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ON SOME PROPERTIES OF SEMI-HAMILTONIAN SYSTEMS ARISING IN THE PROBLEM OF INTEGRABLE GEODESIC FLOWS ON THE TWO-DIMENSIONAL TORUS

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Abstract: Bialy and Mironov demonstrated in a recent series of works that the search for polynomial first integrals of a geodesic flow on the 2-torus reduces to the search for solutions to a system of quasilinear equations which is semi-Hamiltonian. We study the various properties of this system.

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1. Introduction and Statement of the Problem

Consider a two-dimensional surface M with coordinates q^1 and q^2 and a Riemannian metric $ds^2 = g_{ij}(q)\,dq^i dq^j$. The geodesic flow of this metric on M is *completely integrable* if the Hamiltonian system

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \quad H = \frac{1}{2} g^{ij}(q) p_i p_j, \quad i, j = 1, 2,$$

admits an additional *first integral*, i.e., a function $F : T^*M \to \mathbb{R}$ such that

$$\dot{F} = \{F, H\} = \sum_{j=1}^{2} \left(\frac{\partial F}{\partial q^j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j} \frac{\partial F}{\partial p_j} \right) = 0$$

and F is functionally independent with H a.e.

The search for Riemannian metrics on two-dimensional surfaces with an integrable geodesic flow is a classical problem of differential geometry. A survey of the known results and numerous references to various papers on this topic can be found in [1].

On the 2-torus, which will be the only one of our interest, two types of metrics are known to exist with an integrable geodesic flow. In the isothermal coordinates, these metrics and the additional integrals look as follows:

(1) $ds^2 = f(x)(dx^2 + dy^2)$, $F_1 = p_2$;

(2) $ds^2 = (f(x) + g(y))(dx^2 + dy^2)$, $F_2 = \frac{g(y)p_1^2 - f(x)p_2^2}{f(x)+g(y)}$.

In the first example, the coordinate y is cyclic and hence there is a first integral linear in momenta; in the second example (the Liouville metric), there is a quadratic integral. The question of the existence of other metrics on the 2-torus with an integrable geodesic flow in the class of analytic functions is still open in the general case though it is actively studied (see, for example, [2–4] and the references therein). We additionally mention the series [5–7] (see also [8]), where it was shown that the search for an additional polynomial integral in this problem reduces to the search for solutions to a certain quasilinear system with a number of remarkable properties. In particular, the following theorems were proved:

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Theorem 1 [5]. *Suppose that the geodesic flow of the Riemannian metric on the* 2*-torus admits a homogeneous integral* F *of degree* n *polynomial in momenta. Then there exist global semigeodesic coordinates* (t, x) *on the covering plane in which the metric has the form*

$$ds^2 = g^2(t, x)\, dt^2 + dx^2$$

and the integral F *looks as*

$$F = \sum_{k=0}^{n} \frac{a_k(t, x)}{g^{n-k}} p_1^{n-k} p_2^k,$$

where $a_{n-1} \equiv g$ *and* $a_n \equiv 1$*. Then the relation* $\{F, H\} = 0$ *is equivalent to the system of quasilinear differential equations on the functions* $a_0, \dots, a_{n-1}$ *of the form*

$$u_t^i + v_j^i(u) u_x^j = 0, \tag{1.1}$$

where $u^i = (a_0, \dots, a_{n-1})^T$ *and the matrix* v_j^i *has the form*

$$v_j^i = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \cdots & 0 & 0 & 2a_2 - na_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \cdots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}. \tag{1.2}$$

The functions a_i *and* g *are periodic in* x *and quasiperiodic in* t*.*

Theorem 2 [5]. *System* (1.1) *is semi-Hamiltonian; i.e.,*

(1) *in the hyperbolic domain (i.e., where all eigenvalues of* v_j^i *are real and distinct), there is a change of variables (Riemann invariants)* $(a_0, \dots, a_{n-1}) \to (r_1, \dots, r_n)$ *that transforms system* (1.1) *to the diagonal form*

$$(r_i)_t + \lambda_i(r_1, \dots, r_n)(r_i)_x = 0, \quad i = 1, \dots, n;$$

(2) *there is a nondegenerate change of variables*

$$(a_0, \dots, a_{n-1}) \to (G_1, \dots, G_n)$$

such that system (1.1) *is written down in the form of conservation laws:*

$$(G_i(a_0, \dots, a_{n-1}))_t + (H_i(a_0, \dots, a_{n-1}))_x = 0, \quad i = 1, \dots, n.$$

The present article studies the various properties of system (1.1). In Section 2, we recall how the generalized godograph method works. In Section 3, we rewrite system (1.1) for $n = 2$ in Riemann invariants and demonstrate for completeness how to construct its general solution by the generalized godograph method. We also study the partial solution to (1.1) for $n = 4$ which was constructed in [9] and prove that the geodesic flow of the constructed metric does not admit polynomial integrals of degree 1 or 2. In Section 4, basing on [10], we prove that system (1.1) is not weakly nonlinear for any $n > 2$. Finally, in Section 5 we study the *symmetries* of system (1.1) for $n = 2$.

2. Semi-Hamiltonian Systems and the Generalized Godograph Method

The diagonal system of quasilinear equations

$$r_t^i = v_i(r) r_x^i, \quad i = 1, \dots, n, \quad v_i \neq v_j, \tag{2.1}$$

is *semi-Hamiltonian* [11] if it satisfies the relations

$$\partial_i \left(\frac{\partial_j v_k}{v_j - v_k} \right) = \partial_j \left(\frac{\partial_i v_k}{v_i - v_k} \right), \quad i \neq j \neq k.$$

Note that if a generally nondiagonal system of quasilinear equations has Riemann invariants and also can be written in the form of conservation laws (see Theorem 2 above) then it is automatically semi-Hamiltonian (see [12]).

