since the orbits of the overwhelming majority of known satellites are fairly far from the orthogonal orientation relative to the ecliptic plane, the fall effect cannot manifest itself for them. The main and inner satellites of Uranus could be an exception. The nearly equatorial (and nearly circular) orbits of these satellites are inclined to the ecliptic plane by angles differing from the right one only by about 8°. As was shown by Lidov (1963), the influence of Uranus's oblateness, which more than compensates for the secular solar perturbations, is an explanation for the real existence of the most distant main satellite, Oberon (and, of course, all of the closer satellites). The papers of Vashkov'yak (2001) and Vashkov'yak and Teslenko (2002) are also devoted to some of the peculiarities of Uranus's satellite system. The evolution models used in the above papers did not include the attraction by the main satellites in the perturbing factors. At the same time, the influence of this factor on the orbital evolution may turn out to be fairly noticeable—it is the subject of our study.

In the first four sections of this paper, we derive the evolution equations for the problem of the secular perturbations of the orbit of a satellite with a negligible mass under the joint influence of three perturbing factors (the oblateness of the central planet, the attraction by its main satellites, and the attraction by the Sun), describe the integrable cases, and map out the possible ways of their investigation. Studying the region of circumplanetary space where the influence of these perturbations on the satellite is pairwise or, in aggregate, is comparable in magnitude is of greatest interest in this new restricted evolution problem.

In the last fifth section, we consider the satellite system of Uranus. Using the derived approximate analytical dependences and numerical estimates, we revealed the influence of Uranus's main satellites on the orbital evolution of some of its real and hypothetical satellites for a wide range of orbital semimajor axes.

The secular part W of the full perturbing function provides a basis for obtaining the evolution system. It is found by its independent averaging over all "fast" variables: the mean planetocentric longitudes of the Sun, the main satellites, and the test (real or hypothetical) satellite, i.e., by eliminating the short-period part. Thus, the function W depends only on five planetocentric Keplerian orbital elements: the semimajor axis a, eccentricity e, inclination i, pericenter argument ω, and ascending-node longitude Ω. The angular elements are referred to the equatorial plane of the planet and the line of its intersection with the plane of the heliocentric orbit (or the planetocentric orbit of the Sun). As follows from the Lagrange equations in elements, since W does not depend on the mean longitude of the test satellite, the semimajor axis of its orbit remains constant, while this function itself gives the first integral of the evolution system $W =$ const.

The function W consists of three terms corresponding to the three above-mentioned perturbing factors, while the first integral takes the form

$$W = W_0 + W_1 + W_2 = \text{const}. \quad (1)$$

In the subsequent formulas, apart from the orbital elements, we use the following notation:

μ_0, μ_j, and μ' are the products of the gravitational constant by the masses of the planet, its jth main satellite, and the Sun, respectively;

a_0, a_j, and a' are, respectively, the mean equatorial radius of the planet and the semimajor axes of the orbit of its jth main satellite and the planetocentric orbit of the Sun;

c_{20} is the coefficient at the second zonal harmonic of the planet's gravitational field;

i_{orb} and ω_{orb} are, respectively, the inclination and pericenter argument of the orbit of the test satellite referred to the planet's orbital plane.

In this paper, the evolution problem is considered by taking into account the principal terms of the secular parts of the perturbing functions the expressions for which are given below.

The function W_0 describes the secular influence of only the second zonal harmonic of the planet's gravitational field,

$$W_0 = \frac{3\mu_0 a_0^2 c_{20}}{4a^3}\left(1 - e^2\right)^{-3/2}\left(\sin^2 i - 2/3\right).$$

The function W_1 describes the influence of secular solar perturbations in the quadratic approximation in a/a' or in the Hill approximation,

$$W_1 = \frac{3\mu' a^2}{16a'^3} \times \left[2\left(e^2 - \sin^2 i_{\text{orb}}\right) + e^2 \sin^2 i_{\text{orb}}\left(5\cos 2\omega_{\text{orb}} - 3\right)\right].$$

The elements i_{orb} and ω_{orb} can be expressed in terms of i, ω, Ω, and the angle between the equatorial plane of the planet and its orbital plane denoted by I in a known way. The function W_1 is defined by the expression

$$W_1 = \frac{3\mu' a^2}{16a'^3}\left\{\begin{aligned} &2e^2 - \left(2 + 3e^2\right)\left[\sin^2 i + \sin^2 I\left(\cos 2i + \sin^2 i \sin^2\Omega\right) - \frac{1}{2}\sin 2I \sin 2i \cos\Omega\right] \\ &+ 5e^2\cos 2\omega\left[\sin^2 i + \sin^2 I\left(\cos 2i + \left(\sin^2 i - 2\right)\sin^2\Omega\right) - \frac{1}{2}\sin 2I \sin 2i \cos\Omega\right] \\ &+ 5e^2 \sin 2\omega\left[-\sin^2 I \cos i \sin 2\Omega + \sin 2I \sin i \sin\Omega\right] \end{aligned}\right\}.$$

The function W_2 describes the secular perturbations from the planet's main satellites, which are assumed in our analysis to be noninteracting between themselves. This function is defined by the formula

$$W_2 = \frac{1}{2\pi}\int_0^{2\pi} V(E)(1 - e\cos E)\,dE, \quad (2)$$

where E is the eccentric anomaly of the test satellite, and V in the dynamical interpretation is the force function of a system of a finite number J of Gaussian rings with masses equal to the masses of the satellites. We use a nontraditional form of the function W_2 proposed previously (Vashkov'yak et al., 2013a). Its unified representation for both internal and external variants of the problem, i.e., for $a/a_j < 1$ and $a/a_j > 1$, is a peculiarity. In our simplified model, we will assume that the orbits of all main satellites lie in the equatorial plane of the planet and have zero eccentricities. In this case, V is expressed in terms of the Gauss hypergeometric function (Vashkovjak, 1976),

$$V = \sum_{j=1}^{J}\frac{\mu_j}{\sqrt{a_j^2 + r^2}} F\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{4a_j^2\left(r^2 - z^2\right)}{\left(a_j^2 + r^2\right)^2}\right), \quad (3)$$

and depends on E only via the squares of the planetocentric coordinates of the test satellite,

$$r = a(1 - e\cos E) \text{ and } z = a\sin i[(\cos E - e)\sin\omega + (1 - e^2)^{1/2}\cos\omega \sin E]. \quad (4)$$

Using Eqs. (2)–(4), it can be shown that the function W_2 depends on e, i, and ω, respectively, only via the combinations e^2, $\sin^2 i$, and $e^2\sin^2 i\cos 2\omega$. These properties are valid for any orbital eccentricity and inclination of the test satellite, while the expression for the function W_2 can be represented in general form as

$$W_2 = \sum_{j=1}^{J} \frac{\mu_j}{\sqrt{a^2 + a_j^2}} \sum_k P_k^{(j)}(a, a_j) \times Q_k^{(j)}\left(e^2, \sin^2 i, e^2 \sin^2 i \cos 2\omega\right), \quad (5)$$

where $P_k^{(j)}$ and $Q_k^{(j)}$ are some rational functions of their arguments. However, the explicit analytical expression (5) for arbitrary e and i is fairly cumbersome. For this reason, for the purposes of an approximate analysis, we will use a simplified formula,

$$W_2 = \sum_{j=1}^{J} \frac{\mu_j}{\sqrt{a^2 + a_j^2}} \left[\begin{aligned} &P_1^{(j)}(a, a_j)\left(e^2 - \sin^2 i\right) + P_2^{(j)}(a, a_j) e^4 + P_3^{(j)}(a, a_j) \sin^4 i \\ &+ P_4^{(j)}(a, a_j) e^2 \sin^2 i + P_5^{(j)}(a, a_j) e^2 \sin^2 i \cos 2\omega + O(e, \sin i)^6 \end{aligned} \right]. \quad (6)$$

Of course, it is valid only for moderate eccentricity and inclination and disregards the terms of the sixth and higher even degrees in e and $\sin i$. In general, the constant coefficients, $P_k^{(j)}(a, a_j)$ ($1 \le j \le J$; $1 \le k \le 5$), can be calculated using the known Laplace coefficients. We use a nonstandard method of calculation (Vashkov'yak et al., 2013a; 2013b; 2015), where the coefficients $P_k^{(j)}(a, a_j)$ are found with the help of power series in the parameters

$$\zeta_j = \left(\frac{2aa_j}{a^2 + a_j^2}\right)^2 \text{ or } \eta_j = 1 - \zeta_j = \left(\frac{a^2 - a_j^2}{a^2 + a_j^2}\right)^2,$$

which do not depend on the relations between the semimajor axes a and a_j. In addition to positive powers, singularities of the form $1/\eta_j$ and $\ln\eta_j$ appear in the expansion of the function W_2 in terms of the parameter η_j. They are related to the possible closeness of the orbits of the perturbed and perturbing satellites. However, the application of such a series allows the number of its terms needed to achieve a specified accuracy of calculating W_2 to be reduced considerably. Our goal is an analytical study and a numerical solution of the new evolution problem or the system of differential Lagrange equations in elements with an averaged perturbing function W. It should be noted that the results obtained from the averaged perturbing function do not exhaust the description of the orbital evolution of planetary satellites. The secular and long-period variations in orbital elements are also obtained when the short-period perturbations of the second order in perturbing factors are calculated. This occurs when multiplying the trigonometric series containing fast variables of the same multiplicity in the cofactors in the arguments of the trigonometric functions. Such terms are discarded immediately in our analysis when averaging the perturbing function. We assume that the pattern of orbital evolution is determined mainly by the averaged part of the perturbing function.

