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# THE INVERSE PROBLEM FOR THE HEAT EQUATION WITH TWO UNKNOWN COEFFICIENTS

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**Abstract**—We solve the simultaneous recovery of thermal conductivity and a high-frequency coefficient of a source in a one-dimensional initial-boundary value problem for the heat equation with Dirichlet boundary conditions and an inhomogeneous initial condition from some information on the partial asymptotics of a solution. We show that the coefficients can be restored from some data on the asymptotics of a solution, which is constructed and justified. This article was inspired by Denisov's research on a variety of inverse problems without accounting for high-frequency oscillations. Also, we continue the research by Levenshtam and his students which firstly addressed the inverse problems for parabolic equations with high-frequency coefficients and developed the relevant methodology. In contrast to the previous research of the case that only the source function or its factors are unknown, we assume that the thermal conductivity and the factor of a source function are unknown simultaneously.

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## Introduction

In the article we consider the initial-boundary value problem for the heat equation in a rectangle with Dirichlet boundary conditions, an inhomogeneous initial condition, heat conductivity depending on time, and the high-frequency source. The source is a product of two functions: One depends on the space variable and the other on time and fast time variables. In Section 1 we construct and justify the asymptotics of a solution to the direct initial-boundary value problem for the large values of the frequency of oscillations. In Section 2 we solve the inverse problem in which heat conductivity and the second factor of a source are the unknowns recovered from the values of a few first coefficients of the asymptotics of a solution which are calculated at a prescribed space point.

This article was inspired by [1–3] that address various inverse problems for parabolic equations (without high-frequency oscillations). Also, the article continues the research of [4–6] which considered the parabolic inverse problems with high-frequency coefficients for the first time and developed the methods for solving similar problems. In contrast to these articles in which the source is an unknown function, in the present article the unknowns are heat conductivity and the source.

Note that problems with high-frequency data simulate many physical phenomena and processes. Among them are high-frequency impacts in mechanical systems, diffusion in a medium subject to high-frequency vibrations, and convection of a fluid in the field of high-frequency forces (see [7–10], etc).

## 1. Constructing and Justifying Asymptotics

Consider the Dirichlet initial-boundary value problem with inhomogeneous initial condition for the heat equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = k(t)\frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t)r(t,\omega t), \\ u(x,0) = \varphi(x), \\ u(0,t) = u(\pi,t) = 0. \end{cases} \tag{1}$$

Here $x \in (0,\pi)$, $t \in (0,T)$, $T = \text{const} > 0$, and $\omega \gg 1$. Put $S_0 = [0,\pi] \times [0,T]$ and $S_1 = [0,T] \times [0,+\infty)$. Assume that the functions $q(x)$, $k(t)$, and $r(t,\tau)$ are defined and continuous on $[0,\pi]$, $[0,T]$,

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and $S_1$; moreover, $r(t,\tau)$ is $2\pi$-periodic in $\tau$ and representable as $r(t,\tau) = r_0(t) + r_1(t,\tau)$, where $\langle r_1(t,\tau)\rangle = 0$ and $k(t)$ is everywhere positive. Let $\langle f(t,\tau)\rangle$ stand for the integral mean over $\tau$ of an arbitrary function $f(t,\tau)$ $2\pi$-periodic (in $\tau$); i.e.,

$$\langle f(t,\tau)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(t,\tau)\,d\tau \tag{2}$$

while $\{f(t,\tau)\}$ designates the function with zero mean:

$$\{f(t,\tau)\} \equiv f(t,\tau) - \langle f(t,\tau)\rangle. \tag{3}$$

Let $q(x) \in C^3([0,\pi])$, $\varphi(x) \in C^0([0,\pi])$, $k(t) \in C^0([0,T])$, $r_0(t) \in C^0([0,T])$, and $r_1(t,\tau) \in C^{1+\gamma,0}(S_1)$, where $\gamma \in (0,1)$. Denote $C^{l,m}(S)$, where $l$ and $m$ are nonnegative integers, the ordinary Hölder function spaces. The matching conditions are written as follows:

$$q(0) = q(\pi) = \frac{d^2q}{dx^2}(0) = \frac{d^2q}{dx^2}(\pi) = 0, \tag{4}$$

$$\varphi(0) = \varphi(\pi) = 0. \tag{5}$$

Represent a solution to problem (1) as

$$u_\omega(x,t) = u_0(x,t) + \omega^{-1}[u_1(x,t) + v_1(x,t,\omega t)] + W_\omega(x,t), \quad \omega \gg 1, \tag{6}$$

where $v_1(x,t,\tau)$ is $2\pi$-periodic in $\tau$ with zero mean.

