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HARNACK'S INEQUALITY FOR HARMONIC FUNCTIONS ON STRATIFIED SETS

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Abstract: We prove Harnack's inequality for nonnegative harmonic functions in the sense of the "soft" Laplacian on a stratified set with flat strata.

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In this article we establish an analog of Harnack's inequality in its standard form

$$\sup_K u \le C \inf_K u$$

in the following situation: u is a nonnegative harmonic function in the sense of the so-called "soft" Laplacian on a stratified set $\Omega = \Omega^\circ \cup \partial\Omega^\circ$ with flat strata, K is a compact subset of Ω°, while $C = C(\Omega^\circ, K)$ is a constant. The Mean Value Theorem for this Laplacian holds only for special admissible spheres, making the proof of Harnack's inequality rather complicated in comparison with the classical analog.

1. The Main Concepts

1.1. Stratified sets. A *stratified set* Ω in a first approximation is a connected subset of the Euclidean space $\mathbb{R}^n$ which consists of finitely many disjoint connected (boundaryless) submanifolds called the *strata* of Ω. Denote by Σ the set of all strata of Ω, while the strata themselves by σ_{kj}; i.e.,

$$\Omega = \bigcup_{\sigma_{kj}\in\Sigma} \sigma_{kj}.$$

The first index shows the dimension of a stratum; and the second, the index of the stratum of this dimension; k ranges from 0 to n. We impose a few requirements on the contact between the strata:

- the closure $\overline{\sigma}_{kj}$ of each stratum is compact, while its boundary $\partial\sigma_{kj} = \overline{\sigma}_{kj} \setminus \sigma_{kj}$ is the union of some strata in Σ;
- given two strata $\sigma_{kj}, \sigma_{mi} \in \Sigma$, the intersection of the closure $\overline{\sigma}_{kj} \cap \overline{\sigma}_{mi}$ is either empty or consists of some strata in Σ.

Henceforth the relation $\sigma_{kj} \prec \sigma_{mi}$ means $\sigma_{kj} \subset \partial\sigma_{mi}$. In this case we say that the strata are *contiguous*.

We will use the *intrinsic metric* d on Ω, defining $d(X, Y)$ in the usual way as the infimum of the lengths of curves passing through $X, Y \in \Omega$ and lying in Ω. It is easy to see that the metric topology coincides with the induced topology, and all topological concepts below relate to this topology.

Assume that Ω is represented as $\Omega^\circ \cup \partial\Omega^\circ$ (the "*interior*" and the "*boundary*" of Ω), where Ω° is a connected open subset of Ω in the topology mentioned above which consists of some strata in Σ and satisfies $\overline{\Omega^\circ} = \Omega$, while the remaining part $\partial\Omega^\circ = \Omega \setminus \Omega^\circ$ is then the usual topological boundary of Ω°.

In this article *all interior strata are assumed flat* in the following sense: Each stratum $\sigma_{kj} \subset \Omega^\circ$ is a subregion of some k-dimensional affine subspace of $\mathbb{R}^n$.

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1.2. Stratified measure. Refer to a set $\omega \subset \Omega$ as *μ-measurable* whenever every intersection $\sigma_{kj} \cap \omega$ is measurable in the sense of the k-dimensional Lebesgue measure on σ_{kj}. It is not difficult to observe that the set $\mathscr{M}$ of all μ-measurable sets is a σ-algebra on Ω. Define the *stratified measure* μ on Ω, or more exactly on $\mathscr{M}$, as

$$\mu(\omega) = \sum_{\sigma_{kj} \in \Sigma} \mu_k(\omega_{kj}),$$

where $\mu_k(\omega_{kj})$ stands for the k-dimensional Lebesgue measure on $\omega_{kj} = \omega \cap \sigma_{kj}$. The measurability of a function $f : \Omega \to \mathbb{R}$ is defined in the standard fashion: f is μ-measurable whenever the Lebesgue sublevel set $L_f(c) = \{X \in \Omega : f(X) \leq c\}$ belongs to $\mathscr{M}$ for all $c \in \mathbb{R}$. The Lebesgue integral of a μ-measurable function over a μ-measurable set ω reduces to the sum

$$\int_{\omega} f \, d\mu = \sum_{\sigma_{kj} \in \Sigma} \int_{\omega_{kj}} f \, d\mu.$$

Note that $d\mu$ is essentially the same as $d\mu_k$; it is convenient to omit k unless this leads to confusion.