THE EVOLUTION SYSTEM IN KEPLERIAN ELEMENTS

For the subsequent analysis, we introduce several constant parameters:

$$\alpha_0 = -\frac{3}{16}\left(\frac{a_0}{a}\right)^2 c_{20}; \quad \alpha_{k>0} = \frac{a}{\mu_0} S_k;$$
$$S_k = \sum_{j=1}^{J} \frac{\mu_j}{\sqrt{a^2 + a_j^2}} P_k^{(j)}(a, a_j); \quad (1 \le k \le 5); \quad (7)$$
$$\beta = \frac{3}{16}\frac{\mu'}{\mu_0}\left(\frac{a}{a'}\right)^3; \quad \gamma_k = \alpha_k/\beta; \quad (0 \le k \le 5).$$

In addition, it is convenient to introduce a new independent variable, $\tau = \beta n(t - t_0)$, where $n = \mu_0^{1/2}/a^{3/2}$ is the mean motion of the test satellite, t_0 and t are the initial and current instants of time, respectively, and to normalize the function W by assuming that $W_N = W/\beta/n^2/a^2 = Wa/\beta/\mu_0$. In order of magnitude, α_0 characterizes the absolute value of the ratio of the averaged perturbing function of the planet's oblateness to the force function of the central attracting point (i.e., μ_0/a). The parameters $\alpha_{k>0}(1 \le k \le 5)$ characterize the same ratio but for the perturbing function of the system of Gaussian rings modeling the attraction by the main satellites S_k. The parameter β characterizes the ratio of the twice-averaged perturbing function of solar attraction to μ_0/a. Finally, the absolute values of γ_k for $0 \le k \le 5$ are of the order of the ratios of the averaged perturbing functions of the oblateness and the system of Gaussian rings to the perturbing function of solar attraction.

The evolution system for the problem is obtained by substituting into the Lagrange equations

$$
\begin{aligned}
\frac{de}{d\tau} &= -\frac{\sqrt{1-e^2}}{e}\frac{\partial W_{\rm N}}{\partial\omega},\\
\frac{di}{d\tau} &= \frac{1}{\sqrt{1-e^2}\sin i}\left(\cos i\frac{\partial W_{\rm N}}{\partial\omega}-\frac{\partial W_{\rm N}}{\partial\Omega}\right),\\
\frac{d\omega}{d\tau} &= \frac{1}{\sqrt{1-e^2}\sin i}\left(\frac{\left(1-e^2\right)\sin i}{e}\frac{\partial W_{\rm N}}{\partial e}-\cos i\frac{\partial W_{\rm N}}{\partial i}\right),\\
\frac{d\Omega}{d\tau} &= \frac{1}{\sqrt{1-e^2}\sin i}\frac{\partial W_{\rm N}}{\partial i}
\end{aligned}
\tag{8}
$$

the partial derivatives of the normalized function

$$
\begin{aligned}
W_{\rm N} &= 4\gamma_0\left(1-e^2\right)^{-3/2}\left(2/3-\sin^2 i\right)\\
&+\gamma_1\left(e^2-\sin^2 i\right)+\gamma_2 e^4+\gamma_3\sin^4 i+e^2\sin^2 i(\gamma_4\\
&+\gamma_5\cos 2\omega)+2e^2-\left(2+3e^2\right)\left[\sin^2 i+\sin^2 I(\cos 2i\right.\\
&\left.+\sin^2 i\sin^2\Omega)-\sin I\cos I\sin 2i\cos\Omega\right]\\
&+5e^2\cos 2\omega\left[\sin^2 i+\sin^2 I\left(\cos 2i+\left(\sin^2 i-2\right)\sin^2\Omega\right)\right.\\
&\left.-\sin I\cos I\sin 2i\cos\Omega\right]+5e^2\sin 2\omega\\
&\times\left[-\sin^2 I\cos i\sin 2\Omega+2\sin I\cos I\sin i\sin\Omega\right]=\text{const}.
\end{aligned}
\tag{9}
$$

In this case, $W_{\rm N} = \text{const}$ is the first integral of system (8). This system of four averaged differential equations in elements has the following explicit form:

$$
\frac{de}{d\tau} = 10e\sqrt{1-e^2}\left\{\begin{aligned}&\left(1+\frac{\gamma_5}{5}\right)\sin^2 i\sin 2\omega-\sin 2I\left[\sin i\sin\Omega\cos 2\omega+\sin i\cos i\cos\Omega\sin 2\omega\right]\\&+\sin^2 I\left[\cos i\sin 2\Omega\cos 2\omega+\left(\cos 2i+\left(\sin^2 i-2\right)\sin^2\Omega\right)\sin 2\omega\right]\end{aligned}\right\},
\tag{10}
$$

$$
\frac{di}{d\tau} = \frac{1}{\sqrt{1-e^2}}\left\{\begin{aligned}&-(5+\gamma_5)e^2\sin 2i\sin 2\omega+\sin 2I\left[\left(2+3e^2+5e^2\cos 2\omega\right)\cos i\sin\Omega+5e^2\cos 2i\cos\Omega\sin 2\omega\right]\\&+\sin^2 I\left[\left(2+3e^2+5e^2\cos 2\omega\right)\sin i\sin 2\Omega+5e^2\sin i\cos i\left(3+\cos 2\Omega\right)\sin 2\omega\right]\end{aligned}\right\},
\tag{11}
$$

$$
\frac{d\omega}{d\tau} = \frac{1}{\sqrt{1-e^2}}\left\{\begin{aligned}&4\gamma_0\left(1-e^2\right)^{-3/2}\left(4-5\sin^2 i\right)+4\left(2+\gamma_1\right)+2e^2\left(1-\gamma_1+2\gamma_2-\gamma_4\right)-4\gamma_2 e^4\\&-2\sin^2 i\left(5+\gamma_1+2\gamma_3-\gamma_4\right)+4\gamma_3\sin^4 i+2\left(5+\gamma_5\right)\left(\sin^2 i-e^2\right)\cos 2\omega\\&+\sin^2 I\left[\begin{aligned}&3\left(5\sin^2 i-4-e^2\right)+\left(5\sin^2 i-2-3e^2\right)\cos 2\Omega+5\left(3\left(e^2-\sin^2 i\right)-\right.\\&\left.-\left(\sin^2 i-2+e^2\right)\cos 2\Omega\right)\cos 2\omega+5\left(e^2-2\right)\cos i\sin 2\Omega\sin 2\omega\end{aligned}\right]\\&+\sin 2I\left[\begin{aligned}&\left(10\sin 2i\sin^2\omega+\left(5e^2\cos 2\omega-3e^2-2\right)\mathrm{ctg}\, i\right)\cos\Omega\\&+5\left(2\sin^2 i-e^2-e^2\sin^2 i\right)\frac{\sin\Omega}{\sin i}\sin 2\omega\end{aligned}\right]\end{aligned}\right\},
\tag{12}
$$

$$
\frac{d\Omega}{d\tau} = \frac{1}{\sqrt{1-e^2}}\left\{\begin{aligned}&2\cos i\left[-4\gamma_0\left(1-e^2\right)^{-3/2}-\gamma_1+2\gamma_3\sin^2 i+(\gamma_4+\gamma_5\cos 2\omega)e^2+5e^2\cos 2\omega-3e^2-2\right]\\&+\sin^2 I\left[2\cos i\left(\sin^2\Omega-2\right)\left(5e^2\cos 2\omega-3e^2-2\right)+5e^2\sin 2\Omega\sin 2\omega\right]\\&+\sin 2I\left[\left(2+3e^2-5e^2\cos 2\omega\right)\frac{\cos 2i}{\sin i}\cos\Omega+5e^2\,\mathrm{ctg}\, i\sin\Omega\sin 2\omega\right]\end{aligned}\right\}.
\tag{13}
$$

At arbitrary angles I, this system with only one first integral $W_{\rm N} = \text{const}$ is apparently nonintegrable. Nevertheless, it seems interesting and useful to us to reveal its main properties, integrable cases, and particular solutions, possibly also periodic ones, with the invariance of the system with respect to the change of variables

$$\tilde\tau = -\tau,\quad \tilde e = e,\quad \tilde i = i,\quad \tilde\omega = -\omega,\quad \tilde\Omega = -\Omega.$$

pointing to their existence.

For arbitrary I and initial conditions e_0, i_0, ω_0, and Ω_0, in particular, those corresponding to the orbits of real (or hypothetical) satellites of a planet, we will use Everhart’s 19th-order numerical method to solve the evolution system and will check the calculations using the integral $W_{\rm N} = \text{const}$. The code for numerical integration by this method was adapted for a system of four first-order equations.

TRABSFORMATION OF THE EVOLUTION SYSTEM TO POINCARE ELEMENTS

In some analytical studies of the evolution of nearly circular satellite orbits with low inclinations to the equatorial plane of the planet, it turns out to be useful to also apply special regular elements, along with the Keplerian elements having a clear geometrical meaning.

Below, we introduce the elements of the second canonical Poincare system normalized to $(\mu_0 a)^{1/4}$:

$$\xi = \sqrt{x_1}\cos g, \quad \eta = -\sqrt{x_1}\sin g, \quad p = \sqrt{x_2}\cos\Omega,$$
$$q = -\sqrt{x_2}\sin\Omega, \quad g = \Omega + \omega\delta, \quad x_1 = 2\left(1 - \sqrt{1-e^2}\right),$$
$$x_2 = 2\sqrt{1-e^2}\,(1 - \delta\cos i), \quad \delta = \mathrm{sign}(\cos i).$$

Here, g is the pericenter longitude, the variables ξ and η are of the order of e when $e \to 0$, and the variable p and q are of the order of $\sin i$ when $\sin i \to 0$. The main combinations of Keplerian elements appearing in function (9) are expressed in terms of these elements, common for both prograde and retrograde satellite orbits.

