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BOOLEAN VALUED ANALYSIS OF BANACH ALGEBRAS

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Abstract: We implement the Boolean valued analysis of Baer C^*-algebras and Jordan–Banach algebras. These algebras transform into AW^*- and JB-factors. Presentation of the factors as operator algebras leads to Kaplansky–Hilbert modules. We overview the basic properties of these objects.

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Introduction

The theory of Banach algebras is one of the most topical and attractive areas of functional analysis. This paper is an invitation to Boolean valued analysis of the involutive and Jordan Banach algebras.

The main idea is as follows: If the center of an algebra is duly qualified then it turns into a one-dimensional subalgebra after ascending to a suitable Boolean valued model, which simplifies analysis. At that, transfer guarantees the coincidence of the contents of the formal theories of the original algebra and its Boolean valued realization.

We will focus on the analysis of AW^*- and JB-algebras, i.e., Baer C^*-algebras and Jordan–Banach algebras. These algebras transform into AW^*- and JB-factors. Presentation of the factors as operator algebras leads to the Kaplansky–Hilbert modules realizable as Hilbert spaces in appropriate Boolean valued models.

The dimension of a Hilbert space in a model is a Boolean valued cardinal called the Boolean dimension of a Kaplansky–Hilbert model. Note the effect of cardinality shift: Isomorphic Kaplansky–Hilbert modules can have bases of different dimensions. This circumstance implies that, in general, there are many ways of decomposing a type I AW^*-algebra in a direct sum of homogeneous subalgebras, which supports the Kaplansky conjecture of 1953.

Using the Boolean valued embedding of Kaplansky–Hilbert modules, we arrive at their functional realizations. Namely, a Kaplansky–Hilbert module is unitarily equivalent to the direct sum of some homogeneous AW^*-modules of continuous Hilbert-space-valued vector functions. An analogous representation is valid for type I AW^*-algebras, but the continuous vector functions are replaced with operator functions continuous in the strong operator topology.

An AW^*-algebra A is *embeddable* provided that A is isomorphic to the bicommutant of some type I AW^*-algebra. Each embeddable AW^*-algebra admits a Boolean valued realization as a Neumann algebra or factor. We will give various characterizations of embeddable AW^*-algebras. In particular, an AW^*-algebra A is embeddable if and only if A has a separating set of center valued normal states. We will also consider similar problems for JB-algebras which are analogs of C^*-algebras.

The structure of the paper is as follows: Section 1 collects the results on Boolean valued realizations of Banach algebras and, in particular, involutive Banach algebras. We also give some elementary applications of these techniques. Section 2 gives the Boolean valued realization of an AW^*-algebra. Section 3 deals with the Boolean dimension of a Kaplansky–Hilbert module. Section 4 addresses the

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functional realizations of Kaplansky–Hilbert modules. Section 5 describes the functional realizations of type I AW^*-algebras. Section 6 exposes the basics of embeddable C^*-algebras. Section 7 is devoted to JB-algebras and their functional realizations. Section 8 presents some applications of the results of Section 7 to the preadjoint JB-algebras. Section 9 contains some comments, a concise guide to the references, and a few remarks on related topics.

Referring the reader to [1, 2] for the techniques and notations of Boolean valued analysis, we follow [3] as regards the prerequisites of Boolean algebras and [4–6] as regards Banach algebras. The needed facts of the theory of dominated operators are taken from [7]. Also, we supply some available results with proofs for the reader's convenience. Note that everywhere in the sequel $\mathbb{V}^{(B)}$ denotes the Boolean valued universe over a complete Boolean algebra B, while a partition of unity in B is a family of pairwise disjoint members of B whose join is the top $\mathbb{1}$ of B.

The theory of Banach algebras is the product of the revolutionary changes in sciences at the turn of the twentieth century. We note the definitive contributions of von Neumann [8] and Kaplansky [9]. Invoked by quantum mechanics, the theory is still at the focus of many scientists and reflected in dozens of books and hundreds of papers. It is impossible to overview the whole field in principle. We will dwell upon some few aspects of the theory that are connected with nonstandard Boolean valued models of set theory. Takeuti pioneered this track of research (see [10, 11]) which was continued by a scarcity of mathematicians among which most contributions belong to Kusraev, Nishimura, and Ozawa. These scholars use rather similar techniques, notwithstanding distinctions in the aims and directions of their research. The omnipotence of the Internet opens many opportunities to acquaintance with their contributions. Therefore, this paper bases primarily on the article listed in [1, Chapter 6]. We distinguish [12] as an easy elementary introduction to Boolean valued analysis.

1. Descents of Banach Algebras

1.1. Start with the prerequisites, confining exposition to complex algebras. Note that, speaking about an algebra, we bear in mind an associative algebra with unity $\mathbb{1}$.

Consider an involutive algebra A. An element $p \in A$ is a *projection* provided that $p^* = p$ and $p^2 = p$. A projection p is *central* whenever $px = xp$ for all $x \in A$. We let $\mathfrak{P}(A)$ stand for the set of projections of A; and $\mathfrak{P}_c(A)$, the set of central projections of A.

Given $M \subset A$, define the *right annihilator* $M^\perp$ and the *left annihilator* ${}^\perp M$ by the formulas

$$M^\perp := \{y \in A : (\forall\, x \in M)\, xy = 0\};$$
$${}^\perp M := \{x \in A : (\forall\, y \in M)\, xy = 0\}.$$

Clearly, the annihilators are polars as defined by Akilov (see [1, 4.1.12]), and so they share the same simple properties as the disjoint complements:

(a) $M \subset N \to N^\perp \subset M^\perp$;
(b) $M \subset {}^\perp(M^\perp)$ and $M \subset ({}^\perp M)^\perp$;
(c) $M^\perp = ({}^\perp(M^\perp))^\perp$ and ${}^\perp M = {}^\perp(({}^\perp M)^\perp)$;
(d) $(\bigcup_\alpha M_\alpha)^\perp = \bigcap_\alpha M_\alpha^\perp$;
(e) $(M^\perp)^* = {}^\perp(M^*)$ and $({}^\perp M)^* = (M^*)^\perp$.

These imply in particular that the inclusion-ordered set of all right (left) annihilators is an order complete lattice with bottom $\mathbb{0} := \{0\}$ and top $\mathbb{1} := A$. The mapping $K \mapsto K^* := \{x^* : x \in K\}$ is an isotone bijection between the lattices of right and left annihilators.

1.2. Of great import are the involutive algebras whose annihilators are generated by projections. A *Baer $*$-algebra* is an involutive algebra A such that to each nonempty $M \subset A$ there corresponds some $p \in \mathfrak{P}(A)$ that satisfies the condition $M^\perp = pA$. As seen from 1.1(e), A is Baer means that to each nonempty $M \subset A$ there is a projection $q \in \mathfrak{P}(A)$ such that ${}^\perp M = Aq$. So, in a Baer $*$-algebra A for each left annihilator L there is a unique projection $q_L \in A$ such that $x = xq_L$ if $x \in L$ and $q_L y = 0$ if $y \in L^\perp$.

The mapping $L \mapsto q_L$ is an isomorphism between the posets of all left annihilators and all projections. The inverse isomorphism has the form $q \mapsto {}^{\perp}(\mathbb{1} - q)$ with $q \in \mathfrak{P}(A)$. The analogous fact holds for the set of right annihilators. This implies in particular that the poset $\mathfrak{P}(A)$ is an order complete lattice. The mapping $p \mapsto p^{\perp} := \mathbb{1} - p$ with $p \in \mathfrak{P}(A)$ enjoys the properties:

$$p^{\perp\perp} = p, \quad p \wedge p^{\perp} = \mathbb{0}, \quad p \vee p^{\perp} = \mathbb{1},$$
$$(p \wedge q)^{\perp} = p^{\perp} \vee q^{\perp}, \quad (p \vee q)^{\perp} = p^{\perp} \wedge q^{\perp},$$
$$p \le q \to p \vee (p^{\perp} \wedge q) = q.$$

In other words, $(\mathfrak{P}(A), \wedge, \vee, \perp)$ is an *orthomodular lattice*.

1.3. An *AW^*-algebra* is a unital C^*-algebra that is simultaneously a Baer $*$-algebra. In more detail, an AW^*-algebra A is a C^*-algebra whose each right annihilator has the form pA for some projection $p \in A$. Call $z \in A$ a *central* element whenever z commutes with every $x \in A$, i.e., $(\forall\, x \in A)xz = zx$. The center of a AW^*-algebra A is the set $\mathscr{Z}(A)$ of all central elements. Clearly, $\mathscr{Z}(A)$ is a commutative AW^*-subalgebra of A and $\lambda\mathbb{1} \in \mathscr{Z}(A)$ for all $\lambda \in \mathbb{C}$. If $\mathscr{Z}(A) = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$ then the AW^*-algebra A is usually referred to as an *AW^*-factor*.

Let Λ be a *complex Kantorovich space*, i.e., a Dedekind complete complex AM-space with a strong unity. It is well known that Λ is linear isometric and order isomorphic to the space $C(Q) := C(Q, \mathbb{C})$ of continuous complex functions on some extremally disconnected compact space Q. Sometimes this Q is called *extremely disconnected*. Therefore, Λ admits the structure of an involutive algebra, and so Λ becomes a commutative C^*-algebra which is often called a *Stone algebra*.

For a C^*-algebra A to be an AW^*-algebra it is necessary and sufficient that the following hold:

(1) *Each orthogonal family in $\mathfrak{P}(A)$ has a supremum.*

(2) *Each maximal commutative $*$-subalgebra of A is Stone.*

The space $\mathscr{L}(H)$ of bounded linear endomorphisms of a complex Hilbert space H exemplifies an AW^*-algebra. Recall that the Banach algebra structure in $\mathscr{L}(H)$ assumes the conventional addition and composition of operators as well as the operator norm. The involution in $\mathscr{L}(H)$ acts as passage to the hermitian adjoint of a bounded operator. Note also that a commutative AW^*-algebra is exactly a Stone algebra.

1.4. Recall that a normed space X is *B-cyclic* with respect to a complete Boolean algebra B of norm one projections provided that for each partition of unity (b_ξ) in B and each family (x_ξ) of the unit ball U of X there exists a unique $x \in U_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ; see [1, 5.5.4 and 5.5.6]. A Banach algebra A is *B-cyclic* (with respect to a complete Boolean algebra of projections in B) provided that A is a B-cyclic Banach space and every member of B is multiplicative; i.e.,

$$\pi(xy) = \pi(x)\pi(y) \quad (x, y \in A,\ \pi \in B);$$

or, which is the same, $\pi(xy) = \pi(x)y = x\pi(y)$ for all $x, y \in A$. A *B-cyclic involutive algebra* is an involutive Banach algebra A whose projections preserve involution:

$$\pi(x^*) = (\pi x)^* \quad (x \in A,\ \pi \in B).$$

Also, obvious meaning is ascribed to the term a *B-cyclic C^*-algebra*.

Recall that we consider only unital algebras. If $\mathbb{1}$ is the unity of A then we may identify each projection $b \in B$ with $b\mathbb{1}$, which yields a central projection in case A is involutive. In this event we will write $B \subset \mathfrak{P}_c(A)$. Note that $B \sqsubset A$ means that A is a B-cyclic Banach algebra. If A is a C^*-algebra, then A is B-cyclic whenever for every partition of unity $(b_\xi)_{\xi\in\Xi}$ and every bounded family $(x_\xi)_{\xi\in\Xi} \subset A$ there is a unique $x \in A$ such that $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$ and $\|x\| \le \sup_{\xi\in\Xi} \|b_\xi x_\xi\|$.

Each complex Kantorovich space with base B and fixed unity exemplifies a B-cyclic C^*-algebra. This algebra, denoted by $B(\mathbb{C})$, is unique up to $*$-isomorphism. Let $\Lambda = \mathscr{R}{\Downarrow}$ be the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$; i.e., Λ is the order dense ideal in $\mathscr{R}{\downarrow}$ generated by the unity $\mathbb{1} := 1^{\wedge} \in \mathscr{R}{\downarrow}$. Take a Banach space $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ and put $\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : [\![x]\!] \in \Lambda\}$. Then $\mathscr{X}{\Downarrow}$ is a Banach–Kantorovich space called the *restricted* or *bounded* descent of $\mathscr{X}$; see [1, Sections 5.2 and 5.4]. Since Λ is an order complete AM-space with unity, $\mathscr{X}{\Downarrow}$ is a Banach mixed normed space over Λ; hence, a B-cyclic Banach space. We will often identify $B(\mathbb{C})$ with the restricted descent $\mathscr{C}{\Downarrow}$, where $\mathscr{C}$ is the complexes within $\mathbb{V}^{(B)}$. Sometimes we will call $B(\mathbb{C})$ a *Stonean algebra with base* B and denote it also by $\mathscr{S}(B)$.

Consider B-cyclic Banach algebras A_1 and A_2. Say that a bounded linear operator $\Phi : A_1 \to A_2$ is a *B-homomorphism* provided that Φ is B-linear and multiplicative: $b \circ T = T \circ b$ for all $b \in B$ and $\Phi(xy) = \Phi(x) \cdot \Phi(y)$ for all $x, y \in A_1$. If A_1 and A_2 are involutive and a B-homomorphism Φ is involution-preserving, i.e., $\Phi(x^*) = \Phi(x)^*$ for all $x \in A_1$; then Φ is a *$*$-B-homomorphism*. So, A_1 and A_2 are B-isomorphic provided that there is an isomorphism from A_1 onto A_2 which commutes with all projections in B. If a B-isomorphism Φ preserves involution then Φ is a *$*$-B-isomorphism*.

1.5. Theorem. *The restricted descent of a Banach algebra within $\mathbb{V}^{(B)}$ is a B-cyclic Banach algebra. Conversely, for each B-cyclic Banach algebra A there is a Banach algebra $\mathscr{A}$ within the Boolean valued universe $\mathbb{V}^{(B)}$ which is unique up to isomorphism and such that A is isometrically B-isomorphic to the restricted descent of $\mathscr{A}$.*

Proof. Assume that $\mathscr{A}$ is a Banach algebra within $\mathbb{V}^{(B)}$ and A is the restricted descent of $\mathscr{A}$. Clearly, A is a B-cyclic Banach space; see [1, Theorem 5.5.7]. If χ is a canonical isomorphism of B onto the algebra of unit elements $\mathfrak{C}(B(\mathbb{R}))$, then $b \leq [\![x = 0]\!] \leftrightarrow \chi(b)x = 0$ for all $x \in A$; see [1, Theorem 5.4.1]. Using the definition of χ and the obvious implication within $\mathbb{V}^{(B)}$

$$\chi(b) = 0 \vee \chi(b) = 1 \to \chi(b)xy = (\chi(b)x)y = x\,(\chi(b)y) \quad (x, y \in \mathscr{A}),$$

for all $x, y \in A$ we have

$$[\![\chi(b)xy = x\chi(b)y = (\chi(b)x)y]\!] \geq [\![\chi(b) = 1]\!] \vee [\![\chi(b) = 0]\!] = b \vee b^* = \mathbb{1}.$$

If follows that the projection $\pi_b : x \mapsto \chi(b)x$ with $x \in A$ is the one required: $\pi_b xy = (\pi_b x)y = x\,(\pi_b y)$ for all $x, y \in A$. Hence, A is a B-cyclic Banach algebra.

Assume now that A is a B-cyclic Banach algebra. By [1, Theorem 5.4.7] there is a Banach space $\mathscr{A}$ within $\mathbb{V}^{(B)}$ such that the restricted descent A_0 of $\mathscr{A}$ is a B-cyclic Banach space isometrically isomorphic to A. Without loss of generality, we may suppose that $A_0 = A$. Multiplication on A is extensional. Indeed, if $b \leq [\![x = u]\!] \wedge [\![y = v]\!]$, where $x, y, u, v \in A$; then by [1, Theorem 5.4.1] we have

$$0 = x\chi(b)\,(y - v) + \chi(b)\,(x - u)v \to \chi(b)\,(xy - uv) = 0$$
$$\to \chi(b)\,(xy) = \chi(b)\,uv \to b \leq [\![xy = uv]\!].$$

Let $\odot$ be the ascent of multiplication $\cdot$ on A. Clearly, $\odot$ is a binary operation in $\mathscr{A}$ and the vector space with $\odot$ is an algebra. If p is a vector norm on A then $\|a\| = \|p(a)\|_\infty$ and $[\![p(a) = \rho(a)]\!] = \mathbb{1}$ for all $a \in \mathscr{A}$, with ρ the norm on $\mathscr{A}$. Show that p is submultiplicative, i.e., $p(xy) \leq p(x)p(y)$ for all $x, y \in A$. Recall that A is a Banach module over the ring $B(\mathbb{R})$, where $B(\mathbb{R})$ is the bounded part of $\mathscr{R}{\downarrow}$, and for p we have

$$p(x) = \inf\{\alpha \in B(\mathbb{R})^+ : x \in \alpha U_A\} \quad (x \in A).$$

Consequently, the submultiplication property of p follows from the fact that by the definition of a Banach algebra A the unit ball U_A is invariant under multiplication; i.e., $xy \in U_A$ for all $x, y \in U_A$. Hence, $p \circ (\cdot) \leq (\cdot) \circ (p \times p)$. Using the rules of ascending from [1, Theorem 3.3.10], we infer that $[\![\rho \circ \odot \leq \odot \circ (\rho \times \rho)]\!] = \mathbb{1}$, i.e., $[\![\rho \text{ is submultiplicative}]\!] = \mathbb{1}$. We conclude that $\mathscr{A}$ is a Banach algebra within $\mathbb{V}^{(B)}$.

Let us turn to demonstrating the uniqueness of $\mathscr{A}$ by the ascending-descending machinery; see [1, Chapter 3]. Assume that $\mathscr{A}_1$ and $\mathscr{A}_2$ are Banach algebras within $\mathbb{V}^{(B)}$, while g is an isometric B-isomorphism of their restricted descents. Then g is extensional and $\psi := g\uparrow$ is a linear isometry of the Banach spaces $\mathscr{A}_1$ and $\mathscr{A}_2$. Note that ψ is multiplicative because

$$\psi \circ \odot = g\uparrow \circ (\cdot)\uparrow = (g \circ (\cdot))\uparrow = ((\cdot) \circ (g \times g))\uparrow = (\cdot)\uparrow \circ (g\uparrow \times g\uparrow) = \odot \circ (\psi \times \psi),$$

where $\odot$ signifies the multiplication in either of the algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ whereas $(\cdot)$ is the multiplication of either of the restricted descents of the algebras. □

1.6. Theorem. *The restricted descent of a C^*-algebra within $\mathbb{V}^{(B)}$ is a B-cyclic C^*-algebra. Conversely, for each B-cyclic C^*-algebra A there is a C^*-algebra $\mathscr{A}$ within $\mathbb{V}^{(B)}$ unique up to $*$-isomorphism and such that the restricted descent of $\mathscr{A}$ is an algebra $*$-B-isomorphic to A.*

Proof. If A is a B-cyclic C^*-algebra then the structure of the Banach $\mathscr{S}(B)$-module enjoys the extra property on A; i.e., $(\alpha x)^* = \alpha x^*$ for all $\alpha \in B(\mathbb{R})$ and $x \in A$) (as usual, $B(\mathbb{R})$ is the real part of the complex Banach algebra $\mathscr{S}(B)$). Indeed, if

$$\alpha := \sum_{k=1}^{n} \lambda_k \pi_k,$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\pi_1, \dots, \pi_n \in \mathfrak{E}(\mathscr{S}(B))$; then

$$(\alpha x)^* = \sum_{k=1}^{n} \lambda_k (\pi_k x)^* = \sum_{k=1}^{n} \lambda_k \pi_k x^* = \alpha x^*.$$

The involution in a C^*-algebra is an isometry, and so $U_A^* = U_A$. The above implies that

$$x \in \alpha U_A \leftrightarrow xx^* \in \alpha^2 U_A \quad (x \in A,\, \alpha \in \mathscr{S}(B)).$$

Hence, $p(xx^*) = p(x)^2$ and, in particular, the involution is an isometry with respect to the vector norm p; i.e., $p(x^*) = p(x)$ for all $x \in A$. Note also that if $(\mathscr{A}, \rho)$ is a Banach algebra within $\mathbb{V}^{(B)}$, while A is the restricted descent of A and p is the restriction of $\rho\downarrow$ to A; then the ascent of the involution in A satisfies the condition $[\![(\forall\, x \in \mathscr{A})\rho(xx^*) = \rho(x)^2]\!] = \mathbb{1}$ if and only if $p(xx^*) = p(x)^2$ for all $x \in A$. It suffices to apply 1.5 and implement some elementary checks. □

1.7. Theorem. *Let A be a B-cyclic Banach algebra such that x is invertible provided that we have $(\forall\, b \in B)(bx = 0 \to b = \mathbb{0})$ for all $x \in A$. Then A is isometrically B-isomorphic to the Stonean algebra with base B.*

Proof. By 1.5, we may assume that A is the restricted descent of some Banach algebra $\mathscr{A} \in \mathbb{V}^{(B)}$. The hypothesis implies that every nonzero element of $\mathscr{A}$ is invertible. Indeed, put

$$\begin{aligned} c :=& [\![(\forall\, x)\, (x \in \mathscr{A} \wedge x \neq \mathbb{0} \to (\exists\, z)(z = x^{-1}))]\!] \\ =& \bigwedge \{[\![(\exists\, z)(z = x^{-1})]\!] :\, x \in A, [\![x \neq 0]\!] = \mathbb{1}\}. \end{aligned}$$

Clearly, $[\![x \neq 0]\!] = \mathbb{1}$ is tantamount to $\chi(b)x = 0 \leftrightarrow b = \mathbb{0}$. Hence, if $[\![x \neq 0]\!] = \mathbb{1}$ then there is $x^{-1} \in A$ and $[\![(\exists\, z)(z = x^{-1})]\!] = \mathbb{1}$. Therefore, $c = \mathbb{1}$. The algebra $\mathscr{A}$ is isometrically isomorphic to the complexes $\mathscr{C}$ by the Gelfand–Mazur Theorem, and so A is isometrically B-isomorphic to the restricted descent of $\mathscr{C}$, i.e., the Stonean algebra with base B. □

1.8. Theorem. *Let A be a B-cyclic Banach algebra with unity e, let $\Lambda := \mathscr{S}(B)$ be the Stonean algebra with base B and unity $\bar{\mathbb{1}}$, and let $\Phi : A \to \Lambda$ be some B-linear operator. Assume that $\Phi(e) = \bar{\mathbb{1}}$ and $e_{\Phi(x)} = \bar{\mathbb{1}}$ for every invertible $x \in A$. Then Φ is multiplicative, i.e., $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in A$.*

Proof. Arguing along the lines of 1.7, put $\phi := \Phi\uparrow$. Then we obtain

$$[\![\phi : \mathscr{A} \to \mathscr{C} \text{ is a linear functional}]\!] = \mathbb{1}$$

and $[\![\phi(e) = 1]\!] = [\![\phi(x) \neq 0 \text{ for every invertible } x \in A]\!] = \mathbb{1}$. By the Gleason–Kahane–Żelazko Theorem [13] $[\![\phi$ is a multiplicative functional$]\!] = \mathbb{1}$. This implies that Φ is multiplicative along the lines of 1.5, which proves that p is submultiplicative. □

1.9. Theorem. *Assume that A and Λ are as in* 1.8, *while A is involutive and commutative. Denote the set of all positive B-linear operators $\Psi : A \to \Lambda$ such that $\Psi(e) \leq \bar{\mathbb{1}}$ by K. If $\Phi \in K$ then the following are equivalent:*[1)]

(1) $\Phi(xy) = \Phi(x)\Phi(y)$ *for all* $x, y \in A$;

(2) $\Phi(xx^*) = \Phi(x)\Phi(x^*)$ *for all* $x \in A$;

(3) $\Phi \in \operatorname{ext}(K)$.