The definitions of a stratified set and related concepts in a more general situation can be found in [1, 2]. These definitions are significantly inspired by [3].

1.3. Divergence and Laplacian. A vector field $\vec{F}$ in $\mathbb{R}^n$ is *tangent* to Ω° whenever, given a stratum $\sigma_{kj} \subset \Omega^\circ$ and $X \in \sigma_{kj}$, the vector $\vec{F}(X)$ lies in the tangent space $T_X\sigma_{kj}$ in the usual sense of differential geometry.

The notation $\vec{C}^1(\Omega^\circ)$ is applied to the space of tangent vector fields F on Ω° such that the restriction $\vec{F}|_{\sigma_{ki}}$ to each interior stratum $\sigma_{ki} \subset \Omega^\circ$ is continuously differentiable and has continuous extensions to all points of each contiguous interior stratum of dimension less by 1. We do not assume that $\vec{F}$ is continuous on the whole Ω°. The latter means that the tangent vector field in $\vec{C}^1(\Omega^\circ)$ amounts to a collection of independent fields of class C^1, one for each stratum of Ω°.

The formal *divergence* of a tangent vector field $\vec{F} \in \vec{C}^1(\Omega^\circ)$ at $X \in \sigma_{kj} \subset \Omega^\circ$ is defined as

$$\nabla \cdot \vec{F}(X) = \nabla_k \cdot \vec{F}(X) + \sum_{\sigma_{k+1i} \succ \sigma_{kj}} \vec{F}(X + 0 \cdot \vec{\nu}_i) \cdot \vec{\nu}_i, \tag{1}$$

where the summation is over all $(k+1)$-dimensional strata σ_{k+1i} contiguous with σ_{kj}. Let us elucidate the notation of (1). Note that ∇_k on the right-hand side of (1) stands for the ordinary k-dimensional divergence operator applied to the restriction $\vec{F}|_{kj}$ to σ_{kj}, the vector $\vec{\nu}_i$ is the unit interior normal to σ_{kj} in σ_{k+1j} at X, while $\vec{F}(X + 0 \cdot \vec{\nu}_i)$ is the limit of $\vec{F}(Y)$ as $Y \in \sigma_{k+1i}$ tends to X inside the stratum $\sigma_{k+1i} \succ \sigma_{kj}$ in the direction of ν_i. Fig. 1 illustrates that.

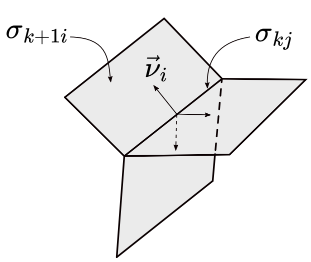

Fig. 1. The unit normal.

It is not clear immediately that the divergence we defined is a genuine analog of the classical one, but we can show that $\nabla \cdot \vec{F}(X)$, as in the usual setup, is the flow density of the vector field at X relative to the stratified measure μ as defined in the previous subsection. For more detail, see [4] as well as [5].

The gradient ∇u of a scalar function u whose restriction to each interior stratum is continuously differentiable is an example of a tangent vector field. In this case ∇u amounts to the collection of gradients of these restrictions. For $\nabla u \in \vec{C}^1(\Omega^\circ)$ it is perfectly natural to define the Laplacian on a stratified set as $\Delta u = \nabla \cdot (\nabla u)$. This Laplacian is often called *hard*. Presently the qualitative theory of harmonic functions in the sense of this Laplacian is poorly developed. In particular, so far no prospect can be discerned of obtaining Harnack's inequality for them. Thus, in this article we restrict exposition to the case of the so-called *soft* Laplacian.