In new variables, the nonlinear evolution equations are written in a standard canonical form,

$$\frac{d\xi}{d\tau} = \frac{\partial W_{\mathrm{N}}}{\partial\eta}, \quad \frac{d\eta}{d\tau} = -\frac{\partial W_{\mathrm{N}}}{\partial\xi},$$
$$\frac{dp}{d\tau} = \frac{\partial W_{\mathrm{N}}}{\partial q}, \quad \frac{dq}{d\tau} = -\frac{\partial W_{\mathrm{N}}}{\partial p}, \qquad (14)$$

while integral (1) takes the form

$$W_N = \sum_{\nu_1+\nu_2+\nu_3+\nu_4=1}^{4} W_{\nu_1\nu_2\nu_3\nu_4}\xi^{\nu_1}\eta^{\nu_2}p^{\nu_3}q^{\nu_4} = \mathrm{const},$$

where the nonzero values of the coefficients $W_{\nu_1\nu_2\nu_3\nu_4}$ are defined by the formulas

$$W_{0010} = 2\mu, \quad W_{2000} = b + 2\sigma, \quad W_{0020} = -b + 4\sigma,$$
$$W_{0002} = -b + 2\sigma, \quad W_{0200} = b - 8\sigma, \quad W_{2010} = -\frac{3}{2}\mu,$$
$$W_{0210} = \frac{17}{2}\mu, \quad W_{0030} = -\frac{5}{4}\mu, \quad W_{0012} = W_{0030},$$
$$W_{1101} = -10\mu, \quad W_{4000} = -\frac{1}{2} + 4\gamma_0 - \frac{1}{4}\gamma_1 + \gamma_2 - \frac{1}{2}\sigma,$$
$$W_{2200} = -1 + 8\gamma_0 - \frac{1}{2}\gamma_1 + 2\gamma_2 + \frac{3}{2}\sigma,$$
$$W_{0400} = -\frac{1}{2} + 4\gamma_0 - \frac{1}{4}\gamma_1 + \gamma_2 + 2\sigma,$$
$$W_{2020} = 1 - 8\gamma_0 - \frac{1}{2}\gamma_1 + \gamma_4 + \gamma_5 - 2\sigma,$$
$$W_{0220} = -9 - 8\gamma_0 - \frac{1}{2}\gamma_1 + \gamma_4 - \gamma_5 + 18\sigma,$$
$$W_{2002} = -9 - 8\gamma_0 - \frac{1}{2}\gamma_1 + \gamma_4 - \gamma_5 + 9\sigma,$$
$$W_{0202} = 1 - 8\gamma_0 - \frac{1}{2}\gamma_1 + \gamma_4 - \gamma_5 - \sigma,$$
$$W_{0040} = \frac{1}{2} + \gamma_0 + \frac{1}{4}\gamma_1 + \gamma_3 - \sigma,$$
$$W_{0022} = 1 + 2\gamma_0 + \frac{1}{2}\gamma_1 + 2\gamma_3 - \frac{3}{2}\sigma,$$
$$W_{0004} = \frac{1}{2} + \gamma_0 + \frac{1}{4}\gamma_1 + \gamma_3 - \frac{1}{2}\sigma,$$
$$W_{1111} = 4(5 + \gamma_5) - 30\sigma,$$
$$b = 2 + 4\gamma_0 + \gamma_1, \quad \sigma = \sin^2 I, \quad \mu = \delta\sin 2I.$$

In the linear approximation, system (14) splits up into two pairs of equations for (ξ, η) and (p, q):

$$\frac{d\xi}{d\tau} = 2W_{0200}\eta, \quad \frac{d\eta}{d\tau} = -2W_{2000}\xi,$$
$$\frac{dp}{d\tau} = 2W_{0002}q, \quad \frac{dq}{d\tau} = -2W_{0020}p - W_{0010}. \qquad (15)$$

The roots of the characteristic equation for the first pair of linear homogeneous equations (15) are defined by the formula $\lambda_{1,2} = \pm 2\sqrt{\left(b + 2\sin^2 I\right)\left(8\sin^2 I - b\right)}$. It is easy to show that the coefficients in the formula for b are positive. Indeed, γ_0 is defined by the formula $\gamma_0 = -c_{20}\frac{\mu_0}{\mu'}\frac{a_0^2 a'^3}{a^5}$ and is greater than zero in the case of an oblate planet. It follows from the paper of Vashkov'yak et al. (2013a) that the coefficient γ_1 is also positive. Therefore, the coefficient b will also take only positive values. The condition for the stability of the zero solution in the linear approximation in eccentricity (or ξ, η) then follows from the expression for $\lambda_{1,2}$:

$$2 + 4\gamma_0 + \gamma_1 > 8\sin^2 I.$$

The critical semimajor axis of a circular equatorial orbit at which it becomes unstable with respect to the eccentricity can be found from this inequality for an arbitrary satellite system, in particular, for the system of Uranus ($I \approx 90°$; see the section "The Satellite System of Uranus").

As follows from the second pair of linear but inhomogeneous equations (15), their zero solution turns out to be unstable in inclination at an arbitrary angle I. Only in the integrable cases I–V considered below, where either $\sin I = 0$ or $\cos I = 0$, i.e., where $\mu = 0$, does it make sense to talk about the conditions for the stability of the solution in the linear approximation in inclination (or p, q). These conditions follow from the formula for the characteristic roots of the second pair of homogeneous equations (15) $\lambda_{3,4} = \pm 2\sqrt{\left(b - 2\sin^2 I\right)\left(4\sin^2 I - b\right)}$.

In the case where $\sin I = 0$, the solution is stable in the linear approximation at any b, because $\lambda_{3,4} = \pm 2\sqrt{-b^2}$.

In the case where $\cos I = 0$, $\lambda_{3,4} = \pm 2\sqrt{(b-2)(4-b)}$, the solution is stable in the linear approximation either at $b < 2$ or at $b > 4$. The corresponding critical semimajor axes of the satellite orbit at which it becomes unstable with respect to the inclination can be found from these inequalities.

THE CASES OF INTEGRABILITY OF THE EVOLUTION SYSTEM

First of all, note that the system of differential equations (10)–(13) at $\gamma_k = 0$ $(1 \le k \le 5)$ describes the evolution of a satellite orbit under the joint influence of the planet's oblateness and a remote attracting point. The integrable cases of this problem were investigated in Lidov and Yarskaya (1974). When describing the cases of integrability of the more general system with $\gamma_k \ne 0$, we will adhere to the numbering established in the above paper. Note at once that no new cases of integrability in the problem under consideration have been found. This is natural, because only a coplanar system of rings located in the equatorial plane of the planet was added to the model.

Case I. The orbit evolves only under the action of solar attraction, i.e., $c_{20} = 0$ and $\mu_j = 0$ $(1 \le j \le J)$ or, as a consequence, $\gamma_k = 0$ $(0 \le k \le 5)$. Setting the arbitrary angle I in this case equal to zero and assuming the angular elements i, ω, and Ω to be referred to the orbital plane of the planet, we will obtain the equations of the twice-averaged Hill problem. Using the additional first integral

$$\left(1-e^2\right)\cos^2 i = c_1 \tag{16}$$

existing in this case and the transformation of the function W_{N}, M.L. Lidov obtained a considerably simpler integral,

$$e^2\left(2/5 - \sin^2 i \sin^2\omega\right) = \text{const.} \tag{17}$$

A comprehensive qualitative analysis of this evolution problem was performed in the already mentioned papers (Lidov, 1961; Kozai, 1962). A general solution of the problem for arbitrary elements of the satellite orbit was obtained by Vashkov'yak (1999) and Kinoshita and Nakai (1999, 2007), while its geometrically clear study was performed in a series of papers starting from 2001 (Prokhorenko, 2001).

Case II. The orbit evolves under the action of the planet's oblateness and attraction by its main satellites, i.e., $\mu' = 0$. Since the problem is symmetric relative to the planet's rotation axis, integral (16), where the inclination i is referred to its equatorial plane, arises just as in case I. It follows from integrals (1) and (16) that

$$\begin{gathered}4\alpha_0\left(2/3 - \sin^2 i\right)\left(1-e^2\right)^{-3/2} \\ + \alpha_1\left(e^2 - \sin^2 i\right) + \alpha_2 e^4 + \alpha_3 \sin^4 i \\ + e^2\sin^2 i\left(\alpha_4 + \alpha_5\cos 2\omega\right) + O(e, \sin i)^6 = c_2,\end{gathered} \tag{18}$$

where the constant parameters α_k $(0 \le k \le 5)$ are defined by Eqs. (7).

Using an independent variable $\nu = n(t - t_0)$, we will obtain

$$\frac{de}{d\nu} = 2\alpha_5 e\sqrt{1-e^2}\sin^2 i \sin 2\omega;$$

$$\frac{di}{d\nu} = -\alpha_5\frac{e^2}{\sqrt{1-e^2}}\sin 2i \sin 2\omega;$$

$$\frac{d\omega}{d\nu} = \frac{1}{\sqrt{1-e^2}}\left\{\begin{aligned}&4\alpha_0\left(1-e^2\right)^{-3/2}\left(4 - 5\sin^2 i\right) \\ &+ \alpha_1\left(3 - e^2 - 2\sin^2 i\right) + 2\alpha_2 e^2 \\ &\times\left(1-e^2\right) - \alpha_3\sin^2 2i + (-2e^2 \\ &+ \sin^2 i + e^2\sin^2 i)\left(\alpha_4 + \alpha_5\cos 2\omega\right)\end{aligned}\right\}. \tag{19}$$

$$\frac{d\Omega}{d\nu} = \frac{2\cos i}{\sqrt{1-e^2}}\left\{\begin{aligned}&-4\alpha_0\left(1-e^2\right)^{-3/2} - \alpha_1 \\ &+ \alpha_3\sin^2 i + e^2\left(\alpha_4 + \alpha_5\cos 2\omega\right)\end{aligned}\right\}.$$

The last equation in (19) allows the dependence of the ascending-node longitude $\Omega(\nu)$ to be found by means of a quadrature, but only after the determination of $e(\nu)$, $i(\nu)$, and $\omega(\nu)$. Note that at $\alpha_5 = 0$ the elements e and i are constant, while ω and Ω are linear functions of time. The eccentricity and inclination do not remain constant at an arbitrary $\alpha_5 \ne 0$ (as in the case where only the planet's oblateness is taken into account), although they are related between themselves in view of integral (16). If $\sin^2 i$ and $\sin^4 i$ are eliminated from Eq. (18) using it, then we will obtain some function that defines a family of phase trajectories in the (ω, e) plane depending on the constants of the integrals c_1 and c_2 and the constant parameters of the problem α_k $(0 \le k \le 5)$.

