ISSN 0037–4466, Siberian Mathematical Journal, 2024, Vol. 65, No. 4, pp. 764–770. © Pleiades Publishing, Ltd., 2024.
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FINITE GROUPS WITHOUT ELEMENTS OF ORDER 10: THE CASE OF SOLVABLE OR ALMOST SIMPLE GROUPS

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Abstract—We find all finite almost simple groups without elements of order 10 and describe finite solvable groups without elements of order $2p$ for an odd prime p.

DOI: 10.1134/S0037446624040037

Keywords: finite group, solvable group, almost simple group, Gruenberg–Kegel graph (prime graph)

1. Introduction

Given a finite group G, denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the set of orders of the elements of G. The Gruenberg–Kegel graph, or the prime graph, of G is the graph $\Gamma(G)$ with vertex set $\pi(G)$ in which two vertices p and q are adjacent if and only if $p \neq q$ and $pq \in \omega(G)$.

The elements of small prime order p are important for studying the structure of finite groups. If $p \in \{2, 3, 5\}$ then elements of order p occur in most nonabelian finite simple groups. If S is a nonabelian finite simple group then $2 \in \pi(S)$ for all S, while $3 \in \pi(S)$ for all S but $S \cong Sz(q)$ and $5 \in \pi(S)$ for all S outside a short list of exceptions; see Lemma 1 below. If G is a finite group of even order and $2 \neq p \in \pi(G)$ then it is important to know whether the vertices 2 and p are adjacent in $\Gamma(G)$; see [1] for instance. Thus, the following problem arises naturally:

Problem 1. *Describe all finite groups without elements of order $2p$ for an odd prime p at least small ones.*

By Lagrange's Theorem, while solving Problem 1 we may assume that $2p$ divides the group order.

The nonabelian finite simple groups without elements of order 6 were determined in 1977 in the three independent articles of Podufalov [2], Fletcher, Stellmacher, and Stewart [3], as well as Gordon [4]. The problem of describing general finite groups without elements of order 6 remained open for more than 40 years before Kondrat'ev and Minigulov [5] solved it without using the classification of finite simple groups. Their results have already been applied to studying the finite groups with certain properties of prime graphs; see [6–8] for instance.

A finite group of order divisible by 5 in which the centralizers of elements of order 5 are 5-groups is called a C_{55}*-group*. The nonabelian finite simple C_{55}-groups were determined in [9–11]. The general description of C_{55}-groups is obtained in [12, 13].

As a particular case of Problem 1 for $p = 5$, we pose the following problem whose solution will substantially generalize the description of C_{55}-groups:

Problem 2. *Describe all finite groups without elements of order* 10.

The nonabelian finite simple groups without elements of order 10 were determined recently in [14, Theorem 1.5]. Using that result (see Lemma 2 below), we substantially generalize it by determining all finite almost simple groups without elements of order 10; see Theorem 1 below. Recall that a finite group X is *almost simple* whenever $S \leq X \leq \mathrm{Aut}(S)$ for some finite nonabelian simple group S; equivalently, provided that the socle of X is a finite nonabelian simple group.

Theorem 1. *If G is a finite almost simple group with socle L and 5 divides $|G|$ then G has no elements of order* 10 *if and only if one of the following holds:*

Original article submitted November 30, 2023; revised April 16, 2024; accepted June 20, 2024.

(1) *the group* $O^{\{2,5\}'}(G)$ *is isomorphic to* $L_2(5^f)$ *with* $f > 1$, *to* $PGL_2(5^f)$ *with* $f > 1$, *or to* $PGL_2^*(5^f)$ *with even* f;

(2) $L \cong L_2(q)$, *where* $q \equiv \varepsilon 1 \pmod 5$ *for* $\varepsilon \in \{+,-\}$ *with* $(q - \varepsilon 1)_2 \le 2$, *and either* $O^{\{2,5\}'}(G) = L$ *or* $O^{\{2,5\}'}(G) \cong PGL_2^*(q)$, *where* q *is the square of an odd integer, or* $O^{\{2,5\}'}(G) = L \rtimes \langle t \rangle$, *where* q *is a square,* $\varepsilon = -$, *and* t *is the field automorphism of order* 2 *of* L;

