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BIRMAN–HILDEN BUNDLES. II

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Abstract—We study the structure of self-homeomorphism groups of fibered manifolds. A fibered topological space is a Birman–Hilden space whenever in each isotopic pair of its fiber-preserving (taking each fiber to a fiber) self-homeomorphisms the homeomorphisms are also fiber-isotopic (isotopic through fiber-preserving homeomorphisms). We prove in particular that the Birman–Hilden class contains all compact connected locally trivial surface bundles over the circle, including nonorientable ones and those with nonempty boundary, as well as all closed orientable Haken 3-manifold bundles over the circle, including nonorientable ones.

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1. Introduction

This article continues studying the structure of self-homeomorphism groups of fibered manifolds and developing the theory of Birman–Hilden bundles based on [1]. A fibered topological space is a *Birman–Hilden space* whenever in each isotopic pair of its fiber-preserving (taking each fiber to a fiber) self-homeomorphisms the homeomorphisms are also fiber-isotopic. Throughout the article, by a *fiber-preserving* mapping we mean a mapping that carries each fiber into some fiber, not necessarily the same, while by an *isotopy* of self-homeomorphisms we mean an isotopy in the class of self-homeomorphisms rather than in the class of embeddings.

A *Birman–Hilden bundle* is a bundle whose total space is a Birman–Hilden space. If a fibered space (bundle) is a Birman–Hilden space (bundle) then we say also that it *has the Birman–Hilden property* or *lies in the Birman–Hilden class*.

The question of membership in the Birman–Hilden class amounts to studying self-homeomorphism groups. Given a fibered space E, denote by $Fib(E)$ the subgroup of fiber-preserving self-homeomorphisms in the group $Homeo(E)$, endowed with the compact-open topology, of all self-homeomorphisms of E; denote by $Fib_1(E)$ and $Homeo_1(E)$ respectively the path-connected components of $Fib(E)$ and $Homeo(E)$ containing the identity mapping id_E. Then the membership in the Birman–Hilden class is equivalent to the coincidence of $Homeo_1(E) \cap Fib(E)$, the subgroup of fiber-preserving self-homeomorphisms isotopic to the identity; and $Fib_1(E)$, the subgroup of self-homeomorphisms with a fiber-preserving isotopy to the identity, or, which is the same, to the path-connectedness of $Homeo_1(E) \cap Fib(E)$. In terms of the path-connected components a fibered space E lies in the Birman–Hilden class if and only if the inclusion $Fib(E) \subset Homeo(E)$ induces a monomorphism on the level of π_0, i.e., the component $Homeo_1(E)$ of $Homeo(E)$ includes no components of $Fib(E)$ other than $Fib_1(E)$.

The membership in the Birman–Hilden class was studied in [2–17] for the case of branched coverings of surfaces; see [15] for a survey and additional references on this topic. This problem is studied in [18, 19] for the case of Seifert fibrations as well as for the case of coverings of three-dimensional manifolds. In knot theory and the theory of three-dimensional manifolds the problem of membership in the Birman–Hilden class arises for the three-dimensional manifolds fibered over the circle. A series of theorems is proved in [1] about sufficient conditions for membership in the Birman–Hilden class for locally trivial bundles over the circle. Here we use the sufficient conditions of [1] to establish the following:

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Theorem 1. *For $n \in \{1, 2, 3\}$ all compact connected n-dimensional manifolds, including nonorientable ones and those with nonempty boundary, which are locally trivial bundles over the circle have the Birman–Hilden property.*

Theorem 2. *All closed four manifolds, including nonorientable ones, which are locally trivial bundles over the circle with orientable Haken fibers have the Birman–Hilden property.*

In addition to Theorem 1, we have

Theorem 3. *For $n \in \{1, 2\}$, if M a closed fibered n-dimensional manifold of Theorem* 1*, then the inclusion $Fib_1(M) \subset Homeo_1(M)$ is a homotopy equivalence.*

The compactness and connectedness requirements in Theorems 1 and 2 are both essential. For instance, a locally trivial bundle over the circle each of whose connected components is a Möbius band has the Birman–Hilden property if and only if it is connected. A locally trivial bundle $\mathbb{R}^n \to S^1$ with fibers consisting of hyperplanes has the Birman–Hilden property if and only if $n = 1$. Apparently, we might relax the compactness requirement in Theorems 1 and 2 to the compactness of the connected components of the fiber, but here we do not pursue this. Theorem 2 can be generalized to a vast class of 4-dimensional manifolds with boundary; however, we omit the description of the class because it is bulky and still incomplete. Within the bounds of the proof methods used, the first obstacles to extrapolating Theorems 1 and 2 to the case of non-Haken fibers and higher dimensions are the manifolds admitting homotopic but not isotopic self-homeomorphisms; see [20] in particular.