If necessary, the investigation of this integrable case can be continued and applied to a satellite system where the influence of a distant external body is negligible. It should only be recalled that the terms proportional to $\alpha_{k>5}$ with the sixth and higher even degrees in e and $\sin i$ remained neglected in integral (18). Therefore, the results of our analysis of case II (along with III–VII) are applicable only for orbits with moderate eccentricities and inclinations.

Case III. The equatorial plane of the planet coincides with the plane of its heliocentric orbit, i.e., $I = 0$. Under this condition, apart from integral (16), it follows from Eq. (9) that

$$\begin{aligned} W_N &= 4\gamma_0\left(1-e^2\right)^{-3/2}\left(2/3-\sin^2 i\right) \\ &+ \gamma_1\left(e^2-\sin^2 i\right)+\gamma_2 e^4+\gamma_3\sin^4 i \\ &+ e^2\sin^2 i(\gamma_4+\gamma_5\cos 2\omega)+2\left(e^2-\sin^2 i\right) \\ &- 3e^2\sin^2 i+5e^2\sin^2 i\cos 2\omega+O(e,\sin i)^6 = c_2. \end{aligned} \tag{20}$$

The evolution equations will take a simpler form,

$$\frac{de}{d\tau} = 2(5+\gamma_5)e\sqrt{1-e^2}\sin^2 i\sin 2\omega, \qquad \frac{di}{d\tau} = -\frac{(5+\gamma_5)e^2\sin 2i\sin 2\omega}{\sqrt{1-e^2}}, \tag{21}$$

$$\frac{d\omega}{d\tau} = \frac{2}{\sqrt{1-e^2}} \times \begin{bmatrix} 2\gamma_0\left(1-e^2\right)^{-3/2}\left(4-5\sin^2 i\right) \\ +2(2+\gamma_1)+e^2(1-\gamma_1+2\gamma_2-\gamma_4) \\ -2\gamma_2 e^4-(5+\gamma_1+2\gamma_3-\gamma_4)\sin^2 i \\ +2\gamma_3\sin^4 i+(5+\gamma_5)\left(\sin^2 i-e^2\right)\cos 2\omega \end{bmatrix}, \tag{22}$$

$$\frac{d\Omega}{d\tau} = \frac{2\cos i}{\sqrt{1-e^2}}\begin{bmatrix} -4\gamma_0\left(1-e^2\right)^{-3/2}-\gamma_1+2\gamma_3\sin^2 i \\ +\gamma_4 e^2+(\gamma_5+5)e^2\cos 2\omega-3e^2-2 \end{bmatrix}.$$

The dependence $\Omega(\tau)$ can be found by a quadrature from the last equation in (22) after the determination of $e(\nu)$, $i(\nu)$, and $\omega(\nu)$. Note that this case at $\gamma_k = 0$ $(0 \le k \le 5)$, which was first investigated by Lidov and Yarskaya (1974), turned out to be richest from the standpoint of a diversity of the structures of the families of phase trajectories and, in addition, was not limited by the smallness of e and $\sin i$. Below we will consider the evolution system (14) in Poincare elements at $I = 0$ but $\gamma_k \ne 0$ $(0 \le k \le 5)$, when the expression for the function W_N is simplified, while integrals (16) and (20), to within terms of order $(e, \sin i)^4$ inclusive, take the form

$$\begin{aligned} &\xi^2+\eta^2+p^2+q^2 = \text{const}, \\ &W_N = W_{2000}\left(\xi^2+\eta^2-p^2-q^2\right) \\ &+ W_{4000}\left(\xi^2+\eta^2\right)^2+W_{0040}\left(p^2+q^2\right)^2 \\ &+ W_{2020}(\xi p+\eta q)^2+W_{2002}(\eta p-\xi q)^2 = \text{const}, \end{aligned}$$

where

$$W_{2000} = 4\gamma_0+\gamma_1+2, \quad W_{4000} = 4\gamma_0-\frac{1}{4}\gamma_1+\gamma_2-\frac{1}{2},$$
$$W_{0040} = \gamma_0+\frac{1}{4}\gamma_1+\gamma_3+\frac{1}{2},$$
$$W_{2020} = -8\gamma_0-\frac{1}{2}\gamma_1+\gamma_4+\gamma_5+1,$$
$$W_{2002} = -8\gamma_0-\frac{1}{2}\gamma_1+\gamma_4-\gamma_5-9.$$

Since no other integrals of the canonical evolution system are directly seen, except for the two obvious first integrals, it is natural to try to simplify it by making a nonlinear (and non-canonical!) change of variables:

$$x_1 = \xi^2+\eta^2, \quad x_2 = p^2+q^2,$$
$$x_3 = \xi p+\eta q, \quad x_4 = \eta p-\xi q.$$

The evolution equations in new variables take the form

$$\frac{dx_1}{d\tau} = \beta_0 x_3 x_4, \quad \frac{dx_2}{d\tau} = -\beta_0 x_3 x_4, \quad \frac{dx_3}{d\tau} = [\beta_1 x_1 + \beta_2 x_2+\beta_5]x_4, \quad \frac{dx_4}{d\tau} = [\beta_3 x_1+\beta_4 x_2-\beta_5]x_3,$$

where

$$\beta_0 = 4(W_{2002}-W_{2020}) = -8(5+\gamma_5),$$
$$\beta_1 = 2(2W_{4000}-W_{2002}) = 2(8+16\gamma_0+2\gamma_2-\gamma_4+\gamma_5),$$
$$\beta_2 = 2(W_{2002}-2W_{0040}) = -2(10+10\gamma_0+\gamma_1+2\gamma_3-\gamma_4+\gamma_5),$$
$$\beta_3 = 2(W_{2020}-2W_{4000}) = 2(2-16\gamma_0-2\gamma_2+\gamma_4+\gamma_5),$$
$$\beta_4 = 2(2W_{0040}-W_{2020}) = 2(10\gamma_0+\gamma_1+2\gamma_3-\gamma_4-\gamma_5),$$
$$\beta_5 = 4W_{2000} = 4(2+4\gamma_0+\gamma_1),$$

with $\beta_1+\beta_2+\beta_3+\beta_4 = 0$ and $\beta_2-\beta_1+\beta_4-\beta_3 = \beta_0$.

As it turned out, the system of equations in variables x_1, x_2, x_3, and x_4 has three independent first integrals:

$$x_1+x_2 = C_1, \quad \beta_0\left(\beta_4 x_3^2-\beta_2 x_4^2\right)+\rho x_1^2 - 2\beta_5(\beta_2+\beta_4)x_1 = C_2, \quad \beta_0\left(\beta_3 x_3^2-\beta_1 x_4^2\right) + \rho x_2^2+2\beta_5(\beta_1+\beta_3)x_2 = C_3,$$

where

$\rho = \beta_2\beta_3 - \beta_1\beta_4 = 8(5+\gamma_5)(-2+6\gamma_0-\gamma_1+2\gamma_2-2\gamma_3) \ne 0$.

The integrals with constants C_2 and C_3 for $\beta_5 = 0$ were derived by Yu.A. Sadov, who kindly reported his result to one of us after a joint discussion of the problem under study.

It is easy to show that the integral $W_N = \text{const}$ (the Hamiltonian of the original canonical evolution system) is composed of a specially selected linear combination of the above three integrals,

$$W_N = \frac{W_{0040} + W_{4000}}{W_{0040} - W_{4000}} W_{2000} C_1 + \frac{1}{\rho}(W_{4000} C_2 + W_{0040} C_3),$$

and an elliptic quadrature is obtained from them to determine the dependence of τ on one of the variables, for example, x_1:

$$\tau = \int \frac{dx_1}{\sqrt{R(x_1)}}, \quad R(x_1) = R_1(x_1) R_2(x_1),$$

$$R_1(x_1) = (\beta_1 - \beta_2)\left(x_1 - x_1^{(1)}\right)\left(x_1 - x_1^{(2)}\right)$$

$$= (\beta_1 - \beta_2) x_1^2 + 2\left[\beta_2 C_1 + \frac{\beta_5}{\rho}(\beta_1 + \beta_3)(\beta_1 + \beta_2)\right] x_1$$

$$+ \frac{1}{\rho}[\beta_2 C_3 - \beta_1 C_2 - 2\beta_2\beta_5(\beta_1 + \beta_3) C_1],$$

$$R_2(x_1) = (\beta_3 - \beta_4)\left(x_1 - x_1^{(3)}\right)\left(x_1 - x_1^{(4)}\right)$$

$$= (\beta_3 - \beta_4) x_1^2 + 2\left[\beta_4 C_1 + \frac{\beta_5}{\rho}(\beta_1 + \beta_3)(\beta_3 + \beta_4)\right] x_1$$

$$+ \frac{1}{\rho}[\beta_4 C_3 - \beta_3 C_2 - 2\beta_4\beta_5(\beta_1 + \beta_3) C_1].$$

This quadrature is analytically found and inverted through a simple calculation of the four roots $x_1^{(1)}, x_1^{(2)}$, $x_1^{(3)}$, and $x_1^{(4)}$ of the polynomial $R(x_1)$, which is the product of two quadratic polynomial in the case under consideration. Two of these roots that do not exceed 1 define the limits $x_{1,\min}$ and $x_{1,\max}$ of variation in the variable x_1 and the related eccentricity of the satellite orbit, the most important evolution characteristic, $[x_{1,\min}(1 - x_{1,\min}/4)]^{1/2} \le e \le [x_{1,\max}(1 - x_{1,\max}/4)]^{1/2}$. The variables x_2, x_3, and x_4 can be found using the same three integrals. We pass from x_1, x_2, x_3, and x_4 to the Keplerian variables using the formulas

$$e^2 = x_1(1 - x_1/4), \quad \sin^2 i = y_1(1 - y_1/4),$$
$$y_1 = x_2/(1 - x_1/2), \quad \cos\omega = x_3/y_2,$$
$$\sin\omega = -\delta x_4/y_2, \quad y_2 = \sqrt{x_1 x_2}.$$

If necessary, the investigation of this integrable case can be continued and applied to a satellite system where the equatorial plane of the planet is close to its orbital plane.