Semi-Hamiltonian systems (2.1) admit infinitely many *symmetries*, i.e., flows of the form $r^i_\tau = w_i(r)r^i_x$, $i = 1, \dots, n$, commuting with (2.1), where w_i and v_i satisfy the relations

$$\frac{\partial_k v_i}{v_k - v_i} = \frac{\partial_k w_i}{w_k - w_i}, \quad i \neq k. \tag{2.2}$$

Suppose that $w_i(r)$, $i = 1, \dots, n$, satisfy (2.2), i.e., they define some symmetry of (2.1). Write the following system of n equations:

$$w_i(r) = v_i(r)t + x. \tag{2.3}$$

It was proved in [11] that if one solves system (2.3) in respect to $r^i(t, x)$, $i = 1, \dots, n$; then these functions automatically satisfy the original semi-Hamiltonian system (2.1). This is the contents of the generalized godograph method.

For a semi-Hamiltonian system written down in the nondiagonal form

$$u^i_t = \sum_{j=1}^{n} v^i_j(u)u^j_x, \quad i = 1, \dots, n, \tag{2.4}$$

we can search for symmetries in the form

$$u^i_\tau = \sum_{j=1}^{n} w^i_j(u)u^j_x, \quad i = 1, \dots, n,$$

taking into account the equality of the mixed derivatives:

$$\partial_\tau\big(u^i_t\big) = \partial_\tau\left(\sum_{j=1}^{n} v^i_j(u)u^j_x\right) = \partial_t\big(u^i_\tau\big) = \partial_t\left(\sum_{j=1}^{n} w^i_j(u)u^j_x\right). \tag{2.5}$$

In this case the solution to (2.4) can be found from the system

$$x\delta^i_k + tv^i_k = w^i_k. \tag{2.6}$$

3. Solutions to System (1.1) for Small Degrees n

In this section, we recall how to construct general solutions to system (1.1) in the cases of $n = 1$ and $n = 2$ and also study the particular solution in case of $n = 4$ which is constructed in [9].

Case $n = 1$. From Theorem 1, for $n = 1$, we have

$$ds^2 = g^2(t, x)\, dt^2 + dx^2, \quad H = \frac{1}{2}\left(\frac{p_1^2}{g^2(t, x)} + p_2^2\right), \quad F = \frac{a_0(t, x)}{g(t, x)}p_1 + a_1(t, x)p_2,$$

where $a_0(t, x) \equiv g(t, x)$ and $a_1(t, x) \equiv 1$. The condition $\{F, H\} = 0$ is equivalent to

$$g_t + g_x = 0; \tag{3.1}$$

i.e.,

$$g(t, x) = f(t - x),$$

where f is an arbitrary function of one argument. In result, we obtain

$$F = p_1 + p_2, \quad H = \frac{1}{2}\left(\frac{p_1^2}{f^2(t - x)} + p_2^2\right), \quad \{F, H\} = 0.$$

CASE $n = 2$. From Theorem 1, for $n = 2$, we have

$$ds^2 = g^2(t,x)\,dt^2 + dx^2, \quad H = \frac{1}{2}\left(\frac{p_1^2}{g^2(t,x)} + p_2^2\right),$$

$$F = \frac{a_0(t,x)}{g^2}p_1^2 + \frac{a_1(t,x)}{g}p_1p_2 + a_2(t,x)p_2^2.$$

Since $a_1(t,x) \equiv g(t,x)$ and $a_2(t,x) \equiv 1$ we see that the condition $\{F,H\} = 0$ is equivalent to the system

$$(a_0)_t + gg_x = 0, \quad g_t + 2(1-a_0)g_x + g(a_0)_x = 0. \tag{3.2}$$

Note that system (3.2) is semi-Hamiltonian. We can rewrite it in the form of conservation laws:

$$(a_0)_t + \left(\frac{g^2}{2}\right)_x = 0, \quad \left(\frac{1}{2g^2}\right)_t + \left(\frac{1-a_0}{g^2}\right)_x = 0.$$

Moreover, (3.2) admits the Riemann invariants r^1 and r^2:

$$a_0(t,x) = 1 - r^1(t,x) - r^2(t,x), \quad g^2(t,x) = -4r^1(t,x)r^2(t,x),$$

in which it takes the diagonal form

$$\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}_t + \begin{pmatrix} 2r^2 & 0 \\ 0 & 2r^1 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}_x = 0; \tag{3.3}$$

i.e.,

$$r_t^i + v_i(r)r_x^i = 0,$$

where $v_1 = 2r^2$ and $v_2 = 2r^1$.