(3) $L \cong L_3^\varepsilon(q)$, *where* $\varepsilon \in \{+,-\}$ *and* $q \equiv -\varepsilon 1 \pmod 5$ *with even* q, *and either* $O^{\{2,5\}'}(G) = L$ *or* q *is a square,* $\varepsilon = +$, *and* $L \rtimes \langle d^{-1}td \rangle \le O^{\{2,5\}'}(G) \le PGL_3(q) \rtimes \langle d^{-1}td \rangle$, *or* $O^{\{2,5\}'}(G) = L \rtimes \langle t \rangle$, *where* t *is respectively a field or graph-field automorphism of order* 2 *of* L, *while* d *is some element of* $\mathrm{Inndiag}(L)$;

(4) $L \cong L_4^\varepsilon(q)$, *where* $\varepsilon \in \{+,-\}$ *and* $q \equiv \pm 2 \pmod 5$ *with* $(q + \varepsilon 1)_2 \le 2$, *as well as* $O^{\{2,5\}'}(G) = L$;

(5) $L \cong L_5^\varepsilon(q)$, *where* $\varepsilon \in \{+,-\}$ *and* $q \equiv \pm 2 \pmod 5$ *with even* q, *as well as* $O^{\{2,5\}'}(G) = L$;

(6) $L \cong S_4(q)$, *where* $2 < q \equiv \pm 2 \pmod 5$, *and* $O^{\{2,5\}'}(G) = L$;

(7) $L \cong Sz(q)$, *where* $q > 2$, *and* $O^{\{2,5\}'}(G) = L$;

(8) $G = L \cong A_7$, M_{11}, M_{22}, *or* M_{23}.

Another result of this article is a solution of Problem 1 for solvable groups:

Theorem 2. *Given a prime* $p > 3$, *if* G *is a finite solvable group without elements of order* $2p$ *and* $2p$ *divides* $|G|$ *then one of the following holds:*

(1) $G/O(G)$ *is isomorphic to a cyclic group or a (generalized) quaternion* 2*-group,* $SL_2(3)$ *or* $SL_2(3){\cdot}2$, *a Sylow* p*-subgroup of* $O(G)$ *is abelian, and* $O(G)$ *is of* p*-length* 1;

(2) $G/O_{p'}(G)$ *is a cyclic* p*-group or a Frobenius group with cyclic core of order* $|G|_p$ *and cyclic complement of order dividing* $p-1$, *the degree of nilpotence of a Sylow* 2*-subgroup of* $O_{p'}(G)$ *is at most* $(p^2-1)/4$ *(and this estimate is sharp for* $p = 5$*), and* $O_{p'}(G)$ *has* 2*-length at most* 1.

As a corollary to Theorem 2 for $p = 5$, we describe all finite solvable groups without elements of order 10. Note that the description of all finite solvable groups without elements of order 6 (the case $p = 3$) is available in [5, Theorem 1]. The key results for proving Theorem 2 are the description of finite groups with (generalized) quaternion Sylow 2-subgroup (see Lemma 3 below) and Higman's description [15, Theorem 1] (see Lemma 4 below) of the finite solvable nonprimary groups, the orders of whose elements are prime powers.

In a subsequent article we intend to study the case of finite nonsolvable groups without elements of order 10 and nontrivial solvable radical relying on the results of the present article.

2. Notation and Auxiliary Results

Our notation and terminology are mostly standard and can be found in [16–18].

Given a positive integer n and a prime p, denote by n_p the *p-part of n* which is the greatest power of p dividing n.

The semidirect product of groups A and B is denoted by $A \rtimes B$, while $A{\cdot}B$ denotes a nonsplit extension of A by B.

Given a finite group G and some set π of primes, denote by $O_\pi(G)$ the largest normal π-subgroup of G, and by $O^\pi(G)$ the smallest normal subgroup of G the quotient of G over which is a π-group. For brevity, put $O(G) = O_{2'}(G)$.