Proof. Using the previous notation, we may assert that $[\![\mathscr{A}$ is an involutive commutative Banach algebra while $\phi : \mathscr{A} \to \mathscr{C}$ is a positive functional and $\phi(e) \leq 1]\!] = \mathbb{1}$. Let $\mathscr{K}$ be the set of all positive functionals ψ on $\mathscr{A}$ such that $\psi(e) \leq 1$. Clearly, $\psi \mapsto (\psi{\downarrow}) \restriction A$ is an affine bijection λ between the convex sets $\mathscr{K}{\downarrow}$ and $\overline{K} := \{\Psi{\uparrow} : \Psi \in K\}$. Showing that $[\![\psi \in \operatorname{ext}(\mathscr{K})]\!] = \mathbb{1} \leftrightarrow \lambda\psi \in \operatorname{ext}(K)$, we are left with applying the scalar version (i.e., the case of $\Lambda = \mathscr{C}$) of the required fact within $\mathbb{V}^{(B)}$. Let $\operatorname{Ext}(K)$ stand for the set of $\Psi \in K$ satisfying the condition: $\alpha_1\Psi = \alpha_1\Psi_1$ and $\alpha_2\Psi = \alpha_2\Psi_2$ for all $\alpha_1, \alpha_2 \in \Lambda^+$ and $\Psi_1, \Psi_2 \in K$ such that $\alpha_1 + \alpha_2 = \bar{\mathbb{1}}$ and $\alpha_1\Psi_1 + \alpha_2\Psi_2 = \Psi$. Straightforward calculation of truth values easily demonstrate that $[\![\psi \in \operatorname{ext}(\mathscr{K})]\!] = \mathbb{1}$ if and only if $\lambda\psi \in \operatorname{Ext}(K)$. Moreover, we obviously have $\operatorname{Ext}(K) \subset \operatorname{ext}(K)$, and so we are to substantiate the reverse inclusion. Take $\Psi \in \operatorname{ext}(K)$, and let $\alpha_1, \alpha_2, \Psi_1$, and Ψ_2 be the same as in the definition of $\operatorname{Ext}(K)$. Then

$$\Psi = \frac{1}{2}(\alpha_1\Psi_1 + \alpha_2\Psi_2) + \frac{1}{2}(\alpha_1\Psi + \alpha_2\Psi) = \frac{1}{2}(\alpha_1\Psi + \alpha_2\Psi_2) + \frac{1}{2}(\alpha_1\Psi_1 + \alpha_2\Psi).$$

Hence, $\alpha_1\Psi = \alpha_1\Psi_1$ and $\alpha_2\Psi = \alpha_2\Psi_2$, i.e., $\Psi \in \operatorname{Ext}(K)$. □

1.10. Denote the set of all homomorphisms from A_1 to A_2 by B-$\operatorname{Hom}(A_1, A_2)$. Assume further that $\operatorname{Hom}^B(\mathscr{A}_1, \mathscr{A}_2)$ is the member of $\mathbb{V}^{(B)}$ which represents the set of all homomorphisms from $\mathscr{A}_1$ to $\mathscr{A}_2$.

1.10(1). *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be Banach algebras within $\mathbb{V}^{(B)}$ and let A_1 and A_2 be the restricted descents of $\mathscr{A}_1$ and $\mathscr{A}_2$. If $\Phi \in B$-$\operatorname{Hom}(A_1, A_2)$ and $\phi := \Phi{\uparrow}$, then $[\![\phi \in \operatorname{Hom}^B(\mathscr{A}_1, \mathscr{A}_2)]\!] = \mathbb{1}$ and $[\![\|\phi\| \leq C^\wedge]\!] = \mathbb{1}$ for some $C \in \mathbb{R}$. The mapping $\Phi \mapsto \phi$ is an isometric bijection between B-$\operatorname{Hom}(A_1, A_2)$ and $\operatorname{Hom}^B(\mathscr{A}_1, \mathscr{A}_2){\Downarrow}$.*

Proof. Everything but the multiplicative property is obvious; see [1, Theorem 5.4.9]. Demonstration that ϕ and Φ are multiplicative proceeds along the lines of proving uniqueness in 1.5. □

1.10(2). *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be involutive Banach algebras within $\mathbb{V}^{(B)}$, while $\Phi \in B$-$\operatorname{Hom}(A_1, A_2)$ and $\phi \in \operatorname{Hom}^B(\mathscr{A}_1, \mathscr{A}_2)$ correspond to each other by the bijection in* 1.10(1). *Then*

$$[\![\phi \text{ is involution-preserving}]\!] = \mathbb{1}$$

holds if and only if Φ is involution-preserving.

Proof. Cp. 1.4 and 1.6. □

1.11. Theorem. *Let $\mathscr{A}$ be an involutive Banach algebra within $\mathbb{V}^{(B)}$ and let A be the restricted descent of $\mathscr{A}$. Then $x \in A$ is hermitian* (*or positive, or a projection, or a central projection*) *if and only if $[\![x$ is hermitian* (*or positive, or a projection, or a central projection*)$]\!] = \mathbb{1}$.

Proof. This is obvious. □

2. AW^*-Algebras

Here we will deal with the Boolean valued realization of AW^*-algebras.

2.1. Recall that an *AW^*-algebra* is defined as a C^*-algebra that is simultaneously a Baer $*$-algebra. In more detail, an AW^*-algebra is a C^*-algebra whose every right annihilator has the form eA, with e a projection. Note in passing that the better name for an AW^*-algebra would be a *Baer C^*-algebra.*

[1)] As usual, $\operatorname{ext}(K)$ is the set of extreme points of a convex set K.

A C^*-algebra A is an AW^*-algebra if and only if the following hold:

(1) *each family of pairwise orthogonal elements of the poset of projections $\mathfrak{P}(A)$ has a supremum;*

(2) *each maximal commutative $*$-subalgebra A_0 of A is a complex Kantorovich space of bounded elements.*

The space $\mathscr{L}(H)$ of bounded linear operators in a complex Hilbert space H exemplifies an AW^*-algebra. The Banach algebra structure in $\mathscr{L}(H)$ involves the usual addition and multiplication of operators together with the classical operator norm. The involution in $\mathscr{L}(H)$ sends an operator to its hermitian adjoint. Note again that a commutative AW^*-algebra called a *Stone algebra* is a complex Kantorovich space of bounded elements and the multiplication unity is a strong order unit. The next theorem is the classical basic tool of operator theory. We sketch a proof for the reader's convenience.

2.2. Spectral Theorem. *To each hermitian element a of an AW^*-algebra A there corresponds the unique resolution of identity $\lambda \mapsto e_\lambda$ with $\lambda \in \mathbb{R}$ in $\mathfrak{P}(A)$ such that*

$$a = \int\limits_{-\|a\|}^{\|a\|} \lambda \, de_\lambda.$$

Moreover, $ax = xa$ for $x \in A$ if and only if $xe_\lambda = e_\lambda x$ for all $\lambda \in \mathbb{R}$.

Proof. In much the same way as in the case of Boolean algebras, by a *resolution of identity* in $\mathfrak{P}(A)$ we mean a function $e : \lambda \mapsto e_\lambda$ with $\lambda \in \mathbb{R}$ and $e_\lambda \in \mathfrak{P}(A)$ which satisfies the three conditions:

(1) $s \le t \ \to \ e_s \le e_t$ for all $s, t \in \mathbb{R}$;

(2) $\bigvee_{t\in\mathbb{R}} e_t = \mathbb{1}$ and $\bigwedge_{t\in\mathbb{R}} e_t = \mathbb{0}$;

(3) $\bigvee_{s\in\mathbb{R}, s<t} e_s = e_t$ for all $t \in \mathbb{R}$.

By 2.1(2) the maximal commutative $*$-subalgebra A_0 of A, containing a and endowed with the induced order, is a complex Kantorovich space. The unity in A_0 is the element $\mathbb{1}$ of A. The components of $\mathbb{1}$ in the Kantorovich space A_0 are projections in A. Indeed, if $e \in \mathfrak{E}(\mathbb{1}) := \mathfrak{E}(A_0)$ then $e(\mathbb{1} - e) \le e\mathbb{1} = e$ and $e(\mathbb{1} - e) \le \mathbb{1} - e$ because the product of two commuting positive elements is positive by 2.1(2). Therefore, $0 \le e(\mathbb{1} - e) \le e \wedge (\mathbb{1} - e) = \mathbb{0}$, which implies that $e(\mathbb{1} - e) = \mathbb{0}$ and $e^2 = e$. If we take as e^a_λ the unity e^a_λ in the Kantorovich space A_0, then the sought representation follows from the Freudenthal Theorem [3, Theorem 6]. Commutativity ensues since a and $\{e_\lambda : \lambda \in \mathbb{R}\}$ generate the same maximal $*$-subalgebra. □

Recall that a subalgebra B_0 of a Boolean algebra B is *regular* provided that B_0 contains all existent joins and meets of arbitrary subsets of B_0.

2.3. Theorem. *Each AW^*-algebra A is a B-cyclic C^*-algebra for every regular subalgebra B of the complete Boolean algebra $\mathfrak{P}_c(A)$.*

Proof. Let U be the unit ball of A. It suffices to check that for every partition of unity $(b_\xi)_{\xi\in\Xi} \subset B$ and every family $(a_\xi)_{\xi\in\Xi} \subset U$ there is a unique $a \in U$ satisfying $b_\xi a_\xi = b_\xi a$ for all $\xi \in \Xi$.

Assume firstly that a_ξ is hermitian for every $\xi \in \Xi$. Then $(b_\xi a_\xi)$ consists of pairwise commuting hermitian elements, since $(b_\xi a_\xi) \cdot (b_\eta a_\eta) = (b_\xi b_\eta) \cdot (a_\xi a_\eta) = 0$ for $\xi \ne \eta$. Assume now that A_0 is the maximal commutative $*$-subalgebra of A, which includes $(b_\xi a_\xi)$. By 2.1(2) A_0 is a complex Kantorovich space of bounded elements. Hence, there exists $a := o\text{-}\sum_{\xi\in\Xi} b_\xi a_\xi$, where the o-sum is calculated in A_0. Clearly, $b_\xi a_\xi = b_\xi a$ for all $\xi \in \Xi$. Also, $-\mathbb{1} \le a_\xi \le \mathbb{1}$ implies that $-\mathbb{1} \le a \le \mathbb{1}$. Consequently, $\|a\| \le 1$.

Let us prove uniqueness. Assume that for some hermitian $d \in A$ we have $b_\xi d = 0$ for all $\xi \in \Xi$. It is well known[2)] that

$$e^{b_\xi d}_\lambda = b^\perp_\xi \vee e^d_\lambda = \mathbb{1} = e^{\mathbb{1}}_\lambda \quad (\lambda \in \mathbb{R},\ \lambda > 0),$$
$$e^{b_\xi d}_\lambda = b_\xi \wedge e^d_\lambda = \mathbb{0} = e^{\mathbb{0}}_\lambda \quad (\lambda \in \mathbb{R},\ \lambda \le 0).$$

[2)] Cp. [1, Theorem 5.2.6(10)].

Note that $b_\xi^\perp \vee e_\lambda^d = \mathbb{1}$ and $b_\xi \wedge e_\lambda^d = \mathbb{0}$ are equivalent to the corresponding inequalities $e_\lambda^d \geq b_\xi$ and $e_\lambda^d \leq b_\xi^\perp$. Hence, $e_\lambda^d = \mathbb{1}$ if $\lambda > 0$ and $e_\lambda^d = \mathbb{0}$ if $\lambda \leq 0$; i.e., the spectral function of d coincides with the spectral function of 0. Therefore, $d = 0$.

In the general case of arbitrary $a_\xi \in U$ we will use the representation $a_\xi = u_\xi + iv_\xi$, where i is the imaginary unity while u_ξ and v_ξ are uniquely defined hermitian elements of U. By the above, there are hermitian $u, v \in U$ such that $b_\xi u = b_\xi u_\xi$ and $b_\xi v = b_\xi v_\xi$ for all $\xi \in \Xi$. So, $a = u + iv$ is the required element. Indeed, $b_\xi a = b_\xi a_\xi$ for all $\xi \in \Xi$. Also, the hermitian elements $a_\xi^* a_\xi$ belong to U and $b_\xi a^* a = b_\xi a_\xi^* a_\xi$ for all $\xi \in \Xi$. Since a^*a, satisfying these conditions, is unique; therefore, $a^*a \in U$. But then $a \in U$, as $\|a\|^2 = \|a^*a\| \leq 1$. □

2.4. Theorem. *Let $\mathscr{A}$ be an AW^*-algebra within $\mathbb{V}^{(B)}$ and let A be the restricted descent of $\mathscr{A}$. Then A is an AW^*-algebra, and $\mathfrak{P}_c(A)$ has a regular subalgebra isomorphic to B. Conversely, if A is an AW^*-algebra such that B is a regular subalgebra of $\mathfrak{P}_c(A)$, then there is an AW^*-algebra $\mathscr{A}$ unique up to $*$-isomorphism within $\mathbb{V}^{(B)}$ whose restricted descent is $*$-B-isomorphic to A.*

Proof. By 1.6 and 2.3 it suffices to check that the C^*-algebras A and $\mathscr{A}$ are Baer, which is elementary on using the Escher rules[3)] for annihilators together with 1.11. □

2.5. The *center* of an AW^*-algebra A is as usual the set of $z \in A$ commuting with all elements of A; i.e., $\mathscr{Z}(A) := \{z \in A : (\forall x \in A)\, xz = zx\}$. Clearly, $\mathscr{Z}(A)$ is a commutative AW^*-subalgebra of A and $\lambda\mathbb{1} \in \mathscr{Z}(A)$ for all $\lambda \in \mathbb{C}$. If $\mathscr{Z}(A) = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$ then A is an *AW^*-factor*.

Theorem. *If $\mathscr{A}$ is an AW^*-factor within $\mathbb{V}^{(B)}$, then the restricted descent of A is an AW^*-algebra whose Boolean algebra of central projections is isomorphic to B. Conversely, if A is an AW^*-algebra and $B := \mathfrak{P}_c(A)$, then there is a factor $\mathscr{A}$ within $\mathbb{V}^{(B)}$ unique up to $*$-isomorphism and such that the restricted descent of $\mathscr{A}$ is $*$-B-isomorphic to A.*

Proof. Use 2.4 on recalling that the descent of the binary Boolean algebra is isomorphic to B. □

2.6. Let us recall the classification of AW^*-algebras into types and show that the type is preserved in the Boolean valued realization. The type of an algebra is determined by the structure of its projection lattice. Hence, we are to track what happens with projections in passage to a Boolean valued universe.

Take an arbitrary AW^*-algebra A. Clearly, the order $\leq$ on the projection set $\mathfrak{P}(A)$, introduced in 1.1 and 1.2, may be given by the formula

$$q \leq p \leftrightarrow q = qp = pq \quad (q, p \in \mathfrak{P}(A)).$$

Projections p and q are *equivalent*, in writhing $p \sim q$, provided that there is $x \in A$ satisfying $x^*x = p$ and $xx^* = q$. If this event x is called a *partial isometry* with *initial projection* p and *final projection* q. Clearly, $\sim$ is indeed an equivalence on $\mathfrak{P}(A)$.

A projection $\pi \in A$ is called

(a) *abelian* provided that the algebra $\pi A \pi$ is commutative;

(b) *finite* provided that $\pi \sim \rho \leq \pi$ implies $\rho = \pi$ for every projection $\rho \in A$;

(c) *infinite* provided that π is not finite;

(d) *purely infinite* provided that π contains no finite projections.[4)]

Recall the definitions of the types of AW^*-algebras. Given an AW^*-algebra A, say that A is of *type I* whenever each nonzero projection in A contains a nonzero abelian projection; A is of *type II* whenever A has no nonzero abelian projection and each nonzero projection in A contains a nonzero finite projection; and, finally, A is *of type III* whenever the unity of A is a purely infinite projection. If the unity of A is a finite projection then A is *finite*. Say that an AW^*-algebra A is *λ-homogeneous*, with λ a cardinal, provided that A has some set $\mathscr{P}$ of pairwise orthogonal equivalent abelian projections such that $\sup \mathscr{P} = \mathbb{1}$ and the cardinality $|\mathscr{P}|$ of $\mathscr{P}$ is λ. Let $\pi \lesssim \rho$ mean that $\pi \sim \pi_0$ for some $\pi_0 \leq \rho$.

[3)] These are often called arrow cancelation rules; cp. [1, 3.3.12(6)].

[4)] The phrase "a projection π contains a projection ρ" means that $\rho \leq \pi$.

2.7. Theorem. *Let $\mathscr{A}$ be an AW^*-algebra within $\mathbb{V}^{(B)}$ and let A be the restricted descent of $\mathscr{A}$. Then, given an arbitrary projection $\pi \in \mathfrak{P}(A)$, we have*

(1) π is abelian $\leftrightarrow$ $[\![\pi$ is abelian$]\!] = \mathbb{1}$;

(2) π is finite $\leftrightarrow$ $[\![\pi$ is finite$]\!] = \mathbb{1}$;

(3) π is purely infinite $\leftrightarrow$ $[\![\pi$ is purely infinite$]\!] = \mathbb{1}$.