A stratum σ_{kj} is *free* whenever σ_{kj} is not contiguous with any stratum of greater dimension; in this case σ_{kj} is obviously interior; i.e., $\sigma_{kj} \subset \Omega^\circ$. The *soft Laplacian* is defined as

$$\widetilde{\Delta} u = \nabla \cdot (p \nabla u), \tag{2}$$

where $p = 1$ on the free strata, while p vanishes on the remaining strata.

The explicit expression for the soft Laplacian at the points of free strata coincides with the usual Laplacian (and for nonflat strata, with the Laplace–Beltrami operator).

However, if a stratum σ_{kj} is not free but there exist free strata $\sigma_{k+1i} \succ \sigma_{kj}$, then we call σ_{kj} *semifree*, and the expression for the soft Laplacian at $X \in \sigma_{kj}$ is

$$\widetilde{\Delta} u(X) = \sum_{\sigma_{k+1i} \succ \sigma_{kj}} \nabla u(X + 0 \cdot \vec{\nu}_i) \cdot \vec{\nu}_i,$$

with the summation over all free strata $\sigma_{k+1i} \succ \sigma_{kj}$.

Finally, if a stratum is not free and does not contiguous with any free stratum of dimension greater by 1 then $\widetilde{\Delta} u = 0$ on this stratum.

Since ∇u appears in (2) with nonzero coefficients only on the free strata, it is natural to consider solutions to this equation in the function class $\widetilde{C}^2(\Omega^\circ)$ consisting of the continuous functions u on Ω° whose restriction to each free stratum is twice continuously differentiable, while the gradient of this restriction extends continuously to each point of the adjoint interior stratum of dimension one less.

By a *harmonic function* u henceforth we will understand a function $u \in \widetilde{C}^2(\Omega^\circ)$ satisfying $\widetilde{\Delta} u = 0$.

2. Harnack's Inequality

The following theorem is the main result of this article:

Theorem 1 (Harnack's inequality). *Given a stratified set Ω and a compact subset K of Ω°, for each nonnegative harmonic function u on Ω° we have*

$$\sup_{X \in K} u(X) \le C \inf_{X \in K} u(X)$$

with some constant $C = C(K, \Omega^\circ)$ independent of u.

We deduce this theorem as a corollary of the corresponding result for sturdy stratified sets. A stratified set Ω is *sturdy* (of dimension d) whenever

- all free strata in Ω° are of the same dimension d;
- $\Omega^\circ \setminus \Omega^\circ_{d-2}$ is connected, where Ω°_{d-2} is the union of all strata $\sigma_{kj} \subset \Omega^\circ$ with $k \le d-2$.

Fig. 2 illustrates this concept.

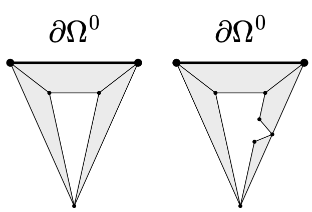

Fig. 2. Stratified sets: (a) sturdy; (b) not sturdy.

Lemma 1 (Harnack's inequality for sturdy sets). *The claim of Theorem* 1 *holds for every sturdy stratified set Ω.*

Deduction of Theorem 1 from Lemma 1. Call two free strata σ_{ki} and σ_{kj} of the same dimension *adjacent* whenever there exists a chain of free strata of the same dimension, $\sigma_{ki_1}, \sigma_{ki_2}, \dots, \sigma_{ki_s}$ with $i_1 = i$ and $i_s = j$, in which the boundaries of each pair of neighboring strata contain a common interior stratum, i.e., a stratum in Ω° of dimension one less. The whole set of free strata splits into the disjoint union of classes of adjacent strata. For each of such classes $\sigma_{ki_1}, \dots, \sigma_{ki_m}$ of adjacent strata consider the stratified set

$$\mathscr{P} = \overline{\sigma}_{ki_1} \cup \dots \cup \overline{\sigma}_{ki_m}.$$

Observe that $\mathscr{P}$ inherits the natural partition $\mathscr{P} = \mathscr{P}^\circ \cup \partial\mathscr{P}^\circ$; the boundary strata which belonging to $\mathscr{P}$ are declared boundary strata in $\mathscr{P}$. It is not difficult to verify that $\mathscr{P}$ is a sturdy stratified set and the restriction $u|_{\mathscr{P}^\circ}$ of an arbitrary harmonic function u on Ω° is harmonic on $\mathscr{P}^\circ$.