We will also use the notation $L_n^\varepsilon(q)$, $PGL_n^\varepsilon(q)$, and $S_{2n}(q)$, where $\varepsilon \in \{+,-\}$ and $L_n^+(q) = L_n(q) = PSL_n(q)$, $L_n^-(q) = U_n(q) = PSU_n(q)$, $PGL_n^+(q) = PGL_n(q)$, $PGU_n^-(q) = PGU_n(q)$, and $S_{2n}(q) = PSp_{2n}(q)$.

Put $S = L_2(q)$, where $q = p^{2k}$ for some odd prime p and $k \in \mathbb{N}$. Put $PGL_2^*(q) = S\langle \delta\varphi \rangle$, where $PGL_2(q) = S\langle \delta \rangle$ and φ is the field automorphism of order 2 of S. It is well known that $PGL_2^*(q) \setminus S$ contains no involutions.

Consider some results used in our proofs:

Lemma 1 [19, Lemma 1]. *Every finite nonabelian simple group of order coprime to* 5 *is isomorphic to one of the groups* $L_2(q)$, *where* $3 < q \equiv \pm 2 \pmod 5$; $L_3^{\varepsilon}(q)$, *where* $\varepsilon \in \{+,-\}$ *and* $2 < q \equiv \pm 2 \pmod 5$; $G_2(q)$, *where* $2 < q \equiv \pm 2 \pmod 5$; ${}^2G_2(q)$, *where* $3 < q \equiv \pm 2 \pmod 5$; *and* ${}^3D_4(q)$, *where* $q \equiv \pm 2 \pmod 5$.

Lemma 2 [14, Theorem 1.5]. *If* G *is a finite nonabelian simple group then* G *has no elements of order* 10 *if and only if* G *is isomorphic to one of the following groups:* $L_2(q)$, *where either* $3 < q \equiv 0, \pm 2 \pmod 5$ *or* $q \equiv \varepsilon 1 \pmod 5$ *for* $\varepsilon \in \{+,-\}$ *and* $(q - \varepsilon 1)_2 \le 2$; $L_3^{\varepsilon}(q)$, *where* $\varepsilon \in \{+,-\}$ *and either* $2 < q \equiv \pm 2 \pmod 5$ *or* $q \equiv -\varepsilon 1 \pmod 5$ *and* q *is even;* $L_4^{\varepsilon}(q)$, *where* $\varepsilon \in \{+,-\}$ *and* $q \equiv \pm 2 \pmod 5$ *with* $(q + \varepsilon 1)_2 \le 2$; $L_5^{\pm}(q)$, *where* q *is even and* $2 < q \equiv \pm 2 \pmod 5$; $S_4(q)$, *where* $2 < q \equiv \pm 2 \pmod 5$; $G_2(q)$, *where* $2 < q \equiv \pm 2 \pmod 5$; ${}^2G_2(q)$, *where* $q > 3$; ${}^3D_4(q)$, *where* $q \equiv \pm 2 \pmod 5$; $Sz(q)$, *where* $q > 2$; A_7; M_{11}; M_{22}; *and* M_{23}.

Lemma 3 [15, Theorem 1]. *If* G *is a finite solvable nonprimary group the orders of whose all elements are prime powers then* G *is biprimary and one of the following holds:*

(a) G *is a Frobenius group;*

(b) G *is a* 2*-Frobenius group, meaning that* $G = ABC$, *where* A *and* AB *are normal subgroups of* G, *while* AB *and* BC *are Frobenius groups with cores* A *and* B *and complements* B *and* C, *respectively.*

Lemma 4 [16, Remark on p. 377]. *If* G *is a finite group whose Sylow* 2*-subgroup is isomorphic to a (generalized) quaternion group and* $\overline{G} = G/O(G)$ *then one of the following holds:*

(a) $\overline{G}$ *is isomorphic to a Sylow* 2*-subgroup of* G;

(b) $\overline{G}$ *is isomorphic to* $2{\cdot}A_7$;

(c) $\overline{G}$ *is an extension of* $SL_2(q)$, *where* q *is odd, by a cyclic group of either an odd order or a doubled odd order.*

Lemma 5 [20, Theorem 1]. *If* G *is a finite primary group possessing an automorphism* α *of prime order* $p \ge 5$ *and* $C_G(\alpha) = 1$ *then the degree of nilpotence of* G *is at most* $(p^2 - 1)/4$, *and this bound is sharp for* $p = 5$.