Proof. (1): This is obvious. Note that, given $\pi, \rho \in \mathfrak{P}(A)$, we may rewrite $\pi \sim \rho$, $\pi \leq \rho$ and $\pi \lesssim \rho$ as the algebraic identities (cp. 2.6):

$$\pi \sim \rho \leftrightarrow xx^* = \pi \wedge x^*x = \rho,$$
$$\pi \leq \rho \leftrightarrow \pi\rho = \rho\pi = \pi,$$
$$\pi \lesssim \rho \leftrightarrow \pi \sim \pi_0 \wedge \pi_0 \leq \rho.$$

Since the multiplication, involution, and equality in A are the descents of the similar objects in $\mathscr{A}$; therefore,

$$\pi \sim \rho \leftrightarrow [\![\pi \sim \rho]\!] = \mathbb{1},$$
$$\pi \leq \rho \leftrightarrow [\![\pi \leq \rho]\!] = \mathbb{1},$$
$$\pi \lesssim \rho \leftrightarrow [\![\pi \lesssim \rho]\!] = \mathbb{1}.$$

(2): Use the formulas

$$[\![(\forall\, x \in \mathscr{A})\, \varphi(x) \to \psi(x)]\!] = \bigwedge \big\{ [\![\psi(x)]\!] : \ x \in \mathscr{A}{\downarrow}, \ [\![\varphi(x)]\!] = \mathbb{1} \big\}$$

and $\mathfrak{P}(\mathscr{A}){\downarrow} = \mathfrak{P}(A)$. Fixing $\pi \in \mathfrak{P}(A)$, take the formulas $\pi \sim \rho \leq \pi$ and $\pi = \rho$ as $\varphi(\rho)$ and $\psi(\rho)$. Then we can write down the chain of equivalences

$$\begin{aligned}[\![\pi \text{ is finite}]\!] = \mathbb{1} &\leftrightarrow [\![(\forall\, \rho \in \mathfrak{P}(\mathscr{A}))\, \pi \sim \rho \leq \pi \to \pi = \rho]\!] = \mathbb{1} \\ &\leftrightarrow (\forall\, \rho \in \mathfrak{P}(A))\, [\![\pi \sim \rho \leq \pi]\!] = \mathbb{1} \to [\![\pi = \rho]\!] = \mathbb{1} \\ &\leftrightarrow (\forall\, \rho \in \mathfrak{P}(A))\, \pi \sim \rho \leq \pi \to \pi = \rho.\end{aligned}$$

(3): Proceed as in proving (2). □

2.8. Theorem. *Let A and $\mathscr{A}$ be as in 2.7. Then*

(1) A is finite $\leftrightarrow$ $[\![\mathscr{A}$ is finite$]\!] = \mathbb{1}$;

(2) A is of type I $\leftrightarrow$ $[\![\mathscr{A}$ is of type I$]\!] = \mathbb{1}$;

(3) A is of type II $\leftrightarrow$ $[\![\mathscr{A}$ is of type II$]\!] = \mathbb{1}$;

(4) A is of type III $\leftrightarrow$ $[\![\mathscr{A}$ is of type III$]\!] = \mathbb{1}$.

Proof. All claims result from definitions and 2.7. □

Recall that a *Kaplansky–Hilbert module* or *AW^*-module* X is a unital module over the Stone algebra Λ with a Λ-valued inner product $\langle\cdot,\cdot\rangle$ such that X is a Banach–Kantorovich space with the Λ-norm defined as $|x| := \sqrt{\langle x, x\rangle}$ for all $x \in X$; see the details in [7, Section 7].

2.9. Theorem. *Let X be a Kaplansky–Hilbert module over a Stone algebra Λ. Then the algebra $\mathscr{L}_\Lambda(X)$ of continuous Λ-linear operators in X is a type I AW^*-algebra with center isomorphic to Λ.*

Proof. Let B be the complete projection algebra of Λ. As known from the Boolean valued analysis of Kaplansky–Hilbert modules, there is a Hilbert space $\mathscr{X}$ within $\mathbb{V}^{(B)}$ such that X is the bounded descent of $\mathscr{X}$ and the algebra $\mathscr{L}_\Lambda(X)$ is $*$-B-isomorphic to the bounded descent $\mathscr{L}^B(\mathscr{X}){\Downarrow}$ of $\mathscr{L}^B(\mathscr{X}) := \mathscr{L}^B(\mathscr{X}, \mathscr{X})$; see [1, Theorems 6.2.7 and 6.2.9]. It suffices to note that $\mathscr{L}^B(\mathscr{X})$ is a type I AW^*-factor within $\mathbb{V}^{(B)}$ and apply 2.4 as well as 2.8(2). □

2.10. Theorem. *Let A be an arbitrary type I AW^*-algebra with center Λ. Then there is some Kaplansky–Hilbert module X over Λ such that A and $\mathscr{L}_\Lambda(X)$ are $*$-B-isomorphic.*

Proof. By 2.5 we may assume that A is a restricted descent of some AW^*-factor $\mathscr{A}$ within $\mathbb{V}^{(B)}$. In this event $\mathscr{A}$ is of type I by 2.8(2). As is known, each type I AW^*-factor is unitarily equivalent to $\mathscr{L}(\mathscr{X})$ for some Hilbert space $\mathscr{X}$ (for instance, [4, Theorem 7.5.8]). So, $\mathscr{A} \simeq \mathscr{L}(\mathscr{X})$, where $\mathscr{X}$ is some Hilbert space within $\mathbb{V}^{(B)}$. This implies that A is $*$-B-isomorphic to $\mathscr{L}_\Lambda(X)$, with X indicating the passage to the restricted descent of $\mathscr{X}$; see [1, Theorem 6.2.9]. $\square$

3. Boolean Dimension of a Kaplansky–Hilbert Module

To each Kaplansky–Hilbert module X we can ascribe some nonstandard cardinal that serves as the *Hilbert dimension* of the Boolean valued realization of X. Deciphering the concept leads to the definition of Boolean dimension.

3.1. Let X be a Kaplansky–Hilbert module over a Stone algebra Λ and let $B := \mathfrak{P}(\Lambda)$ which amounts to $\Lambda = \mathscr{S}(B)$. A subset $\mathscr{E}$ of X is *orthonormal* provided that

(1) $\langle x \,|\, y\rangle = 0$ for every two different $x, y \in \mathscr{E}$;

(2) $\langle x \,|\, x\rangle = \mathbb{1}$ for all $x \in \mathscr{E}$.

Call an orthonormal subset $\mathscr{E}$ of X a *basis* for X whenever

(3) $(\forall\, e \in \mathscr{E})\langle x \,|\, e\rangle = 0$ implies that $x = 0$.

A Kaplansky–Hilbert module X is λ-*homogeneous* if λ is a cardinal and X has a basis of cardinality λ and *homogeneous* if X is λ-homogeneous for some λ. Given $b \in B$ such that $\mathbb{0} \neq b$, denote by $\varkappa(b)$ the least cardinal γ such that the Kaplansky–Hilbert module $bX := \{bx : x \in X\}$ over $b\Lambda := \{b\lambda : \lambda \in \Lambda\}$ is γ-homogeneous. If X is homogeneous then $\varkappa(b)$ is defined for all $b \in B$ satisfying $\mathbb{0} \neq b$. It is convenient to agree that $\varkappa(\mathbb{0}) = 0$. Call a Kaplansky–Hilbert module X *strictly* γ-*homogeneous* provided that X is homogeneous and $\gamma = \varkappa(b)$ for all nonzero $b \in B$; and X *strictly homogeneous* provided that X is strictly λ-homogeneous for some cardinal λ.

If γ is a finite cardinal then the γ-homogeneity and strict γ-homogeneity coincide for every Kaplansky–Hilbert module. As usual, we let $|M|$ stand for the cardinality of a set M; i.e., $|M|$ is the cardinal bijective with M. Agree that throughout this section $(\mathscr{X}, (\cdot|\cdot))$ is the Boolean valued realization of a Kaplansky–Hilbert module $(X, \langle\cdot|\cdot\rangle)$.

3.2. Theorem. *For a Kaplansky–Hilbert module X to be λ-homogeneous it is necessary and sufficient that $[\![\dim(\mathscr{X}) = |\lambda^\wedge|]\!] = \mathbb{1}$.*

Proof. Using Escher rules we may assume that $X = \mathscr{X}{\Downarrow}$. If $x, y \in X$ and $a \in \Lambda$ then the relations $\langle x|y\rangle = a$ and $[\![(x|y) = a]\!] = \mathbb{1}$ are equivalent, since the mapping $\langle \cdot \,|\, \cdot\rangle$ and the descent of the form $(\cdot \,|\, \cdot)$ coincide on $X \times X$. This implies in particular that orthogonality in X is the restriction to X of the descent of orthogonality in $\mathscr{X}$.

It follows that $\mathscr{E} \subset X$ is orthonormal if and only if $[\![\mathscr{E}{\uparrow}$ is orthonormal in $\mathscr{X}]\!] = \mathbb{1}$. Using the Escher rules for polars to the orthogonal complements in X and $\mathscr{X}$, we see that $(\mathscr{E}{\uparrow})^{\perp}{\downarrow} = (\mathscr{E}{\uparrow}{\downarrow})^{\perp}$. Note also that $\mathscr{E}^{\perp} = (\mathscr{E}{\uparrow}{\downarrow})^{\perp}$. Hence, $\mathscr{E}^{\perp}{\uparrow} = (\mathscr{E}{\uparrow})^{\perp}$. In particular, $\mathscr{E}^{\perp} = \{0\}$ if and only if $[\![(\mathscr{E}{\uparrow})^{\perp} = \{0\}]\!] = \mathbb{1}$. Therefore, $\mathscr{E}$ is a basis for X only if $[\![\mathscr{E}{\uparrow}$ is a basis for $\mathscr{X}]\!] = \mathbb{1}$. In the case that $|\mathscr{E}| = \lambda$ and $\varphi : \lambda \to \mathscr{E}$ are bijections, the modified ascent $\varphi{\uparrow}$ is a bijection from $\lambda^\wedge$ onto $\mathscr{E}{\uparrow}$; i.e.,

$$[\![\dim(\mathscr{X}) = |\mathscr{E}{\uparrow}| = |\lambda^\wedge|]\!] = \mathbb{1}.$$

Conversely, assume that $\mathscr{D}$ is a basis for $\mathscr{X}$ and $[\![\psi : \lambda^\wedge \to \mathscr{D}$ is a bijection$]\!] = \mathbb{1}$ for some cardinal λ. Then the modified descent $\varphi := \psi{\downarrow} : \lambda \to \mathscr{D}{\downarrow}$ is injective. Consequently, the cardinality of $\mathscr{E} := \operatorname{im}(\varphi)$ is λ and, as noted above, $\mathscr{E}$ is orthonormal. We are done on observing that $\mathscr{D}{\downarrow} = \operatorname{mix}(\mathscr{E}) = \mathscr{E}{\uparrow}{\downarrow}$; i.e., $[\![\mathscr{E}{\uparrow} = \mathscr{D}]\!] = \mathbb{1}$, and so $\mathscr{E}$ is a basis for X. $\square$

3.3. Theorem. *For the strict λ-homogeneity of a Kaplansky–Hilbert module X it is necessary and sufficient that $[\![\dim(\mathscr{X}) = \lambda^\wedge]\!] = \mathbb{1}$.*

Proof. If X is strictly λ-homogeneous then X is λ-homogeneous and by 3.2 we have $[\![\dim(\mathscr{X}) = |\lambda^\wedge|]\!] = \mathbb{1}$. Also, there are a cardinal β and a partition of unity $(b_\alpha)_{\alpha\in\beta}$ of the Boolean algebra B such that $|\lambda^\wedge| = \operatorname{mix}_{\alpha\in\beta}(b_\alpha \alpha^\wedge)$. Since $b_\alpha \le [\![\mathscr{X} = b_\alpha \mathscr{X}]\!]$, we see that $b_\alpha \le [\![\dim(b_\alpha\mathscr{X}) = \alpha^\wedge]\!]$.

Put

$$B_\alpha := [\mathbb{0}, b_\alpha] := \{b' \in B : \ b' \le b_\alpha\}.$$

If $b_\alpha \ne \mathbb{0}$; then B_α is a complete Boolean algebra and $\mathbb{V}^{(B_\alpha)} \models$ "$b_\alpha\mathscr{X}$ is a Hilbert space and $\alpha^\wedge = \dim(b_\alpha\mathscr{X})$." The restricted descent of $b_\alpha\mathscr{X}$ in $\mathbb{V}^{(B_\alpha)}$ is $b_\alpha X$. Therefore, $b_\alpha X$ is an α-homogeneous Kaplansky–Hilbert module. Moreover, $\mathbb{V}^{(B_\alpha)} \models$ "$\alpha^\wedge$ is a cardinal." Consequently, α is a cardinal too as the standard name of α is a cardinal (see [1, 3.1.13]). By the definition of strict homogeneity, $\lambda \le \alpha$. Hence, $b_\alpha = \mathbb{0}$ if $\alpha < \lambda$ and so $[\![\lambda^\wedge \le |\lambda^\wedge|]\!] = \mathbb{1}$. Therefore, $[\![\lambda^\wedge = |\lambda^\wedge|]\!] = \mathbb{1}$, since $[\![|\lambda^\wedge| \le \lambda^\wedge]\!] = \mathbb{1}$ holds by the definition of cardinality. Finally, we may conclude that $[\![\dim(\mathscr{X}) = \lambda^\wedge]\!] = \mathbb{1}$.

Assume now that the last equality is valid. Then λ is a cardinal because $\lambda^\wedge$ is a cardinal within $\mathbb{V}^{(B)}$. By 3.2 X is a λ-homogeneous module. Provided that X is γ-homogeneous with some cardinal γ, we again use 3.2 to conclude that $[\![\dim(\mathscr{X}) = |\gamma^\wedge|]\!] = \mathbb{1}$. Therefore,

$$[\![\lambda^\wedge = |\gamma^\wedge| \le \gamma^\wedge]\!] = \mathbb{1}$$

and so $\lambda \le \gamma$. The same arguments apply to the AW^*-algebra bX, with $\mathbb{0} \ne b \in B$, provided that we replace $\mathbb{V}^{(B)}$ with $\mathbb{V}^{([\mathbb{0},b])}$. Thus, the Kaplansky–Hilbert module X is strictly λ-homogeneous. $\square$

3.4. Theorem. *Let X be a Kaplansky–Hilbert module over Λ. The mapping $\varkappa$ preserves the suprema of nonempty sets; i.e., $\varkappa(\sup(D)) = \sup(\varkappa(D))$ for each nonempty $D \subset \operatorname{dom}(\varkappa) \subset B$.*

Proof. Put $\bar b := \sup D$. Note that $\varkappa$ is increasing by definition; i.e., $b_1 \le b_2 \ \to\ \varkappa(b_1) \le \varkappa(b_2)$. So $\sup_{b\in D}\varkappa(b) \le \varkappa(\bar b)$. Let us demonstrate the reverse inequality. If $b \in B$ then the set of cardinals $\{\varkappa(b') : \mathbb{0} \ne b' \le b\}$ has a least element, say, $\gamma := \varkappa(b_0)$. The choice of b_0 shows that $b_0 \ne \mathbb{0}$ and $\varkappa(b_0) = \varkappa(b')$ for all nonzero $b' \le b_0$. Hence, D', consisting of all $b \in B$ such that bX is strictly homogeneous, is coinitial to D. By the exhaustion principle (see [3, 2.1]) there is a disjoint decomposition $(b_\xi)_{\xi\in\Xi}$ of $\bar b$ such that $b_\xi X$ is a strictly $\varkappa(b_\xi)$-homogeneous Kaplansky–Hilbert module over $b_\xi\Lambda$. Let $\mathscr{E}_\xi := (e_{\gamma,\xi})_{\gamma<\varkappa(b_\xi)}$ be a basis for $b_\xi X$. Put $\lambda := \sup_{\xi\in\Xi}\varkappa(b_\xi)$ and $e_{\gamma,\xi} = 0$ if $\varkappa(b_\xi) \le \gamma < \lambda$. Define the family $\mathscr{E} := (e_\gamma)_{\gamma\in\lambda}$, where

$$e_\gamma := bo\text{-}\sum_{\xi\in\Xi} e_{\gamma,\xi} \quad (\gamma \in \lambda).$$

Note that $\mathscr{E}$ is orthonormal since

$$\begin{aligned}\langle e_\gamma \,|\, e_\beta\rangle &= \left\langle bo\text{-}\sum_{\xi\in\Xi} e_{\gamma,\xi} \,\middle|\, bo\text{-}\sum_{\eta\in\Xi} e_{\beta,\eta}\right\rangle = bo\text{-}\sum_{\xi,\eta\in\Xi}\langle e_{\gamma,\xi} \,|\, e_{\beta,\eta}\rangle \\ &= bo\text{-}\sum_{\xi,\eta\in\Xi}\langle b_\xi e_{\gamma,\xi} \,|\, b_\eta e_{\beta,\eta}\rangle = bo\text{-}\sum_{\xi,\eta\in\Xi} b_\xi b_\eta \langle e_{\gamma,\xi} \,|\, e_{\beta,\eta}\rangle =: e,\end{aligned}$$

while $e = 0$ if $\gamma \ne \beta$, and $e = \mathbb{1}$ if $\gamma = \beta$. The family $\mathscr{E}$ is a basis for $\bar b X$. Indeed, if $x \in X$ and $\langle x \,|\, e_\gamma\rangle = 0$ for all $\gamma \in \lambda$, then $\langle x \,|\, e_{\gamma,\xi}\rangle = 0$ for all $\xi \in \Xi$ and $\gamma < \varkappa(b_\xi)$. Thus, $b_\xi x \perp \mathscr{E}_\xi$, implying that $b_\xi x = 0$ and so $x = 0$. Since $|\mathscr{E}| \le \lambda$; using the definition of $\varkappa$, we infer that

$$\varkappa(\bar b) \le \lambda = \sup_{\xi\in\Xi}\varkappa(b_\xi) \le \sup_{b\in D}\varkappa(b). \quad \square$$

3.5. We introduce the main concept of Section 3.

A partition of unity $(b_\gamma)_{\gamma\in\Gamma}$ in B is the *B-dimension* of a Kaplansky–Hilbert module X provided that Γ is a nonempty set of cardinals, $b_\gamma \neq 0$ for all $\gamma \in \Gamma$, and $b_\gamma X$ is a strictly γ-homogeneous AW^*-module for every $\gamma \in \Gamma$. In this event we write $B\text{-}\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$. Note that the elements of a B-dimension are pairwise distinct by the definition of strict homogeneity. We will say that the B-dimension of X is γ (in symbols, $B\text{-}\dim(X) = \gamma$) provided that $\Gamma = \{\gamma\}$ and $b_\gamma = \mathbb{1}$. Observe that $B\text{-}\dim(X) = \gamma$ implies that X is strictly γ-homogeneous.

The function $\varkappa$ in 3.1 may be defined on the whole Boolean algebra $B := \mathfrak{P}(\Lambda)$. Let B' consist of $b' \in B$ such that $b'X$ is homogeneous. Extend $\varkappa$ from B' to the whole of B by letting $\varkappa(b) := \sup\{\varkappa(b') : b' \in B',\, b' < b\}$. This definition is sound in view of 3.4. The mapping $\varkappa$ is usually called the *multiplicity function* of X. Clearly, if $B\text{-}\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$ then $\varkappa(b) = \sup\{\gamma \in \Gamma : b \wedge b_\gamma \neq \mathbb{0}\}$.

3.6. Theorem. *Let $(b_\gamma)_{\gamma\in\Gamma}$ be a partition of unity in B, with Γ a nonempty set of cardinals and $b_\gamma \neq \mathbb{0}$ for all $\gamma \in \Gamma$. Then $B\text{-}\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$ if and only if $[\![\dim(\mathscr{X}) = \mathrm{mix}_{\gamma\in\Gamma}(b_\gamma\gamma^\wedge)]\!] = \mathbb{1}$.*

Proof. As was mentioned above, we may identify $b_\gamma X$ with the restricted descent of the Hilbert space $b_\gamma\mathscr{X}$ within $\mathbb{V}^{(B_\gamma)}$, where $B_\gamma := [\mathbb{0}, b_\gamma]$. By 3.3 the strict γ-homogeneity of $b_\gamma X$ means that

$$b_\gamma = [\![\dim(b_\gamma\mathscr{X}) = \gamma^\wedge]\!]^{B_\gamma} \le [\![\dim(\mathscr{X}) = \gamma^\wedge]\!]^{B}.$$

However, $B\text{-}\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$ if and only if $b_\gamma \le [\![\dim(\mathscr{X}) = \gamma^\wedge]\!]$ for all $\gamma \in \Gamma$, since

$$b_\gamma \le [\![\mathscr{X} = b_\gamma\mathscr{X}]\!] = [\![\dim(\mathscr{X}) = \dim(b_\gamma\mathscr{X})]\!].$$

Thus, $[\![\dim(\mathscr{X}) = \mathrm{mix}_{\gamma\in\Gamma}(b_\gamma\gamma^\wedge)]\!] = \mathbb{1}$ and $B\text{-}\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$ are equivalent. $\square$

3.7. Let us find out the partitions of unity that can serve as the B-dimensions of Kaplansky–Hilbert modules. To this end, we need the definition: If we are given $b \in B$ and $\beta \in \mathrm{On}$, where On stands for the class of all ordinals; then denote by $b(\beta)$ the set of all partitions of b of the form $(b_\alpha)_{\alpha\in\beta}$. We then introduce the $[\mathbb{0}, b]$-valued metric d on $b(\beta)$ by the formula

$$d(u, v) := \neg\Big(\bigvee_{\alpha\in\beta} u_\alpha \wedge v_\alpha\Big) \quad \big(u = (u_\alpha),\ \ v = (v_\alpha) \in b(\beta)\big).$$

This means that $\big(b(\beta), d\big)$ is a B-set.[5] The record $b(\beta) \simeq b(\gamma)$ for $\gamma \in \mathrm{On}$ means that there is a bijection between $b(\beta)$ and $b(\gamma)$ which preserves the *canonical Boolean metric* $d(x, y) := [\![x \neq y]\!] = \neg[\![x = y]\!]$; i.e., the bijection is a B-isometry.