This procedure leads to the representation of Ω as a union of sturdy stratified sets:

$$\Omega = \mathscr{P}_1 \cup \cdots \cup \mathscr{P}_l;$$

furthermore, $\Omega^\circ = \mathscr{P}_1^\circ \cup \cdots \cup \mathscr{P}_l^\circ$ and $\partial\Omega^\circ = \partial\mathscr{P}_1^\circ \cup \cdots \cup \mathscr{P}_l^\circ$.

By Lemma 1, the claim of Theorem 1 holds for the harmonic functions on each sturdy stratified set $\mathscr{P}_i$. From this, it is not difficult to deduce the claim of Theorem 1. □

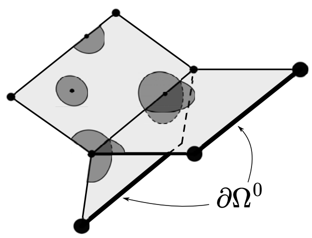

Fig. 3. Admissible balls.

In turn, Harnack's inequality for sturdy sets follows from its validity for admissible balls in the sets. We omit the standard proof; see [6] for instance.

Call the ball $B_r(X_0) = \{X \in \Omega : d(X, X_0) < r\}$ *admissible*, or in more detail, an open ball of *admissible radius* $r > 0$ centered at $X_0 \in \Omega^\circ$, whenever r is at most the distance from X_0 to all strata whose closures avoid X_0. Fig. 3 depicts some examples of admissible balls. If the radius of a ball is not essential then we use the simpler notation $B(X_0)$.

Lemma 2 (Harnack's inequality for admissible balls). *Given a sturdy stratified set Ω, if $B_{4r}(X_0)$ is an admissible ball in Ω° then there exists a constant $C > 0$ such that*

$$\frac{1}{C} u(X) \le u(X_0) \le C u(X) \tag{3}$$

for all $X \in B_r(X_0)$ and nonnegative harmonic functions u on Ω°.

Lemma 2 is equivalent to the following:

Lemma 3. *Given a sturdy stratified set Ω, if $B_{4r}(X_0)$ is an admissible ball in Ω°, while $\{u_i\}$ for $i = 1, 2, \dots$ is a sequence of nonnegative harmonic functions on Ω° then*

(i) *if $\{X_i\}$ is a sequence in $B_r(X_0)$ such that $\{u_i(X_i)\}$ is bounded below by a positive constant then so is some subsequence of $\{u_i(X_0)\}$;*

(ii) *if the sequence $\{u_i\}$ is uniformly bounded on $B_r(X_0)$, while $\{X_i\}$ is a sequence in $B_r(X_0)$ such that $u_i(X_i) \to 0$ as $i \to \infty$, then the same holds for some subsequence of $\{u_i(X_0)\}$.*

Proof of the equivalence. Assume that Lemma 2 is valid. Then (3) easily implies claims (i) and (ii) even without extracting subsequences and without the condition of uniform boundedness on a ball.

Conversely, assuming that Lemma 3 is valid and arguing by contradiction, find a sequence of nonnegative harmonic functions u_i on Ω° and a sequence of $X_i \in B_r(X_0)$ satisfying either

$$u_i(X_i) > i u_i(X_0) \tag{4}$$

or

$$u_i(X_0) > i u_i(X_i). \tag{5}$$

At least one of these inequalities must hold infinitely many times. If (4) holds for some subsequence $\{i_j\}$ for $j = 1, 2, \dots$; then for the sequence $u_j^*(X) = u_{i_j}(X)/u_{i_j}(X_{i_j})$ we obtain

$$u_j^*(X_{i_j}) = 1, \quad u_j^*(X_0) \to 0 \quad \text{as } j \to \infty$$

in contradiction with claim (i). However, if (5) holds for some subsequence $\{i_j\}$ for $j = 1, 2, \dots$; then for the sequence $u_j^*(X) = u_{i_j}(X)/u_{i_j}(X_0)$ we have

$$u_j^*(X_0) = 1, \quad u_j^*(X_{i_j}) \to 0 \quad \text{as } j \to \infty.$$

Observe that claim (i) is equivalent to

(i′) if $\{u_i(X_0)\}$ is bounded then some subsequence of $\{u_i\}$ is uniformly bounded on $B_r(X_0)$.