Theorem 1 has a series of actual corollaries. In particular, Theorem 1 can be applied to show that in every compact connected three-dimensional manifold which is a locally trivial bundle over the circle all isotopic transversal links are transversally isotopic. This substantially generalizes the well-known result that closed braids in the solid torus are isotopic if and only if they represent the same conjugacy class of the braid group; see [21; 22, Theorem 1; 23, Proposition 10.16; 24, Theorem 2.1]. We intend to present a proof of this generalization separately.

The rest of this article deals with the proofs of Theorems 1–3 and has the following structure: In Section 2 we collect a series of preliminary facts, including the statement of sufficient conditions, proved in [1], for membership in the Birman–Hilden class. In Section 3 we give some propositions that are used in the subsequent proofs to reduce the situation to the case of bundles with connected fibers. In Section 4 Theorems 1 and 3 are proved in the case $n = 1$. In Section 5 Theorem 1 is proved in the case $n = 2$. In Section 6 Theorem 3 is proved in the case $n = 2$. In Section 7 Theorem 1 is proved in the case $n = 3$. Theorem 2 is proved in Section 8.

2. Preliminaries

2.1. Sufficient conditions. Let us recall the statements, established in [1], of the sufficient conditions for membership in the Birman–Hilden class used to prove Theorems 1–3.

Given a topological space X, denote by $Map(X, X)$ the space of continuous mappings $X \to X$ with its compact-open topology. Given a subspace Z of X, denote by $Map(X, X; [Z])$ the subspace of $Map(X, X)$ consisting of the mappings which restrict to the identity on Z. Denote by $Homeo(X; [Z])$ the subgroup $Homeo(X) \cap Map(X, X; [Z])$. Denote by $Map_1(X, X)$, $Map_1(X, X; [Z])$, and $Homeo_1(X; [Z])$ the path-connected components of the identity mapping in $Map(X, X)$, $Map(X, X; [Z])$, and $Homeo(X; [Z])$ respectively.

Theorem 4 [1]. *Suppose that a path-connected topological space X admits no homotopic but not isotopic self-homeomorphisms, and that either the group $Homeo_1(X)$ is simply-connected or its inclusion in the monoid $Map_1(X, X)$ induces an isomorphism of fundamental groups. Then every locally trivial bundle over the circle with fiber X has the Birman–Hilden property.*

Definition. Say that a bundle $p : E \to B$ *has the epimorphism property* whenever the inclusion $Fib_1(E) \subset Homeo_1(E)$ induces an epimorphism on the level fundamental groups.

Theorem 5 [1]. *Consider a locally trivial bundle $p : E \to S^1$ over the circle with fiber X, where X is a compact connected manifold with nonempty boundary ∂X. Suppose that the following hold:*

(1) *X admits no pair of self-homeomorphisms related by a pointwise boundary fixing homotopy, but not related by a pointwise boundary fixing isotopy;*

(2) *either $Homeo_1(X; [\partial X])$ is simply-connected or its inclusion into $Map_1(X, X; [\partial X])$ induces an isomorphism of fundamental groups;*

(3) *the restriction of the bundle p to each connected component of the boundary ∂E has the Birman–Hilden property and the epimorphism property.*

Then p has the Birman–Hilden property.

2.2. Sporadic facts. Let us present a series of facts use below.

Definition. A topological group is *Polish* whenever it is separable and admits a complete metric.

For convenience of references, recall the following well-known statements (see [25, Corollary 2] and [26, Section 5, I] for instance):

Assertion 1. *The self-homeomorphism groups of metric compact sets with the compact-open topology are Polish. The closed subgroups of Polish groups are Polish.*

Assertion 2. *If M is a compact manifold then $Homeo_1(M)$ is closed in $Homeo(M)$ endowed with the compact-open topology.*

Proof. Chernavskii's Theorem shows [27, 28] that $Homeo(M)$ is locally contractible, and so it is locally path-connected. This implies that the path-connected components of $Homeo(M)$ are closed. $\square$

Assertion 3. *If $p : E \to B$ is a bundle of compact metric sets, then the subgroup $Fib(E)$ of fiber-preserving self-homeomorphisms is closed in $Homeo(E)$.*

Proof. Define the mapping $f : Homeo(E) \to \mathbb{R}$ by putting the value of $f(h)$ at $h \in Homeo(E)$ equal to the supremum over all fibers of the diameters of the projections to B of the images $h(F)$ of the fibers F. In other words,

$$f(h) := \sup_{b \in B} \operatorname{diam}(p(h(p^{-1}(b)))),$$

where $\operatorname{diam}(Y)$ is the diameter of $Y \subset B$ in the fixed metric on B.

It is not difficult to see then that f is continuous and $Fib(E) = f^{-1}(\{0\})$, so that $Fib(E)$ is closed as the preimage of a closed set under a continuous mapping. $\square$

The following lemma is proved in [29, Theorem A.3] for instance; see also [30, 27.Vx] for the case of semidirect products:

Lemma 1. *If G is a Polish group, while A and B are closed subgroups of G with $G = AB$ and $A \cap B = 1_G$, then the group operation induces a homeomorphism between $A \times B$ and G.*

Definition. Given a class $\mathscr{C}$ of topological spaces, say that a topological space X is an *absolute neighborhood retract* (*ANR*) *for* $\mathscr{C}$ whenever X is a neighborhood retract for every space of class $\mathscr{C}$ with X as a closed subspace.