Recall the obvious functional description of the B-set $(b(\beta), d)$. Note that Q_b is the clopen subset of the Stone space $\mathrm{St}(B)$ which corresponds to $b \in B$. Define $C_\infty(Q_b, \beta)$ as the set of continuous functions $f : \mathrm{dom}(f) \to \beta$ densely defined in Q_b and endow β with the discrete topology. Clearly, for every $f \in C_\infty(Q_b, \beta)$ there is a family of pairwise disjoint clopen sets (Q_α) with the dense union in Q_b and such that f is constant on every Q_α. Define the Boolean distance $d'(f, g)$ between $f, g \in C_\infty(Q_b, \beta)$ as the closure of the open set $\{q \in Q_b : f(q) \neq g(q)\}$. We establish some bijection between $(b(\beta), d)$ and $(C_\infty(Q_b, \beta), d')$ by assigning the constant α function on the clopen set corresponding to b_α to each entry of the family $(b_\alpha)_{\alpha\in\beta}$. Furthermore, $b(\beta) \simeq b(\gamma)$ means that there is a bijection $\jmath : C_\infty(Q_b, \beta) \to C_\infty(Q_b, \gamma)$ such that if f and g in $C_\infty(Q_b, \beta)$ coincide on some clopen set Q_0 then $\jmath(f)$ and $\jmath(g)$ coincide on Q_0 too.

Take some cardinal λ. Say that a Boolean algebra B is *λ-stable* provided that $b(\lambda) \simeq b(\alpha)$ implies $\lambda \le \alpha$ for all $\alpha \in \mathrm{On}$ and nonzero $b \in B$. The Stone space of a λ-stable B is said to be *λ-stable*. We call a nonzero $b \in B$ *λ-stable* provided that so is the Boolean algebra $[\mathbb{0}, b]$.

3.8. Theorem. *A pairwise disjoint partition of unity $(b_\gamma)_{\gamma\in\Gamma}$ in a complete Boolean algebra B is the B-dimension of some Kaplansky–Hilbert module if and only if Γ is a nonempty set of cardinals and b_γ is γ-stable for all $\gamma \in \Gamma$.*

Proof. Put $\lambda := \mathrm{mix}_{\gamma\in\Gamma}(b_\gamma\gamma^\wedge)$. There is a Hilbert space $\mathscr{X}$ within $\mathbb{V}^{(B)}$ such that

$$[\![\dim(\mathscr{X}) = |\lambda|]\!] = \mathbb{1}.$$

[5] Cp. [1, 2.5.7] and [12, 5.1].

By 3.6 B-$\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$ if and only if $[\![|\lambda| = \lambda]\!] = \mathbb{1}$. The latter amounts to the simultaneous inequalities

$$b_\gamma \le [\![|\gamma^\wedge| = \gamma^\wedge]\!] \quad (\gamma \in \Gamma).$$

Note that $b_\gamma \le [\![|\gamma^\wedge| = \gamma^\wedge]\!]$ with nonzero b_γ means that $\mathbb{V}^{([\mathbb{0},b_\gamma])} \models \gamma^\wedge = |\gamma^\wedge|$. Therefore, we are left with demonstrating that the γ-stability of the Boolean algebra $B_0 := [\mathbb{0}, b]$ and the event that $\mathbb{V}^{(B_0)} \models \gamma^\wedge = |\gamma^\wedge|$ are valid or invalid together.

Note that

$$\begin{aligned}[\![\gamma^\wedge = |\gamma^\wedge|]\!] &= [\![(\forall\, \alpha \in \mathrm{On})\,(\gamma^\wedge \sim \alpha \to \gamma^\wedge \le \alpha)]\!] \\ &= \bigwedge \big\{ [\![\gamma^\wedge \sim \alpha^\wedge]\!] \Rightarrow [\![\gamma^\wedge \le \alpha]\!] : \alpha \in \mathrm{On} \big\}.\end{aligned}$$

Clearly, $[\![\gamma^\wedge = |\gamma^\wedge|]\!] = \mathbb{1}$ only if $c := [\![\gamma^\wedge \sim \alpha^\wedge]\!] \le [\![\gamma^\wedge \le \alpha^\wedge]\!]$ for every $\alpha \in \mathrm{On}$. If $c \ne \mathbb{0}$ then $\gamma \le \alpha$. Also, $c \le [\![\gamma^\wedge \sim \alpha^\wedge]\!]$ means that $c(\gamma) \simeq c(\alpha)$. Thus, $[\![\gamma^\wedge = |\gamma^\wedge|]\!] = \mathbb{1}$ is tantamount to the γ-stability of B_0. □

3.9. Kaplansky–Hilbert modules X and Y over Λ are *unitarily equivalent* provided that there is a Λ-linear operator U from X to Y which keeps the inner product; i.e., $\langle Ux_1|Ux_2\rangle = \langle x_1|x_2\rangle$.

3.10. Theorem. *Kaplansky–Hilbert modules are unitarily equivalent if and only if they have the same Boolean dimension.*

Proof. Let $\mathscr{X}$ and $\mathscr{Y}$ be Boolean valued realization of X and Y. By transfer we see that the Kaplansky–Hilbert modules X and Y are unitarily equivalent if and only if $\mathscr{X}$ and $\mathscr{Y}$ are unitarily equivalent as Hilbert spaces within $\mathbb{V}^{(B)}$. It suffices to recall 3.6 and use the fact that Hilbert spaces are unitarily equivalent whenever they have the same Hilbert dimension. □

4. Functional Representation of Kaplansky–Hilbert Modules

In this section we will establish that each Kaplansky–Hilbert module may be represented as the direct sum of a family of modules of continuous vector functions and, moreover, this representation is unique in some sense.

Denote by $C_\#(Q,H)$ the subspace of $C_\infty(Q,H)$ which consists of the vector functions z such that $|z| \in C(Q)$, where $|z| : q \in Q \mapsto \|z(q)\|$ is the vector norm of z; see [1, 5.3.7(5, 6)].

4.1. Theorem. *Assume that Q is an extremally disconnected compact space, and H is a Hilbert space of dimension λ. The space $C_\#(Q,H)$ is a λ-homogeneous Kaplansky–Hilbert over the algebra $\Lambda := C(Q,\mathbb{C})$.*

Proof. Note firstly that $C_\#(Q,H)$ is a faithful unitary Λ-module with the pointwise multiplication of any vector function $u : \mathrm{dom}(u) \to H$ and any scalar function $\lambda \in \Lambda$; i.e., $\lambda u : q \mapsto \lambda(q)u(q)$ for all $q \in \mathrm{dom}(u)$. Let $(\cdot\,|\,\cdot)$ stand for the inner product in H. Then we introduce the Λ-valued inner product on $C_\#(Q,H)$ as follows: Take continuous vector functions $u : \mathrm{dom}(u) \to H$ and $v : \mathrm{dom}(v) \to H$. In this event $q \mapsto \langle\, u(q)|v(q)\,\rangle$ with $q \in \mathrm{dom}(u) \cap \mathrm{dom}(v))$ is continuous and has the unique continuation $z \in C_\infty(Q)$ on the whole Q. If x and y are the equivalence classes containing u and v, then we put $z := \langle x\,|\,y\rangle$. Clearly, $\langle\cdot\,|\,\cdot\rangle$ is a Λ-valued inner product and $|x| = \sqrt{\langle x\,|\,x\rangle}$ for all $x \in C_\#(Q,H)$. The pair $(C_\#(Q,H), |\cdot|)$ is a Banach–Kantorovich space, and so $C_\#(Q,H)$ is a Banach space under the mixed norm involving the Chebyshev norm $\|\cdot\|_\infty$; i.e.,

$$\|x\| = \||x|\|_\infty = \sqrt{\|(x\,|\,x)\|_\infty} \quad \big(x \in C_\#(Q,H)\big).$$

Consequently, $C_\#(Q,H)$ is a Kaplansky–Hilbert module over Λ. Assume that $\mathscr{E}$ is a basis for H. Given $e \in \mathscr{E}$, introduce the vector function $\bar{e} : q \mapsto e$ with $q \in Q$ and put $\overline{\mathscr{E}} := \{\bar{e} : e \in \mathscr{E}\}$. It is easy that $\overline{\mathscr{E}}$ is a basis for the module $C_\#(Q,H)$, which proves the λ-homogeneity of $C_\#(Q,H)$ with $\lambda = \dim(H)$. □

4.2. We will need another auxiliary fact. Denote the set of all linear combinations of elements of A with coefficients in a field $\mathbb{P}$ by $\mathbb{P}$-lin(A).

Theorem. *Let X be a vector space over a field $\mathbb{F}$ and let $\mathbb{P}$ be a subfield of $\mathbb{F}$. Then $X^\wedge$ is a vector space over the standard name $\mathbb{F}^\wedge$ and $(\mathbb{P}\text{-lin}(A))^\wedge = \mathbb{P}^\wedge\text{-lin}(A^\wedge)$ for all $A \subset X$.*

Proof. The first claim is obvious, since the proposition "X is a vector space over $\mathbb{F}$" is a restricted formula. Also, $(\mathbb{P}\text{-lin}(A))^\wedge$ is a $\mathbb{P}^\wedge$-linear subspace of $X^\wedge$ including $A^\wedge$. Hence, $\mathbb{P}^\wedge\text{-lin}(A^\wedge) \subset (\mathbb{P}\text{-lin}(A))^\wedge$. Conversely, let $x \in X$ be of the form $\sum_{k\in n} \alpha(k)\, u(k)$, with $n := \{0, 1, \dots, n-1\}$, $\alpha : n \to \mathbb{P}$, and $u : n \to A$. Then $\alpha^\wedge : n^\wedge \to \mathbb{P}^\wedge$, $u^\wedge : n^\wedge \to A^\wedge$, and $x^\wedge = \sum_{k\in n^\wedge} \alpha^\wedge(k) u^\wedge(k)$. Consequently $x^\wedge \in \mathbb{P}^\wedge$-lin$(A^\wedge)$, which proves that $(\mathbb{P}\text{-lin}(A))^\wedge \subset \mathbb{P}^\wedge\text{-lin}(A^\wedge)$. $\square$

4.3. Theorem. *Assume that H is a Hilbert space and $\lambda = \dim(H)$. Assume further that $\mathscr{H}$ is the completion of the metric space $H^\wedge$ within $\mathbb{V}^{(B)}$. Then $[\![\mathscr{H}$ is a Hilbert space and $\dim(\mathscr{H}) = |\lambda^\wedge|]\!] = \mathbb{1}$.*

Proof. By definition $\mathscr{H}$ is a Banach space. If $b(\cdot\,,\cdot)$ is the inner product on H, then $b^\wedge : H^\wedge \times H^\wedge \to \mathbb{C}^\wedge$ is a uniformly continuous function with the unique continuous extension to the whole of $\mathscr{H} \times \mathscr{H}$ which we will denote by $(\cdot\,|\,\cdot)$. So, $(\cdot\,|\,\cdot)$ is the inner product in $\mathscr{H}$, and it is easy to see that

$$\mathbb{V}^{(B)} \models \|x\| = \sqrt{(x\,|\,x)} \quad (x \in \mathscr{H}).$$

Hence, $[\![\mathscr{H}$ is a Hilbert space$]\!] = \mathbb{1}$. Let $\mathscr{E}$ be a Hilbert basis for H. Show that $[\![\mathscr{E}^\wedge$ is a basis for $\mathscr{H}]\!] = \mathbb{1}$. The definition of the inner product in $\mathscr{H}$ implies that $\mathscr{E}^\wedge$ is orthonormal as demonstrated as follows:

$$[\![(\forall\, x \in \mathscr{E}^\wedge)\ (x\,|\,x) = 1]\!]$$

$$= \bigwedge_{x\in\mathscr{E}} [\![(x^\wedge\,|\,x^\wedge) = 1]\!] = \bigwedge_{x\in\mathscr{E}} [\![b(x,x)^\wedge = \mathbb{1}^\wedge]\!] = \mathbb{1};$$

$$[\![(\forall\, x, y \in \mathscr{E}^\wedge)\ (x \neq y \to (x\,|\,y) = 0)]\!]$$

$$= \bigwedge_{x,y\in\mathscr{E}} [\![x^\wedge \neq y^\wedge]\!] \Rightarrow [\![(x^\wedge\,|\,y^\wedge) = 0]\!]$$

$$= \bigwedge_{x,y\in\mathscr{E};x\neq y} [\![b^\wedge(x^\wedge, y^\wedge) = 0]\!] = \bigwedge_{x,y\in\mathscr{E};x\neq y} [\![b(x,y)^\wedge = 0^\wedge]\!] = \mathbb{1}.$$

Since $H^\wedge$ is dense in $\mathscr{H}$ and $\mathbb{C}^\wedge\text{-lin}(\mathscr{E}^\wedge) \subset \mathscr{C}\text{-lin}(\mathscr{E}^\wedge)$, we are left with proving that $\mathbb{C}^\wedge\text{-lin}(\mathscr{E}^\wedge)$ is dense in $H^\wedge$. Take $x \in H$ and a real $\varepsilon > 0$. As $\mathscr{E}$ is a basis for H, find some $x_\varepsilon \in \mathbb{C}\text{-lin}(\mathscr{E})$ such that $\|x - x_\varepsilon\| < \varepsilon$. Hence, $[\![\|x^\wedge - x_\varepsilon{}^\wedge\| < \varepsilon^\wedge]\!] = \mathbb{1}$ and $[\![x_\varepsilon^\wedge \in (\mathscr{C}\text{-lin}(\mathscr{E}))^\wedge]\!] = \mathbb{1}$. Using 4.2, we see that the formula

$$(\forall\, x \in H)\, (\forall\, 0 < \varepsilon \in \mathbb{R}^\wedge)\, (\exists\, x_\varepsilon \in \mathbb{C}^\wedge\text{-}\operatorname{lin}(\mathscr{E}^\wedge)\, (\|x - x_\varepsilon\| < \varepsilon)$$

is valid within $\mathbb{V}^{(B)}$; i.e., $[\![\mathbb{C}^\wedge\text{-lin}(\mathscr{E}^\wedge)$ is dense in $H^\wedge]\!] = \mathbb{1}$. It suffices to observe that if φ is a bijection between $\mathscr{E}$ and some cardinal λ, then $\varphi^\wedge$ is a bijection between $\mathscr{E}^\wedge$ and $\lambda^\wedge$ within $\mathbb{V}^{(B)}$. $\square$

4.4. We will list a few useful consequences:

4.4(1). Corollary. *In the hypotheses of 4.3 the restricted descent of a Hilbert space $\mathscr{H}$ within $\mathbb{V}^{(B)}$ is unitarily equivalent to the Kaplansky–Hilbert module $C_\#(Q, H)$, where Q is the Stone space of B.*

Proof. This follows from 4.1 and [1, 5.4.10]. $\square$

4.4(2). Corollary. *Let M be a nonempty set. The restricted descent of the Hilbert space $l_2(M^\wedge)$ within $\mathbb{V}^{(B)}$ is unitarily equivalent to the Kaplansky–Hilbert module $C_\#(Q, l_2(M))$, where Q is the Stone space of B.*

Proof. Put $H = l_2(M)$ in 4.3 and recall that $[\![\dim(\mathscr{H}) = |M^\wedge|]\!] = \mathbb{1}$. It is clear now that $[\![\mathscr{H}$ and $l_2(M^\wedge)$ are unitarily equivalent$]\!] = \mathbb{1}$. Boundedly descending, we complete the proof. $\square$

4.4(3). Corollary. *Let $\lambda = \dim(H)$ be an infinite cardinal. The Kaplansky–Hilbert module $C_\#(Q, H)$ is strictly λ-homogeneous if and only if Q is a λ-stable compact space.*

Proof. It suffices to apply 3.3, 3.8, and 4.3. $\square$

4.4(4). Corollary. *If H_1 and H_2 are infinite-dimensional Hilbert spaces then there is an extremally disconnected compact space Q such that the Kaplansky–Hilbert modules $C_\#(Q, H_1)$ and $C_\#(Q, H_2)$ are unitarily equivalent.*

Proof. Put $\lambda_k := \dim(H_k)$ with $k := 1, 2$. There is a complete Boolean algebra B such that the ordinals $\lambda_1^\wedge$ and $\lambda_2^\wedge$ are of the same cardinality within $\mathbb{V}^{(B)}$ by the *cardinal shift*; see [1, 3.1.13(1)]. We are done on recalling 4.3 and 4.4(1). $\square$

4.4(5). Corollary. *Let H_k be a Hilbert space and $\lambda_k := \dim(H_k) \geq \omega$ with $k := 1, 2$. Assume that the Kaplansky–Hilbert modules $C_\#(Q, H_k)$ are strictly λ_k-homogeneous. If $C_\#(Q, H_1)$ and $C_\#(Q, H_2)$ are unitarily equivalent then so are H_1 and H_2.*

Proof. From 3.3, 4.3, and 4.4(1) it follows that $[\![\lambda_1^\wedge = |\lambda_1^\wedge| = |\lambda_2^\wedge| = \lambda_2^\wedge]\!] = \mathbb{1}$. Hence, $\lambda_1 = \lambda_2$. $\square$

Say that a Kaplansky–Hilbert module X is *B-separable* provided that there is a sequence $(x_n) \subset X$ such that the Kaplansky–Hilbert submodule, generated by $\{bx_n : n \in \mathbb{N}, b \in B\}$, coincides with X. Obviously, if H is a separable Hilbert space then the Kaplansky–Hilbert module $C_\#(Q, H)$ is B-separable.

4.4(6). Corollary. *To each infinite-dimensional Hilbert space H there is an extremally disconnected compact space Q such that the Kaplansky–Hilbert module $C_\#(Q, H)$ is B-separable, with B the Boolean algebra of the characteristic functions of clopen subsets of Q.*

Proof. Recalling 4.4(4) with $H_1 := l_2(\omega)$ and $H_2 := H$, proceed with using the separability of $l_2(\omega)$ within $\mathbb{V}^{(B)}$. $\square$

4.5. Theorem. *To every Kaplansky–Hilbert module X there is a family of nonempty extremally disconnected compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$, with Γ a set of cardinals and Q_γ a γ-stable space for all $\gamma \in \Gamma$, such that we have the unitary equivalence*

$$X \simeq \sum_{\gamma\in\Gamma}^{\oplus} C_\#\big(Q_\gamma, l_2(\gamma)\big).$$

If some family $(P_\delta)_{\delta\in\Delta}$ of extremally disconnected compact spaces satisfies the above properties, then $\Gamma = \Delta$ and P_γ is homeomorphic to Q_γ for every $\gamma \in \Gamma$.

Proof. By transfer we may assume that X is a restricted descent of some Hilbert space $\mathscr{X}$ in $\mathbb{V}^{(B)}$; see [1, Theorem 6.2.8]. Assume that $B\text{-}\dim(X) = (b_\gamma)_{\gamma\in\Gamma}$ and Q_γ is the clopen subset of the Stone space of B which corresponds to $b_\gamma \in B$ in the Stone representation; see [3, Chapter 3]. Recall that X is the direct sum of the modules of the form $b_\gamma X$, with $b_\gamma X$ unitarily equivalent to the restricted descent of the Hilbert space $b_\gamma \mathscr{X}$ within $\mathbb{V}^{(B_\gamma)}$, where $B_\gamma = [\mathbb{0}, b_\gamma]$. By 3.8 $b_\gamma \leq [\![\dim(b_\gamma \mathscr{X}) = \gamma^\wedge]\!]$. Therefore, if b_γ is nonzero then $\mathbb{V}^{(B_\gamma)} \models$ "$b_\gamma \mathscr{X}$ is a Hilbert space of dimension $\gamma^\wedge$." By transfer $\mathbb{V}^{(B_\gamma)} \models$ "$b_\gamma \mathscr{X}$ is unitarily equivalent to $l_2(\gamma^\wedge)$." By 4.4(2) the restricted descent of the Hilbert space $l_2(\gamma^\wedge)$ within $\mathbb{V}^{(B_\gamma)}$ is unitarily equivalent to the Kaplansky–Hilbert module $C_\#(Q_\gamma, l_2(\gamma))$. Assume that $u_\gamma \in \mathbb{V}^{(B_\gamma)}$ is a unitary isomorphism from $b_\gamma \mathscr{X}$ onto $l_2(\gamma^\wedge)$ within $\mathbb{V}^{(B_\gamma)}$ and U_γ is the restricted descent of u_γ. Then U_γ establishes a unitary equivalence between the Kaplansky–Hilbert modules $b_\gamma X$ and $C_\#(Q_\gamma, l_2(\gamma))$. By definition, $b_\gamma \in B$ as well as the compact space Q_γ is γ-stable in view of 3.8.