This is easy to prove, arguing by contradiction.

In this case (i′) implies the uniform boundedness of $\{u_j^*\}$ on $B_r(X_0)$, and we arrive at a contradiction with claim (ii) applied to u_j^*. □

Summing up the above, we see that Lemma 3 will yield our main result.

3. Some Auxiliary Tools

3.1. The Mean Value Theorem. For the harmonic functions corresponding to the soft Laplacian the analog of the classical Mean Value Theorem looks as follows; see [7].

Theorem 2. *Given a stratified set Ω all of whose free strata are of the same dimension, if u is a harmonic function on Ω° and $B_r(X_0)$ is an admissible ball in Ω° then*

$$u(X_0) = \frac{1}{\mu_p(B_r(X_0))} \int\limits_{B_r(X_0)} pu(X)\,d\mu,$$

where $\mu_p(B_r(X_0)) = \int_{B_r(X_0)} p\,d\mu$.

Note the following useful corollary of the Mean Value Theorem which we will use a few times.

Remark 1. If $B_r(X_0)$ is an admissible ball and $B_\rho(X) \subset B_r(X_0)$ is another one, while u is a nonnegative harmonic function on $B_r(X_0)$; then $u(X_0) \geq Cu(X)$, where $C = \mu_p(B_\rho(X))/\mu_p(B_r(X_0))$.

3.2. Harnack's inequality for special compact sets. When the center X_0 of an admissible ball $B_r(X_0)$ lies on a free stratum σ_{kj} or on a stratum σ_{kj} adjoined by only free strata $\sigma_{k+1i} \succ \sigma_{kj}$ of dimension greater by 1, we can obtain an analog of the "spherical" Harnack's inequality. It is obvious in the first case, when σ_{kj} is a free stratum, because then $\widetilde{\Delta}$ is the standard Laplacian. As for the second case (recall that σ_{kj} is then called *semifree*), we can assert the following

Lemma 4. *Given a stratified set Ω, if $\sigma_{kj} \subset \Omega^\circ$ is a semifree stratum and $X_0 \in \sigma_{kj}$ then for every admissible ball $B_r(X_0)$ and an arbitrary ρ with $0 < \rho < r$ we have*

$$\frac{(r-\rho)r^{k-1}}{(r+\rho)^k} u(X) \leq u(X_0) \leq \frac{(r+\rho)r^{k-1}}{(r-\rho)^k} u(Y)$$

for all $X, Y \in B_\rho(X_0)$ and all nonnegative harmonic functions u on Ω°.

This property is a corollary of the Poisson formula in [4] for the ball in question presented a proof based on the formula appeared in [8]. The latter article deals with a slightly more general situation: the stratified constant p can take arbitrary positive values on the free strata, though still $p = 0$ on the remaining strata.

As we can see, this "spherical" version of the inequality looks precisely as the classical one (see [6]) if we assume that the dimension of $B_r(X_0)$ equals $k+1$, meaning the dimension of the free strata σ_{k+1i} contiguous with σ_{kj}.

The validity of the spherical Harnack's inequality for the admissible balls centered on d- and $(d-1)$-dimensional strata enables us to prove Harnack's inequality for special compact sets in the case of a sturdy stratified set Ω; cf. [9].

Lemma 5. *Given a sturdy stratified set $\Omega = \Omega^\circ \cup \partial\Omega^\circ$, if $K \subset \Omega^\circ$ is a compact set containing only some points in the $(d-1)$-dimensional and d-dimensional strata then each nonnegative harmonic function u on Ω° satisfies*

$$\sup_K u \leq C \inf_K u$$

with some constant $C = C(K, \Omega^\circ)$ independent of u.