As is known, problem (1) has the unique classical solution $u_\omega(x,t)$.

**Theorem 1.** *A solution $u_\omega(x,t)$ to problem* (1) *is representable as* (6), *where*

$$\|W_\omega(x,t)\|_{C(S_0)} = o(\omega^{-1}) \quad \text{as } \omega \to \infty. \tag{7}$$

Remark 1. It is possible to construct and justify the complete asymptotics of a solution to problem (1) in the norm of $C(S_0)$ for the functions $q(x)$, $\varphi(x)$, $k(t)$, and $r(t,\tau)$ infinitely differentiable with respect to $x$ and $t$ and satisfying some conditions of the type (4) of each even order. However, to solve the inverse problem, we need only the asymptotics of the form (6).

Remark 2. We can construct the asymptotics of a solution to problem (1) by replacing the source $q(x)r(t,\tau)$ with some function $f(x,t,\tau)$ $2\pi$-periodic in $\tau$ of more general form that is defined and continuous on $[0,\pi]\times[0,T]\times[0,+\infty)$. Note that to simplify exposition, we choose the above particular case.

Proof. Indeed, insert (6) in system (1) and equate the coefficients of the powers of $\omega^0$ and $\omega^{-1}$ in the above equalities and apply the averaging operation in $\tau = \omega t$. In result, we come to the problems

$$\begin{cases} \frac{\partial u_0(x,t)}{\partial t} = k(t)\frac{\partial^2 u_0(x,t)}{\partial x^2} + q(x)r_0(t), \\ u_0(x,0) = \varphi(x), \\ u_0(0,t) = u_0(\pi,t) = 0, \end{cases} \tag{8}$$

$$\begin{cases} \frac{\partial v_1(x,t,\tau)}{\partial \tau} = q(x)r_1(t,\tau), \\ \langle v_1(x,t,\tau)\rangle = 0, \end{cases} \tag{9}$$

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = k(t)\frac{\partial^2 u_1(x,t)}{\partial x^2}, \\ u_1(x,0) = -v_1(x,0,0), \\ u_1(0,t) = u_1(\pi,t) = 0. \end{cases} \tag{10}$$

Next, the Fourier method yields (according to the matching conditions (4) $v_1(0,t,\tau) = v_1(\pi,t,\tau) = 0$):

$$u_0(x,t) = \sum_{n=1}^{\infty} \sin(nx)\varphi_n e^{-n^2 \int\limits_0^t k(s)\,ds}$$

$$+\sum_{n=1}^{\infty} \sin(nx) \int\limits_0^t q_n(s) r_0(s) e^{-n^2 \int\limits_0^{t-s} k(\xi)\,d\xi}\,ds, \tag{11}$$

where

$$\varphi_n = \frac{2}{\pi}\int\limits_0^{\pi} \varphi(y)\sin(ny)\,dy, \quad q_n = \frac{2}{\pi}\int\limits_0^{\pi} q(y)\sin(ny)\,dy, \tag{12}$$

$$v_1(x,t,\tau) = q(x)\left\{\int\limits_0^{\tau} r_1(t,s)\,ds\right\}, \tag{13}$$

$$u_1(x,t) = \left\langle \int\limits_0^{\tau} r_1(0,s)\,ds \right\rangle \sum_{n=1}^{\infty} \sin(nx) q_n e^{-n^2 \int\limits_0^t k(s)\,ds}. \tag{14}$$

One problem remains for the summand $W_\omega$ which is as follows:

$$\begin{cases} \frac{\partial W_\omega(x,t)}{\partial t} = k(t)\frac{\partial^2 W_\omega(x,t)}{\partial x^2} + \omega^{-1}\left[k(t)\frac{\partial^2 v_1(x,t,\omega t)}{\partial x^2} - \frac{\partial v_1(x,t,\omega t)}{\partial t}\right], \\ W_\omega(x,0) = 0, \\ W_\omega(0,t) = W_\omega(\pi,t) = 0. \end{cases} \tag{15}$$