Lemma 6 [6, Lemma 11]. *Given three distinct primes* p, q, *and* r, *consider a finite group* G *of the form* $G = P \rtimes (T \rtimes \langle x \rangle)$, *where* P *is a nontrivial* p*-group,* T *is a* q*-group,* $|x| = r$, *and* $C_G(P) = Z(P)$. *If* C *is a critical subgroup of* T *(see* [16, 5.3.11]*) and* $[T, \langle x \rangle] \ne 1$ *then either* $C_P(x) \ne 1$ *or* $Z(T) \le Z(C) \le C_T(x)$, $q = 2$, $r = 1 + 2^n$ *is a Fermat prime, and* $[C, \langle x \rangle]$ *is an extraspecial group of order* 2^{2n+1}.

3. Proof of Theorem 1

Consider a finite almost simple group G with socle L such that 5 divides $|G|$.

Necessity: Suppose that G has no elements of order 10 and justify the claim of Theorem 1. The group L is isomorphic to one of the groups of Lemma 2.

If L is isomorphic to some sporadic group then $G = L$ because [17] yields $\mathrm{Out}(M_{11}) = \mathrm{Out}(M_{23}) = 1$ and $\mathrm{Aut}(M_{22}) \cong M_{22}.2$ contains an element of order 10.

If $L \cong A_7$ then $G = L$ because $\mathrm{Aut}(A_7) \cong S_7$ contains an element of order 10, namely, the product of some independent 5-cycle and transposition. Therefore, claim (8) of Theorem 1 holds in the considered cases.

Thus, we will assume that $L = \Phi(q)$ is a simple group of Lie type Φ over the field of order $q = p^f$, where p is a prime and $f \in \mathbb{N}$. By [18, Theorem 2.5.12], we have $\mathrm{Aut}(L) = \mathrm{Inndiag}(L) \rtimes (\Phi_L \Gamma_L)$, where $\mathrm{Inndiag}(L)$, Φ_L, and Γ_L are respectively the groups of inner-diagonal, standard field, and standard graph automorphisms of L, while Φ_L is a cyclic group of order either f or $2f$, $|\Gamma_L| \le 6$, and $\Phi_L \Gamma_L$ is an abelian group.

Suppose that 5 divides $|G : L|$. Then [18, Theorem 2.5.12] implies that 5 divides $|\,\mathrm{Outdiag}(L)|$ or $|\Phi_L|$. If 5 divides $|\,\mathrm{Outdiag}(L)|$ then $L \cong L_5^{\pm}(q)$, where $5 = (q \pm 1, 5)$ and $q \equiv \pm 2 \pmod 5$, which is contradictory. Therefore, 5 divides $|\Phi_L|$, and consequently $G \setminus L$ contains an element x of order 5 which induces a field automorphism on L. By [18, Proposition 4.9.1], the centralizer $C_L(x)$ has a subgroup isomorphic to $\Phi(q_0)$, where q is a square and $q = q_0^2$. However, $\Phi(q_0)$ is of even order, and so G contains an element of order 10; this is a contradiction.

Thus, 5 does not divide $|G : L|$, and so 5 divides $|L|$.

Suppose that $p = 5$. Then Lemma 2 yields $L \cong L_2(5^f)$, and consequently $G \leq \mathrm{Inndiag}(L) \rtimes \Phi_L$. If $G \cap O_2(\Phi_L) \neq 1$ then by [18, Proposition 4.9.1] for an involution $x \in G \cap O_2(\Phi_L)$ the centralizer $C_L(x)$ has a subgroup isomorphic to $\Phi(q_0)$, where q is a square and $q = q_0^2$. However, $|\Phi(q_0)|$ is divisible by 5, and so G contains an element of order 10; this is a contradiction. Thus, $G \cap O_2(\Phi_L) = 1$ and consequently $O^{\{2,5\}'}(G)$ is isomorphic to $PGL_2(5^f)$ or $PGL_2^*(5^f)$ for even f. If $f = 1$ then $L \cong L_2(5) \cong L_2(4)$. For this reason, we may assume that $f > 1$, meaning that claim (1) of Theorem 1 is valid.