4. Proof of Lemma 3

In this section we prove Lemma 3 and, as we mentioned at the end of Section 2, thereby complete the proof of our main result, Theorem 1.

In accordance with the hypotheses of Lemma 3, henceforth we assume that the stratified set Ω is sturdy of dimension d.

Lemma 6. *If $B_{3r}(X_0)$ is an admissible ball and $X_0 \in \sigma_{kj}$ then there exists a constant $C > 0$ such that*
$$\frac{1}{C}u(X) \le u(X_0) \le Cu(X)$$
for all $X \in B_r(X_0) \cap \sigma_{kj}$ and all nonnegative harmonic functions u on Ω°.

We obtain one side of the inequality by applying Remark 1 to the pair $B_r(X) \subset B_{2r}(X_0)$ of admissible balls, and the other, to the pair $B_r(X_0) \subset B_{2r}(X)$ of admissible balls.

By analogy with the equivalence of Lemmas 2 and 3, Lemma 6 is equivalent to the following lemma which differs from Lemma 3 by the additional condition on the points X_i and has no effect on the proof of equivalence.

Lemma 7. *Consider an admissible ball $B_{3r}(X_0)$ with $X_0 \in \sigma_{kj}$, a sequence $\{u_i\}$ for $i = 1, 2, \dots$ of nonnegative harmonic functions on Ω°, and a sequence of points $\{X_i\}$ in $B_r(X_0) \cap \sigma_{kj}$. Then*

(i) *if $\{u_i(X_i)\}$ is bounded below by some constant then so is some subsequence of $\{u_i(X_0)\}$;*

(ii) *if $\{u_i\}$ is uniformly bounded on $B_r(X_0)$, while $u_i(X_i) \to 0$ as $i \to \infty$; then the same holds for some subsequence of $\{u_i(X_0)\}$.*

The next lemma differs substantially from Lemma 3 by the additional requirement that all points X_i coincide.

Lemma 8. *Consider an admissible ball $B_{2r}(X_0)$, a sequence $\{u_i\}$ of nonnegative harmonic functions on Ω°, and $X \in B_r(X_0)$. Then*

(i) *if $\{u_i(X)\}$ is bounded below by a positive constant then so is $\{u_i(X_0)\}$;*

(ii) *if $\{u_i\}$ is uniformly bounded on the ball $B_r(X_0)$, while $u_i(X) \to 0$ as $i \to \infty$; then $u_i(X_0) \to 0$ as $i \to \infty$.*

Proof. Take some admissible ball $B(X) \subset B_r(X_0)$.

Claim (i) trivially follows from Remark 1. Furthermore, there is no need to extract a subsequence.

Let us justify claim (ii). The sturdyness of Ω ensures that $B(X) \cap \sigma_{dl}$ is nonempty for some free stratum σ_{dl}. Fix some $Y \in B(X) \cap \sigma_{dl}$ (see Fig. 4) and an admissible ball $B(Y) \subset B(X)$. Remark 1 applied to X and Y leads to the inequality $u_i(Y) \le Cu_i(X)$ with some constant C depending only on the ratio of radii. Then $u_i(Y) \to 0$ as $i \to \infty$.

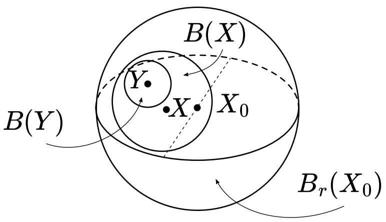

Fig. 4.

Fix an arbitrary $\varepsilon > 0$. Take an open subset G_ε of $B_r(X_0)$ containing all points of $B_r(X_0)$ lying on the strata of dimension at most $d-2$, avoiding Y, and satisfying
$$\int_{G_\varepsilon} p\,d\mu < \frac{\varepsilon\mu_p(B_r(X_0))}{2M},$$
where $M = \sup\{u_i(Z) : Z \in B_r(X_0),\ i \in \mathbb{N}\}$. Such set obviously exists, because the measure $p\,d\mu$ in this case is simply the d-dimensional Lebesgue measure. After that, define K as $\overline{B_r(X_0)} \setminus G_\varepsilon$.