To construct solutions to system (3.3) we apply the generalized godograph method. We will search for the symmetries of system (3.3) in the form

$$\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}_\tau = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}_x,$$

where $w_1(r)$ and $w_2(r)$ are still unknown functions. Relations (2.2) take the form

$$\frac{\partial w_1}{\partial r^2} = \frac{w_1 - w_2}{r^2 - r^1}, \quad \frac{\partial w_2}{\partial r^1} = \frac{w_2 - w_1}{r^1 - r^2},$$

which implies in particular that $\partial w_1/\partial r^2 = \partial w_2/\partial r^1$. Consequently, there exists a function $\Psi(r^1, r^2)$ such that $\Psi_{r^1} = w_1$ and $\Psi_{r^2} = w_2$; moreover, Ψ satisfies the Euler–Poisson–Darboux equation

$$\Psi_{r^1r^2} + \frac{\Psi_{r^1} - \Psi_{r^2}}{r^1 - r^2} = 0.$$

The general solution to this equation has the form (see, for instance, [13])

$$\Psi(r^1, r^2) = 2u(r^1) + 2v(r^2) + (r^1 - r^2)(v'(r^2) - u'(r^1)),$$

where $u(r^1)$ and $v(r^2)$ are two arbitrary functions of one argument. Finding w_1 and w_2 and inserting them in (2.3), we obtain the general solution to system (3.3) (and to system (1.1) at the same time) for $n = 2$, in the implicit form

$$t = -\frac{1}{2}(u''(r^1) + v''(r^2)), \quad x = u'(r^1) + v'(r^2) - r^1u''(r^1) - r^2v''(r^2). \tag{3.4}$$

CASE $n = 4$. The exact local solutions to system (1.1) were constructed in [9] in case of $n = 4$ by the generalized godograph method.

Theorem 3 [9]. *For $n = 4$, system* (1.1) *has the solution*

$$a_0 = \frac{3\big(c_2 + t + 3c_3^2\big)}{5c_3^2}, \quad a_1 = -\frac{3\sqrt{c_3^2\big(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x\big) - 12(c_2 + t)^2}}{5c_3^2},$$

$$a_2 = \frac{-6\big(2c_2 + 2t + c_3^2\big)}{5c_3^2}, \quad g = \frac{2\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2};$$

where c_1, c_2, and c_3 are arbitrary constants.

Let us check whether the constructed integral of degree 4 is irreducible. Namely, we check whether the geodesic flow of the metric in [9] admits an additional polynomial integral of degree 1 or 2.

Theorem 4. *Under the conditions of Theorem* 1, *the geodesic flow of the metric*

$$ds^2 = g^2(t,x)\,dt^2 + dx^2, \tag{3.5}$$

$$g(t,x) = \frac{2\sqrt{c_3^2(-5c_1 - 4(3c_2 + 8t) - 18c_3^2 + 5x) - 12(c_2 + t)^2}}{5c_3^2}$$

admits no first integrals linear or quadratic in momenta.

Proof. As shown above (the case $n = 1$), a linear integral exists only for metrics of the form $ds^2 = g^2(t,x)\,dt^2 + dx^2$ satisfying (3.1). Metric (3.5) does not satisfy this equation, which implies the absence of linear integrals.

In the case of a quadratic integral (the case $n = 2$), the conditions of (3.2) hold true. Insert (3.5) in (3.2) to obtain

$$a_0(t,x) = -\frac{2t}{5c_3^2} + A(x),$$

where the function $A(x)$ must satisfy the equation

$$12c_2 + 11c_3^2 + 10t + 5c_3^2A(x)$$
$$+\big(5c_1c_3^2 + 2\big(6c_2^2 + 9c_3^4 + 16c_3^2t + 6t^2 + 6c_2\big(c_3^2 + 2t\big)\big) - 5c_3^2x\big)A'(x) = 0.$$

Rewrite this equation as

$$t^2\,\big(12A'(x)\big) + t\big(10 + 8\big(3c_2 + 4c_3^2\big)A'(x)\big) + 5c_1c_3^2A'(x) + 12c_2^3A'(x)$$
$$+18c_3^4A'(x) + 12c_2c_3^2A'(x) - 5c_3^2xA'(x) + 5c_3^2A(x) + 12c_2 + 11c_3^2 = 0.$$

For this expression to be identically zero, it is necessary that the coefficients at all degrees of t vanish. We see that this is impossible for any $A(x)$. Consequently, (3.5) admits no quadratic integrals.

Theorem 4 is proved.

4. Weakly Nonlinear Systems

The diagonal system (2.1) is called *weakly nonlinear* if it satisfies the condition

$$\frac{\partial v_i}{\partial r^i} = 0$$

for each $i = 1, \dots, n$ (see, for instance, [14]). The following fact demonstrates the remarkable feature of such systems: If a solution to a weakly nonlinear system is bounded on every finite time interval then the derivatives are also bounded. Thus, the absence of a gradient catastrophe for solutions is typical for the solutions to such systems (see [15]).

Weakly nonlinear semi-Hamiltonian systems (2.1) written down in Riemann invariants were completely described in [10]. In particular, it was shown in [10] that the characteristic velocities $v_i(r)$ of such system can be expressed explicitly in terms of the Riemann invariants r. In [10] were described various methods for constructing solutions to the systems (see also [16, 17]).

System arising in applications usually have nondiagonal form (2.4). Therefore, the natural question appears how we can understand, without finding Riemann invariants explicitly, whether a system of the form (2.4) is weakly nonlinear or not. The following procedure for checking weak nonlinearity was proposed in [10].

Consider the nondiagonal system

$$u_t^i + v_j^i(u)u_x^j = 0, \quad i, j = 1, \dots, n. \tag{4.1}$$

Calculate the characteristic polynomial of v_j^i; i.e.,

$$\det\bigl(\lambda E - v_j^i\bigr) = \lambda^n + f_1(u)\lambda^{n-1} + f_2(u)\lambda^{n-2} + \dots + f_n(u), \tag{4.2}$$

and consider the covector

$$(\nabla f_1)v^{n-1} + (\nabla f_2)v^{n-2} + \dots + (\nabla f_n), \tag{4.3}$$

where

$$\nabla f_k = \left(\frac{\partial f_k}{\partial u^1}, \dots, \frac{\partial f_k}{\partial u^n}\right)$$

and v^n means the nth power of v_j^i.