Assume now that $p \neq 5$ and $L < G$.

Suppose firstly that $G \cap \mathrm{Inndiag}(L) \neq L$.

Assume further that $L \cong L_3^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$ and $q \equiv -\varepsilon 1 \pmod 5$ with even q. Then $G \cap \mathrm{Inndiag}(L) \cong PGL_3^\varepsilon(q)$, where $(q - \varepsilon 1, 3) = 3$ and $G = PGL_3^\varepsilon(q) \rtimes (G \cap (\Phi_L \Gamma_L))$. Recall that $G \cap (\Phi_L \Gamma_L)$ is an abelian $5'$-group. Since $O^{\{2,5\}'}(G) = G$; therefore, G/L is not abelian, and $G \cap (\Phi_L \Gamma_L)$ is a nontrivial abelian 2-subgroup. Thus, $G \cap (\Phi_L \Gamma_L)$ contains some involution t which is either a field, graph, or graph-field automorphism of L.

If t is a graph automorphism of L then by [18, Proposition 4.9.2] the centralizer $C_L(t)$ is isomorphic to $L_2(q)$. Since $q + \varepsilon 1$ is divisible by 5, while the subgroup $L_2(q)$ has a cyclic subgroup of order $q + \varepsilon 1$; the group G contains an element of order 10. This contradiction shows that $G \cap (\Phi_L \Gamma_L)$ is a cyclic group.

If t is a graph-field automorphism of L then $G \cap (\Phi_L \Gamma_L) = \langle t \rangle$, and consequently [18, Theorem 2.5.12] shows that G/L is an abelian group; this is a contradiction.

If t is a field automorphism of L then [18, Proposition 4.9.1] yields $O^{2'}(C_L(t)) \cong L_3^\varepsilon(q_0)$, where $q = q_0^2$. If $q_0 \equiv \pm 1 \pmod 5$ then $q \equiv 1 \pmod 5$, and so $\varepsilon = -$ and $O^{2'}(C_L(t)) \cong U_3(q_0)$. However, then by Lemma 1 the group $L\langle t \rangle$ contains an element of order 10; this is a contradiction. Therefore, $q_0 \equiv \pm 2 \pmod 5$, whence $\varepsilon = +$. Consequently, by Lemma 1 the group $G = PGL_3(q)\langle t \rangle$ has no elements of order 10, meaning that claim (3) of Theorem 1 is valid.

Let us exclude all other possibilities for L.

Assume that $L \cong L_2(q)$, where $q \equiv \varepsilon 1 \pmod 5$ for $\varepsilon \in \{+, -\}$ with $(q - \varepsilon 1)_2 \leq 2$. Then $G = \mathrm{Inndiag}(L) \cong PGL_2(q)$ and $(q - \varepsilon 1)_2 = 2$. By [21, Proposition 7], G has abelian subgroups (maximal tori) of orders $q - 1$ and $q + 1$. But one of them contains an element of order 10; this is a contradiction.

Assume further that $L \cong L_4^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$ and $q \equiv \pm 2 \pmod 5$ with $(q + \varepsilon 1)_2 \leq 2$. Then

$$L < G \cap \mathrm{Inndiag}(L) \leq \mathrm{Inndiag}(L) \cong PGL_4^\varepsilon(q)$$

and $(q + \varepsilon 1)_2 = 2$. This yields $(q - \varepsilon 1, 4) = 4$, meaning that $\mathrm{Outdiag}(L) \cong \mathbb{Z}_4$. By [21, Propositions 7 and 8], the group $PGL_4^\varepsilon(q)$ has an abelian subgroup (maximal torus) of order $(q^2 + 1)(q + \varepsilon 1)$, which is divisible by 20. Therefore, the subgroup $G \cap \mathrm{Inndiag}(L)$ is of index at most 2 in $PGL_4^\varepsilon(q)$, and so it contains an element of order 10; this is a contradiction.