Since the compact set K consists of points in some strata of dimension $d-1$ and d, we can apply Lemma 5 and obtain
$$\sup_{Z\in K} u_i(Z) \le C \inf_{Z\in K} u_i(Z) \le Cu_i(Y) \to 0 \quad (\text{as } i \to \infty).$$
As a corollary, $\sup_{Z\in K} u_i(Z)$ is at most $\frac{\varepsilon}{2}$ for i sufficiently large. Finally, appreciating the Mean Value Theorem (Theorem 2), we infer that
$$u_i(X_0) = \frac{1}{\mu_p(B_r(X_0))}\int_{B_r(X_0)} pu_i(Z)\,d\mu = \frac{1}{\mu_p(B_r(X_0))}\left(\int_K pu_i(Z)\,d\mu + \int_{G_\varepsilon} pu_i(Z)\,d\mu\right)$$
$$\le \frac{\varepsilon}{2\mu_p(B_r(X_0))}\int_K p\,d\mu + \frac{M}{\mu_p(B_r(X_0))}\int_{G_\varepsilon} p\,d\mu \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary, we conclude that $u_i(X_0) \to 0$ as $i \to \infty$. $\square$

The following lemma in fact establishes the validity of Lemma 3 under the additional assumption that $X_i \to X_0$.

Lemma 9. *Consider an admissible ball $B(X_0)$, a sequence $\{u_i\}$ of nonnegative harmonic functions on $B(X_0)$, and a sequence $\{X_i\}$ of points in $B(X_0)$ converging to X_0. Then*

(i) *if $\{u_i(X_i)\}$ is bounded below by a positive constant then so is some subsequence of $\{u_i(X_0)\}$;*

(ii) *if $\{u_i\}$ is uniformly bounded on $B(X_0)$, while $u_i(X_i) \to 0$ as $i \to \infty$, then the same holds for some subsequence of $\{u_i(X_0)\}$.*

Proof. Induct on the decreasing dimension of the stratum containing the center of $B(X_0)$. The maximal possible dimension is d, precisely the dimension of the free strata. If X_0 lies in some free stratum σ_{dl} then all terms of the sequence $\{X_i\}$ will eventually lie in σ_{dl}. Then the conclusion of the lemma follows from Lemma 7. We do not even have to extract a subsequence.

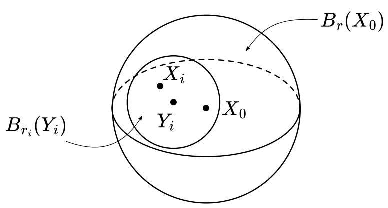

Fig. 5.

Assume now that all claims are valid in the case that the center X_0 lies in a stratum of dimension greater than k. We must show that all these claims are also valid in the case that X_0 lies in a stratum σ_{kj} of dimension k. We can suppose that infinitely many entries of $\{X_i\}$ lie outside σ_{kj}; otherwise we are under the hypotheses Lemma 7 for some subsequence of $\{X_i\}$, and therefore we can assert that the claim of Lemma 9 is valid. Consequently, we can pass to the case that infinitely many entries of the sequence lie outside σ_{kj}. Assume for simplicity that all entries of $\{X_i\}$ lie outside σ_{kj}.

Denote by L the k-dimensional affine plane in $\mathbb{R}^n$ such that $L \supset \sigma_{kj}$. The orthogonal projection Y_i of X_i onto L lies in the stratum $\sigma_{kj} \cap B(X_0)$ because all X_i lie in $B(X_0)$ which projects entirely inside σ_{kj}. Take the ball $B_{r_i}(Y_i)$ with $r_i = 3d(Y_i, X_i)$. Since $X_i \to X_0$; therefore, $Y_i \to X_0$ and so $r_i \to 0$ as $i \to \infty$. Fig. 5 illustrates our geometric construction.