Proposition 1 [10]. *System* (4.1) *is weakly nonlinear if and only if covector* (4.3) *is identically zero.*

Apply this procedure to checking whether system (1.1) is weakly nonlinear. We have

$$(u^1, \dots, u^n) = (a_0, \dots, a_{n-1}), \quad \nabla f = \left(\frac{\partial f}{\partial a_0}, \dots, \frac{\partial f}{\partial a_{n-1}}\right).$$

Assertion 1. *System* (1.1) *is weakly nonlinear for* $n = 2$.

PROOF. The assertion holds since for $n = 2$, system (1.1) in Riemann invariants has the form (3.3), which obviously satisfies weak nonlinearity. We will verify however that the above criterion gives the correct answer as well. For $n = 2$, matrix (1.2) has the form

$$v_j^i = \begin{pmatrix} 0 & a_1 \\ a_1 & 2 - 2a_0 \end{pmatrix}.$$

Find the characteristic polynomial

$$\det(\lambda E - v_j^i) = \left|\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & a_1 \\ a_1 & 2 - 2a_0 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} \lambda & -a_1 \\ -a_1 & \lambda + 2a_0 - 2 \end{pmatrix}\right| = \lambda^2 + (2a_0 - 2)\lambda - a_1^2,$$

where $f_1(u) = 2a_0 - 2$ and $f_2(u) = -a_1^2$; $\nabla f_1 = (2, 0)$ and $\nabla f_2 = (0, -2a_1)$. Construct covector (4.3)

$$\nabla f_1 \cdot \begin{pmatrix} 0 & a_1 \\ a_1 & 2 - 2a_0 \end{pmatrix} + \nabla f_2 \cdot \begin{pmatrix} 0 & a_1 \\ a_1 & 2 - 2a_0 \end{pmatrix}^0$$
$$= (2, 0) \cdot \begin{pmatrix} 0 & a_1 \\ a_1 & 2 - 2a_0 \end{pmatrix} + (0, -2a_1) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 2a_1) + (0, -2a_1) = (0, 0).$$

Therefore, system (1.1) is weakly nonlinear for $n = 2$.

Assertion 2. *System* (1.1) *is not weakly nonlinear for* $n = 3$.

Proof. For $n = 3$, matrix (1.2) has the form

$$v^i_j = \begin{pmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 2a_2 - 3a_0 \\ 0 & a_2 & 3 - 2a_1 \end{pmatrix}.$$

Find the characteristic polynomial

$$\det\bigl(\lambda E - v^i_j\bigr) = \left|\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 2a_2 - 3a_0 \\ 0 & a_2 & 3 - 2a_1 \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} \lambda & 0 & -a_1 \\ -a_2 & \lambda & 3a_0 - 2a_2 \\ 0 & -a_2 & \lambda + 2a_1 - 3 \end{pmatrix}\right|$$
$$= \lambda^3 + (-3 + 2a_1)\lambda^2 + (3a_0a_2 - 2a_2^2)\lambda - a_1a_2^2;$$

i.e.,

$$f_1(u) = -3 + 2a_1, \quad f_2(u) = 3a_0a_2 - 2a_2^2, \quad f_3 = -a_1a_2^2.$$

Consequently,

$$\nabla f_1 = (0, 2, 0), \quad \nabla f_2 = (3a_2, 0, 3a_0 - 4a_2), \quad \nabla f_3 = \bigl(0, -a_2^2, -2a_1a_2\bigr).$$

Construct covector (4.3)

$$\nabla f_1 \cdot \begin{pmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 2a_2 - 3a_0 \\ 0 & a_2 & 3 - 2a_1 \end{pmatrix}^2 + \nabla f_2 \cdot \begin{pmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 2a_2 - 3a_0 \\ 0 & a_2 & 3 - 2a_1 \end{pmatrix}^1 + \nabla f_3 \cdot \begin{pmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 2a_2 - 3a_0 \\ 0 & a_2 & 3 - 2a_1 \end{pmatrix}^0$$
$$= (0, -a_2(3a_0 + a_2), a_0(-9 + 6a_1) + 3a_1a_2).$$

Therefore, system (1.1) is not weakly nonlinear for $n = 3$.

Theorem 5. *System* (1.1) *is not weakly nonlinear for* $n > 2$.

Proof. It suffices to prove that covector (4.3) has a nonzero component. For convenience, put

$$A^l_n = la_l - (n - l + 2)a_{l-2}, \quad l = \overline{1, n};$$

here $a_j = 0$ for $j < 0$.

Lemma 1. *For* $1 < m \le n - 1$, *the matrix* $\bigl(v^i_j\bigr)^m$ *has the block form*

$$\bigl(v^i_j\bigr)^m = \left(\frac{O_{m\times l}}{D_{l\times l}} \,\middle|\, A \cdot a^{m-1}_{n-1} \,\middle|\, \begin{matrix} \dots \\ \dots \end{matrix}\right).$$

Here $l = n - m$, $O_{m\times l}$ *is the zero matrix of* m *rows and* l *columns,* $D_{l\times l}$ *is the diagonal matrix with an element* a^m_{n-1} *on the diagonal, and* A *is the column vector* $A = \bigl(A^1_n, A^2_n, \dots, A^n_n\bigr)^T$.