Assume now that some family $(P_\delta)_{\delta\in\Delta}$ of extremally disconnected compact spaces enjoys the same properties as $(Q_\gamma)_{\gamma\in\Gamma}$. Then P_δ is homeomorphic to some clopen subset P'_δ of the Stone space of B. Note that P'_δ is δ-stable. If $P_{\delta\gamma} := P'_\delta \cap Q_\gamma$ and $b_{\gamma\delta} \in B$ corresponds to $P_{\delta\gamma}$, then the Kaplansky–Hilbert modules $C_\#(P_{\delta\gamma}, l_2(\delta))$ and $C_\#(P_{\delta\gamma}, l_2(\gamma))$ are unitarily equivalent to the same band $b_{\delta\gamma}X$. Moreover, $P_{\delta\gamma}$ must be δ- and γ-stable simultaneously. Using 4.4(3), we see that either $P_{\delta\gamma} = \varnothing$ or $l_2(\delta) \sim l_2(\gamma)$. Since the latter holds only if $\delta = \gamma$; therefore, $P'_\gamma = Q_\gamma$ for all $\gamma \in \Gamma$. $\square$

5. Functional Representation of Type I AW^*-Algebras

Using the results of Section 4, we will obtain the functional realization of type I AW^*-algebras. Throughout this section, A is an arbitrary type I AW^*-algebra, while Λ is the center of A and B is the complete Boolean algebra of central projections in A. Hence, $B \subset \Lambda \subset A$.

5.1. Let B_h consist of $b \in B$ such that bA is a homogeneous algebra. Given $b \in B_h$, denote by $\varkappa(b)$ the least cardinal λ such that bA is a λ-homogeneous AW^*-algebra. Put $\varkappa(b) := \sup\{\varkappa(b') : b' \leq b, b' \in B_h\}$ for every $b \in B$. We thus define the function $\varkappa$ from B to some set of cardinals. Call $\varkappa$ the *multiplicity function* of A. Call $b \in B$ as well as bA *strictly λ-homogeneous* whenever $\varkappa(b') = \lambda$ provided that $0 \neq b' \leq b$. We also say that b and bA have *strict multiplicity* λ. There is a unique mapping $\overline{\varkappa} : \Gamma \to B$ such that Γ is some set of cardinals not greater than $\varkappa(\mathbb{1})$, while $(\overline{\varkappa}(\gamma))_{\gamma\in\Gamma}$ is a partition of unity in B and $\overline{\varkappa}(\gamma)$ has strict multiplicity γ for all $\gamma \in \Gamma$. The partition of unity $(\overline{\varkappa}(\gamma))_{\gamma\in\Gamma}$ is the *strict decomposition series* of an AW^*-algebra A. Clearly, if $A = \mathscr{L}_\Lambda(X)$ (see 2.10) for some Kaplansky–Hilbert module X, then the strict decomposition series of A coincides with B-$\dim(X)$ and $\varkappa$ is the multiplicity function introduced in 3.1. The multiplicity functions $\varkappa$ and $\varkappa'$ on the Boolean algebras B and B' as well as the corresponding partitions of unity $\overline{\varkappa}$ and $\overline{\varkappa'}$ are *congruent* provided that there is an isomorphism π from B onto B' such that $\varkappa' \circ \pi = \varkappa$. Clearly, $\overline{\varkappa}$ and $\overline{\varkappa'}$ are congruent if and only if both are defined on the same set and $\pi \circ \overline{\varkappa} = \overline{\varkappa'}$.

5.2. Theorem. *Let X be a Kaplansky–Hilbert module over the Stone algebra Λ. If X is λ-homogeneous then so is the AW^*-algebra $\mathscr{L}_\Lambda(X)$.*

Proof. We saw in 2.9 that $\mathscr{L}_\Lambda(X)$ is a type I AW^*-algebra. Assume that X is homogeneous with some basis $\mathscr{E}$ and $|\mathscr{E}| = \lambda$. Given $e, d \in \mathscr{E}$, define the operators π_e and π_{ed} by the formulas

$$\pi_e x := \langle x \,|\, e\rangle e, \quad \pi_{ed} x := \langle x \,|\, e\rangle d \quad (x \in X).$$

Show that π_e is an abelian projection. Indeed,

$$\langle \pi_e x \mid y\rangle = \langle x \mid e\rangle\langle e \mid y\rangle = \langle x, \pi_e y\rangle,$$
$$\pi_e^2 x = \langle x \mid e\rangle\langle e \mid e\rangle e = \langle x \mid e\rangle e = \pi_e x,$$

where the first line means that π_e is hermitian and the second says that π_e is idempotent. Furthermore, $\pi_e \circ \pi_d = 0$ in case $e \neq d$. If a nonzero projection $\pi \in \mathscr{L}_\Lambda(X)$ is orthogonal to all π_e with $e \in \mathscr{E}$, then there is a nonzero $x \in X$ such that $\pi x = x$, while we simultaneously have $0 = \pi_e x = \langle x \,|\, e\rangle e$ and $\langle x \,|\, e\rangle = 0$ for all $e \in \mathscr{E}$. This contradiction proves that $\sup_{e\in\mathscr{E}} \pi_e = I_X$. Since $\pi_{ed} \circ \pi_{de} = \pi_d$ and $\pi_{de} \circ \pi_{ed} = \pi_e$; therefore, π_e and π_d are equivalent, which yields the λ-homogeneity of $\mathscr{A}$. $\square$

5.3. Consider an extremally disconnected compact space Q and a Hilbert space H. As usual, let $\mathscr{L}(H)$ stand for the space of all bounded linear endomorphisms of H.

Denote by $\mathfrak{C}(Q, \mathscr{L}(H))$ the set of all operator-functions $u : \mathrm{dom}(u) \to \mathscr{L}(H)$ on the comeager subset $\mathrm{dom}(u)$ of Q which are continuous in the strong operator topology.

If $u \in \mathfrak{C}(Q, \mathscr{L}(H))$ and $h \in H$; then $uh : q \mapsto u(q)h$ with $q \in \mathrm{dom}(u)$ is continuous and so defines the unique $\widetilde{uh} \in C_\infty(Q, H)$ such that $uh \in \widetilde{uh}$; see [1, 5.3.12]. Introduce the equivalence in $\mathfrak{C}(Q, \mathscr{L}(H))$ by letting $u \sim v$ if and only if u and v coincide on $\mathrm{dom}(u) \cap \mathrm{dom}(v)$. If $\tilde{u}$ is the equivalence class of $u : \mathrm{dom}(u) \to \mathscr{L}(H)$ then $\tilde{u}h := \widetilde{uh}$ $(h \in H)$ by definition.

Denote by $SC_\infty(Q, \mathscr{L}(H))$ the set of the equivalence classes $\tilde{u}$ such that $u \in \mathfrak{C}(Q, \mathscr{L}(H))$ and the set $\{|\tilde{u}h| : \|h\| \leq 1\}$ is order bounded in $C_\infty(Q)$.

Since $|\tilde{u}h|$ coincides with the function $q \mapsto \|u(q)h\|$ with $q \in \mathrm{dom}(u)$ on some comeager subset, $\tilde{u} \in SC_\infty(Q, \mathscr{L}(H))$ means that the function $q \mapsto \|u(q)\|$ with $q \in \mathrm{dom}(u)$ is continuous on a comeager subset. Consequently, there are $|\tilde{u}| \in C_\infty(Q)$ and a comeager subset Q_0 of Q such that $|\tilde{u}|(q) = \|u(q)\|$ with $q \in Q_0$. Moreover, $|\tilde{u}| = \sup\{|\tilde{u}h| : \|h\| \leq 1\}$, where the supremum is taken in $C_\infty(Q)$. There are unique natural structures of a $*$-algebra in $SC_\infty(Q, \mathscr{L}(H))$ and a unitary $C_\infty(Q)$-module in concordance with the formulas

$$(u + v)(q) := u(q) + v(q) \quad (q \in \mathrm{dom}(u) \cap \mathrm{dom}(v)),$$
$$(uv)(q) := u(q) \circ v(q) \quad (q \in \mathrm{dom}(u) \cap \mathrm{dom}(v)),$$
$$(av)(q) := a(q)v(q) \quad (q \in \mathrm{dom}(a) \cap \mathrm{dom}(v)),$$
$$u^*(q) := u(q)^* \quad (q \in \mathrm{dom}(u)),$$

where $u, v \in \mathfrak{C}(Q, \mathscr{L}(H))$ and $a \in C_\infty(Q)$. Note also that the following hold:

$$|\tilde{u} + \tilde{v}| \le |\tilde{u}| + |\tilde{v}|, \quad |\tilde{u}\tilde{v}| \le |\tilde{u}| \cdot |\tilde{v}|,$$
$$|a\tilde{v}| = |a||\tilde{v}|, \quad |\tilde{u} \cdot \tilde{u}^*| = |\tilde{u}|^2.$$

If $\tilde{u} \in SC_\infty(Q, \mathscr{L}(H))$ and $\tilde{x} \in C_\infty(Q, H)$ is defined by some continuous vector-function $x : \mathrm{dom}(x) \to H$; then we may put $\tilde{u}\tilde{x} := \widetilde{ux} \in C_\infty(Q, H)$, where $ux : q \mapsto u(q)x(q)$ with $q \in \mathrm{dom}(u) \cap \mathrm{dom}(x)$. This definition is sound since ux is continuous. In this event

$$|\tilde{u}x| \le |\tilde{u}| \cdot |x| \quad (x \in C_\infty(Q, H)).$$

In particular, this implies the formula

$$|\tilde{u}| = \sup \big\{ |\tilde{u}x| : x \in C_\infty(Q, H), \, |x| \le \mathbb{1} \big\}.$$

We will denote the operator $x \mapsto \tilde{u}x$ in $C_\infty(Q, H)$ by $S_{\tilde{u}}$. Let us introduce the normed $*$-algebra $SC_\#(Q, \mathscr{L}(H))$ by the formulas

$$SC_\#(Q, \mathscr{L}(H)) := \big\{ v \in SC_\infty(Q, \mathscr{L}(H)) : |v| \in C(Q) \big\},$$
$$\|v\| = \| |v| \|_\infty \quad (v \in SC_\#(Q, \mathscr{L}(H))).$$

Put $\Lambda := C(Q, \mathbb{C})$. Recall that $C_\#(Q, H)$ is a λ-homogeneous Kaplansky–Hilbert module over Λ, where $\lambda := \dim(H)$; see 4.1. By 2.9 $\mathscr{L}_\Lambda(C_\#(Q, H))$ is a λ-homogeneous type I AW^*-algebra with center isomorphic to Λ. The next theorem asserts that $\mathscr{L}_\Lambda(C_\#(Q, H))$ can be presented as the algebra of dominated operators in $C_\#(Q, H)$. We will preserve the notation $S_{\tilde{u}}$ for the restriction of this operator to $C_\#(Q, H)$.

5.4. Theorem. *Let H be a Hilbert space and $\lambda = \dim(H)$. To each $U \in \mathscr{L}_\Lambda(C_\#(Q, H))$ there corresponds the unique $u \in SC_\#(Q, \mathscr{L}(H))$ such that $U = S_u$. The mapping $U \mapsto u$ is a $*$-B-isomorphism of $\mathscr{L}_\Lambda(C_\#(Q, H))$ onto $A := SC_\#(Q, \mathscr{L}(H))$. In particular, A is a λ-homogeneous AW^*-algebra. If the compact space Q is λ-stable then A is a strictly λ-homogeneous AW^*-algebra.*

Proof. Recall once again that S_u satisfies the inequality $|S_u x| \le |u| \cdot |x|$ for all $x \in C_\#(Q, H)$; see 5.3. Consequently, if $u \in SC_\#(Q, \mathscr{L}(H))$ then S_u acts from $C_\#(Q, H)$ to $C_\#(Q, H)$ and is bounded with respect to the vector norm; see [1, Section 5.3]. Moreover,

$$\|S_u\| = \sup_{\|x\| \le 1} \| |S_u x| \|_\infty = \sup_{|x| \le 1} \sup_{q \in Q} |ux|(q) = \sup_{q \in Q} |u|(q) = \|u\|.$$

The definition of S_u shows that $S_{au} = aS_u$ and $S_{u^*} = {S_u}^*$ for all $a \in \Lambda$ and $u \in SC_\#(Q, \mathscr{L}(H))$. So, $u \mapsto S_u$ is a $*$-B-isomorphic embedding of $SC_\#(Q, \mathscr{L}(H))$ to $\mathscr{L}_\Lambda(C_\#(Q, H))$. Show that the embedding is a surjection. Note that $U \in \mathscr{L}_\Lambda(C_\#(Q, H))$ is bounded with respect to the vector norm; i.e., $|Ux| \le f \cdot |x|$ for all $x \in C_\#(Q, H)$, where $f := \sup\{|Ux| : |x| \le \mathbb{1}\} \in C(Q)$. Consequently, there is an operator function $u : \mathrm{dom}(u) \to \mathscr{L}(H)$ such that

(a) $q \mapsto \langle u(q)h | g \rangle$ $(q \in \mathrm{dom}(u))$ is continuous for all $g, h \in H$;
(b) there is $\varphi \in C_\infty(Q)$ such that $\|u(q)\| \le \varphi(q)$ $(q \in \mathrm{dom}(u))$;
(c) $Ux = \tilde{u}x$ for all $x \in C_\#(Q, H)$ and $|u| = f$.

So, $U = S_{\tilde{u}}$ and we are left with proving that u is continuous in the strong operator topology. Considering the well-known forms of joints and meets in the Kantorovich space $C_\infty(Q)$, observe that $\|u(q)\| = |u|(q)$ with $q \in Q_0$, where Q_0 is a comeager subset of Q. Replacing $\mathrm{dom}(u)$ with $Q_0 \cap \mathrm{dom}(u)$, if need be, we may assume that the function $q \mapsto \|u(q)\|$ with $q \in \mathrm{dom}(u)$ is continuous. Together with (a) this implies the continuity of u with the strong operator topology; i.e., $u \in SC_\#(Q, \mathscr{L}(H))$. Appealing to 2.9 and 4.1 completes the proof. □

We will say that some families $(Q_\gamma)_{\gamma \in \Gamma}$ and $(P_\delta)_{\delta \in \Delta}$ of nonempty compact spaces are *congruent* provided that $\Gamma = \Delta$ while Q_γ and P_γ are homeomorphic for all $\gamma \in \Gamma$.

5.5. Theorem. *For an arbitrary type I AW^*-algebra A there is a family $(Q_\gamma)_{\gamma\in\Gamma}$ of nonempty extremally disconnected compact spaces which is unique up to congruence and satisfies the conditions*

(a) Γ *is a nonempty set of cardinals and* Q_γ *is* γ*-stable for every* $\gamma \in \Gamma$*;*

(b) *we have the* $*$*-isomorphism*

$$A \simeq \sum_{\gamma\in\Gamma}^{\oplus} SC_{\#}(Q_\gamma, \mathscr{L}(l_2(\gamma))).$$

Proof. By 2.5 we may assume that A is the bounded descent of an AW^*-factor $\mathscr{A}$ within $\mathbb{V}^{(B)}$. In this event $\mathscr{A}$ is of type I, and so $\mathscr{A} \simeq \mathscr{L}(\mathscr{X})$, with $\mathscr{X}$ some Hilbert space within $\mathbb{V}^{(B)}$. It follows that A and $\mathscr{L}_\Lambda(X)$, with X the bounded descent of $\mathscr{X}$, are $*$-B-isomorphic algebras.

Assume that B-dim$(X) = (b_\gamma)_{\gamma\in\Gamma}$, and Q_γ is a clopen subset of the Stone space of B which corresponds to $b_\gamma \in B$. By 3.8 Q_γ is γ-stable, which implies (a).

By 4.5 we have the unitary equivalence $X \simeq \sum_{\gamma\in\Gamma}^{\oplus} C_{\#}(Q_\gamma, l_2(\gamma))$ which leads to the $*$-isomorphism of the AW^*-algebras

$$\mathscr{L}_\Lambda(X) \simeq \sum_{\gamma\in\Gamma}^{\oplus} \mathscr{L}_\Lambda(C_{\#}(Q_\gamma, l_2(\gamma))).$$

Using 5.4, we arrive at (b). The required uniqueness results from 4.5. □

5.6. We list another three consequences of the results of this section.

5.6(1). Corollary. *Every type I AW^*-algebra decomposes into the direct sum of strictly homogeneous terms. This decomposition is unique to within $*$-B-isomorphism.*

Proof. Cp. 4.5. □

5.6(2). Corollary. *Two type I AW^*-algebras are $*$-isomorphic if and only if they have the isomorphic centers and the congruent multiplicity functions or, in other words, the congruent decomposition series.*

Proof. Everything follows from 5.6(1) on observing that if in 5.5 the B-dimension of A is congruent to the partition of unity $(\chi_\gamma)_{\gamma\in\Gamma}$, where χ_γ is the characteristic function of Q_γ in the disjoint sum Q of the family (Q_γ), and the center of A is $*$-isomorphic to $C(Q, \mathbb{C})$. □

5.6(3). Corollary. *Let Γ be a set of cardinals and let (b_γ) be a partition of unity in B with nonzero terms. Then $(b_\gamma)_{\gamma\in\Gamma}$ is a strict decomposition series of some AW^*-algebra if and only if b_γ is γ-stable for every $\gamma \in \Gamma$.*

Proof. The claim follows from 3.8 and 5.4. □

6. Embeddable C^*-Algebras

Type I members have the simplest structure among the whole class of AW^*-algebras. Of a natural interest are the algebras realizable as bicommutants of type I AW^*-algebras. These algebras are referred to as *embeddable*. The results of Section 2 shows that these algebras turn in von Neumann algebras by embedding in an appropriate Boolean valued model. This opens the opportunity to translate the available facts concerning von Neumann algebras to some statements about embeddable algebras. In the current section we will illustrate this approach with a few examples.

6.1. We start with prerequisites.

6.1(1). Let H be a Hilbert space and let $\mathscr{L}(H)$ be the space of bounded linear endomorphisms of H. If $M \subset \mathscr{L}(H)$ then the *commutant* M' of M is the set of all members of $\mathscr{L}(H)$ that commute with every operator in M. Clearly, M' is a Banach algebra with unity the identity operator $\mathbb{1} := I_H$. The *bicommutant* of M is $M'' := (M')'$. A *von Neumann algebra* or W^**-algebra* in H is a $*$-subalgebra A of $\mathscr{L}(H)$ which contains unity and coincides with the bicommutant of A; i.e., $\mathbb{1} \in A$ and $A = A''$. The *center* of a von Neumann algebra A is defined by the formula $\mathscr{Z}(A) = A \cap A'$. A von Neumann algebra is a *factor* provided that the center of A is trivial; i.e., $\mathscr{Z}(A) = \mathbb{C} \cdot \mathbb{1} := \{x \cdot I_H : \lambda \in \mathbb{C}\}$; cp. 2.5.

6.1(2). Bicommutant Theorem. *Let A be an involutive algebra of operators in a Hilbert space H and $I_H \in A$. Then A coincides with the bicommutant A'' if and only if A is closed in the strong or, which is the same, weak operator topology of $\mathscr{L}(H)$.*

6.1(3). A C^*-algebra A is *B-embeddable* provided that there are a type I AW^*-algebra N and a $*$-monomorphism $\imath : A \to N$ such that $B = \mathfrak{P}_c(N)$ and $\imath(A) = \imath(A)''$, where $\imath(A)''$ is the bicommutant of $\imath(A)$ in N. Note that in this event A is an AW^*-algebra and B is a regular subalgebra of $\mathfrak{P}_c(A)$. In particular, A is a B-cyclic algebra; see 2.3.

Say that a C^*-algebra A is *embeddable* provided that A is B-embeddable for some regular subalgebra $B \subset \mathfrak{P}_c(A)$. If $B = \mathfrak{P}_c(A)$ and A is B-embeddable then A is *centrally embeddable*.

Recall that we always assume that we consider only unital C^*-algebras. Furthermore, $B \sqsubset A$ means as in the above that A is B-cyclic.