Note that after a modification of the formulas for W_{2000}, W_{4000}, W_{0040}, W_{2020}, and W_{2002}, the equations describing the changes in x_1, x_2, x_3, and x_4 can also be integrated in quadratures in the previous case II as well.

Case IV. The plane of the planet's heliocentric orbit is orthogonal to its equatorial plane, i.e., $\cos I = 0$, while the satellite moves in this plane ($\sin i_{\rm orb} = 0$), so that $\cos i = 0$ and $\sin\Omega = 0$ in the particular solution. In this case, $di/d\tau = d\Omega/d\tau = 0$. However, to confirm the last two equalities, we cannot rely only on Eq. (9) but should use the previously described properties of the function W_2 and its general expression (5). Note that the expressions for the functions W_0 and W_1 were derived for arbitrary eccentricity and inclination of the satellite orbit, while the proposed formulas (6) and (9) for the functions W_2 and $W_{\rm N}$, respectively, are valid only at moderate e and $\sin i$. Therefore, this case can be legitimately considered in our formulation of the evolution problem if an expression for the function W_2 will be derived for polar orbits. Note that its expression for arbitrary i, which is also valid, in particular, for $i = 90°$, was derived by Vashkovjak (1976), but only to within e^2 inclusive.

Case V. Just as in case IV, the plane of the planet's heliocentric orbit is orthogonal to its equatorial plane, i.e., $\cos I = 0$, but the satellite in the particular solution moves in the equatorial plane ($\sin i = 0$, $di/d\tau = 0$). The adopted assumptions simplify the evolution system. For equatorial orbits, since the node longitude is uncertain, the system of four equations is reduced to two equations for the eccentricity and the previously introduced pericenter longitude g.

The system of these equations

$$\delta\frac{de}{d\tau} = 10e\sqrt{1 - e^2}\sin 2g, \quad \delta\frac{dg}{d\tau} = 8\gamma_0\left(1 - e^2\right)^{-2} + 2\sqrt{1 - e^2}\left(\gamma_1 + 2\gamma_2 e^2 + 5\cos 2g - 1\right) \tag{23}$$

has the first integral

$$\gamma_2 e^4 + (\gamma_1 + 5\cos 2g - 1)e^2 + \frac{8}{3}\gamma_0\left(1 - e^2\right)^{-3/2} = \text{const}.$$

At zero initial eccentricity e_0, there exists a particular solution where the orbit remains circular $(de/d\tau = 0)$.

At $e_0 \ne 0$, the stationary solutions of system (23) under consideration are defined by the conditions

$$g = 0, 180°; \quad \sqrt{1 - e^2}\left(\gamma_1 + 2\gamma_2 e^2 + 4\right) + 4\gamma_0\left(1 - e^2\right)^{-2} = 0; \quad g = \pm 90°; \quad \sqrt{1 - e^2} \times \left(\gamma_1 + 2\gamma_2 e^2 - 6\right) + 4\gamma_0\left(1 - e^2\right)^{-2} = 0. \tag{24}$$

In what follows, we will assume that $\gamma_0 > 0$ ($c_{20} < 0$) and, in addition, γ_1 and γ_2 are positive. As follows from the calculations presented in the next section, these two conditions hold, in particular, for the system of Uranus's main satellites. In view of the adopted assumptions with regard to the parameters γ_0, γ_1, and γ_2, the first equation in (24) is not satisfied at any eccentricities $e < 1$. Therefore, the stationary solutions exist only at $g = g^* = \pm 90°$ and are determined from the second algebraic equation in (24). Since the eccentricity in case V under consideration (just as in all cases except case I) is assumed to be low, it is natural to obtain the solution of this equation only in the biquadratic approximation in e,

$$e = e^* = \sqrt{\frac{5+q_1}{2q_2}}, \quad (25)$$

where $q_1 = 1 - \gamma_1 - 4\gamma_0$ and $q_2 = \gamma_2 + 5\gamma_0$.

Thus, singular points exist in the (g, e) plane only under the condition $5 + q_1 > 0$ or

$$f(a) = -(5+q_1) = 4\gamma_0(a) + \gamma_1(a) - 6 \le 0. \quad (26)$$

If the semimajor axis of the satellite orbit (or γ_0, γ_1) is changed continuously, then the function $f(a)$ will unavoidably become zero, which will imply a bifurcation (the disappearance or appearance of a singular solution) at point $(g^*, 0)$. The inequality $f(a) > 0$ corresponds only to a circulational variation of the pericenter longitude (when $\frac{dg}{d\tau}$ is a sign-constant function of τ), while at $f(a) \le 0$ it can also vary librationally relative to g^* (when $\frac{dg}{d\tau}$ is an alternating function of τ). The appearance of a libration region in the (g, e) phase plane, along with a center-type singular point, entails the appearance of a saddle singular point as well, which makes the circular orbits unstable with respect to the eccentricity.

The integral of Eqs. (23) at low e takes a simpler form,

$$e^2\left(5\cos 2g - q_1 + q_2 e^2\right) = c_2. \quad (27)$$

An equation for the family of phase trajectories in the (g, e) plane is derived directly from this expression, where the terms of order e^6 were discarded:

$$e^2 = \frac{1}{2q_2}\left[q_1 - 5\cos 2g \pm \sqrt{(q_1 - 5\cos 2g)^2 + 4q_2c_2}\right]. \quad (28)$$

The constant of the integral c_2 is determined from the initial values e_0 and g_0,

$$c_2 = e_0^2\left[5\cos 2g_0 - q_1 + q_2 e_0^2\right],$$

$c_2 > 0$ correspond to a circulational variation of g, $c_2 < 0$ correspond to its librational variation, and $c_2 = 0$ corresponds to a singular trajectory (separatrix) whose equation is

$$e^2 = \frac{q_1 - 5\cos 2g}{q_2}. \quad (29)$$

This trajectory limits the eccentricity variation at $g = \pm 90°$ by $e_s = \sqrt{2}e^*$. The simplified expression of the integral (27) allows the quadrature to determine the dependence $\tau(e)$ to be obtained by eliminating the pericenter longitude g from the first equation in (23). For the change in new variable $z = e^2$, we will have the equation

$$\delta\frac{dz}{d\tau} = \pm 4\sqrt{(1-z)P(z)}, \quad (30)$$

where the sign of the right-hand side is determined by the sign of sin sin $2g$,

$$P(z) = p_0 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4,$$
$$p_0 = -c_2^2, \quad p_1 = -2c_2 q_1,$$
$$p_2 = 25 + 2c_2 q_2 - q_1^2, \quad p_3 = 2q_1 q_2, \quad p_4 = -q_2^2,$$

From (30) we will obtain a fairly complex ultra-elliptic integral,

$$\tau = \pm\frac{\delta}{4}\int_{z_0}^{z}\frac{d\zeta}{\sqrt{(1-\zeta)P(\zeta)}}, \quad (31)$$

where $z_0 = e_0^2 = e^2(\tau = 0)$.

The extrema of the eccentricity (or z) are determined by two positive roots of the polynomial $P(z)$ less than one. Depending on the sign of the constant c_2, they are found in different ways from Eq. (28) for the phase trajectory.

$$\text{For } c_2 \ge 0 \; e_{\min} = e(g = 0) = \sqrt{z_{\min}},$$
$$e_{\max} = e(g = \pm 90°) = \sqrt{z_{\max}},$$
$$\text{where } z_{\min} = \frac{1}{2q_2}\left[q_1 - 5 + \sqrt{(q_1-5)^2 + 4q_2c_2}\right], \quad (32)$$
$$z_{\max} = \frac{1}{2q_2}\left[q_1 + 5 + \sqrt{(q_1+5)^2 + 4q_2c_2}\right].$$

In this case, the pericenter longitude g changes monotonically with time, while its circulation period T_C can be found by calculating the quadrature

$$T_C = \frac{1}{\beta n}\int_{z_{\min}}^{z_{\max}}\frac{d\zeta}{\sqrt{(1-\zeta)P(\zeta)}}.$$

For $c_2 \le 0$ $e_{extr} = e(g = \pm 90°) = \sqrt{z_{extr}}$, with the minus and plus corresponding, respectively, to the minimum and maximum in (28), i.e.,

$$z_{\min} = \frac{1}{2q_2}\left[q_1 + 5 - \sqrt{(q_1+5)^2 + 4q_2c_2}\right],$$
$$z_{\max} = \frac{1}{2q_2}\left[q_1 + 5 + \sqrt{(q_1+5)^2 + 4q_2c_2}\right]. \quad (33)$$

Table 1. Physical and orbital parameters of Uranus

μ', km^3/s^2	a', km	μ_0, km^3/s^2	a_0, km	c_{20}	I, deg
132712440000	2875038596	5793939.3	25559	-3343.46×10^{-6}	97.86

Table 2. Orbital and physical parameters of Uranus's main satellites

j	1	2	3	4	5
Name	Miranda (U V)	Ariel (U I)	Umbriel (U II)	Titania (U III)	Oberon (U IV)
a_j, km	129872	190945	265998	436298	583519
μ_j, km^3/s^2	4.4	90.3	78.2	235.3	201.1

In this case, the pericenter longitude g oscillates within a limited range relative to $g^* = \pm 90°$; its extrema are

$$g^{(1)}_{\min} = \frac{1}{2}\arccos\frac{q_1 - 2e_0\sqrt{q_2\left(5 + q_1 - q_2 e_0^2\right)}}{5},$$
$$g^{(1)}_{\max} = 180° - g^{(1)}_{\min}, \tag{34}$$
$$g^{(2)}_{\min} = 180° + g^{(1)}_{\min}, \quad g^{(2)}_{\max} = 360° - g^{(1)}_{\min},$$

while its libration period T_L can be found by calculating the quadrature

$$T_L = \frac{1}{2\beta n}\int_{z_{\min}}^{z_{\max}} \frac{d\zeta}{\sqrt{(1-\zeta)P(\zeta)}}.$$

In the case of a limitational motion, where $c_2 = 0$, the period is infinite,

$$e_{\min} = 0, \quad e_{\max} = e_s, \quad g_{\min} = g_s = \frac{1}{2}\arccos\frac{q_1}{5}. \tag{35}$$

When applied to a satellite system where the equatorial plane of the central planet is nearly orthogonal to its orbital plane, this integrable case V, if necessary, can be investigated in more detail. However, it should be kept in mind that the corresponding results will also be applicable only for orbits with low eccentricities and inclinations.