Proof. We will proceed by induction.

Step 1. Check the claim for $m = 2$. We have

$$\bigl(v^i_j\bigr)^2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & A^1_n \\ a_{n-1} & 0 & \cdots & 0 & A^2_n \\ 0 & a_{n-1} & \cdots & 0 & A^3_n \\ 0 & 0 & \cdots & 0 & A^4_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A^{n-2}_n \\ 0 & 0 & \cdots & 0 & A^{n-1}_n \\ 0 & 0 & \cdots & a_{n-1} & A^n_n \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 & A^1_n \\ a_{n-1} & 0 & \cdots & 0 & A^2_n \\ 0 & a_{n-1} & \cdots & 0 & A^3_n \\ 0 & 0 & \cdots & 0 & A^4_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A^{n-2}_n \\ 0 & 0 & \cdots & 0 & A^{n-1}_n \\ 0 & 0 & \cdots & a_{n-1} & A^n_n \end{pmatrix}$$

$$
= \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & A_n^1 \cdot a_{n-1} & * \\
0 & 0 & \cdots & 0 & 0 & A_n^2 \cdot a_{n-1} & * \\
a_{n-1}^2 & 0 & \cdots & 0 & 0 & A_n^3 \cdot a_{n-1} & * \\
0 & a_{n-1}^2 & \cdots & 0 & 0 & A_n^4 \cdot a_{n-1} & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & A_n^{n-2} \cdot a_{n-1} & * \\
0 & 0 & \cdots & a_{n-1}^2 & 0 & A_n^{n-1} \cdot a_{n-1} & * \\
0 & 0 & \cdots & 0 & a_{n-1}^2 & A_n^{n} \cdot a_{n-1} & *
\end{pmatrix}
$$

$$
= \left(\frac{O_{2\times(n-2)}}{D_{(n-2)\times(n-2)}} \middle| A \cdot a_{n-1} \middle| \begin{matrix} * \\ * \end{matrix} \right).
$$

Step 2. Suppose that the lemma holds for $m = k$, i.e.,

$$
\left(v_j^i\right)^k = \left(\frac{O_{k\times(n-k)}}{D_{(n-k)\times(n-k)}} \middle| A \cdot a_{n-1}^{k-1} \middle| \begin{matrix} \cdots \\ \cdots \end{matrix} \right).
$$

Step 3. Prove that the lemma holds for $m = k+1$. It is easy to see that

$$
\left(v_j^i\right)^{k+1} = \left(v_j^i\right)^k \cdot \left(v_j^i\right) = \left(\frac{O_{k\times(n-k)}}{D_{(n-k)\times(n-k)}} \middle| A \cdot a_{n-1}^{k-1} \middle| \begin{matrix} \cdots \\ \cdots \end{matrix} \right) \cdot \left(\frac{O_{1\times(n-1)}}{D_{(n-1)\times(n-1)}} \middle| A \right).
$$

Lemma 1 is proved. $\square$

Note that, by Lemma 1, the matrix $\left(v_j^i\right)^{n-1}$ has the form

$$
\left(v_j^i\right)^{n-1} = \begin{pmatrix}
0 & A_n^1 \cdot a_{n-1}^{n-2} & \cdots \\
0 & A_n^2 \cdot a_{n-1}^{n-2} & \cdots \\
0 & A_n^3 \cdot a_{n-1}^{n-2} & \cdots \\
\cdots & \cdots & \cdots \\
0 & A_n^{n-2} \cdot a_{n-1}^{n-2} & \cdots \\
0 & A_n^{n-1} \cdot a_{n-1}^{n-2} & \cdots \\
a_{n-1}^{n-1} & A_n^{n} \cdot a_{n-1}^{n-2} & \cdots
\end{pmatrix} = \left(\frac{O_{(n-1)\times 1}}{D_{1\times 1}} \middle| A \cdot a_{n-1}^{n-2} \middle| \begin{matrix} \cdots \\ \cdots \end{matrix} \right).
$$

Let us find the characteristic polynomial of (1.2).

Lemma 2. *The characteristic polynomial of* (1.2) *is*

$$
\lambda^n + \sum_{k=1}^{n-1} \left(f_k \cdot \lambda^{n-k}\right) + f_n,
$$

where $f_k = (k+1)a_{n-(k+1)}a_{n-1}^{k-1} - (n-(k-1))a_{n-(k-1)}a_{n-1}^{k-1}$ *for all* $k = 1, \ldots, n-1$, *and* $f_n = -a_{n-1}^{n-1}a_1$.

Proof. Calculate the determinant decomposing it by the last row

$$
\det(\lambda E - v_j^i) = \left| \begin{pmatrix}
\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & -A_n^1 \\
-a_{n-1} & \lambda & 0 & \cdots & 0 & 0 & 0 & -A_n^2 \\
0 & -a_{n-1} & \lambda & \cdots & 0 & 0 & 0 & -A_n^3 \\
0 & 0 & -a_{n-1} & \cdots & 0 & 0 & 0 & -A_n^4 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \lambda & 0 & 0 & -A_n^{n-3} \\
0 & 0 & 0 & \cdots & -a_{n-1} & \lambda & 0 & -A_n^{n-2} \\
0 & 0 & 0 & \cdots & 0 & -a_{n-1} & \lambda & -A_n^{n-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & -a_{n-1} & \lambda - A_n^n
\end{pmatrix} \right|
$$