Each ball $B_{r_i}(Y_i)$ can be transformed into $B(X_0)$ by the homothety

$$X = X_0 + \frac{r}{r_i}\,\overrightarrow{Y_iY} \quad (Y \in B_{r_i}(Y_i)), \quad \text{where } r \text{ is the radius of } B(X_0).$$

This transformation preserves harmonicity for the new functions $u_i^\star(X) = u_i(Y)$.

Denote by $X_i^\star$ the image of X_i under this transformation. Then we have

$$u_i(Y_i) = u_i^\star(X_0), \quad u_i(X_i) = u_i^\star(X_i^\star). \tag{6}$$

It is not difficult to observe that the images $X_i^\star$ of X_i cannot lie in σ_{kj}. We can extract some subsequence of $\{X_i^\star\}$ converging to some $X_0^\star$ which cannot lie in σ_{kj} because $d(X_i^\star, X_0) = (r/r_i)d(X_i, Y_i) = r/3$. But then all these points and their partial limits lie in a stratum of dimension greater than k because $B(X_0)$ is an admissible ball. All assumptions of the lemma hold if we take this subsequence of $\{X_i^\star\}$ and the corresponding functions $u_i^\star$ on some admissible ball $B(X_0^\star)$.

Without loss of generality assume that $B(X_0^\star)$, the sequence $\{u_i^\star\}$ of functions, and the sequence $\{X_i^\star\}$ satisfy the hypotheses of the lemma. Thus, resting on the base of induction and the inductive assumption, we can conclude that

(i*) if $\{u_i^\star(X_i^\star)\}$ is bounded below by a positive constant then so is some subsequence of $\{u_i^\star(X_0^\star)\}$;

(ii*) if $\{u_i^\star\}$ is uniformly bounded on the ball $B(X_0^\star)$, while $u_i^\star(X_i^\star) \to 0$ as $i \to \infty$, then the same holds for some subsequence of $\{u_i^\star(X_0^\star)\}$.

Lemma 8 applied to $B(X_0)$, the sequence $\{u_i^\star\}$, and $X_0^\star$ as X implies that

(i**) if $\{u_i^\star(X_0^\star)\}$ is bounded below by some positive constant then so is some subsequence of $\{u_i^\star(X_0)\}$;

(ii**) if $\{u_i^\star\}$ is uniformly bounded on $B(X_0)$, while $u_i^\star(X_0^\star) \to 0$ as $i \to \infty$, then the same holds for some subsequence of $\{u_i^\star(X_0)\}$.

Combining (i*) and (ii*) with (i**) and (ii**), as well as taking (6) into account, we infer that

(i$'$) if $\{u_i(X_i)\}$ is bounded below by some positive constant then so is some subsequence of $\{u_i(Y_i)\}$;

(ii$''$) if $\{u_i\}$ is uniformly bounded on $B(X_0)$, while $u_i(X_i) \to 0$ as $i \to \infty$, then the same holds for some subsequence of $\{u_i(Y_i)\}$.

By construction, X_0 and all points Y_i lie in one stratum, and furthermore, $Y_i \to X_0$. Applying Lemma 7 to $B(X_0)$, the sequence $\{u_i\}$, and the sequence $\{Y_i\}$, while using (i$'$) and (i$''$), we justify claims (i) and (ii), and thus fulfill the step of induction and the proof of Lemma 9. $\square$

Let us finish the proof of Lemma 3, thus completing the proof of our main Theorem 1.

PROOF OF LEMMA 3. Without loss of generality we may assume that $\{X_i\}$ converges to some $X^\star \in \overline{B_r(X_0)}$. Then we can take some admissible ball $B(X^\star)$ and, applying Lemma 2, conclude that some subsequence of $\{u_i(X^\star)\}$ is bounded below by a positive constant provided that the same holds for some subsequence of $\{u_i(X_i)\}$. Then we can apply Lemma 8 and assert the same about some subsequence of $\{u_i(X_0)\}$, which proves claim (i). Claim (ii) is justified similarly. $\square$

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