Case VI. The equatorial plane of the planet is inclined to its orbital plane at an arbitrary angle I, while the plane of the satellite orbit is orthogonal to the line of their intersection, i.e., $\cos i = 0$ and $\cos\Omega = 0$. In this case, $di/dt = d\Omega/dt = 0$. However, the properties of the function W_2 and its general expression (5) should be used to confirm these equalities, just as in the above case IV. All of the remaining remarks concerning the function W_2 made in case IV fully pertain to case VI as well.

Case VII. The equatorial plane of the planet is inclined to its orbital plane at an arbitrary angle I, while the satellite in the particular solution moves in a circular orbit, i.e., $e = 0$. In this case, as follows from Eq. (10), $\frac{de}{d\tau} = 0$ and integral (9) takes the form

$$\sin 2I \sin i \cos i \cos\Omega - \sin^2 i\left[1 + 2\gamma_0 + \frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_3 \sin^2 i + \sin^2 I\left(\sin^2\Omega - 2\right)\right] = c_2.$$

Hence we can obtain a family of phase trajectories in the (Ω, i) plane depending on the constant of the integral c_2 and the constant parameters of the problem γ_k $(k = 0, 1, 3)$ and find the dependences $i(\tau)$ and $\Omega(\tau)$ from Eqs. (11) and (13). Thereafter, we obtain a linear system of two differential equations with periodic coefficients for the Lagrangian elements $h = e\cos\omega$ and $k = e\sin\omega$ at low e. If necessary, the investigation of this integrable case can be continued. It should only be recalled the terms proportional to $\gamma_{k>5}$ with the sixth and higher even degrees in $\sin i$ remained neglected in this integral, so that the results of our analysis of case VII (along with case II) are applicable only for orbits with low inclinations. Note that the spatial evolution of circular satellite orbits at $\gamma_k = 0$ $(1 \le k \le 5)$ was investigated by Sekuguchi (1961), Allan and Cook (1964), and Vashkov'yak (1983) using geosynchronous orbits of artificial Earth satellites arbitrarily inclined to the Earth's equator as an example.

THE SATELLITE SYSTEM OF URANUS

Parameters of Gravitational Perturbations

In this section, we will consider the satellite system of the planet Uranus. Physical and orbital parameters of the planet itself are given in Table 1.

Here, I is the inclination of Uranus's retrograde rotation axis to its orbital plane. Analogous parameters of Uranus's main satellites are given in Table 2, and they are numbered in order of increasing orbital semimajor axis. The data for both tables were taken from the site of the Sternberg Astronomical Institute, to be more precise, from the section supervised by

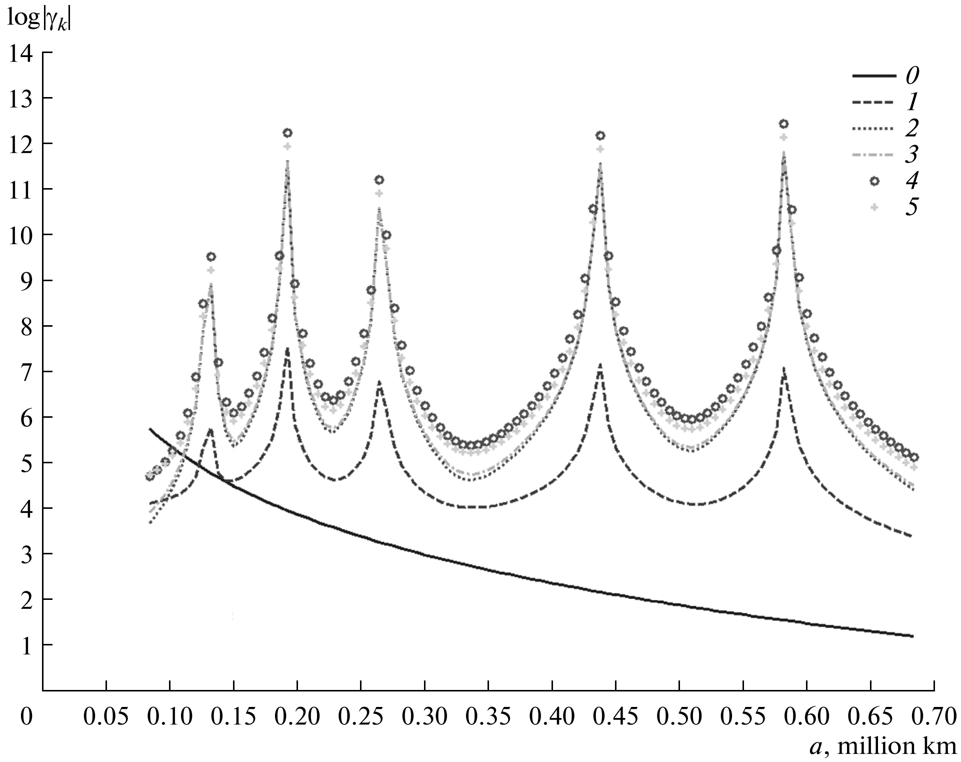

Fig. 1. Coefficients γ_k versus semimajor axis for the range 0.09 million km < a < 0.69 million km.

V.S. Uralskaya: http://www.sai.msu.ru/neb/rw/natsat/index.htm.

In this paper, remaining within a limited formulation, we did not plan to consider the problem of the mutual perturbations of Uranus's main satellites. It seems to us that the most complete investigation of its mathematical and astronomical aspects is contained in Nikonchuk (2012, 2013) and Emelyanov and Nikonchuk (2013).

We will consider a fairly wide range of orbital semimajor axes for Uranus's satellites (mostly hypothetical ones). For this purpose, we performed preliminary calculations of the coefficients $\gamma_k(a)$ $(0 \le k \le 5)$ for a from 90 thousand to 6.9 million km with a step of 6 thousand km. To find the coefficients $P_k^{(j)}(a, a_j)$ in Eq. (6), we used a previously developed special computational code. Since the semimajor axis of the satellite orbit in the evolution problem under consideration is constant, the coefficients γ_k at the fixed physical and orbital parameters of the system specified by Tables 1 and 2 will also be constant.

It is easy to calculate these coefficients for a specific value of a by interpolating the nodal values in the specified range. To make the dependences of the coefficients $\gamma_k(a)$ more detailed, we present them on a logarithmic scale in two ranges of semimajor axes. In Fig. 1, these dependences are shown for 0.09 million km < a < 0.69 million km.

Since one of the coefficients (namely γ_4) is negative, $\log|\gamma_k|$ are plotted along the vertical axis for all k. The solid, dashed, dotted, and dash–dotted curves indicate the dependences for $k = 0, 1, 2$, and 3, respectively. The circles and crosses indicate the dependences for $k = 4$ and 5, respectively. The range in Fig. 1 includes the orbital semimajor axes a_j for Uranus's five main satellites; therefore, all of the dependences except $\log|\gamma_0|$ have discontinuities (appearing as peaks on the graph) at $a = a_j$, when the function W_1, naturally, tends to infinity. Turning to the previously described physical meaning of the parameters γ_k, it can be made sure that the perturbing influence of oblateness ($k = 0$) at a less than ~0.1 million km turns out to prevail over the attraction by the main satellites ($k > 0$). At larger a, this influence weakens considerably, nevertheless exceeding appreciably the influence of solar attraction even near the orbit of Oberon ($\log|\gamma_0| \approx 1.6$ for $a = a_5$). The corresponding dependences are shown in Fig. 2 for 0.5 million km < a < 6 million km.

In this range including the orbital semimajor axes of only two Uranian satellites, Oberon ($a = a_5 \approx$ 0.58 million km) and Francisco ($a \approx 4.3$ million km), the influence of oblateness ($k = 0$) is weaker than the attraction by the main satellites ($k = 1, 4, 5$) approximately by two orders of magnitude. At the same time, the influence of solar attraction increases noticeably with a against the background of a natural weakening

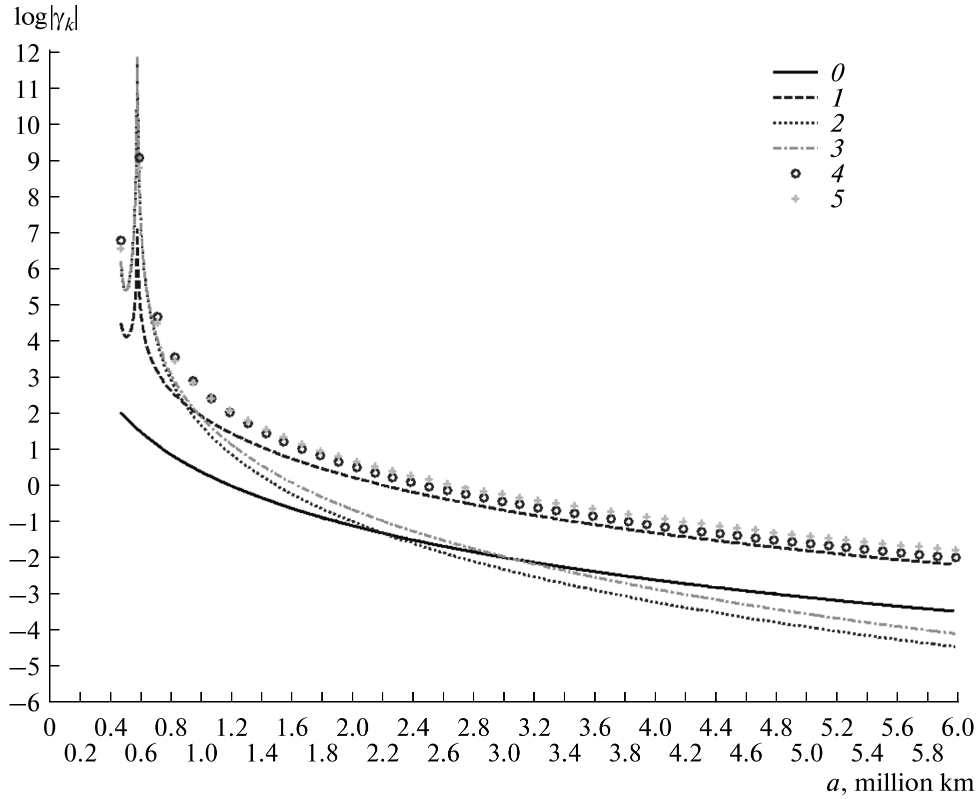

Fig. 2. Coefficients γ_k versus semimajor axis for the range 0.5 million km < a < 6 million km.

of the two remaining perturbing factors, the oblateness of Uranus and the attraction by its main satellites. An approximate equality, when $\log|\gamma_k| \approx 0$, is achieved at $a \approx 1.2$ million km ($\log|\gamma_0| = 0$) and $a \approx 2.6$ million km ($\log|\gamma_5| = 0$), respectively. Near the orbit of Francisco, the solar perturbations exceed the perturbations from the attraction by the main satellites by more than an order of magnitude and even more so from the oblateness of Uranus.