Assertion 4. *If $f_1 : A \to B$ is a homotopy equivalence and $f_2 : B \to C$ is a mapping such that the composition $f_2 \circ f_1$ is a homotopy equivalence, then f_2 is a homotopy equivalence as well.*

Proof. Observe that f_2 is homotopic to the composition of homotopy equivalences:

$$f_2 = f_2 \circ \mathrm{id}_B \simeq f_2 \circ (f_1 \circ g) = (f_2 \circ f_1) \circ g,$$

where $g : B \to A$ is a homotopy equivalence such that $g \circ f_1 \simeq \mathrm{id}_A$ and $f_1 \circ g \simeq \mathrm{id}_B$. $\square$

Assertion 4 relates the standard statement of the classical Whitehead Theorem about the homotopy equivalence of cellular complexes with the following version of the theorem:

Theorem 6 [31]. *A continuous mapping between topological spaces with the homotopy type of cellular complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Theorem 7 [32, 33]. *A topological space has the homotopy type of a countable cellular complex if and only if it has the homotopy type of a separable metric space which is an ANR for the class of separable metric spaces.*

3. Reduction to the Case of Connected Fibers

In this section we establish the propositions which reduce the general case of Theorems 1–3 to the case of bundles with connected fibers.

Assertion 5. *Every locally trivial bundle with locally connected sequentially compact total space* (*a compact manifold for instance*) *is the composition of a locally trivial bundle with connected fibers and a finite covering of the base.*

Proof. Observe firstly that the bundle of the said form has only finitely many connected components in each fiber. Indeed, from the definition of direct product topology we can infer in the standard fashion that in every locally trivial bundle with locally connected sequentially compact total space both the base and fibers are also locally connected and sequentially compact, while every space with these properties has only finitely many connected components. Indeed, using sequential compactness and taking a converging sequence of points running over an infinite set of components, we would otherwise obtain the limit point of this sequence without connected neighborhoods.

Furthermore, given an arbitrary bundle $p : E \to B$, define the quotient space Q_p and the quotient mapping $p_1 : E \to Q_p$ corresponding to the identification of all points in every connected component of each fiber of p. Then p factors as the composition of bundles

$$E \overset{p_1}{\to} Q_p \overset{p_2}{\to} B;$$

furthermore, p_1 and p_2 are locally trivial provided that so is p. The fibers of p_1 are connected by definition. Observe that if the number of connected components in each fiber of p is finite then the fibers of p_2 are finite and discrete because the finite number of connected components of the fiber F of p makes the subsets corresponding to these components in an arbitrary trivially fibered neighborhood $U \times F$ of F not only closed, but also open in $U \times F$, which implies that the corresponding fiber of p_2 is discrete. Therefore, the fibers of p_1 are always connected, while by hypothesis the fibers of p_2 are finite and discrete, meaning that p_2 is a finite covering. □

Corollary 1. *Every locally trivial bundle $p : M \to S^1$ over the circle with compact connected manifold as the total space factors as the composition*

$$M \overset{p_1}{\to} S^1 \overset{c}{\to} S^1,$$

where p_1 is a locally trivial bundle with connected fibers and c is a covering.

Proof. The claim follows from Assertion 5 because manifolds are locally connected, while every quotient of a connected space is connected. □

Considering simultaneously two bundles p and $c \circ p$ with the same total space M in the sequel, we will denote the subgroup of fiber-preserving self-homeomorphisms in $Homeo(M)$ corresponding to the fibers of $c \circ p$ and p by $Fib^{c \circ p}(M)$ and $Fib^{p}(M)$ respectively, while the path-connected components of these subgroups containing the identity mapping id_M by $Fib_1^{c \circ p}(M)$ and $Fib_1^{p}(M)$.

Proposition 1. *If $p : M \to S^1$ is a locally trivial bundle with connected metrizable compact fiber F and $c : S^1 \to S^1$ is a covering then $Fib^{c \circ p}(M)$ lies in $Fib^{p}(M)$, while the inclusions $Fib^{c \circ p}(M) \subset Fib^{p}(M)$ and $Fib_1^{c \circ p}(M) \subset Fib_1^{p}(M)$ are homotopy equivalences.*

Proof. The property that $Fib^{c \circ p}(M)$ lies in $Fib^{p}(M)$ follows because the fiber F is connected: the fibers of p are precisely the connected components of the fibers of $c \circ p$, so that each $c \circ p$-fiber-preserving self-homeomorphism is also p-fiber-preserving.