A solution to problem (15) is of the form

$$W_\omega(x,t) = \frac{1}{\omega}\sum_{n=1}^{\infty} \sin(nx)\left[q_n^{(2)} \int\limits_0^t k(s) p_0(s,\omega s) e^{-n^2 \int\limits_0^{t-s} k(\xi)\,d\xi}\,ds\right.$$

$$\left. -q_n \int\limits_0^t \frac{\partial p_0(t,\omega s)}{\partial t}\bigg|_{t=s} e^{-n^2 \int\limits_0^{t-s} k(\xi)\,d\xi}\,ds\right], \tag{16}$$

where

$$q_n^{(2)} = \frac{2}{\pi}\int\limits_0^{\pi} \frac{d^2 q(x)}{dx^2}\bigg|_{x=y} \sin(ny)\,dy, \quad p_0(t,\tau) = \left\{\int\limits_0^{\tau} r_1(t,s)\,ds\right\}. \tag{17}$$

To prove Theorem 1, it suffices to demonstrate that

$$\|W_\omega(x,t)\|_{C(S_0)} = o(\omega^{-1}) \tag{18}$$

as $\omega \to \infty$. We use the same scheme as that in [5]; i.e., we divide (16) into two series, establish estimates for the Fourier coefficients, apply to either of the series the Cauchy–Bunyakovsky inequality, and estimate the integrals in the series.

Since $q(x) \in C^3([0,\pi])$ and the matching conditions (4) hold at the endpoints of $[0,\pi]$, we can extend $\frac{d^2q}{dx^2}(x)$ as an odd $2\pi$-periodic function on preserving the smoothness class $C^1$ on the whole real axis. The extension will keep the same symbol. In this case

$$\frac{d^2q(x)}{dx^2} = \sum_{n=1}^{\infty} q_n^{(2)} \sin(nx), \quad x \in R, \tag{19}$$

$$q_n^{(2)} = \frac{a_n}{n}, \quad \sum_{n=1}^{\infty} a_n^2 < \infty. \tag{20}$$

Similar arguments can be used for the function $q(x)$ itself.

The above considerations and the Cauchy–Bunyakovsky inequality suffice to show the estimate uniform in $n \in \mathbb{N}$:

$$\left\| \int_0^t p(s,\omega s) e^{-n^2 \int_0^{t-s} k(\xi)\, d\xi} ds \right\|_{C([0,T])} = o(1) \quad \text{as } \omega \to \infty, \tag{21}$$

where $p(t,\tau)$ is $2\pi$-periodic in $\tau$ with zero mean in $\tau$.

The proof of the estimate is divided into three steps. First, we take an arbitrary $\varepsilon > 0$ and find $\delta > 0$ such that, for all $t \in [0,T]$, $n \in \mathbb{N}$ and $\omega > 0$,

$$\left| \int_{t_0}^t p(s,\omega s) e^{-n^2 \int_0^{t-s} k(\xi)\, d\xi} ds \right| < \frac{\varepsilon}{3}, \tag{22}$$

where $t_0 = \max(0, t-\delta)$. Next, taking the positivity of $k(t)$ into account, we find $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $t > \delta$, and $\omega > 0$, we have

$$\left| \int_0^{t-\delta} p(s,\omega s) e^{-n^2 \int_0^{t-s} k(\xi)\, d\xi} ds \right| < \frac{\varepsilon}{3}. \tag{23}$$

Finally, divide the segment $[0, t-\delta]$ for every $t > \delta$ into $m$ equal parts $[t_j, t_{j+1}]$, with $j = 0, \ldots, m-1$, and apply the representation

$$\begin{aligned} &\int_0^{t-\delta} p(s,\omega s) e^{-n^2 \int_0^{t-s} k(\xi)\, d\xi} ds \\ &= \sum_{j=0}^{m-1} \left[ \int_{t_j}^{t_{j+1}} p(s,\omega s) e^{-n^2 \int_0^{t-s} k(\xi)\, d\xi} ds - \int_{t_j}^{t_{j+1}} p(t_j,\omega s) e^{-n^2 \int_0^{t-t_j} k(\xi)\, d\xi} ds \right] \\ &\quad + \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} p(t_j,\omega s) e^{-n^2 \int_0^{t-t_j} k(\xi)\, d\xi} ds. \end{aligned} \tag{24}$$