6.2. Theorem. *Let $\mathscr{A}$ be a C^*-algebra within $\mathbb{V}^{(B)}$, while A is the restricted descent of $\mathscr{A}$. Then A is a B-embeddable AW^*-algebra if and only if $\mathscr{A}$ is a von Neumann algebra within $\mathbb{V}^{(B)}$. An algebra A is centrally embeddable if and only if $\mathscr{A}$ is a factor within $\mathbb{V}^{(B)}$.*

Proof. Assume that A is a bicommutant in a type I AW^*-algebra N and $\mathfrak{P}_c(N) = B$. By 2.5 and 2.8, we may view N as the restricted descent of some type I AW^*-factor $\mathscr{N}$ within $\mathbb{V}^{(B)}$. From $A'' \subset N$ and $A'' = A$ we immediately infer that $[\![\mathscr{A} = A{\uparrow} \subset N]\!] = \mathbb{1}$ and $[\![\mathscr{A}'' = (A{\uparrow})'' = A''{\uparrow} = \mathscr{A}]\!] = \mathbb{1}$. Hence, $\mathscr{A}$ is a bicommutant in $\mathscr{N}$. So, we are done on observing that the type I AW^*-factor $\mathscr{N}$ is isomorphic to $\mathscr{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

Conversely, assume that $[\![\mathscr{A}$ is a von Neumann algebra$]\!] = \mathbb{1}$. This means that

$$[\![\mathscr{A} \text{ is a bicommutant in } \mathscr{L}(\mathscr{H})]\!] = \mathbb{1}$$

for some Hilbert space $\mathscr{H}$ within $\mathbb{V}^{(B)}$.

Let N be the restricted descent of $\mathscr{L}(\mathscr{H})$. Then N is a type I AW^*-algebra by 2.8(2), while A is a bicommutant in N and $\mathfrak{P}_c(N) = B$; see 2.5. The second claim follows from Theorem 2.5 asserting that $\mathscr{A}$ is a factor within $\mathbb{V}^{(B)}$ if and only if $\mathfrak{P}_c(A) = B$. $\square$

6.3. We now turn to characterizing embeddable C^*-algebras. Given a normed B-space X, we will let $X^{\#}$ stand for the B-dual space; cp. [1, 5.5.8]. Say that a C^*-algebra A is *B-dual* provided that A includes the Boolean algebra B of central projections and A is B-isometric to the B-dual space $X^{\#}$ of some normed B-space X. In this event X is *B-predual* to A, which we express by the formula $A_{\#} = X$.

Sakai Theorem. *A C^*-algebra $\mathscr{A}$ is a von Neumann algebra up to $*$-isomorphism if and only if $\mathscr{A}$ is a dual Banach space.*[6)]

6.4. Theorem. *A C^*-algebra is B-embeddable if and only if A is B-dual. The B-predual space is unique up to B-isometry in the class of B-cyclic Banach spaces.*

Proof. Assume that A is a C^*-algebra and $B \sqsubset \mathfrak{P}_c(A)$. By 1.6 we may assume also that A is the restricted descent of some C^*-algebra $\mathscr{A}$ within $\mathbb{V}^{(B)}$. Using the Sakai Theorem and transfer, we see that $[\![\mathscr{A}$ is a von Neumann algebra$]\!] = [\![\mathscr{A}$ is linear isometric to the dual Banach space $\mathscr{X}']\!]$. If X is the restricted descent of a Banach space $\mathscr{X}$ then $X^{\#}$ is B-linear isometric to the restricted descent of $\mathscr{X}'$; cp. [1, 5.5.8]. Using 6.2 we see that if A is B-embeddable then A is B-dual too. Moreover, $A_{\#} = X$ is B-cyclic.

Conversely, assume that A is B-dual and $A_{\#} = X_0$ is a normed B-space. If X is a B-cyclic completion of X_0 then $X_0^{\#} = X^{\#}$, i.e., $A_{\#} = X$; see [1, 5.5.6 and 5.5.10]). Denote the Boolean valued realization of X by $\mathscr{X}$. Then $\mathscr{A} \simeq \mathscr{X}^{\#}$. By 6.2 A is B-embeddable.

6) Cp. [14, Chapter 1].

Suppose now that B-cyclic spaces X and Y are B-predual to A. Denote the realizations of X and Y within $\mathbb{V}^{(B)}$ by $\mathscr{X}$ and $\mathscr{Y}$. Then $[\![\mathscr{X}$ and $\mathscr{Y}$ are predual to $\mathscr{A}]\!] = \mathbb{1}$. Since each von Neumann algebra has the unique predual up to linear isometry; therefore,

$$[\![\mathscr{X} \text{ and } \mathscr{Y} \text{ are linear isometric}]\!] = \mathbb{1}.$$

Note that X and Y are the restricted descents of $\mathscr{X}$ and $\mathscr{Y}$, and so X and Y are B-isometric. □

6.5. Theorem. *Let N be a type I AW^*-algebra and let A be an AW^*-subalgebra of N and $\mathscr{Z}(N) \subset A$. Then both A and the commutant A' of A in N are of the same type I, or II, or III.*

Proof. By 2.5 and 2.8 we may assume that N and A are the restricted descents of $\mathscr{N}$ and $\mathscr{A}$ within $\mathbb{V}^{(B)}$, where $B = \mathfrak{P}_c(N)$ and

$$[\![\mathscr{N} = \mathscr{L}(\mathscr{H}) \text{ for some Hilbert space } \mathscr{H}]\!] = \mathbb{1},$$

and

$$[\![\mathscr{A} \text{ is an } AW^*\text{-subalgebra of } \mathscr{N}]\!] = \mathbb{1}.$$

So, $\mathscr{A}$ is a von Neumann algebra within $\mathbb{V}^{(B)}$. We know that the claim holds for Neumann algebras (see [9, 2.5.6]); i.e., $\mathscr{A}$ and $\mathscr{A}'$ are of the same type I, or II, or III. Also, A' coincides with the restricted descent of $\mathscr{A}'$ because $\mathscr{A}'{\downarrow} = (\mathscr{A}{\downarrow})^\circ$, where $^\circ$ stands for the passage to the commutant in $\mathscr{N}{\downarrow}$. It suffices to apply 2.8 once again. □

6.6. Theorem. *Let a C^*-algebra A be B_0-embeddable for some regular subalgebra B_0 of $\mathfrak{P}_c(A)$. Then A is B-embeddable for every regular subalgebra B such that $B_0 \subset B \subset \mathfrak{P}_c(A)$.*

Proof. Assume that A is a bicommutant of a type I AW^*-algebra N and $\mathfrak{P}_c(N) = B_0$. Let B be a regular subalgebra of $\mathfrak{P}_c(A)$ and $B_0 \subset B$. Let $\mathscr{C}(B)$ stand for the C^*-algebra generated by B. Since B is a regular subalgebra, $\mathscr{C}(B)$ is an AW^*-subalgebra in N (see 2.1(1) and 2.1(2)). Moreover, $\mathscr{C}(B)$ includes the center of N because of the equality $B_0 = \mathfrak{P}_c(N)$. By 6.5 the commutant $\mathscr{C}(B)' = B'$ of $\mathscr{C}(B)$ in N is of the same type as $\mathscr{C}(B)$. However, $\mathscr{C}(B)$ is a commutative AW^*-algebra, and so $\mathscr{C}(B)'$ is of type I. For the same reason, the center of $\mathscr{C}(B)'$ coincides with $\mathscr{C}(B)$. Note that $\mathscr{C}(B)$ lies in the center of A, and so the commutant A' in N is included in $\mathscr{C}(B)'$. Consequently, the bicommutant of A in $\mathscr{C}(B)'$ coincides with the bicommutant of A; i.e., A is a bicommutant in $\mathscr{C}(B)$. Thus, A is B-embeddable. □

6.7. We list the next two propositions:

6.7(1). *A C^*-algebra A is embeddable if and only if A is centrally embeddable.*

6.7(2). *A von Neumann algebra A is B-embeddable for every regular subalgebra B of $\mathfrak{P}_c(A)$.*

6.8. Let A be a C^*-algebra and $B \sqsubset A$. A linear operator $T : A \to B(\mathbb{C})$ is *positive* whenever $T(x^*x) \geq 0$ for all $x \in A$. A positive B-linear operator T is a *state* of A provided that $\|T\| = 1$. A state T is *normal* if $T(\sup(x_\alpha)) = \sup(T(x_\alpha))$ for every increasing net (x_α) of hermitian elements which has a supremum. Say that A has a *separating set* of $B(\mathbb{C})$-*valued normal states* whenever the positivity of $x \in A$ is the same as $Tx \geq 0$ for every normal $B(\mathbb{C})$-valued state T of A. In case $B(\mathbb{C})$ coincides with $\mathbb{C}$, we speak about normal states.

The *monotone completeness* of a C^*-algebra A means that each upper bounded increasing set of hermitian elements of A has a supremum. By transfer it is clear that the monotone completeness of A is equivalent to the monotone completeness of the Boolean valued realization of A; cp. [1, p. 187].

Kadison Theorem. *An arbitrary C^*-algebra is isomorphic to a von Neumann algebra if and only if A is monotone complete and has a separating set of normal states.*[7)]

[7)] Cp. [15, p. 38].

6.9. Theorem. *Let $\mathscr{A}$ be a C^*-algebra within $\mathbb{V}^{(B)}$ and let A be the restricted descent of $\mathscr{A}$.*

(1) *$[\![\phi := \Phi\!\uparrow$ is a state of $\mathscr{A}]\!] = \mathbb{1}$ for every $B(\mathbb{C})$-valued state Φ of A; and each state of $\mathscr{A}$ is of the form $\Phi\!\uparrow$, where Φ is some $B(\mathbb{C})$-valued state of A.*

(2) *Φ is normal if and only if $[\![\phi := \Phi\!\uparrow$ is normal$]\!] = \mathbb{1}$.*

Proof. (1): This results by transfer on recalling that the mapping $\Phi \mapsto \phi := \Phi\!\uparrow$ preserves positivity as

$$\Phi(A^+)\!\uparrow = \phi(A^+\!\uparrow) = \phi(\mathscr{A}^+).$$

(2): It suffices to use the Escher rules for polars; see [1, 3.3.12]. □

6.10. Theorem. *If A is a B-cyclic C^*-algebra then the following are equivalent:*

(1) *A is B-embeddable;*

(2) *A is monotone complete and has a separating set of $B(\mathbb{C})$-valued states.*

Proof. By 1.6 we may assume that A is the restricted descent of some C^*-algebra $\mathscr{A}$ within $\mathbb{V}^{(B)}$. By 6.2 A is B-embeddable if and only if $[\![\mathscr{A}$ is a von Neumann algebra$]\!] = \mathbb{1}$. Note that we will use the Kadison Theorem for proving the existence of normal states without plunging into the details.

Let $\mathscr{S}_n(\mathscr{A})$ be the set of all normal states of $\mathscr{A}$ within $\mathbb{V}^{(B)}$, and let $\mathscr{S}_n(A,B)$ be the set of all normal $B(\mathbb{C})$-valued states of A. The mapping $\Phi \mapsto \phi := \Phi\!\uparrow$ is a bijection between $\mathscr{S}_n(\mathscr{A})\!\downarrow$ and $\mathscr{S}_n(A,B)$; cp. 6.9.

Assume that $\mathscr{S}_n(A,B)$ is a separating set. Given a nonzero $x \in A$, find $\Phi_0 \in \mathscr{S}_n(A,B)$ satisfying $\Phi_0 x \neq 0$. Since Φ is B-linear, we have $[\![0 \neq x]\!] \leq [\![\Phi_0(x) \neq 0]\!]$. Using the rules of calculations truth values, we may write

$$[\![\mathscr{S}_n(\mathscr{A}) \text{ is a separating set}]\!]$$
$$= [\![(\forall\, x \in \mathscr{A})\ (x \neq 0 \to (\exists\, \phi \in \mathscr{S}_n(\mathscr{A}))\ \phi(x) \neq 0)]\!]$$
$$= \bigwedge_{x\in A} [\![x \neq 0]\!] \Rightarrow \bigvee_{\Phi\in\mathscr{S}_n(A,B)} [\![\Phi\!\uparrow\!(x) \neq 0]\!] \geq \bigwedge_{x\in A} [\![x \neq 0]\!] \Rightarrow [\![\Phi_0\!\uparrow\!(x) \neq 0]\!] = \mathbb{1}.$$

Therefore, $\mathscr{S}_n(\mathscr{A})$ is a separating set within $\mathbb{V}^{(B)}$.

Conversely, assume that $\mathscr{S}_n(\mathscr{A})$ is a separating set within $\mathbb{V}^{(B)}$. Given a nonzero $x \in A$, we have $b := [\![x \neq 0]\!] > \mathbb{0}$. By the maximum principle[8] there is $\phi \in \mathscr{S}_n(\mathscr{A})\!\downarrow$ satisfying $b \leq [\![\phi(x) \neq 0]\!]$. Let Φ be the restriction of $\phi\!\downarrow$ to $A \subset \mathscr{A}\!\downarrow$. Then $\Phi \in \mathscr{S}_n(A,B)$ and $b \leq [\![\Phi(x) \neq 0]\!]$. Note that the trace $e_{\Phi(x)}$ of $\Phi(x)$ is at least b (cp. [1, 5.2.3(5)]), and so $\Phi(x) \neq 0$. □

6.11. Theorem. *If A is an AW^*-algebra then the following are equivalent:*

(1) *A is embeddable;*

(2) *A is centrally embeddable;*

(3) *A has a separating set of center valued normal states;*

(4) *A is a $\mathfrak{P}_c(A)$-dual space.*

Proof. Use 6.4, 6.7(1), and 6.10. □

7. *JB*-Algebras

This section addresses the possibility of Boolean valued realization of the real nonassociative analogs of C^*-algebras.

7.1. Let A be a vector space over some field $\mathbb{F}$. Note that A is a *Jordan algebra* provided that A is equipped with some possibly nonassociative binary operation $A \times A \ni (x,y) \mapsto xy \in A$, called multiplication, such that for all $x, y, z \in A$ and $\alpha \in \mathbb{F}$ we have

(1) $xy = yx$;

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[8] Cp. [1, 2.3.3].

(2) $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y$;
(4) $(x^2y)x = x^2(yx)$.

Say that e in a Jordan algebra A is *unity* provided that $e \neq 0$ and $ea = a$ for all $a \in A$.

Jordan algebras are tied with associative algebras as follows: Given an associative algebra A over a field of characteristic not 2, define the new multiplication $a \circ b := 1/2(ab + ba)$ with $a, b \in A$. Denote the resulting new algebra by A^J. Note that A^J is Jordan. If some subspace $A_\circ$ of A is closed under $\circ$, then $A_\circ$ with multiplication $\circ$ is a subalgebra of A^J; hence, $A_\circ$ is Jordan algebra. We call a Jordan algebra *special* if it results by the above procedure from some associative algebra. Jordan algebras that are not special are called *exceptional*.

7.2. Consider some key examples of Jordan algebras.

(1) Take an associative algebra A with involution $*$. The set $\mathrm{H}(A,*)$ of the hermitian elements $\{h \in A : h^* = h\}$ is closed under the Jordan multiplication $a \circ b = 1/2(ab + ba)$, and so $\mathrm{H}(A,*)$ is a special Jordan algebra.

(2) Let $\mathbb{O}$ be the Cayley algebra known also as the algebra of *octonions*. Consider the algebra $M_n(\mathbb{O})$ of $n \times n$-matrices with entries in $\mathbb{O}$. Endow $M_n(\mathbb{O})$ with the involution $*$ that is the composition of conjugation and transpose which is called *hermitian transpose*. The set $M_n(\mathbb{O})_{\mathrm{sa}} := \{x \in M_n(\mathbb{O}) : x^* = x\}$ of hermitian matrices is closed under the Jordan multiplication $x \circ y := 1/2(xy + yx)$ in $M_n(\mathbb{O})$. However, the vector space $M_n(\mathbb{O})_{\mathrm{sa}}$ with $\circ$ is a Jordan algebra only if $n \leq 3$. The Jordan algebra $M_3(\mathbb{O})_{\mathrm{sa}}$, denoted by M_3^8, is special.

(3) Let X be a vector space over a field $\mathbb{F}$. Assume given a nondegenerate symmetrical bilinear form $\langle \cdot, \cdot \rangle$. Define the multiplication of the direct sum $\mathbb{F} \oplus X$ by the formula

$$(s, x) \circ (t, y) := (st + \langle x, y \rangle, sy + tx) \quad (s, t \in \mathbb{F};\ x, y \in X).$$

Then $\mathbb{F} \oplus X$ is a special Jordan algebra.

7.3. A Jordan algebra A with unity $\mathbb{1}$ is a *JB-algebra* if A is simultaneously a real Banach space whose norm enjoys the conditions

(1) $\|xy\| \leq \|x\| \cdot \|y\|$ for all $x, y \in A$;
(2) $\|x^2\| = \|x\|^2$ for all $x \in A$;
(3) $\|x^2\| \leq \|x^2 + y^2\|$ for all $x, y \in A$.

The intersection $\mathscr{Z}(A)$ of all maximal associative subalgebras of A is the *center* of A. Note that a belongs to $\mathscr{Z}(A)$ if and only if $(ax)y = a(xy)$ for all $x, y \in A$. If $\mathscr{Z}(A) = \mathbb{R} \cdot \mathbb{1}$ then A is a *JB-factor*.

7.4. Recall a few well-known properties of JB-algebras; for instance, see [15, 16–19] and elsewhere.

(1) Let A be a JB-algebra. The set $A^+ := \{x^2 : x \in A\}$ is a pointed convex cone making A an ordered vector space such that the element $\mathbb{1}$ of A is a strong order unity and the order interval $[-\mathbb{1}, \mathbb{1}] := \{x \in A : -\mathbb{1} \leq x \leq \mathbb{1}\}$ is the unit ball. In this event $-\mathbb{1} \leq x \leq \mathbb{1}$ and $0 \leq x^2 \leq \mathbb{1}$ mean the same.

(2) Let A be an ordered Banach space with strong unit $\mathbb{1}$ such that the unit ball of A is the order interval $[-\mathbb{1}, \mathbb{1}]$. If A is endowed with some Jordan multiplication so that $-\mathbb{1} \leq x \leq \mathbb{1}$ and $0 \leq x^2 \leq \mathbb{1}$ mean the same then A is a JB-algebra.

(3) If A_0 is a closed associative subalgebra of a JB-algebra A then A_0 is order and algebraic isomorphic to the real Banach algebra $C(Q)$ for some compact space Q. (Recall that we always assume that a compact space is Hausdorff.) In particular, the center $\mathscr{Z}(A)$ of A is a real Banach space isometrically isomorphic to $C(Q)$.

(4) Given a Jordan algebra A and $a \in A$, we introduce the operator $U_a : A \to A$ by the rule $U_a x := 2a(ax) - a^2 x$. Then U_a is positive, i.e., $U_a(A^+) \subset A^+$.

7.5. Idempotents of a JB-algebra A are usually called *projections*, and we denote the set of all projections by $\mathfrak{P}(A)$. The set of projections belonging to the center of A is the Boolean algebra denoted by $\mathfrak{P}_c(A)$. Assume that $\mathbb{B}$ is a subalgebra of $\mathfrak{P}_c(A)$; in other words, $\mathbb{B}(\mathbb{R})$ is a unital subalgebra of $\mathscr{Z}(A)$. Call A a $\mathbb{B}$-*JB-algebra* provided that for each partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and each norm bounded family $(x_\xi)_{\xi\in\Xi}$ in A there exists $\mathbb{B}$-*mixing* $x := \operatorname{mix}_{\xi\in\Xi}(e_\xi x_\xi)$; i.e., the only $x \in A$ such that $e_\xi x_\xi = e_\xi x$ for all $\xi \in \Xi$. If $\mathbb{B}(\mathbb{R}) = \mathscr{Z}(A)$ then A is called a *centrally extended* JB-algebra.

7.5(1). *The unit ball of a $\mathbb{B}$-JB-algebra is closed under mixing.*

Proof. The unit ball of a JB-algebra is the order interval $[-\mathbb{1}, \mathbb{1}]$. So, we have to show that if $x \in A$ and a partition of unity $(e_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ are such that $e_\xi x \geq 0$ for all $\xi \in \Xi$, then $x \geq 0$. But this is clear because if $e_\xi x = a_\xi^2$ for some $a_\xi \in A$, then $x = a^2$ for $a = \operatorname{mix}(e_\xi a_\xi)$. □

7.5(2). *Each $\mathbb{B}$-JB-algebra is a $\mathbb{B}$-cyclic Banach space.*

Proof. Apply 7.5(1) and [1, 5.4]. □

By 7.5(2) we may apply the Boolean valued analysis of Banach algebras to $\mathbb{B}$-JB-algebras.