Verify that the inclusion $Fib^{c\circ p}(M) \subset Fib^p(M)$ is a homotopy equivalence. Denote by S^1_p and S^1_c the base circles of p and $c\circ p$ respectively. Take an arbitrary (compatible with the topology) intrinsic metric ρ on S^1_c. Denote by $I^\rho(M)$ the subgroup of those elements of $Fib^{c\circ p}(M)$ whose induced automorphisms $S^1_c \to S^1_c$ are isometries with respect to ρ. This yields the chain of inclusions

$$I^\rho(M) \subset Fib^{c\circ p}(M) \subset Fib^p(M) \subset Homeo(M).$$

To verify that $I^\rho(M) \subset Fib^{c\circ p}(M)$ is a homotopy equivalence, fix an arbitrary fiber $\widetilde{F}_0$ of $c \circ p$ and an arbitrary continuous surjection

$$\phi : \widetilde{F}_0 \times [0,1] \to M$$

homeomorphically carrying each fiber $\widetilde{F}_0 \times \{t\}$ with $t \in [0,1]$ into some fiber of $c \circ p$ such that $c \circ p \circ \phi$ induces a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and S^1_c. Denote by L the subgroup $\mathrm{id}_{\widetilde{F}_0} \times Homeo_1([0,1])$ of $Homeo_1(\widetilde{F}_0 \times [0,1])$. Observe that L is isomorphic to the contractible group $Homeo_1([0,1])$, while ϕ induces the monomorphism $\phi_* : L \to Fib_1^{c\circ p}(M)$. It is not difficult to verify that the subgroups $I^\rho(M)$ and $\phi_*(L)$ are closed in $Fib^{c\circ p}(M)$, while we can uniquely express each $g \in Fib^{c\circ p}(M)$ as the product $g = ab$ with $a \in I^\rho(M)$ and $b \in \phi_*(L)$. These properties imply by Assertion 1 and Lemma 1 that the mapping

$$I^\rho(M) \times \phi_*(L) \to Fib^{c\circ p}(M), \quad a \times b \mapsto ab,$$

is a homeomorphism. Since $\phi_*(L)$ is contractible, we infer that $I^\rho(M) \subset Fib^{c\circ p}(M)$ is a homotopy equivalence.

Verify that $I^\rho(M) \subset Fib^p(M)$ is a homotopy equivalence. Consider the metric $\widetilde{\rho}$ on S^1_p induced by the covering $c : S^1_p \to S^1_c$ from the metric ρ on S^1_c and the subgroup $I^{\widetilde{\rho}}(M)$ of the elements of $Fib^p(M)$ whose projections to S^1_p are isometries with respect to $\widetilde{\rho}$. Since the fiber F is connected, some elementary arguments show that $I^{\widetilde{\rho}}(M)$ coincides with $I^\rho(M)$. Applying the same arguments to the inclusion $I^\rho(M) = I^{\widetilde{\rho}}(M) \subset Fib^p(M)$ as to $I^\rho(M) \subset Fib^{c\circ p}(M)$, we see that it is a homotopy equivalence.

Assertion 4 implies that $Fib^{c\circ p}(M) \subset Fib^p(M)$ is a homotopy equivalence as well.

Since $Fib^{c\circ p}(M) \subset Fib^p(M)$ is a homotopy equivalence, it follows that so is $Fib_1^{c\circ p}(M) \subset Fib_1^p(M)$ because $Fib_1^{c\circ p}(M)$ and $Fib_1^p(M)$ are defined as the identity components of the subgroups $Fib^{c\circ p}(M)$ and $Fib^p(M)$. $\square$

Corollary 2. *If $p : M \to S^1$ is a locally trivial bundle with connected metrizable compact fibers and $c : S^1 \to S^1$ is a covering then the following hold:*

(1) *p has the Birman–Hilden property if and only if so does $c \circ p$;*

(2) *$Fib_1^p(M) \subset Homeo_1(M)$ is a homotopy equivalence if and only if so is $Fib_1^{c\circ p}(M) \subset Homeo_1(M)$.*

Proof. Since a fibered space E lies in the Birman–Hilden class if and only if the component $Homeo_1(E)$ of $Homeo(E)$ includes no component of $Fib(E)$ other than $Fib_1(E)$, the first claim follows because $Fib^{c\circ p}(M)$ lies in $Fib^p(M)$ and $Fib^{c\circ p}(M) \subset Fib^p(M)$ is a homotopy equivalence by Proposition 1, which indicates, in particular, the presence of a natural bijection between the components of $Fib^{c\circ p}(M)$ and $Fib^p(M)$.

The second claim follows from Assertion 4 because $Fib_1^{c\circ p}(M)$ lies in $Fib_1^p(M)$ and the inclusion $Fib_1^{c\circ p}(M) \subset Fib_1^p(M)$ is a homotopy equivalence by Assertion 1 again. $\square$

4. Proofs of Theorems 1 and 3 in the Case $n = 1$

For the bundle-homeomorphism $p : S^1 \to S^1$, when the fiber is a point, Theorems 1 and 3 hold because every isotopy between self-homeomorphisms of a circle fibered into points is automatically fiber-preserving; i.e., in this case $Homeo_1(M) = Fib_1(M)$. The case of covering $S^1 \to S^1$ with multiple sheets follows by Corollary 2: Theorems 1 and 3 in the case of a covering follow from claims (1) and (2) of Corollary 2 respectively.