Find $m \in \mathbb{N}$ such that, for all $n \le n_0$, $\omega > 0$, we have

$$\left| \sum_{j=0}^{m-1} \left[ \int_{t_j}^{t_{j+1}} p(s,\omega s) e^{-n^2 \int_0^{t-s} k(\xi)\, d\xi} ds - \int_{t_j}^{t_{j+1}} p(t_j,\omega s) e^{-n^2 \int_0^{t-t_j} k(\xi)\, d\xi} ds \right] \right| < \frac{\varepsilon}{6}. \tag{25}$$

Next, using the equality

$$\sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} p(t_j, \omega s) e^{-n^2 \int_0^{t-t_j} k(\xi)\, d\xi} ds$$

$$= \frac{1}{\omega} e^{-n^2 \int_0^{t-t_j} k(\xi)\, d\xi} \sum_{j=0}^{m-1} \Big[ \int_0^{\omega t_{j+1}} p(t_j, \tau)\, d\tau - \int_0^{\omega t_j} p(t_j, \tau)\, d\tau \Big] \tag{26}$$

and the fact that $p(t, \tau)$ has zero mean in $\tau$, we can find $\omega_0 > 0$ such that

$$\Big| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} p(t_j, \omega s) e^{-n^2 \int_0^{t-t_j} k(\xi)\, d\xi} ds \Big| < \frac{\varepsilon}{6} \tag{27}$$

for all $\omega > \omega_0$ and $n \le n_0$. Hence, estimate (21) holds as well. Theorem 1 is proved. □

## 2. Solving the Inverse Problem

Our aim is to define some functions $k(t)$ and $r(t, \tau)$ of the above-pointed class for which a solution $u_\omega(x, t)$ to problem (1) satisfies the condition

$$\|u_\omega(x_0, t) - \psi(t, \omega t)\||_{C([0,T])} = o(\omega^{-1}) \quad \text{as } \omega \to \infty, \tag{28}$$

where

$$\psi(t, \tau) = \varphi_0(t) + \frac{1}{\omega}(\varphi_1(t) + \psi_1(t, \tau)). \tag{29}$$

Here $x_0 \in (0, \pi)$ is a point at which $q(x_0) \ne 0$, while $\varphi_0(t)$, $\varphi_1(t)$, and $\psi_1(t, \tau)$ are known functions such that $\varphi_0(t) \in C^1([0, T])$, $\varphi_0(0) = \varphi(x_0)$, and $\psi_1(t, \tau)$ is $2\pi$-periodic in $\tau$ with zero mean, and $\psi_1(t, \tau) \in C^{1+\gamma,1}(S_1)$.

Consider the operator equation

$$Ay(t) = \varphi_1(t), \tag{30}$$

where

$$Ay(t) = \left\langle \int_0^\tau r_1(0, s) ds \right\rangle \sum_{n=1}^{\infty} q_n \sin(n x_0) e^{-n^2 \int_0^t y(s)\, ds}. \tag{31}$$

Assume that (30) is uniquely solvable.

**Theorem 2.** *For every collection of functions $\varphi_0(t)$, $\varphi_1(t)$, and $\psi_1(t, \tau)$ and a point $x_0$ satisfying the above conditions, there exists a unique pair of functions $k(t)$ and $r(t, \tau)$ from the above-pointed classes for which a solution to problem* (1) *satisfies* (28) *and* (29).

Remark 3. If (30) is solvable and $\frac{d^2 q(x)}{d^2 x} > 0$ ($\frac{d^2 q(x)}{d^2 x} < 0$) for all $x \in [0, \pi]$; then, in accord with the results of [1], (30) has the unique solution. Note that [1] contains some necessary solvability conditions as well as some examples of solvable equations.

Proof. Theorem 1 implies that if $k(t)$ and $r(t, \tau)$ are from the above classes, then (1) has the unique classical solution representable as (6).

Assume that a pair $k(t)$, $r(t, \tau)$ is a solution to the inverse problem and $u_\omega$ is the corresponding solution to (1). Using Theorem 1, together with the conditions of the inverse problem, (6), (7), (28), and (29), we have

$$u_0(x_0, t) + \omega^{-1} [u_1(x_0, t) + v_1(x_0, t, \omega t)] = \varphi_0(t) + \omega^{-1} [\varphi_1(t) + \psi_1(t, \omega t)] + o(\omega^{-1}) \tag{32}$$

for $\omega \gg 1$ uniformly in $t \in [0, T]$.