7.6. Theorem. *The restricted descent of a JB-algebra within $\mathbb{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-JB-algebra. Conversely, for each $\mathbb{B}$-JB-algebra A there is a JB-algebra $\mathscr{A}$ within $\mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism and whose restricted descent is isometrically $\mathbb{B}$-isomorphic to A. In this event $[\![\mathscr{A}$ is a JB-factor$]\!] = \mathbb{1}$ if and only if $\mathbb{B}(\mathbb{R}) = \mathscr{Z}(A)$.*

Proof. Let A be a $\mathbb{B}$-JB-algebra. Note that A as a Banach space is the restricted descent of some Banach space $\mathscr{A}$ within $\mathbb{V}^{(\mathbb{B})}$; see [1, Theorem 5.5.7]. To introduce the Jordan algebra structure in $\mathscr{A}$, we will show that multiplication on A is extensional.

Given $x, y, x', y' \in A$, put $e := [\![x = x']\!] \wedge [\![y = y']\!]$. Since $e \leq [\![u = v]\!]$ and $eu = ev$ means the same, $ex = ex'$ and $ey = ey'$. If e is a central projection then

$$e(xy) = (ex)y = (ex')y = (ey)x' = (ey')x' = e(x'y').$$

Consequently,

$$[\![x = x']\!] \wedge [\![y = y']\!] = e \leq [\![xy = x'y']\!];$$

i.e., multiplication on A is extensional.

Define the binary operator $(x, y) \mapsto x \circ y$ on $\mathscr{A}$ as the ascent of multiplication on A. Thus, to $x, y \in A$ there is a unique $x \circ y \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![x \circ y \in \mathscr{A}]\!] = [\![x \circ y = xy]\!] = \mathbb{1}.$$

Show that $(\mathscr{A}, \circ)$ is a JB-algebra within $\mathbb{V}^{(\mathbb{B})}$. Recall that the linear endomorphism $T_a : x \mapsto ax$ with $x \in A$ is extensional. If $\mathscr{T}_a : x \mapsto a \circ x$, where $x \in \mathscr{A}$, is an operator within $\mathbb{V}^{(\mathbb{B})}$; then it is clear that $[\![\mathscr{T}_a = T_a{\uparrow}]\!] = \mathbb{1}$. Consequently, the operators T_x and T_y commute if and only if $\mathscr{T}_x$ and $\mathscr{T}_y$ commute within $\mathbb{V}^{(\mathbb{B})}$. In particular, in case $y = x^2$ we see that $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ for $x, y \in \mathscr{A}$. Moreover, the above shows that $x \in A$ belongs to $\mathscr{Z}(A)$ if and only if $[\![x \in \mathscr{Z}(\mathscr{A})]\!] = \mathbb{1}$, which means the same as

$$[\![\mathscr{Z}(A){\uparrow} = \mathscr{Z}(\mathscr{A})]\!] = \mathbb{1}.$$

We are left with proving that 7.1(1)–7.1(3) are valid in $\mathscr{A}$. To this end it suffices to establish that the vector norm on A satisfies the similar conditions. Note firstly that

$$\|x\| \leq 1 \leftrightarrow \|{\mid}x{\mid}\|_\infty \leq 1 \leftrightarrow {\mid}x{\mid} \leq \mathbb{1}.$$

Now, take $x, y \in A$ and $0 < \varepsilon \in \mathbb{R}$. Put $x_0 := \alpha^{-1}x$ and $y_0 := \beta^{-1}y$, where $\alpha := {\mid}x{\mid} + \varepsilon\mathbb{1}$ and $\beta := {\mid}y{\mid} + \varepsilon\mathbb{1}$. Since ${\mid}x_0{\mid} = |\alpha^{-1}|{\mid}x{\mid} \leq \mathbb{1}$; therefore, $\|x_0\| \leq 1$. By analogy, $\|y_0\| \leq 1$. Hence, $\|x_0 y_0\| \leq 1$ or ${\mid}x_0 y_0{\mid} \leq \mathbb{1}$. This implies that

$${\mid}xy{\mid} \leq {\mid}x{\mid} \cdot {\mid}y{\mid} + \varepsilon({\mid}x{\mid} + {\mid}y{\mid}) + \varepsilon^2\mathbb{1}.$$

Vanishing ε, conclude that $|xy| \leq |x| \cdot |y|$. Then we put $\gamma^2 := |x^2| + \varepsilon\mathbb{1}$ and $x' := \gamma^{-1}x$. In this event $|x'^2| = \gamma^{-2}|x^2|$, implying that $\|x'\|^2 = \|x'^2\| \leq 1$ or $\|x'\| \leq 1$. Consequently, $|x'| \leq \mathbb{1}$. Also, $|x'|^2 \leq \mathbb{1}$ and $|x|^2 \leq \gamma^2$. So, $|x|^2 \leq |x^2| + \varepsilon\mathbb{1}$, and $|x|^2 \leq |x^2|$ as $\varepsilon \to 0$. The above implies the reverse inequality and so $|x|^2 = |x^2|$. Putting $\delta^2 := |x^2 + y^2| + \varepsilon\mathbb{1}$, we easily see that $|\delta^{-2}x^2| \leq \mathbb{1}$, because

$$\|\delta^{-2}x^2\| \leq \|\delta^{-2}x^2 + \delta^{-2}y^2\| = \|\delta^{-2}|x^2 + y^2|\|_\infty \leq 1.$$

In this event $|x^2| \leq \delta^2$ and we arrive at the inequality $|x^2| \leq |x^2 + y^2|$ as $\varepsilon \to 0$.

Since $[\![\|x\|_{\mathscr{A}} = |x|]\!] = \mathbb{1}$; using the above properties of the vector norm and calculating truth values, we infer that

$$[\![\text{the norm on } \mathscr{A} \text{ satisfies 7.1(1)–7.1(3)}]\!] = \mathbb{1}.$$

Put $\Lambda := \mathbb{B}(\mathbb{R})$. If $\Lambda = \mathscr{Z}(A)$ then

$$\mathbb{1} = [\![\mathscr{Z}(\mathscr{A})\!\uparrow = \Lambda\!\uparrow = \mathscr{R} \cdot 1]\!] \wedge [\![\mathscr{Z}(A) = \mathscr{Z}(\mathscr{A})]\!] \leq [\![\mathscr{Z}(\mathscr{A}) = \mathscr{R} \cdot 1]\!].$$

Consequently, $[\![\mathscr{A}$ is a JB-factor$]\!] = \mathbb{1}$.

Assume conversely that $[\![\mathscr{Z}(\mathscr{A}) = \mathscr{R} \cdot 1]\!] = \mathbb{1}$. Then $[\![\mathscr{Z}(A)\!\uparrow = \mathscr{R} \cdot 1]\!] = \mathbb{1}$ and so

$$\operatorname{mix}(\mathscr{Z}(A)) = \mathscr{Z}(A)\!\uparrow\downarrow = \mathscr{R}\!\downarrow \cdot \mathbb{1} = \operatorname{mix}(\Lambda).$$

Passing to the bounded parts, we see that $\Lambda = \mathscr{Z}(A)$. $\square$

7.7. We will consider the interesting subclass of AJW-algebras of the class of all $\mathbb{B}$-JB-algebras. By an *AJW-algebra* we will mean a JB-algebra A satisfying the two conditions:

(1) Each set of pairwise orthogonal elements has a supremum in the ordered set $\mathfrak{P}(A)$.

(2) Each maximal strongly associative subalgebra[9)] is generated by its projections, i.e., coincides with the least closed subalgebra containing the projections.

The definition implies that each maximal strongly associative subalgebra of an AJW-algebra is a Kantorovich space of bounded elements and so it is isomorphic to the algebra and lattice $C(Q, \mathbb{R})$ for some Stone space Q.

Let A be an AJW-algebra and let $\mathbb{B}$ be the Boolean algebra of the central projections of A. Then A is a $\mathbb{B}$-JB-algebra: Given a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a bounded family $(x_\xi)_{\xi\in\Xi}$ in A, there is a unique $x \in A$ satisfying $b_\xi x = b_\xi x_\xi$ for all $\xi \in \Xi$.

Indeed, the family $(b_\xi x_\xi)$ consists of the pairwise commuting elements and so it lies in a maximal strongly associative subalgebra A_0 with unity. Since A_0 is a Kantorovich space and $(b_\xi x_\xi)$ is order bounded in A_0, the element $x := o\text{-}\sum_{\xi\in\Xi} b_\xi x_\xi$ exists in A_0. Clearly, $b_\xi x = b_\xi x_\xi$ for all ξ.

The above implies that we may apply 7.6 to AJW-algebras. However, some elaborations are possible.

7.8. Theorem. *The restricted descent A of an AJW-algebra $\mathscr{A}$ within $\mathbb{V}^{(\mathbb{B})}$ is an AJW-algebra such that $\mathfrak{P}_c(A)$ has a regular subalgebra isomorphic to $\mathbb{B}$. Conversely, if A is an AJW-algebra and $\mathfrak{P}_c(A)$ has a regular subalgebra isomorphic to $\mathbb{B}$, then there is an AJW-algebra $\mathscr{A}$ unique up to isomorphism within $\mathbb{V}^{(\mathbb{B})}$ and such that the restricted descent of $\mathscr{A}$ is $\mathbb{B}$-isomorphic to A. In this event $\mathscr{A}$ is an AJW-factor within $\mathbb{V}^{(\mathbb{B})}$ if and only if $\mathbb{B} = \mathfrak{P}_c(A)$.*

Proof. Note that by 7.6 the claim is valid provided that we replace an AJW-algebra $\mathscr{A}$ with a JB-algebra and an AJW-algebra A with a $\mathbb{B}$-JB-algebra. Thus we show only that a $\mathbb{B}$-JB-algebra A is an AJW-algebra if and only if the Boolean valued realization $\mathscr{A}$ of A is an AJW-algebra. In other words, it suffices to establish the equivalence

$$\mathscr{F}_1(A) \wedge \mathscr{F}_2(A) \leftrightarrow [\![\mathscr{F}_1(\mathscr{A})]\!] = \mathbb{1} \wedge [\![\mathscr{F}_2(\mathscr{A})]\!] = \mathbb{1},$$

where $\mathscr{F}_1(A)$ and $\mathscr{F}_2(A)$ are the properties (1) and (2) in 7.7.

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9) Cp. [13, p. 38].

(1): Start with checking that $\mathscr{F}_1(A) \leftrightarrow [\![\mathscr{F}_1(\mathscr{A})]\!] = \mathbb{1}$. We will need the auxiliary identity $\mathfrak{P}(\mathscr{A}){\downarrow} = \mathfrak{P}(A)$. If e is a projection in A, i.e., $[\![e \in \mathfrak{P}(\mathscr{A})]\!] = \mathbb{1}$; then

$$[\![e \in \mathscr{A}]\!] = [\![e^2 = e]\!] = \mathbb{1}$$

by definition. Hence, $e \in \mathscr{A}$ and $e^2 = e$. Since $[\![\|e\| = 1]\!] = \mathbb{1}$; therefore, $|e| = \mathbb{1}$ implying that $e \in A$ and $e \in \mathfrak{P}(A)$. Thus, $\mathfrak{P}(\mathscr{A}){\downarrow} \subset \mathfrak{P}(A)$. The reverse inclusion is obvious.

Take a set of pairwise orthogonal projections $\mathscr{E} \subset \mathfrak{P}(\mathscr{A})$ and put $E := \mathscr{E}{\downarrow}$. The above shows that $E \subset \mathfrak{P}(A)$. The fact that $\mathscr{E}$ consists of the pairwise disjoint elements may be written down as follows:

$$[\![(\forall\, e \in \mathfrak{P}(\mathscr{A}))\,(\forall\, c \in \mathfrak{P}(\mathscr{A}))\ (e \neq c \to ec = 0)]\!] = \mathbb{1}.$$

Using the above and calculating the truth values of quantifiers, we infer that $b^*ec = 0$ for all $e, c \in \mathfrak{P}(A)$ and the projection $b := \bigvee\{b \in \mathbb{B} : be = bc\}$. The elements of E are not pairwise orthogonal in general, and so $\mathscr{F}_1(A)$ is unapplicable. We have to adjust E by replacing E with E'. If $\gamma := |E|$ then we can enumerate the elements of E by cardinals in γ; i.e., $E = (e_\beta)_{\beta\in\gamma}$. Put $e'_1 := e_1$ and

$$e'_\alpha := b^*_\alpha e_\alpha, \quad b_\alpha := \bigvee_{\beta<\alpha} [\![e_\alpha = e_\beta]\!] \quad (1 < \alpha < \gamma).$$

If $d_{\alpha\beta} := [\![e_\alpha = e_\beta]\!]$, then the above property of $\mathscr{E}$ yields $d_{\alpha\beta}e_\alpha e_\beta = 0$. Using this together with the definition of e'_α and given $\beta < \alpha$, we conclude that

$$e'_\alpha e'_\beta = b^*_\alpha e_\alpha b^*_\beta e_\beta = \Big(\bigvee_{\nu<\alpha} d_{\alpha\nu}\Big)^* e_\alpha b^*_\beta e_\beta = \bigwedge_{\nu<\alpha} d^*_{\alpha\nu} e_\alpha e_\beta b^*_\beta \leq d^*_{\alpha\beta} e_\alpha e_\beta = \mathbb{0}.$$

Thus, $E' := (e'_\alpha)_{\alpha\in\gamma}$ consists of pairwise orthogonal projections. By the hypothesis that $\mathscr{F}_1(A)$ is valid for every $\alpha \in \gamma$, there exists $e''_\alpha := \bigvee_{\beta\leq\alpha} e'_\beta$. Induct on α to show that $e_\alpha \leq e''_\alpha$ for all $\alpha \in \gamma$). If $\alpha = 1$ then $e_1 = e'_1 = e''_1$. Assume that $e_\beta \leq e''_\beta$ for all $\beta < \alpha$. Considering the above, we infer that

$$e_\alpha = e'_\alpha \vee b_\alpha e_\alpha = e'_\alpha \vee \bigvee_{\beta<\alpha} d_{\alpha\beta} e_\alpha = e'_\alpha \vee \bigvee_{\beta<\alpha} e_\beta \leq e'_\alpha \vee \bigvee_{\beta<\alpha} e''_\beta = e''_\alpha,$$

i.e., $e_\alpha \leq e''_\alpha$.

Note that $\mathscr{F}_1(A)$ implies the existence of $e := \sup E' = \sup_{\alpha<\gamma} e'_\alpha$. However, $e'_\alpha \leq e_\alpha \leq e''_\alpha \leq e$ for all $\alpha \in \gamma$ and so E has a supremum too and $\sup E = \sup \mathscr{E}{\downarrow} = e$. It is clear now that $[\![\sup \mathscr{E} = e]\!] = \mathbb{1}$. Consequently, $\mathscr{F}_1(A) \to [\![\mathscr{F}_1(\mathscr{A})]\!] = \mathbb{1}$.

The converse implication ensues easily from the following observation: If E is a set of pairwise orthogonal projections in A; then $\mathscr{E} := E{\uparrow}$ is a set of pairwise orthogonal projections in $\mathscr{A}$ and the existence of $\sup \mathscr{E} \in \mathscr{A}$ implies the existence of $\sup E$ in A by the maximum principle.

(2): Show now that $\mathscr{F}_2(A) \to [\![\mathscr{F}_2(\mathscr{A})]\!] = \mathbb{1}$ and

$$[\![\mathscr{F}_1(\mathscr{A})\ \wedge\ \mathscr{F}_2(\mathscr{A})]\!] = \mathbb{1} \to \mathscr{F}_2(A).$$

Note first of all that the restricted descent mapping $\mathscr{A}_0 \mapsto \mathscr{A}_0{\downarrow} \cap A$ with $\mathscr{A}_0 \subset \mathscr{A}$ is a bijection between the sets $\mathscr{M}(\mathscr{A})$ and $\mathscr{M}(A)$ of maximal strongly associative subalgebras of $\mathscr{A}$ and A.

Take $x \in \mathscr{A}_0{\downarrow}$. Since $\mathscr{A}{\downarrow} = \mathrm{mix}(A)$; therefore, $x = \mathrm{mix}(b_\xi a_\xi)$ for some partition of unity $(b_\xi) \subset \mathbb{B}$ and some family (a_ξ) in A. In particular, $[\![a_\xi \in \mathscr{A}_0]\!] \geq b_\xi$. If a'_ξ is the mixing of the elements a_ξ and 0 with probabilities b_ξ and $\mathbb{1} - b_\xi$, then we still have that $x = \mathrm{mix}(b_\xi a'_\xi)$, whereas $a'_\xi \in \mathscr{A}_0{\downarrow} \cap A$. Hence, $\mathscr{A}_0{\downarrow} = \mathrm{mix}(\mathscr{A}_0{\downarrow} \cap A)$, which is equivalent to the formula $[\![\mathscr{A}_0{\uparrow} = (\mathscr{A}_0{\downarrow} \cap A){\uparrow}]\!] = \mathbb{1}$.

Assume that $\mathscr{A}_0 \in \mathscr{M}(\mathscr{A})$ and $A_0 := \mathscr{A}_0{\downarrow} \cap A$. For the associative subalgebra $A_0 \subset A_1 \subset A$ we have $[\![\mathscr{A}_0 = A_0{\uparrow} \subset A_1{\uparrow}]\!] = \mathbb{1}$ and $[\![\mathscr{A}_0 = A_1{\uparrow}]\!] = \mathbb{1}$ since $A_1{\uparrow}$ is associative. Hence,

$$A_0 = \mathscr{A}_0{\downarrow} \cap A = A_1{\uparrow}{\downarrow} \cap A \supset A_1,$$

implying that $A_0 \in \mathscr{M}(A)$. Conversely, take $A_0 \in \mathscr{M}(A)$ and put $\mathscr{A}_0 := A_0{\uparrow}$. If $\mathscr{A}_1 \subset \mathscr{A}$ is a strongly associative subalgebra including $\mathscr{A}_0$, then $\mathscr{A}_1{\downarrow}\cap A$ is a strongly associative subalgebra of A which includes $\mathscr{A}_0{\downarrow}\cap A = A_0{\uparrow}{\downarrow}\cap A \supset A_0$. Therefore, $A_0 = \mathscr{A}_1{\downarrow}\cap A$ and the ascent yields $\mathscr{A}_0 = A_0{\uparrow} = (\mathscr{A}_1{\downarrow}\cap A){\uparrow} = \mathscr{A}_1$, which proves that $\mathscr{A}_0$ is maximal. In the sequel $\mathscr{A}_0$ and A_0 correspond to each other by the above bijection. Note also that $\mathfrak{P}(\mathscr{A}_0){\downarrow} = \mathfrak{P}(A_0)$, which follows from the arguments in (1).

Assume that $\mathscr{F}_2(A)$ is valid. Take a closed subalgebra $\overline{\mathscr{A}}$ of $\mathscr{A}_0$ which includes $\mathfrak{P}(\mathscr{A}_0)$. Then $\overline{A} := \overline{\mathscr{A}}{\downarrow}\cap A$ is a closed subalgebra of A_0. Consequently, $A_0 = \bar{A}$. This implies that $\overline{A} = \overline{A}{\uparrow} = A_0{\uparrow} = \mathscr{A}_0$; i.e., $\mathscr{F}_2(\mathscr{A})$ is valid within $\mathbb{V}^{(\mathbb{B})}$.

Assume now that $[\![\mathscr{F}_1(\mathscr{A})]\!] = [\![\mathscr{F}_2(\mathscr{A})]\!] = \mathbb{1}$. Let $\overline{A}$ be the least closed subalgebra A_0 which includes $\mathfrak{P}(A_0)$. As shown in (1) $\mathscr{F}_1(A)$ is valid, and so $\overline{A}$ is a Kantorovich space of bounded elements. Moreover, $\mathbb{B} \subset \mathfrak{P}(A_0) \subset \overline{A}$, and so $\overline{A} = \mathrm{mix}(\overline{A}) \cap A$. If $\overline{\mathscr{A}} := \overline{A}{\uparrow}$ then

$$[\![\overline{\mathscr{A}} \text{ is a closed subalgebra of } \mathscr{A}_0]\!] = [\![\mathfrak{P}(\mathscr{A}_0) \subset \overline{\mathscr{A}}\,]\!] = \mathbb{1}.$$

Since $\mathscr{F}_2(\mathscr{A})$ is valid within $\mathbb{V}^{(\mathbb{B})}$; therefore, $\overline{\mathscr{A}} = \mathscr{A}_0$ and $\overline{\mathscr{A}}{\downarrow} = \mathscr{A}_0{\downarrow}$. Restricting the descents to A, we see that $\overline{A} = (A{\uparrow}{\downarrow}) \cap A = \mathscr{A}_0{\downarrow} \cap A = A_0$. So, $\mathscr{F}_2(A)$ is valid. $\square$

8. Preadjoint *JB*-Algebras

We will apply Boolean valued realization to studying the structure of $\mathbb{B}$-JB-algebras. So the new results appear by transferring the relevant facts about JB-algebras. We start with the Boolean valued realization of homomorphisms of JB-algebras.