5. Proof of Theorem 1 in the Case $n = 2$

Corollaries 1 and 2 (claim (1)) reduce the situation to the case of bundles with connected fibers, and so to prove Theorem 1 for $n = 2$, it suffices to consider the case of a fiber homeomorphic to a circle, and the case of a fiber homeomorphic to a segment.

5.1. The fiber homeomorphic to a circle. We deduce this case from Theorem 4. The property that the homotopic self-homeomorphisms of a circle are isotopic, i.e.,

$$Map_1(S^1, S^1) \cap Homeo(S^1) = Homeo_1(S^1),$$

is well known and implied by many classical constructions; see [34, 35] for instance. As for the second condition of Theorem 4 the second alternative in this condition holds: The inclusion $Homeo_1(S^1) \subset Map_1(S^1, S^1)$ induces an isomorphism of fundamental groups, see Proposition 2 below.

Proposition 2. *$Homeo_1(S^1) \subset Map_1(S^1, S^1)$ is a homotopy equivalence.*

Proof. Consider the natural embedding

$$SO(2) \subset Homeo_1(S^1) \subset Map_1(S^1, S^1),$$

where $SO(2)$ stands for the group of Euclidean rotations of a Euclidean circle. The property that the embedding $SO(2) \subset Homeo_1(S^1)$ is a homotopy equivalence is obtained, for instance, in [36, Proposition 4.2]; see also [37, Lemma 3.3]. The property that the embedding $SO(2) \subset Map_1(S^1, S^1)$ is a homotopy equivalence follows from the results about H_*-spaces established in [38]; see also [39; 40, Theorem (2.2); 41, 42; 43, Theorem 5.1]. Thus, Assertion 4 implies the claim. $\square$

5.2. The fiber homeomorphic to a segment. In this part we deduce Theorem 1 from Theorem 5. Let us mention the results from which it follows that if in Theorem 1 the fiber is a segment then the hypotheses of Theorem 5 hold.

The property that the segment lacks a pair of self-homeomorphisms related by a boundary fixing homotopy, but not related by a boundary fixing isotopy, can be established by Alexander's trick; see [44] and [45, Theorem 1.1.1]. This trick shows that the underlying space of $Homeo([0, 1])$ consists of the two components: The component $Homeo_1([0, 1])$ contains the self-homeomorphisms which fix the endpoints, and the second component contains the self-homeomorphisms which exchange the endpoints. It remains to observe that the self-homeomorphisms in distinct components are obviously not related by a pointwise boundary fixing homotopy.

The property that the underlying space of $Homeo_1([0, 1])$ is simply-connected (and even contractible) can also be proved using Alexander's trick; see [45, Theorem 1.1.1] and [44].

The claim that the restriction of the bundle p to each connected component of the boundary ∂E has the Birman–Hilden property follows respectively from the case $n = 1$ of Theorems 1 and 3.

6. Proof of Theorem 3 in the Case $n = 2$

For $n = 2$ the closed fibered n-dimensional manifold of Theorem 3 is either the torus or the Klein bottle, while the fiber is a collection of circles. Corollaries 1 and 2 (claim (2)) reduce the situation to the case of connected fibers, i.e., it suffices to give the proof only in the case that the fiber is a circle.

Therefore, take a bundle $p : E \to S^1$ with circle fibers (while the total space E is either the torus or the Klein bottle). Introduce on E some locally Euclidean metric such that all fibers of p are geodesics. Denote by $I_1(E)$ the identity component of the isometry group of E and consider the identical inclusions

$$I_1(E) \subset Fib_1(E) \subset Homeo_1(E).$$

In the case of the torus $E = T = S^1 \times S^1$ the underlying space of $I_1(E)$ is homeomorphic to the torus, while in the case of the Klein bottle $E = K$ the group $I_1(K)$ is isomorphic to $SO(2)$; going around the circle $SO(2)$ corresponds to a double rotation of the base of the bundle. As [46] and [47] show,

$I_1(T) \subset Homeo_1(T)$ and $I_1(K) \subset Homeo_1(K)$ are weak homotopy equivalences; the appearing spaces are separable, metrizable (Proposition 1) and ANRs for the class of metric spaces; see [48]. Hence, Theorems 6 and 7 imply that the inclusions are homotopy equivalences. Let us verify that $I_1(E) \subset Fib_1(E)$ is a homotopy equivalence in both cases $E = T$ and $E = K$. The claim that $Fib_1(E) \subset Homeo_1(E)$ is a homotopy equivalence would then follow by Assertion 4.