In studying the evolution of satellite orbits in the system of Uranus, we will rely on the averaged differential equations (10)–(13). Idealizing the problem, instead of the real inclination of Uranus's equatorial plane to its orbital plane we will assume that $I = 90°$ to obtain approximate estimates of the evolution parameters.

Table 3. Change in the pericenter distance of Francisco's orbit on a time scale of 50 thousand years

Perturbing factor	$q_{\min}$, million km	$q_{\max}$, million km
Sun, oblateness of Uranus and its five main satellites	3.53	3.79
Sun	3.58	3.79

On the Orbit of Francisco (U XXII)

Francisco is the closest of Uranus's outer satellites discovered to date. The planes of the noticeably elliptical orbits of these satellites are fairly far from the equatorial ones and, consequently, from those orthogonal to the orbit of Uranus. Therefore, the influence of solar attraction, which is the main perturbing factor, leads only to such oscillations of the orbital eccentricities of outer satellites at which the pericenter distances remain considerably larger than the orbital radius of Oberon. In this case, the main satellites turn out to have a negligible influence on the orbital evolution of Caliban ($a \approx 7.2$ million km) and more distant satellites of Uranus. However, the orbit of Francisco ($a \approx 4.3$ million km) is located in the region of circumplanetary space where the influence of the main satellites is already noticeable. Since this orbit is highly inclined to the equatorial plane of Uranus, it turns out to be impossible to use any of the integrable cases to study the orbital evolution of Francisco. This forces us to resort to the numerical integration of the rigorous evolution system. Such integration performed on a time scale of 50 thousand years showed that during the evolution of the orbit the attraction by the main satellites of Uranus (mainly Oberon and Titania) lowers its pericenter approximately to 3.53 million km (Table 3). Although this value is far from the orbital radius of Oberon, it nevertheless turns out to be smaller than

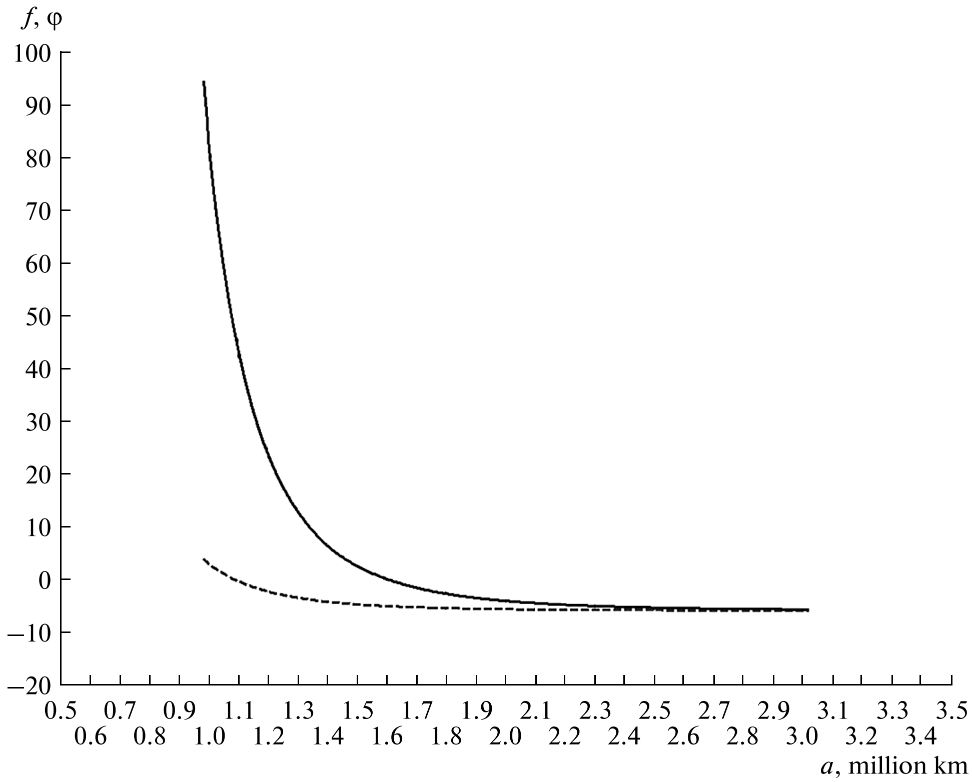

Fig. 3. Functions f and φ versus semimajor axis.

that under the hypothetical influence of only the solar perturbations approximately by 50 thousand km.

Note, in addition, that the orbital inclination of Francisco to the orbital plane of Uranus is ~150°. Such an orientation is far from the orthogonal one at which the Lidov–Kozai effect of a satellite's fall to a central body could manifest itself.

On the Orbits of Uranus's Hypothetical Equatorial Satellites

Now we will dwell on the condition for the appearance of singular points in the (g, e) plane obtained in the section "The Cases of Integrability... (case V)", which leads to the loss of stability of circular orbits with respect to the eccentricity. Consider the function $f(a) = f[\gamma_0(a), \gamma_1(a)]$ defined by Eq. (26) and represented by the solid curve in Fig. 3. For comparison, the dashed line in the same figure indicates the function $\varphi(a) = f[\gamma_0(a), 0]$ calculated without any influence of the main satellites. The zeros of these functions are the bifurcation values of the semimajor axis, $a_f \approx 1.6$ million km and $a_\varphi \approx 1.1$ million km, respectively, corresponding to the loss of stability of circular orbits with respect to the eccentricity. Obviously, the influence of inner satellites noticeably shifts the boundary of the region of instability of circular orbits toward larger semimajor axes.

To illustrate the qualitatively different behavior of the pericenter longitude at two, close to a_f, semimajor axes of the nearly circular orbits of Uranus's hypothetical satellites and to check the validity of the derived formulas (case V), we numerically integrated the evolution system at $e_0 = 0.05$ in an idealized model ($I = 90°$). The initial angular elements referred to its equatorial plane were taken to be $i_0 = 0°.1$, $\omega_0 = 90°$, and $\Omega_0 = 0$.

Our calculations performed on a time scale of 150 thousand years showed that at $a = 1.595$ million km $< a_f$ the pericenter longitude g changed monotonically with time, while the eccentricity oscillated within a narrow range, from $e_{min} \approx 0.00534$ to $e_{max} = e_0 = 0.05$ (in agreement with Eqs. (32)).

By contrast, at $a = 1.605$ million km $> a_f$ the pericenter longitude experienced oscillations within the range from $g^{(1)}_{min} = 85.8°$ to $g^{(1)}_{max} = 94.2°$ (Eqs. (34)). The eccentricity experienced considerably larger (than those at $a = 1.595$ million km) oscillations from $e_{min} = e_0 = 0.05$ to $e_{max} = 0.227$ (Eqs. (33)). The libration region of the eccentricity and pericenter argument relative to the point ($g^* = g_0 = 90°$, $e^* = 0.168$) is bounded by $e_s = 0.237$, $g_s = 84.5°$, and $180° - g_s$ belonging to the separatrix (Eq. (35)). The solid and dashed curves in Fig. 4 indicate the libration trajectory in the (g, e) plane and the separatrix, respectively. The circle marks the libration center; the arrows indicate the

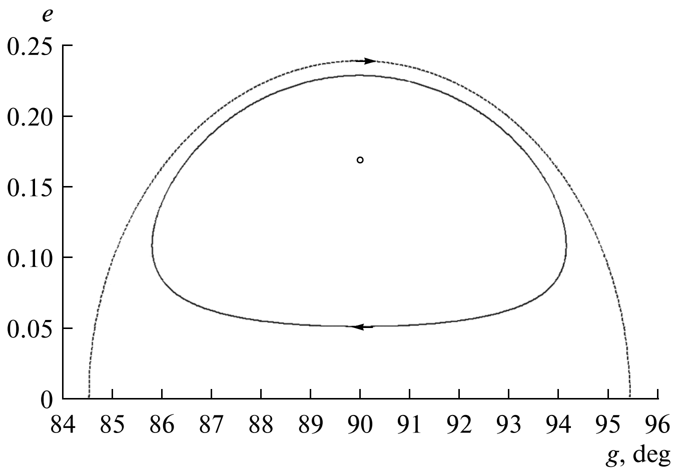

Fig. 4. Phase (g, e) plane for a = 1.605 million km.

direction of motion of the phase point in trajectories with finite and infinite periods.