$$= (\lambda - (na_n - 2a_{n-2}))\lambda^{n-1} + a_{n-1}((-(n-1)a_{n-1} + 3a_{n-3})\lambda^{n-2}$$
$$+a_{n-1}((-(n-2)a_{n-2} + 4a_{n-4})\lambda^{n-3} + \cdots + a_{n-1}((-3a_3 + (n-1)a_1)\lambda^2$$
$$+a_{n-1}((-2a_2 + na_0)\lambda - a_{n-1}a_1))\cdots))$$
$$= \lambda^n + (2a_{n-2} - na_n)\lambda^{n-1} + a_{n-1}(3a_{n-3} - (n-1)a_{n-1})\lambda^{n-2}$$
$$+a_{n-1}^2(4a_{n-4} - (n-2)a_{n-2})\lambda^{n-3} + \cdots + a_{n-1}^{n-4}((n-2)a_2 - 4a_4)\lambda^3$$
$$+a_{n-1}^{n-3}((n-1)a_1 - 3a_3)\lambda^2 + a_{n-1}^{n-2}(na_0 - 2a_2)\lambda - a_{n-1}^{n-1}a_1$$
$$= \lambda^n + \sum_{k=1}^{n-1}\big(a_{n-1}^{k-1}((k+1)a_{n-(k+1)} - (n-(k-1))a_{n-(k-1)})\lambda^{n-k}\big) - a_{n-1}^{n-1}a_1$$
$$= \lambda^n + \sum_{k=1}^{n-1}(f_k \cdot \lambda^{n-k}) + f_n,$$

where $f_k = (k+1)a_{n-(k+1)}a_{n-1}^{k-1} - (n-(k-1))a_{n-(k-1)}a_{n-1}^{k-1}$ for all $k = 1, \ldots, (n-1)$, and $f_n = -a_{n-1}^{n-1}a_1$.

Lemma 2 is proved. $\square$

Lemma 3. *The gradients ∇f_i, $i = 1, \ldots, n$, look as*

$$\nabla f_1 = (\underbrace{0,\ \ldots,\ 0}_{n-2}, 2, 0), \quad \nabla f_2 = (\underbrace{0,\ \ldots,\ 0}_{n-3},\ 3a_{n-1},\ 0,\ 3a_{n-3} - 2(n-1)a_{n-1}),$$

$$\nabla f_k = \big(\underbrace{0,\ldots,0}_{n-(k+1)},\ (k+1)a_{n-1}^{k-1},\ 0,\ -(n-(k-1))a_{n-1}^{k-1},\ \underbrace{0,\ldots,0}_{k-3},$$
$$(k^2-1)a_{n-(k+1)}a_{n-1}^{k-2} - (k-1)(n-(k-1))a_{n-(k-1)}a_{n-1}^{k-2}\big)$$

for $2 < k < n$,

$$\nabla f_n = \big(0,\ -a_{n-1}^{n-1},\ \underbrace{0,\ \ldots,\ 0}_{n-3},\ -(n-1)\cdot a_1 \cdot a_{n-1}^{n-2}\big).$$

Proof. 1. If $k = 1$ then $a_n \equiv 1$ which implies $f_1 = 2a_{n-2} - n$, and hence

$$\nabla f_1 = \left(\frac{\partial f_1}{\partial a_0}, \frac{\partial f_1}{\partial a_1}, \ldots, \frac{\partial f_1}{\partial a_{n-2}},\ \frac{\partial f_1}{\partial a_{n-1}}\right) = (\underbrace{0,\ldots,0}_{n-2}, 2, 0).$$

2. If $k = 2$ then $n - (k-1) = n - 1$. In this case, $f_2 = 3a_{n-3}\cdot a_{n-1} - (n-1)\cdot a_{n-1}^2$ and

$$\nabla f_2 = \left(\frac{\partial f_2}{\partial a_0}, \ldots, \frac{\partial f_2}{\partial a_{n-4}}, \frac{\partial f_2}{\partial a_{n-3}}, \frac{\partial f_2}{\partial a_{n-2}}, \frac{\partial f_2}{\partial a_{n-1}}\right) = (\underbrace{0,\ldots,0}_{n-3},\ 3a_{n-1},\ 0,\ 3a_{n-3} - 2(n-1)a_{n-1}).$$

3. If $2 < k < n$ then

$$f_k = (k+1)a_{n-(k+1)}a_{n-1}^{k-1} - (n-(k-1))a_{n-(k-1)}a_{n-1}^{k-1},$$
$$\frac{\partial f_k}{\partial a_{n-1}} = (k-1)(k+1)a_{n-(k+1)}a_{n-1}^{k-2} - (k-1)(n-(k-1))a_{n-(k-1)}a_{n-1}^{k-2},$$
$$\frac{\partial f_k}{\partial a_{n-(k-1)}} = -(n-(k-1))a_{n-1}^{k-1}, \quad \frac{\partial f_k}{\partial a_{n-(k+1)}} = (k+1)a_{n-1}^{k-1},$$
$$\frac{\partial f_k}{\partial a_i} = 0 \quad \text{for} \quad i \neq n-1 \quad \text{and} \quad i \neq n-(k\pm 1).$$