THE TORUS CASE. In order to verify that $I_1(T) \subset Fib_1(T)$ is a homotopy equivalence, observe firstly that $Fib_1(T)$ is closed in $Homeo(T)$ because $Fib_1(T) = Homeo_1(T) \cap Fib(T)$ by the case $n = 2$ of Theorem 1 established above,[1)] while $Homeo_1(T)$ and $Fib(T)$ are closed in $Homeo(T)$ by Assertions 2 and 3. This implies that $Fib_1(T)$ is a Polish group because it is a closed subgroup of the Polish group $Homeo(T)$; see Assertion 1.

Furthermore, choose an arbitrary point x in T and denote by $Fib_1(T, x)$ the subgroup of $Fib_1(T)$ formed by the homeomorphisms fixing x. It is not difficult to verify that $I_1(T)$ and $Fib_1(T, x)$ are closed in $Fib_1(T)$, while we can uniquely express each $g \in Fib_1(T)$ as the product $g = ab$ with $a \in I_1(T)$ and $b \in Fib_1(T, x)$, i.e.,

$$I_1(T)Fib_1(T, x) = Fib_1(T), \quad I_1(T) \cap Fib_1(T, x) = \{\mathrm{id}_T\}.$$

Hence, Lemma 1 shows that the mapping

$$I_1(T) \times Fib_1(T, x) \to Fib_1(T), \quad a \times b \mapsto ab,$$

is a homeomorphism.

In order to verify that the underlying space of $Fib_1(T, x)$ is contractible, fix some section γ of the bundle $T = E \to S^1$ such that γ contains x and consider the following subgroups G_1, G_2, and G_3 of $Fib_1(T, x)$:

G_1 is the subgroup of all fiberwise[2)] self-homeomorphisms in $Fib_1(T, x)$ which are the identity on γ;

G_2 is the subgroup of all fiberwise self-homeomorphisms in $Fib_1(T, x)$ whose restriction to each fiber is an isometry (with respect to the restrictions of the original locally Euclidean metric on E to the fibers),

G_3 is the intersection of the subgroup of all elements of $Fib_1(T, x)$ carrying γ to γ with the subgroup of all elements of $Fib_1(T, x)$ whose restriction to each fiber is an isometry (with respect to the restriction of the original locally Euclidean metric on E to the fibers) between the fibers.

It is clear from the definitions that the underlying spaces of G_1, G_2, and G_3 are homeomorphic to (obviously contractible) spaces: the space of free loops in $Homeo_1([0, 1])$, the space of based loops in $\mathbb{R}^1$ with the base point at the origin, and the space $Homeo_1([0, 1])$ respectively. Therefore, G_1, G_2, and G_3 are contractible. Moreover, simple reasoning shows that each of the subgroups G_1, G_2, and G_3 is closed in $Fib_1(T, x)$, while we can uniquely express each $g \in Fib_1(T, x)$ as the product $g = abc$ with $a \in G_1$, $b \in G_2$, and $c \in G_3$. Hence, Assertion 1 and Lemma 1 imply that the underlying space of the group $Fib_1(T, x)$ is contractible.

Thus, the group $Fib_1(T)$ trivially fibers over $I_1(T)$ with contractible fibers, so that $I_1(T) \subset Fib_1(T)$ is a homotopy equivalence.

THE KLEIN BOTTLE CASE. Verify that $I_1(K) \subset Fib_1(K)$ is a homotopy equivalence. The proof strategy repeats the torus case.

Observe firstly that $Fib_1(K)$ is a Polish group because it is a closed subgroup of the Polish group $Homeo(K)$; see Proposition 1. As in the torus case, $Fib_1(K)$ is closed in $Homeo(K)$ because $Fib_1(K) =$

[1)] In the case of a manifold N locally trivially fibered over the circle with compact fiber M the subgroup $Fib_1(N)$ is closed regardless of the bundle lying in the Birman–Hilden class. We can justify this by using the local contractibility (Chernavskii's Theorem [27]) of the underlying space of the group of self-homeomorphisms of M: since $Homeo(M)$ is locally simply-connected, we conclude that $Fib_1(N)$ is locally path-connected.

[2)] A fiber-preserving self-homeomorphism of a fibered space is called *fiberwise* whenever it takes each fiber to the same fiber.

$Homeo_1(K) \cap Fib(K)$ by the case $n = 2$ of Theorem 1 considered above, while $Homeo_1(K)$ and $Fib(K)$ are closed in $Homeo(K)$ by Assertions 2 and 3.

Furthermore, choose and denote by m some fiber of the bundle p. Denote by $Fib_1(K, m)$ the subgroup of $Fib_1(K)$ formed by the orientation-preserving elements that send m to m. It is not difficult to verify that $I_1(K)$ and $Fib_1(K, m)$ are closed in $Fib_1(K)$, while we can uniquely express each $g \in Fib_1(K)$ as the product $g = ab$ with $a \in I_1(K)$ and $b \in Fib_1(K, m)$.