The possibility $L \cong L_5^\varepsilon(q)$ is excluded because 5 does not divide $|G : L|$.

Assume that $L \cong S_4(q)$, where $q \equiv \pm 2 \pmod 5$. Then q is odd and $G = \mathrm{Inndiag}(L)$. By [21, Proposition 9] G has an abelian subgroup (maximal torus) of order $q^2 + 1$, which is divisible by 10, and so G contains an element of order 10; this is a contradiction.

Thus, we may assume that $G \not\leq \mathrm{Inndiag}(L)$; and, moreover, $G \cap \mathrm{Inndiag}(L) = L$. Put $\overline{G} = G/L$. Since $G \cap \mathrm{Inndiag}(L) = L$, we have

$$\begin{aligned}\overline{G} &= G/(G \cap \mathrm{Inndiag}(L)) \cong \mathrm{Inndiag}(L)G/\mathrm{Inndiag}(L) \\ &\leq (\mathrm{Inndiag}(L) \rtimes \Phi_L \Gamma_L)/\mathrm{Inndiag}(L) \cong \Phi_L \Gamma_L.\end{aligned}$$

Hence, $\overline{G}$ is isomorphic to a subgroup of the abelian group $\Phi_L \Gamma_L$, and so

$$\overline{G} = \overline{O^{\{2,5\}'}(G)} = O^{\{2,5\}'}(\overline{G}) = O_2(\overline{G}) \neq 1.$$

Inspect all three possibilities for L which arise in Lemma 2.

Assume that $L \cong L_2(q)$, where $q = p^f \equiv \varepsilon 1 \pmod 5$ for $\varepsilon \in \{+,-\}$ and $(q-\varepsilon 1)_2 \leq 2$. Therefore, $\overline{G}$ is isomorphic to a subgroup of even order in the cyclic group Φ_L of order f. Then f is even, and so q is the square of $q_0 = p^{f/2}$. If $G \cap \Phi_L = 1$ then q is odd and G is isomorphic to $PGL_2^*(q)$; consequently, claim (2) of Theorem 1 is valid. Suppose that $G \cap \Phi_L \neq 1$ and take an involution t in $G \cap \Phi_L$. Then [18, Proposition 4.9.1] yields $O^{2'}(C_L(t)) \cong L_2(q_0)$. Since $L_2(q_0)$ has no elements of order 5, by Lemma 1 we see that $q_0 \equiv \pm 2 \pmod 5$ and $\varepsilon = -$, while the number $f/2$ is odd. Thus, $G = L \rtimes \langle t \rangle$ has no elements of order 10, and so claim (2) of Theorem 1 is valid.

Assume that $L \cong L_3^\varepsilon(q)$, where $\varepsilon \in \{+,-\}$ and $q \equiv -\varepsilon 1 \pmod 5$ with even q.

Assume at first that $(q-\varepsilon 1, 3) = 3$. Then $|\operatorname{Outdiag}(L)| = 3$ and $\operatorname{Inndiag}(L)G$ has no elements of order 10. The argument above shows that $G < \operatorname{Inndiag}(L) \rtimes \langle t \rangle$, where t is the field automorphism of order 2 of L. Therefore, $\operatorname{Inndiag}(L)\langle t \rangle / L \cong S_3$ and so $G = L \rtimes \langle d^{-1}td \rangle$ for some $d \in \operatorname{Inndiag}(L)$, meaning that claim (3) of Theorem 1 is valid.

Assume now that $(q-\varepsilon 1, 3) = 1$. Then $\operatorname{Outdiag}(L) = 1$ and consequently $G = L \rtimes (G \cap (\Phi_L \Gamma_L))$. Arguing as above, we see that $G = L \rtimes \langle t \rangle$, where q is a square, $\varepsilon = +$, and t is the field or graph-field automorphism of order 2 of L. If t is the field automorphism; then, as we showed above, $L \rtimes \langle t \rangle$ has no elements of order 10. Assume that t is the graph-field automorphism. Then [18, Proposition 4.9.1] implies that $\varepsilon = +$ and $O^{2'}(C_L(t)) \cong U_3(q_0)$, where $q = q_0^2$. Since $q \equiv -1 \pmod 5$, we have $q_0 \equiv \pm 2 \pmod 5$ and therefore Lemma 1 shows that $L\langle t \rangle$ has no elements of order 10. Thus, in both cases claim (3) of Theorem 1 is valid.