Equating in (32) the coefficients of the same powers of $\omega$ and averaging in $\tau$, we infer

$$u_0(x_0,t) = \varphi_0(t), \tag{33}$$

$$u_1(x_0,t) = \varphi_1(t), \tag{34}$$

$$v_1(x_0,t,\tau) = \psi_1(t,\tau). \tag{35}$$

Using (14), (34) and inserting $x = x_0$, we obtain the nonlinear operator equation for $k(t)$; i.e.,

$$Ak(t) = \varphi_1(t). \tag{36}$$

By conditions, (36) is uniquely solvable and $k(t)$ is found.

The function $u_0(x,t)$ in accord with Section 1 is defined by (11). Differentiating $u_0(x,t)$ with respect to $t$ and putting $x = x_0$, we see that

$$\frac{\partial u_0(x_0,t)}{\partial t} = f(t) + q(x_0)r_0(t) + \int\limits_0^t K(t,s)r_0(s)\,d\xi, \tag{37}$$

where

$$K(t,s) = -\sum_{n=1}^{\infty} n^2 k(t-s) q_n e^{-n^2 \int\limits_0^{t-s} k(\xi)\,d\xi} \sin(nx_0), \tag{38}$$

$$f(t) = -\sum_{n=1}^{\infty} n^2 k(t) \varphi_n e^{-n^2 \int\limits_0^{t} k(\xi)\,d\xi} \sin(nx_0). \tag{39}$$

Now, by (33), we derive that the unknown $r_0(t)$ satisfies the Volterra equation of the second kind:

$$q(x_0,t)r_0(t) + \int\limits_0^t K(t,s)r_0(s)\,d\xi = \frac{d\varphi_0(t)}{dt} - f(t). \tag{40}$$

In view of the conditions on $q(x)$ and $\varphi(x)$ in Section 1, the functions $K(t,s)$ and $f(t)$ are continuous and, thereby, equation (40) has the unique continuous solution $r_0(t)$.

In view of (9), putting $x = x_0$ and (35) finally yields

$$q(x_0)r_1(t,\tau) = \frac{\partial \psi_1(t,\tau)}{\partial \tau}, \tag{41}$$

and, therefore, $r_1(t,\tau) = \frac{\partial \psi_1(t,\tau)}{\partial \tau}/q(x_0)$.

Since the above functions $r(t,\tau) = r_0(t) + r_1(t,\tau)$ and $k(t,\tau)$ satisfy the conditions of Section 1, Theorem 1 holds for them and, in particular, a solution to problem (1) is representable as (6) and (7). Demonstrate that the functions $u_\omega(x,t)$ satisfy conditions (28) and (29). It suffices to validate (33)–(35).

From the previous part of the proof of Theorem 2, we know that $k(t)$ is a solution to equation (36); using (14) and (36), we infer that $u_1(x_0,t) = \varphi_1(t)$. The function $r_0(t)$ is a solution to (40), equalities (37) and (40) imply that

$$\frac{\partial u_0(x_0,t)}{\partial t} = \frac{d\varphi_0(t)}{dt}.$$

Since $u_0(x_0,0) = \varphi_0(0) = \varphi(x_0)$, we conclude that $u_0(x_0,t) = \varphi_0(t)$. The function $r_1$ is defined by (41). Using (9), (41) and considering that $\psi_1(t,\tau)$ and $v_1(x,t,\tau)$ are $2\pi$-periodic in $\tau$ with zero mean, we derive that $v_1(x_0,t,\tau) = \psi_1(t,\tau)$. Theorem 2 is proved. □

## Conclusion

We solved the problem on simultaneous recovering the heat conductivity and high-frequency coefficient of the source in the one-dimensional initial-boundary value problem for the heat equation with the Dirichlet boundary condition and inhomogeneous initial condition from partial asymptotics of a solution. We show that the coefficients can be completely recovered from the data on the partial asymptotics of a solution. Furthermore, we construct and justify the asymptotics of a solution to the direct initial-boundary value problem.

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**CONFLICT OF INTEREST**

As author of this work, I declare that I have no conflicts of interest.

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