To verify that $Fib_1(K, m)$ is contractible, by analogy with the torus case fix some section γ of the bundle $K = E \to S^1$ and consider the following subgroups of $Fib_1(K, m)$:

G'_1 is the subgroup of all fiberwise self-homeomorphisms in $Fib_1(K, m)$ which are the identity on γ;

G'_2 is the subgroup of all fiberwise self-homeomorphisms in $Fib_1(K, m)$ whose restriction to each fiber is an isometry;

G'_3 is the intersection of the subgroup of all elements of $Fib_1(K, m)$ carrying γ to γ with the subgroup of all elements of $Fib_1(K, m)$ whose restriction to each fiber is an isometry between fibers.

The underlying space of G'_1 is homeomorphic to the space of fiber-preserving pointwise boundary fixing self-homeomorphisms of the Möbius band fibered over the circle (with segments as fibers). The underlying space of G'_2 is homeomorphic to the space of sections of the open Möbius band fibered over the circle (with fibers homeomorphic to the real line arising here as the universal covering of the circle, which is the fiber of the bundle p). The underlying space of G'_3 is homeomorphic to $Homeo_1([0, 1])$. It is a simple exercise to verify that these spaces are contractible.

The rest of the argument repeats verbatim the torus case with the replacement of $Fib_1(T, x)$ by $Fib_1(K, m)$.

7. Proof of Theorem 1 in the Case $n = 3$

Corollaries 1 and 2 (claim (1)) reduce the situation to the case of bundles with connected fibers. Proceed to inspect the subcases.

7.1. The fiber is a closed surface. If the fiber of the bundle in Theorem 1 is a connected closed surface then the claim follows from Theorem 4. Let us indicate why the hypotheses of the latter hold.

The property is established in [35] that homotopic self-homeomorphisms of a closed surface are isotopic.[3)] The exposition in [35] is in the piecewise linear category, but the result extends to the topological category. Indeed, take two homotopic self-homeomorphisms of a closed surface; since every self-homeomorphism of a surface is isotopic to a piecewise linear one [35, Theorem A4], the problem reduces to the case of homotopic piecewise linear self-homeomorphisms; homotopic piecewise linear self-homeomorphisms are piecewise linearly homotopic, and consequently piecewise linearly isotopic [35, Theorems 6.3 and 6.4].

The property that for every connected closed surface X the inclusion

$$Homeo_1(X) \subset Map_1(X, X)$$

induces an isomorphism of fundamental groups follows from the next proposition.

Proposition 3. *If a connected closed surface F is neither the sphere S^2 nor the projective plane P^2 then the inclusion*

$$Homeo_1(F) \subset Map_1(F, F)$$

is a homotopy equivalence. If $F \in \{S^2, P^2\}$ then the inclusion induces an isomorphism on the level of fundamental groups but is neither a homotopy equivalence nor a weak homotopy equivalence.

Proof. Demonstration amounts to comparing the available results about the underlying spaces of $Homeo_1(F)$ and $Map_1(F, F)$. The necessary results about $Homeo_1(F)$ lie mainly in [46, 47, 50–52],

[3)] For the case of a closed orientable surface of genus above 1 this fact is already justified in [34, 49].

while those about $Map_1(F,F)$, in [53–55], which cover the cases of aspherical surfaces, the sphere, and the projective plane respectively. Below we give a more detailed information for each class.

Recall that, since the spaces $Homeo_1(F)$ and $Map_1(F,F)$ for a compact surface F are separable, metrizable (see Assertion 1 and, for instance, [56] respectively) and are ANR for the class of metric spaces (respectively, see [48] and, for instance, [57, Theorem 2.4, p. 186]), by Theorems 6 and 7 $Homeo_1(F) \subset Map_1(F,F)$ is a homotopy equivalence if and only if it is a weak homotopy equivalence.

The case $\chi(F) < 0$. In the case of a connected closed surface F of negative Euler characteristic $Homeo_1(F)$ and $Map_1(F,F)$ are homotopically trivial, and so $Homeo_1(F) \subset Map_1(F,F)$ is a weak homotopy equivalence. For $Homeo_1(F)$, we refer to [50].[4)] For $Map_1(F,F)$, we use [53, Corollary III.2] stating that for every linearly connected aspherical polyhedron X whose fundamental group has trivial center the underlying space of $Map_1(X,X)$ is contractible.

The torus case. In the case of the torus $T = S^1 \times S^1$ the natural embedding

$$T \subset Homeo_1(T) \subset Map_1(T,T)$$

is supplied by the isometries of the torus endowed with a locally Euclidean metric. As [46] shows, $T \subset Homeo_1(T)$ is a weak homotopy equivalence. A weak homotopy equivalence for $T \subset Map_1(T,T)$ follows from [53, Theorem III.2] which describes the weak homotopy type for aspherical (locally finite linearly connected simplicial) polyhedra.[5)] (For mappings to the Eilenberg–Mac Lane spaces with an abelian group the description of the weak homotopy type was previously available, see [58]; for details, see the surveys [59, Section 2.1; 60, Section 2.1.2].) Consequently, $Homeo_1(T) \subset Map_1(T,T)$ is a weak homotopy equivalence.