Then

$$\nabla f_k = \left(\frac{\partial f_k}{\partial a_0}, \dots, \frac{\partial f_k}{\partial a_{n-(k+1)}}, \frac{\partial f_k}{\partial a_{n-k}}, \frac{\partial f_k}{\partial a_{n-(k-1)}}, \dots, \frac{\partial f_k}{\partial a_{n-2}}, \frac{\partial f_k}{\partial a_{n-1}}\right)$$
$$= \big(\underbrace{0, \dots, 0}_{n-(k+1)}, (k+1)a_{n-1}^{k-1}, 0, -(n-(k-1))a_{n-1}^{k-1}, \underbrace{0, \dots, 0}_{k-3},$$
$$(k^2-1)a_{n-(k+1)}a_{n-1}^{k-2} - (k-1)(n-(k-1))a_{n-(k-1)}a_{n-1}^{k-2}\big).$$

4. For $k = n$, we have $f_n = -a_{n-1}^{n-1}a_1$ and

$$\nabla f_n = \left(\frac{\partial f_n}{\partial a_0}, \frac{\partial f_n}{\partial a_1}, \frac{\partial f_n}{\partial a_2}, \dots, \frac{\partial f_n}{\partial a_{n-2}}, \frac{\partial f_n}{\partial a_{n-1}}\right) = \big(0,\ -a_{n-1}^{n-1}, \underbrace{0, \dots, 0}_{n-3}, -(n-1)\cdot a_1 \cdot a_{n-1}^{n-2}\big).$$

Lemma 3 is proved. □

For proving Theorem 5, it suffices to show that covector (4.3) has at least one nonzero component. Let us show that its second component is nonzero. By Lemmas 1 and 3, for $2 < k < n-1$ we have

$$\nabla f_k \cdot (v_j^i)^{n-k} = \big(\underbrace{0, \dots, 0}_{n-(k+1)}, (k+1)a_{n-1}^{k-1}, 0, -(n-(k-1))a_{n-1}^{k-1}, \underbrace{0, \dots, 0}_{k-3},$$
$$(k^2-1)a_{n-(k+1)}a_{n-1}^{k-2} - (k-1)\big(n-(k-1)\big)a_{n-(k-1)}a_{n-1}^{k-2}\big) \cdot \left(\frac{O_{(n-k)\times k}}{D_{k\times k}} \;\middle|\; A \cdot a_{n-1}^{n-k-1} \;\middle|\; \begin{matrix}\cdots\\ \cdots\end{matrix}\right)$$
$$= \big(0, \big(-(n-(k-1))a_{n-1}^{k-1}\big) \cdot a_{n-1}^{n-k}, \dots\big).$$

Consequently, covector (4.3) looks as

$$(\nabla f_1)v^{n-1} + (\nabla f_2)v^{n-2} + \sum_{k=3}^{n-1} \nabla f_k \cdot \big(v_j^i\big)^{n-k} + (\nabla f_n) = (\underbrace{0, \dots, 0}_{n-2}, 2, 0) \cdot \left(\frac{O_{(n-1)\times 1}}{D_{1\times 1}} \;\middle|\; A \cdot a_{n-1}^{n-2} \;\middle|\; \begin{matrix}\cdots\\ \cdots\end{matrix}\right)$$
$$+(\underbrace{0, \dots, 0}_{n-3}, 3a_{n-1}, 0, 3a_{n-3} - 2(n-1)a_{n-1}) \cdot \left(\frac{O_{(n-2)\times 2}}{D_{2\times 2}} \;\middle|\; A \cdot a_{n-1}^{n-3} \;\middle|\; \begin{matrix}\cdots\\ \cdots\end{matrix}\right)$$
$$+\left(0, \sum_{k=3}^{n-1}\big(\big(-(n-(k-1))a_{n-1}^{k-1}\big) \cdot a_{n-1}^{n-k}\big), \dots\right) + \big(0,\ -a_{n-1}^{n-1}, \underbrace{0, \dots, 0}_{n-3}, -(n-1)\cdot a_1 \cdot a_{n-1}^{n-2}\big)$$
$$= \big(0, 2a_{n-1}^{n-2} \cdot ((n-1)a_{n-1} - 3a_{n-3}), \dots\big) + \big(0, a_{n-1}^{n-2} \cdot (3a_{n-3} - 2(n-1)a_{n-1}), \dots\big)$$
$$+\left(0, \sum_{k=3}^{n-1}\big(\big(-(n-(k-1))\cdot a_{n-1}^{k-1}\big) \cdot a_{n-1}^{n-k}\big), \dots\right) + \big(0, -a_{n-1}^{n-1}, \underbrace{0, \dots, 0}_{n-3}, -(n-1)\cdot a_1 \cdot a_{n-1}^{n-2}\big)$$
$$= \left(0,\ \ -a_{n-1}^{n-2} \cdot \left(3a_{n-3} + \frac{(n-1)(n-2)}{2} \cdot a_{n-1}\right), \dots\right).$$

Thus, for every $n > 2$, the second component of covector (4.3) is equal to

$$-a_{n-1}^{n-2} \cdot \left(3a_{n-3} + \frac{(n-1)(n-2)}{2} \cdot a_{n-1}\right),$$

i.e., the component is nonzero in general.

Theorem 5 is proved. □

5. Description of Commuting Flows

Let us study the structure of commuting flows (symmetries) of system (1.1). In case of $n = 2$, for the system written in Riemann invariants (3.3), symmetries were described in Section 3. For $n > 2$, the search for Riemann invariants and an explicit diagonalization of the system in general become a difficult problem. Therefore, in this case, for constructing solutions by the generalized godograph method, it is reasonable to try to describe the symmetries of the initial nondiagonal system (1.1). In this section, we demonstrate how it can be done for $n = 2$.