In view of Lemma 2 we may assume henceforth that $q = p^f \equiv \pm 2 \pmod 5$ and so f is odd. Then, since $G \cap \operatorname{Inndiag}(L) = L$ and $\overline{G} = O_2(\overline{G}) \neq 1$, we obtain $G = L\langle t \rangle$, where t is some involution in $\operatorname{Aut}(L) \setminus \operatorname{Inndiag}(L)$.

Assume that $L \cong S_4(q)$. Then q is even, $\operatorname{Inndiag}(L) = L$, and we may assume that t is an involution in the cyclic group $\Phi_L \Gamma_L$. By [22, (19.5)] the centralizer $C_L(t)$ is isomorphic to the group $Sz(q)$. Since $5 \in \pi(Sz(q))$ by Lemma 1; G contains an element of order 10, which is a contradiction.

Assume that $L \cong L_5^\varepsilon(q)$ with $\varepsilon \in \{+,-\}$ and even q. Then $\operatorname{Inndiag}(L) = 1$ and we may assume that t is an involution in the cyclic group $\Phi_L \Gamma_L$. By [18, Proposition 4.9.2] the centralizer $C_L(t)$ is isomorphic to $S_4(q)$. Since $5 \in \pi(S_4(q))$ by Lemma 1, the group G contains an element of order 10, which is a contradiction.

Thus, $L \cong L_4^\varepsilon(q)$ with $\varepsilon \in \{+,-\}$ and $(q+\varepsilon 1)_2 \leq 2$. For q even we arrive at a contradiction as in the previous paragraph. Hence, q is odd, and so $(q+\varepsilon 1)_2 = 2$, whence $(q-\varepsilon 1)_2 \geq 4$. Therefore, $\operatorname{Outdiag}(L) \cong \mathbb{Z}_4$. By [18, Theorem 2.5.12] we have $\operatorname{Outdiag}(L)\langle t \rangle \cong D_8$. Since $q \equiv \pm 2 \pmod 5$, the well-known formula for the order of L (see [17]) shows that $|L|_5 = (q^2+1)_5 > 1$. According to [23, Tables 8.8 and 8.10], there is a maximal subgroup M of L isomorphic to an extension of $L_2(q^2)$ by the cyclic group of order $q+\varepsilon 1$. The socle $Soc(M)$ of M is isomorphic to $L_2(q^2)$, and so it contains a maximal dihedral subgroup D of order q^2+1. Take some Sylow 5-subgroup T of D. Since $(q^2+1)_5 = |L|_5 = |D|_5$, it follows that T is a cyclic Sylow 5-subgroup of G. Frattini's Lemma yields $M = Soc(M)N_M(T) = Soc(M)N_M(\Omega_1(T))$. This implies that

$$M/Soc(M) \cong N_M(\Omega_1(T))/N_{Soc(M)}(\Omega_1(T)) \cong N_M(\Omega_1(T))/D,$$

and so the Sylow 2-subgroup of $N_M(\Omega_1(T))$ is of order 4. Since $C_M(\Omega_1(T))$ is of odd order and $N_M(\Omega_1(T))/C_M(\Omega_1(T))$ is isomorphic to a subgroup of $\operatorname{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$, we find that $N_M(\Omega_1(T)) = C_M(\Omega_1(T))Z$, where $C_M(\Omega_1(T)) = O(N_M(\Omega_1(T)))$ and $Z \cong \mathbb{Z}_4$. Furthermore, Frattini's Lemma yields

$$G = LN_G(\Omega_1(T)) = LZC_G(\Omega_1(T)),$$

but $Z < L$, and hence $G = LC_G(\Omega_1(T))$. Since G has no elements of order 10, the subgroup $C_G(\Omega_1(T))$ is of odd order. However, this contradicts the evenness of the index $|G : L|$.