The Klein bottle case. In the case of the Klein bottle K we have the natural embedding

$$SO(2) \subset Homeo_1(K) \subset Map_1(K,K),$$

where the embedding $SO(2) \subset Homeo_1(K)$ corresponds to the double rotation about the base in the representation of the Klein bottle as a circle bundle over the circle; see [47, Section 4]. As [47] shows, $SO(2) \subset Homeo_1(K)$ is a weak homotopy equivalence. The construction in the proof of Theorem III.2 of [53] implies that $SO(2) \subset Map_1(K,K)$ is a weak homotopy equivalence. Thus, so is $Homeo_1(K) \subset Map_1(K,K)$.

The sphere case. In the case of S^2 we have the natural embedding

$$SO(3) \subset Homeo_1(S^2) \subset Map_1(S^2,S^2),$$

where by $SO(3) \subset Homeo_1(S^2)$ we understand Euclidean rotations. Kneser showed in [61] that the image of $SO(3) \subset Homeo_1(S^2)$ is a deformation retract for $Homeo_1(S^2)$, while Hansen showed [54, p. 364; 62, p. 44] that $SO(3) \subset Map_1(S^2,S^2)$ induces an isomorphism of fundamental groups without being a homotopy equivalence (and consequently, a weak homotopy equivalence either because we deal with ANR; see Theorems 6 and 7). This implies that $Homeo_1(S^2) \subset Map_1(S^2,S^2)$ also induces an isomorphism of fundamental groups without being a weak homotopy equivalence.

The projective plane case. In the case of the projective plane P^2 we have the natural embedding

$$SO(3) \subset Homeo_1(P^2) \subset Map_1(P^2,P^2).$$

[4)] Thus, in the case of a closed surface of negative Euler characteristic both alternative hypotheses of Theorem 4 hold: $Homeo_1(X) \subset Map_1(X,X)$ induces an isomorphism of fundamental groups and $Homeo_1(X)$ is simply-connected.

[5)] Henceforth in similar cases the mentioned statements, as a rule, deal only with isomorphisms of groups, but it is clear from the constructions that these isomorphisms are induced by the embeddings of spaces in question.

The embedding $SO(3) \subset Homeo_1(P^2)$ is a weak homotopy equivalence; see the proof of Theorem 3.2 and Section 5 of [47]. As [55] shows, $SO(3) \subset Map_1(P^2, P^2)$ is not a homotopy equivalence (and consequently, not a weak homotopy equivalence because we deal with ANR; see Theorems 6 and 7). However, the results of [55] imply (see the explanations with calculations of the fundamental group $\pi_1(Map_1(P^2, P^2))$ in [63, Remark 3.2]) that $SO(3) \subset Map_1(P^2, P^2)$ induces an isomorphism of fundamental groups. Thus, $Homeo_1(P^2) \subset Map_1(P^2, P^2)$ induces an isomorphism of fundamental groups, although it is not a weak homotopy equivalence. $\square$

7.2. The fiber is a surface with boundary. This case of Theorem 1 follows from Theorem 5. Let us indicate why the hypotheses of the latter hold in the former in the case that the fiber is a connected compact surface with nonempty boundary.

The property that all self-homeomorphisms of a surface with boundary related by a pointwise boundary fixing homotopy are also related by a pointwise boundary fixing isotopy is proved in [35, Theorems 6.3 and 6.4].[7)]

In the case that X is a connected compact surface with nonempty boundary, [50–52, 46, 47] show that the underlying space of $Homeo_1(X; [\partial X])$ is simply-connected and even contractible; the case of the disk follows already from the results of Alexander [44].

The assertion that the restriction of the bundle p to each connected component of the boundary ∂E has the Birman–Hilden property and the epimorphism property follows from the case $n = 2$ of Theorems 1 and 3 respectively.

8. Proof of Theorem 2

Corollaries 1 and 2 (claim (1)) reduce the general case of Theorem 2 to the case of bundles with connected fibers. In the case of connected fibers Theorem 2 follows from Theorem 4 in view of the results of [64, 65]. In particular, [64] shows that homotopic self-homeomorphisms of every orientable closed Haken manifold M are isotopic, which is the first condition on the fiber in Theorem 4, while [65] shows that $Homeo_1(M) \subset Map_1(M, M)$ induces an isomorphism of fundamental groups, which is the second condition on the fiber in Theorem 4; see also [66], where a stronger result about homotopy equivalence is established. These articles use the piecewise linear and smooth categories [66, p. 343], and transition between them and the topological category is guaranteed by the triangulation theorems of Bing and Moise and the Smale conjecture, proved by Hatcher in [67].

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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[7)] The statements of Theorems 6.3 and 6.4 in [35] refer to homotopies and isotopies without restrictions on the boundary, but in their proofs presented in [35] the situation reduces precisely to the pointwise boundary fixing cases, and the required property is established.

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