For $n = 2$, system (1.1) has the form

$$u^i_t + v^i_j(u)u^j_x = 0, \quad v^i_j = \begin{pmatrix} 0 & a_1 \\ a_1 & 2-2a_0 \end{pmatrix}.$$

We will search for symmetries of this system in the form

$$u^i_\tau + b^i_j(u)u^j_x = 0, \quad b^i_j = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{5.1}$$

where $b^i_j(u) = b^i_j(a_0, a_1)$. By the definition of symmetries,

$$\frac{\partial}{\partial\tau}\big(v^i_j(u)u^j_x\big) = \frac{\partial}{\partial t}\big(b^i_j(u)u^j_x\big)$$

for all $i = 1, 2$. Straightforward calculations give the relations

$$b_{21} = b_{12}, \quad b_{22} = b_{11} - \frac{2(-1+a_0)b_{12}}{a_1},$$

$$(b_{11})_{a_1} - (b_{12})_{a_0} = 0, \quad b_{12} - a_1(b_{12})_{a_1} + a_1(b_{11})_{a_0} - 2(-1+a_0)(b_{12})_{a_0} = 0.$$

Therefore, there exists a function $\Psi(a_0, a_1)$ such that

$$b_{11} = \Psi_{a_0}, \quad b_{12} = \Psi_{a_1}; \tag{5.2}$$

here $\Psi(a_0, a_1)$ satisfies the second-order linear partial differential equation

$$a_1\Psi_{a_0a_0} - 2(a_0-1)\Psi_{a_0a_1} - a_1\Psi_{a_1a_1} + \Psi_{a_1} = 0. \tag{5.3}$$

This equation has the hyperbolic type everywhere but one point:

$$D = (a_0-1)^2 + a_1^2 \geq 0.$$

Reduce (5.3) to a canonical form. The characteristic equation looks as

$$a_1(da_1)^2 + 2(a_0-1)\,da_0da_1 - a_1(da_0)^2 = 0,$$

i.e.,

$$\frac{da_1}{da_0} = \frac{1-a_0 \pm \sqrt{(1-a_0)^2 + a_1^2}}{a_1}. \tag{5.4}$$

After the change of variables

$$1 - a_0 = r\cos\phi, \quad a_1 = r\sin\phi, \tag{5.5}$$

equation (5.4) takes the form

$$r\sin\phi\,d\phi = (\cos\phi \pm 1)\,dr.$$

The variables are separated, and after integration we obtain

$$r = \frac{C}{\cos\phi \pm 1}. \tag{5.6}$$

Taking into account (5.5) and (5.6), find the general solution to (5.4) in the form

$$a_1 = \pm\sqrt{C^2 + 2C(a_0 - 1)}.$$

Squaring both sides of this equation and solving the resulting quadratic equation in C, we obtain

$$C_1 = 1 - a_0 - \sqrt{(1-a_0)^2 + a_1^2}, \quad C_2 = 1 - a_0 + \sqrt{(1-a_0)^2 + a_1^2}. \tag{5.7}$$

Note that the so-found C_1 and C_2 are in fact Riemann invariants of (1.1) (see Section 3, the case $n = 2$). Now, performing the corresponding change of variables in (5.3), write it down in the canonical form. We obtain the Euler–Poisson–Darboux equation

$$\Psi_{C_1C_2} + \frac{\Psi_{C_1} - \Psi_{C_2}}{C_1 - C_2} = 0. \tag{5.8}$$

The general solution to (5.8) has the form (see [13]):

$$\Psi(C_1, C_2) = 2u(C_1) + 2v(C_2) + (C_1 - C_2)(v'(C_2) - u'(C_1)),$$

where u and v are arbitrary functions of one argument. Owing to (5.2), we obtain the final form of the symmetries of (5.1):

$$b_{11} = -2(u'(C_1) + v'(C_2) - C_1u''(C_1) - C_2v''(C_2)),$$

$$b_{12} = b_{21} = -2a_1(u''(C_1) + v''(C_2)),$$

$$b_{22} = -2(u'(C_1) + v'(C_2) + C_2u''(C_1) + C_1v''(C_2)).$$

The general solution to (1.1) is defined by (2.6) and, in view of the above-found symmetries, takes the form

$$x = -2(u'(C_1) + v'(C_2) - C_1u''(C_1) - C_2v''(C_2)), \quad t = -2(u''(C_1) + v''(C_2)),$$

which agrees completely with the general solution (3.4). Here C_1 and C_2 are of the form (5.7).

Thus, for system (1.1) for $n = 2$, we have the general description of the symmetries and the general solution constructed by the generalized godograph method.

6. Conclusion

In the present article, we consider the problem of integrable geodesic flows on the 2-torus. In accordance with the fundamental observation of [5], the search for an additional polynomial integral of this flow reduces to the search for solutions to the quasilinear system of differential equations (1.1) which possesses a number of remarkable properties. In particular, it was proved in [5] that this system is semi-Hamiltonian.

The aim of this article is to study various properties of system (1.1). In particular, we obtained the following results:

1. We proved that the solution to (1.1) for $n = 4$ in [9] is nontrivial; i.e., the geodesic flow of the metric in [9] admits no polynomial integrals of degree 1 and 2.
2. We proved that system (1.1) is weakly nonlinear only for $n = 2$.
3. We described the symmetries of system (1.1) for $n = 2$.

It would be very interesting to construct the solution to system (1.1) for $n = 3$ or $n = 5$. Maybe this can be done by describing the symmetries of (1.1) in general form and using the generalized godograph method (by analogy with Section 5 of this article in case $n = 2$).

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