Necessity is verified.

Sufficiency follows from Lemmas 1 and 2 on arguing as in the proof of necessity.

The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Given a group G satisfying the hypotheses of Theorem 2, take $S \in Syl_2(G)$ and $T \in Syl_p(G)$. The Hall–Chunikhin Theorem [16, Theorem 6.4.1] shows that G contains a biprimary Hall $\{2, p\}$-subgroup U; and, moreover, we may assume that $U = ST$. All elements of U are of primary orders, and so U is either a Frobenius group or a 2-Frobenius group by Lemma 3. It is clear that either $O_p(U) \neq 1$ or $O_2(U) \neq 1$.

Assume that $O_p(U) \neq 1$. Then by [16, Theorem 10.3.1] S is either a cyclic group or a (generalized) quaternion group. Burnside's Theorem (see [16, Theorem 7.4.3]) and Lemma 4 show that $U = O(U)S$. Hence, U is a Frobenius group with core T and complement S. Since $C_T(s) = 1$ for the (unique) involution s in S, this involution inverts T, and so T is abelian. Therefore, by Lemma 4 and [16, Theorem 6.3.2], claim (1) of Theorem 2 is valid.

Assume now that $O_2(U) \neq 1$. Then the subgroup T is cyclic. By [16, Theorem 6.3.2] the centralizer $C_G(T)$ is contained in $O_{p',p}(G)$. Hence,

$$O_{p',p}(G) = O_{p'}(G)T,$$

and so Frattini's Lemma yields $G = O_{p'}(G)N_G(T)$. By [16, Theorem 5.2.4] $C_{N_G(T)}(\Omega_1(T)) = C_G(T)$. Since $\mathrm{Aut}(\Omega_1(T))$ is a cyclic group of order $p-1$, this implies that $G/O_{5'}(G)$ is either a cyclic p-group or a Frobenius group with cyclic core of order $|G|_p$ and cyclic complement of order dividing $p-1$. Put $K = O_{p'}(G)$. Since $O_2(U) \neq 1$, it follows that K is of even order. Without loss of generality we may assume that $O_{\{2,p\}'}(G) = O(K) = 1$. However, then $O_2(K) = O_2(G) \neq 1$ and $C_G(O_2(K)) \leq O_2(G)$ by [16, Theorem 6.3.2]. If $K = O_2(K)$, then by Lemma 5 claim (2) of Theorem 2 is valid. Assume that $O_2(K) < K$. Then

$$O_2(K) < O_{\{2,2'\}}(K).$$

Denote by R some 2-complement in $O_{\{2,2'\}}(K)$. By [16, Theorem 6.3.2] we have

$$C_K(R) \leq O_2(K)R.$$

Frattini's Lemma yields $G = O_2(K)N_G(R)$, and so we may assume that T normalizes R. Consider the subgroup $O_2(K) \rtimes (R \rtimes \langle t \rangle)$, where t is an element of order p of T. Since $C_{O_2(K)}(t) = 1$, Lemma 6 implies that $[R, \langle t \rangle] = 1$. Since t lies in $C_G(R)$, which is a normal subgroup of $N_G(R)$, we have

$$[K, \langle t \rangle] \leq C_K(R) \leq O_2(K)R.$$

It is then clear that $K = O_2(K)R$, and so the 2-length of $O_{p'}(G)$ is 1. Since $C_{O_2(K)}(t) = 1$, Lemma 5 implies that claim (2) of Theorem 2 is valid.

The proof of Theorem 2 is complete.

Acknowledgment. The authors are grateful to the anonymous referee for the valuable comments that helped improve this article.

FUNDING

This research was supported by the NSFC (Grants nos. 11961017 and 12171126), the Hainan Provincial Natural Science Foundation of China (Grant no. 122RC543), and the Ministry of Science and Higher Education of the Russian Federation (Project "Ural Mathematical Center"; Agreement 075–02–2024–1428).

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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