However, the existence of singular points and a g libration zone does not yet rule out the initial conditions under which the eccentricity of the orbit during its evolution will be unable to reach its critical value, $\bar{e} = 1 - a_0/a$. And only from the equality

$$\bar{e}(a) = e^*(a) \tag{36}$$

do we find the "threshold" semimajor axis starting from which (by analogy with Lidov's theorem in the twice-averaged Hill problem) the eccentricity reaches its critical values $\bar{e}$ in a finite time at any initial e_0 and g_0. Vashkov'yak (2001) and Vashkov'yak and Teslenko (2002) obtained an estimate of 1.3 million km for such a threshold value exceeding $a_\varphi \approx 1.1$ million km. This estimate corresponds to the determination of e^* as a rigorous (without assuming the eccentricity to be low) solution of the second equation in (24) at $\gamma_1 = \gamma_2 = 0$, i.e., the condition $\bar{e}^2 = 1 - (2\gamma_0/3)^{2/5}$. In the evolution problem under consideration, where all γ_k are nonzero, to obtain the corresponding estimate in equality (36), we should use Eq. (25), i.e., the condition $\bar{e}^2 = 0.5(5 + q_1)/q_2$ or, in an explicit form,

$$\begin{aligned}\psi(a, a_0) &= 2[\gamma_2(a) + 5\gamma_0(a)] \\ &\times (1 - a_0/a)^2 - 6 + \gamma_1(a) + 4\gamma_0(a) = 0.\end{aligned} \tag{37}$$

An approximate numerical solution of this equation for the semimajor axis gives its threshold value that slightly exceeds a_f and is ~1.72 million km. This approximate value will slightly decrease if the orbital radii of the main satellites are substituted into Eq. (37) for a_0. The solution of each of the j equations $\psi(a, a_j) = 0$ will then define its threshold semimajor axis corresponding to the contact of the orbits of hypothetical and one of the main satellites of Uranus. For the contact with Oberon's orbit, this value is ~1.66 million km.

Thus, at a greater than 1.66 million km, the initially nearly circular orbits of Uranus's hypothetical satellites should have evolved in such a way that their pericenter distances became smaller than the orbital radius of Oberon (and other main satellites) in a finite time, which would lead to their mutual collisions with a high probability. At a greater than 1.72 million km, the collision with Uranus itself is also unavoidable. As an illustration, we will consider its hypothetical satellite that has an orbit with the semimajor axis a = 1.8 million km, $e_0 = 0.05$, $i_0 = 3°$, and $\omega_0 = \Omega_0 = 0$. Figure 5 shows the dependences of the pericenter distance q on time t obtained by numerically integrating the rigorous ($I \neq 90°$) evolution system (10)–(13). The solid and dashed curves correspond to the complete set of perturbing factors under consideration and only the solar perturbations, respectively. The horizontal straight lines numbered on the right mark the radius of Uranus (0) and the orbital radii of its main satellites (j = 1–5). It can be seen from this figure that in the central gravitational field of Uranus and without allowance for the influence of its main satellites, the solar perturbations lower the pericenter almost to a distance smaller than the planet's radius.

At the same time, when the complete set of perturbations, including the main satellites, is taken into account, the pericenter is lowered only to a distance slightly smaller than the orbital radius of Umbriel. Nevertheless, the orbit of this satellite crosses the orbits of the two most distant main satellites of Uranus already in approximately 10–11 thousand years. As has already been pointed out in Vashkov'yak (2001), their noticeable massiveness compared to the remaining main satellites (Table 2) apparently suggests a fatal role of the "main absorbers" of outer equatorial small bodies.

On the Orbit of Oberon (U IV)

In the Introduction, we described the cause of the real existence of both the most distant main satellite, Oberon, and all of the closer satellites established by M.L. Lidov. At those, relatively small, Uranocentric distances at which these satellites are located, the planet's oblateness prevents a catastrophic decrease in the pericenter radii of their nearly equatorial orbits to the radius of Uranus. Because of the solar perturbations, such a decrease would inevitably occur if the gravitational field of Uranus were (hypothetically) central. The loss of stability of circular satellite orbits with increasing a, i.e., the possibility of a dramatic

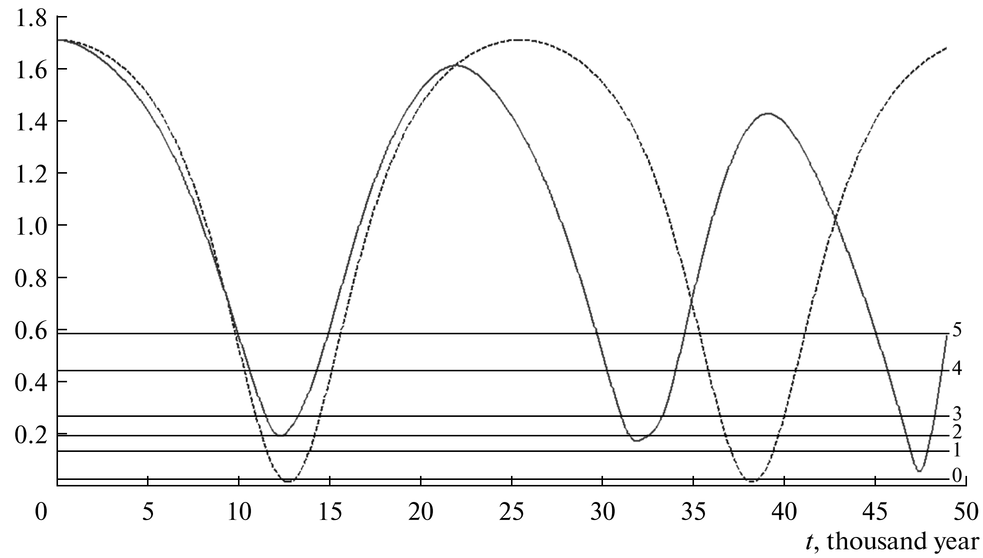

Fig. 5. Variation of the pericenter distance with time for $a = 1.8$ million km.

increase in their eccentricities, a corresponding decrease in the pericenter distances, and a collision of the satellite with the "surface" of Uranus, serves as a precursor of this phenomenon. A mathematical reflection of the possibility of such a collision is the emergence of the Lidov–Kozai resonance, the appearance of singular points and a libration region in the (ω, e) or (g, e) plane. As, in general, might be expected, in addition to the solar perturbations and the oblateness of Uranus, the attraction by the main satellites also manifests itself noticeably in its satellite system. It follows from Figs. 1 and 2 that its influence exceeds the influence of oblateness for a greater than ~0.1 million km.

In view of the geometrical and dynamical properties of this satellite system, below we will use the integrable case V as the most suitable one for an approximate analytical study of the evolution of the nearly circular equatorial orbits of Uranus's satellites. To estimate the influence of its main satellites on this evolution, we will dwell on the oscillations of the orbital eccentricity of Oberon, the most distant main satellite of Uranus. In this case, naturally, by excluding it from the perturbing bodies, we will assume it to be a perturbed body of an infinitesimal mass, i.e., we will consider a restricted evolution problem using it as an example. The dependences shown in Figs. 1 and 2 will undergo changes, the most significant of which will be the "disappearance" of the discontinuity at $a = a_5$. Since the family of phase trajectories in the (g, e) plane has no singular points for this semimajor axis, the orbital eccentricity of Oberon experiences small long-period oscillations $\Delta e = e_{max} - e_{min}$. Our estimates of Δe using Eqs. (32) for various perturbation models are presented in Table 4. The first row gives our estimate without allowance for the influence of Uranus's main satellites that coincides in order of magnitude with that from Lidov (1963). The second and third rows give our estimates without allowance for the influence of Uranus's oblateness and with allowance for the complete set of perturbations under consideration, respectively. It can be seen from this table that the attraction by the main satellites reduces the amplitude of the oscillations in Oberon's orbital eccentricity by an order of magnitude.

To illustrate the influence of the Sun on the orbital evolution of Oberon in the hypothetical "absence" of the remaining perturbations, we numerically integrated the evolution system with $\gamma_k = 0$ ($0 \le k \le 5$) following M.L. Lidov's computational experiment (Lidov, 1961) with an "orthogonal Moon". It turned out that in approximately 120 thousand years the orbital eccentricity of Oberon would reach its critical value, about 0.96, while the corresponding pericenter distance would become equal to the radius of Uranus.

Table 4. Estimates of Δe for the orbit of Oberon

Perturbing factor	Δe
Sun and Uranus's oblateness	5×10^{-5}
Sun and four main Uranian satellites	6×10^{-6}
Sun, Uranus's oblateness, and its four main satellites	5×10^{-6}

CONCLUSIONS

In this paper, we investigated a more general celestial-mechanics problem of the evolution of orbits than those considered previously under the joint influence of three different perturbing factors: the non-centrality of the planet's gravitational field and the attraction by the Sun and its main satellites. The goal of this paper was to describe the integrable cases of the problem and (based on one of them) to estimate the influence of attraction by the main satellites of Uranus on the orbital evolution its real (Oberon, Francisco) and hypothetical satellites.

Analysis of the evolution problem revealed that the influence of the main satellites for the orbit of the closest of the distant satellites (Francisco) leads to a lowering of its pericenter compared to the case where only the solar perturbations are taken into account. For the orbit for the most distant main satellite (Oberon), allowance for the attraction by the remaining main satellites reduces the amplitude of its eccentricity oscillations by an order of magnitude compared to the model that takes into account the perturbations only from Uranus's oblateness and solar attraction.

In addition, we refined the location of the boundary of the region where the existence of Uranus's equatorial satellites is hypothetically possible that separates this region from the region "populated" by its distant satellites. The influence of the main satellites leads to a shift of this boundary to a distance of ~1.7 million km (instead of 1.3 million km in the case where only Uranus's oblateness and solar attraction are taken into account). A manifestation of the Lidov–Kozai mechanism, which, in the long run, leads to a collision either with the planet itself or with its massive main satellites, is unavoidable beyond this boundary.

The same factor is apparently responsible for the existence of zones of "avoidance" by small celestial bodies in the satellite systems of all giant planets. Whereas the integrable case V turned out to be useful for Uranus ($I \approx 98°$), case III may turn out to be useful for Jupiter ($I \approx 3°$). For Saturn ($I \approx 25°$) and Neptune ($I \approx 28°$), it is apparently possible to obtain information about the fairly smeared boundaries of such zones by numerically integrating the presented evolution system.

ACKNOWLEDGMENTS

We wish to thank Yu.A. Sadov for his interest in the problem and the report on his result.

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Translated by V. Astakhov