8.1. Consider two JB-algebras A and D with the respective unities $\mathbb{1}$ and $\bar{\mathbb{1}}$. A linear operator $\Phi : A \to D$ is a Jordan homomorphism, i.e., a homomorphism of Jordan algebras, only if $\Phi(a^2) = \Phi(a)^2$ $(a \in A)$. If $\Phi(\mathbb{1}) = \bar{\mathbb{1}}$ and Φ is injective then $\|a\| = \|\Phi(a)\|$ for all $a \in A$. In particular, a Jordan isomorphism of JB-algebras is an isometry. If $\mathbb{B}$ is a complete Boolean algebra and either of the algebras $\mathfrak{P}_c(A)$ and $\mathfrak{P}_c(D)$ has a regular subalgebra isomorphic to $\mathbb{B}$, we will take the liberty to assume that $\mathbb{B} \subset \mathfrak{P}_c(A)$ and $\mathbb{B} \subset \mathfrak{P}_c(D)$. In this event a homomorphism (or isomorphism) is called a $\mathbb{B}$-*homomorphism* (or $\mathbb{B}$-*isomorphism*) provided that Φ is $\mathbb{B}$-linear, i.e., $b\Phi(a) = \Phi(ba)$ for all $a \in A$ and $b \in \mathbb{B}$. A homomorphism Φ is *normal* whenever $\Phi(x) = \sup_\alpha \Phi(x_\alpha)$ for every increasing net (x_α) in A which has the supremum $x = \sup_\alpha x_\alpha$.

8.2. Theorem. *Let $\mathscr{A}$ and $\mathscr{D}$ be JB-algebras within $\mathbb{V}^{(\mathbb{B})}$, and let A and D be the restricted descents of $\mathscr{A}$ and $\mathscr{D}$. If Φ is a $\mathbb{B}$-linear operator from A to D and $\varphi := \Phi{\uparrow}$ then*

(1) Φ is a $\mathbb{B}$-homomorphism $\leftrightarrow [\![\varphi$ is a homomorphism$]\!] = \mathbb{1}$;

(2) Φ is positive $\leftrightarrow [\![\varphi$ is positive$]\!] = \mathbb{1}$;

(3) Φ is normal $\leftrightarrow [\![\varphi$ is normal$]\!] = \mathbb{1}$.

Proof. All follow from 1.10. $\square$

8.3. Let us state a few facts concerning the structure of JB-algebras. The existence of exceptional JB-algebras implies that some JB-algebras are not isomorphic to any operator algebra in a Hilbert space. Thus, it is impossible to introduce the concept of a closed operator algebra in the class of all JB-algebras as in the case of C^*-algebras. However, we can adjust the characterizations of weakly closed operator JB-algebras which are given in the Kadison or Sakai Theorems. It turns out that these characterizations are equivalent for JB-algebras, which is so for C^*-algebras.

Assume that A is a $\mathbb{B}$-JB-algebra and $\Lambda := \mathbb{B}(\mathbb{R})$. An operator $\Phi \in A^\#$ is a Λ-*valued state* whenever $\Phi \geq 0$ and $\Phi(\mathbb{1}) = \mathbb{1}$. In case $\Lambda := \mathbb{R}$ we simple speak about states rather than Λ-valued states. If $\mathscr{A}$ is a Boolean valued realization of A, then $\phi := \Phi{\uparrow}$ is a bounded linear functional on $\mathscr{A}$ by 7.8. Moreover, ϕ is positive and order continuous; i.e., ϕ is a normal state of $\mathscr{A}$. Conversely, if $[\![\phi$ is a normal state on $\mathscr{A}]\!] = \mathbb{1}$, then the restriction of $\phi{\downarrow}$ to A is a Λ-valued normal state.

Let us characterize the $\mathbb{B}$-JB-algebras that are $\mathbb{B}$-dual spaces.

8.4. Theorem. *A JB-algebra is isometrically isomorphic to a dual Banach space if and only if A is monotone complete and has a separating set of normal states.*

Proof. Cp. [14, Theorem 2.3]. $\square$

A *JBW-algebra* is a JB-algebra satisfying the equivalent conditions of 8.4.

8.5. Theorem. *If A is a $\mathbb{B}$-JB-algebra then the following are equivalent:*

(1) *A is a $\mathbb{B}$-dual space;*

(2) *A is monotone complete and has a separating set of Λ-valued normal states.*

In the presence of (1) *and* (2) *the set of order continuous operators in $A^{\#}$ is the $\mathbb{B}$-predual space of A.*

Proof. By 7.6 A may be viewed as the restricted descent of some JB-algebra $\mathscr{A}$ within $\mathbb{V}^{(\mathbb{B})}$. Using transfer and 8.4 shows that it suffices to check that

(a) A and $\mathscr{A}$ are monotone complete simultaneously;

(b) A has a separating normal Λ-valued states if and only if

$$[\![\mathscr{A} \text{ is a separating set of normal states}]\!] = \mathbb{1}.$$

(a): The claim is valid since ascents and descents preserve polars; see [1, Theorems 3.2.13 and 3.3.12]. In this event it is clear that the polar $\pi_{\le}(M)$ of $\le$, with $\le$ standing for the orders on $\mathscr{A}$ and A, is the set of all upper bounds of M, and if there exists $\sup M$ then $\{\sup M\} = \pi_{\le}(M) \cap \pi_{\le}^{-1}(\pi_{\le}(M))$.

(b): Assume that $\mathscr{S}(\mathscr{A})$ is the set of states of $\mathscr{A}$ within $\mathbb{V}^{(\mathbb{B})}$ and $\mathscr{S}_{\mathbb{B}}(A)$ is the set of all Λ-valued states of A. Since each $\Phi \in \mathscr{S}_{\mathbb{B}}(A)$ is $\mathbb{B}$-linear, Φ is extensional and has the ascent $\phi := \Phi{\uparrow}$ that is some functional $\phi : \mathscr{A} \to \mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$. Ascending preserves linearity and positivity, implying that $[\![\|\phi\| = |\!|\Phi|\!|]\!] = \mathbb{1}$. Hence, $\Phi \mapsto \phi$ is a bijection between $\mathscr{S}_{\mathbb{B}}(A)$ and $\mathscr{S}(\mathscr{A}){\downarrow}$. In this event Φ is a normal state if and only if $[\![\phi \text{ is a normal state}]\!] = \mathbb{1}$; cp. 7.8. Suppose that $\mathscr{S}_{\mathbb{B}}(A)$ is a separating set of states. Take a nonzero $x \in A$ and choose $\Phi_0 \in \mathscr{S}_{\mathbb{B}}(A)$ so that $\Phi_0(x) \neq 0$. Since Φ is extensional; therefore, $[\![x \neq 0]\!] \le [\![\Phi_0(x) \neq 0]\!]$. Calculating truth values and using the above, we infer

$$\begin{aligned} &[\![\mathscr{S}(\mathscr{A}) \text{ is a separating set of states}]\!] \\ &= [\![(\forall\, x \in \mathscr{A})(x \neq 0 \to (\exists\, \phi \in \mathscr{S}(\mathscr{A}))\phi(x) \neq 0]\!] \\ &= \bigwedge_{x\in A} [\![x \neq 0]\!] \Rightarrow \bigvee_{\Phi\in\mathscr{S}_{\mathbb{B}}(A)} [\![\Phi\uparrow(x) \neq 0]\!] \ge \bigwedge_{x\in A} [\![x \neq 0]\!] \Rightarrow [\![\Phi_0(x) \neq 0]\!] = \mathbb{1}. \end{aligned}$$

Thus, $\mathscr{S}(\mathscr{A})$ is a separating set of states within $\mathbb{V}^{(\mathbb{B})}$. Conversely, if the latter holds then $b := [\![x \neq 0]\!] > 0$ for all x such that $0 \neq x \in A$. By the maximum principle there is $\phi \in \mathscr{S}(\mathscr{A}){\downarrow}$ satisfying $b \le [\![\phi(x) \neq 0]\!]$. Denote the restriction of $\phi{\downarrow}$ to $A \subset \mathscr{A}{\downarrow}$ by Φ. In this event $\Phi \in \mathscr{S}_{\mathbb{B}}(A)$ and $b \le [\![\Phi(x) \neq 0]\!]$. So the *carrier*[10)] of $\Phi(x)$ is at least b, and so $\Phi(x) \neq 0$. $\square$

8.6. If an algebra A satisfies either of the equivalent conditions (1) and (2) in 8.5, then A is a $\mathbb{B}$-*JBW-algebra*. If $\mathbb{B}$ is the set of all central projections then A is a $\mathbb{B}$-*JBW-factor*. From 7.6 and 8.5 it follows that A is a $\mathbb{B}$-JBW-algebra or a $\mathbb{B}$-JBW-factor if and only if the Boolean valued realization $\mathscr{A}$ of A is a JBW-algebra or a JBW-factor within $\mathbb{V}^{(\mathbb{B})}$.

Example. Let X be a Kaplansky–Hilbert module over $\overline{\Lambda} := \mathbb{B}(\mathbb{C})$. Then X is a $\mathbb{B}$-cyclic Banach space, and $\mathscr{L}_{\overline{\Lambda}}(X)$ is a type I AW^*-algebra; see 2.9. Given $x, y \in X$, define the seminorm

$$p_{x,y}(a) := \|\langle ax \mid y\rangle\|_\infty \quad (a \in \mathscr{L}_{\overline{\Lambda}}(X)),$$

where $\langle\cdot \mid \cdot\rangle$ is the $\overline{\Lambda}$-valued inner product on X. Let σ_∞ stand for the topology on $\mathscr{L}_{\overline{\Lambda}}(X)$ which is generated by the collection of all $p_{x,y}$ with $x, y \in X$. We can show (see the proof of 8.8) that each

[10)] This is defined as $[\lambda] := \bigwedge\{b \in \mathbb{B} : b\lambda = \lambda\}$ for all $\lambda \in \Lambda$; cp. [1, 4.2.6].

σ_∞-closed $\mathbb{B}$-JB-algebra of selfadjoint operators in $\mathscr{L}_{\overline{\Lambda}}(X)$ is a monotone closed subalgebra of $\mathscr{L}_{\overline{\Lambda}}(X)_{\rm sa}$. Also, the latter algebra is monotone complete and has a separating set of Λ-valued normal states. So, each σ_∞-closed $\mathbb{B}$-JB-algebra of selfadjoint operators exemplifies a $\mathbb{B}$-JBW-algebra.

8.7. Assume that A is a Jordan algebra of selfadjoint operators. If A is norm closed then A is a *JC-algebra*. If A is weakly closed then A is a *JW-algebra*.

8.7(1). *A JC-algebra A is a JW-algebra if and only if A is monotone complete and has a separating set of normal $\mathbb{C}$-valued states or, in other words, A is linearly isometric to some dual Banach space.*

Proof. This follows from 8.3. $\square$

8.7(2). Kaplansky Density Theorem. *Let A be a strongly closed subalgebra of a JBW-algebra M. Then the unit ball of A is strongly closed in the unit ball of M.*[11)]

8.8. Theorem. *A special $\mathbb{B}$-JB-algebra A is a $\mathbb{B}$-JBW-algebra if and only if A is isomorphic to a σ_∞-closed $\mathbb{B}$-JB-subalgebra of $\mathscr{L}_{\overline{\Lambda}}(X)_{\rm sa}$, where X is some Kaplansky–Hilbert module.*

Proof. Sufficiency holds by 8.4 and so we have to demonstrate necessary. Let A be a special $\mathbb{B}$-JBW-algebra. We may assume again that A is the restricted descent of some JB-algebra $\mathscr{A}$ within $\mathbb{V}^{(\mathbb{B})}$. In this event, it is easy to show that $\mathscr{A}$ is special.

Using 8.7(1) we see that the special JBW-algebra $\mathscr{A}$ is a JW-algebra; i.e., A is isomorphic to a weakly closed subalgebra of some algebra of the form $\mathscr{L}(\mathscr{X})_{\rm sa}$, where $\mathscr{X}$ is a complex Hilbert space within $\mathbb{V}^{(\mathbb{B})}$. So we may assume that $\mathscr{A}$ is uniformly closed Jordan subalgebra of $\mathscr{L}(\mathscr{X})_{\rm sa}$. Therefore, it suffices to prove only that A is a σ_∞-closed subalgebra of $\mathscr{L}_{\overline{\Lambda}}(X)_{\rm sa}$ if and only if $\mathbb{V}^{(\mathbb{B})}\models$ "$\mathscr{A}$ is a weakly closed subalgebra of $\mathscr{L}(\mathscr{X})_{\rm sa}$."

The algebraic claim is obvious. Let the formula $\psi(\mathscr{A},u)$ says that some operator u belongs to the weak closure of $\mathscr{A}$. Then the formula can be expressed as follows:

$$(\forall\, n\in\omega)(\forall\,\theta_1,\theta_2\in\mathscr{P}_{\rm fin}(\mathscr{X}))(\exists\, v\in\mathscr{A})(\forall\, x\in\theta_1)(\forall\, y\in\theta_2)\ |\langle u(x)-v(x),y\rangle|\le n^{-1},$$

with ω the set positive integers, $\langle\cdot,\cdot\rangle$ the inner product on $\mathscr{X}$, and $\mathscr{P}_{\rm fin}(\mathscr{X})$ the set of finite subsets of X. Assume that $[\![\psi(\mathscr{A},u)]\!]=\mathbb{1}$. Calculating truth values and applying the maximum principle together with the equality

$$\mathscr{P}_{\rm fin}(\mathscr{X})=\{\theta{\uparrow}:\ \theta\in\mathscr{P}_{\rm fin}(X)\}{\uparrow}$$

implies the following: Given $n\in\omega$ together with finite sets $\theta_1:=\{x_1,\dots,x_n\}\subset X$ and $\theta_2:=\{y_1,\dots,y_m\}\subset X$, we have $v\in\mathscr{A}{\downarrow}$ such that

$$[\![(\forall\, x\in\theta_1{\uparrow})(\forall\, y\in\theta_2{\uparrow})|\langle u(x)-v(x),y\rangle|\le n^{-1}]\!]=\mathbb{1}.$$

By the Kaplansky Density Theorem we may choose v so that $[\![\|v\|\le\|u\|]\!]=\mathbb{1}$. If U and V are the restrictions to X of $u{\downarrow}$ and $v{\downarrow}$ then

$$|V|\le|U|,\quad |\langle(U-V)(x_k)\mid y_l\rangle|<n^{-1}\mathbb{1}\quad (k:=1,\dots,n;\ l:=1,\dots,m).$$

There is a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ which depends only on u and such that $e_\xi|U|\in\Lambda$ for all ξ. Hence, $e_\xi U\in A$ and $e_\xi V\in A$. Moreover,

$$\|\langle e_\xi(U-V)(x_k)\mid y_l\rangle\|_\infty<n^{-1}\quad (k:=1,\dots,n;\ l:=1,\dots,m).$$

Repeating the above arguments in the reverse order, we conclude that $\psi(\mathscr{A},u)$ is valid within $\mathbb{V}^{(\mathbb{B})}$ if and only if there are a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(U_\xi)_{\xi\in\Xi}$, with U_ξ in the σ_∞-closure of A, such that $e_\xi\le[\![u=U_\xi{\uparrow}]\!]$ for all ξ; i.e., $u=\operatorname{mix}(e_\xi U_\xi{\uparrow})$.

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11) Cp. [4, vol. 1, p. 329].

Assume now that A is σ_∞-closed and $\psi(\mathscr{A}, u)$ is valid within $\mathbb{V}^{(\mathbb{B})}$. Then U_ξ lies in A by hypothesis and $[\![U_\xi\!\uparrow \in \mathscr{A}]\!] = \mathbb{1}$. Hence, $e_\xi \le [\![u \in \mathscr{A}]\!]$ for all ξ; i.e., $[\![u \in \mathscr{A}]\!] = \mathbb{1}$. Thus

$$\mathbb{V}^{(\mathbb{B})} \models (\forall\, u \in \mathscr{L}(\mathscr{X}))\psi(\mathscr{A}, u) \to u \in \mathscr{A}.$$

Conversely, let us assume that $\mathscr{A}$ is weakly closed. If U lies in the σ_∞-closure of A then $u = U\!\uparrow$ belongs to the weak closure of $\mathscr{A}$. By hypothesis $[\![u \in \mathscr{A}]\!] = \mathbb{1}$ and so $u \in \mathscr{A}\!\downarrow$. The restriction of $u\!\downarrow$ to X coincides with U and, hence, belongs to A. $\square$

8.9. Let $M_3^8 := M_3(\mathbb{O})$ be the algebra of hermitian 3×3-matrices over octonions as in 7.2(2). If $(\cdot)^\wedge$ is the canonical standard name embedding in $\mathbb{V}^{(\mathbb{B})}$ then $[\![\mathbb{O}^\wedge$ is a normed algebra over $\mathbb{R}^\wedge]\!] = \mathbb{1}$ and

$$[\![(M_3^8)^\wedge \text{ is the } \mathbb{R}^\wedge\text{-algebra of hermitian } 3 \times 3\text{-matrices over } \mathbb{O}^\wedge]\!] = \mathbb{1}.$$

Let $\mathscr{O}$ and $\mathscr{M}_3^8$ be the norm completions of the algebras $\mathbb{O}^\wedge$, while $(M_3^8)^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. The Hurwitz Theorem[12)] implies that $[\![\mathscr{O}$ is a Cayley algebra$]\!] = \mathbb{1}$ and

$$[\![\mathscr{M}_3^8 \text{ is the algebra of hermitian } 3 \times 3\text{-matrices over Cayley numbers}]\!] = \mathbb{1}.$$

By 7.6 the restricted descent of $\mathscr{M}_3^8$ is a $\mathbb{B}$-JB-algebra. Also, the restricted descent of the JB-algebra $\mathscr{M}_3^8$ is isometrically isomorphic to $C(Q, M_3^8)$, where Q is the Stone space of $\mathbb{B}$. Using the above, we will give the Boolean valued interpretation of the following fact: *Each JBW-factor is isomorphic to M_3^8 or a JC-algebra.*

8.10. Theorem. *Each $\mathbb{B}$-JBW-factor A admits the unique decomposition $A = eA \oplus e^*A$ with a central projection $e \in \mathbb{B}$ and $e^* := \mathbb{1} - e$ such that eA is special and e^*A is purely exceptional.*[13)] *Moreover, eA is $\mathbb{B}$-isomorphic of a σ_∞-closed subalgebra of selfadjoint endomorphisms of some Kaplansky–Hilbert module, and e^*A is isomorphic to $C(Q, M_3^8)$, where Q is a Stone space of $e^*\mathbb{B} := [0, e^*]$.*

Proof. If $\mathscr{A}$ is a Boolean valued realization of A then $[\![\mathscr{A}$ is a JBW-factor $= \mathbb{1}]\!]$ by 8.4. By transfer

$$[\![\mathscr{A} \text{ is isomorphic either to a } JC\text{-algebra or } M_3^8]\!] = \mathbb{1}.$$

Put $e := [\![\mathscr{A}$ is special$]\!]$. Then

$$e^* = [\![\mathscr{A} \text{ is isomorphic to } \mathscr{M}_3^8]\!].$$

Furthermore,

$$\mathbb{V}^{(e\mathbb{B})} \models [\![e\mathscr{A} \text{ is a special } JBW\text{-factor}]\!];$$

$$\mathbb{V}^{(e^*\mathbb{B})} \models [\![e^*\mathscr{A} \text{ is an algebra isomorphic to } \mathscr{M}_3^8]\!].$$

The restricted descent of $e\mathscr{A}$ is a special $e\mathbb{B}$-algebra. We are done on recalling 8.8 and the remark in 8.9. $\square$

9. Comments

9.1. C^*-Algebras and AW^*-algebras. The Boolean valued transfer principle for C^*-algebras and von Neumann algebras was discovered by Takeuti and started with Theorem 1.6. Takeuti contributed many important applications of the Boolean valued models of set theory to analysis. Theorems 1.7 and 1.8 are some Boolean valued interpretations of the classical results of Banach algebra theory: the Gelfand–Mazur Theorem and the Gleason–Kahane–Żelazko Theorem. Note a special instance of Theorem 1.8 in

[12)] Cp. [20, Section 3].

[13)] Cp. [14, p. 361].

[21, Theorem 3.1]. A brief survey of some extensions of the Gleason–Kahane–Żelazko Theorem to certain Banach function spaces that are not algebras can be found in [13].

The modern structure theory of AW^*-algebras started with the research by Kaplansky. These objects arise naturally by the way of algebraization of the theory of von Neumann operator algebras. Theorems 2.9 (clarified in 5.5) and 2.10 are taken from the Kaplansky paper [22]. The Boolean valued representations of AW^*-algebras (Theorems 2.4 and 2.5) belong to Ozawa. Takeuti proved that the Boolean valued representation of AW^*-algebras keeps the classification into types (Theorems 2.7 and 2.8); see [1] for more details.

9.2. Heuristics of Kaplansky–Hilbert modules. The concept of Kaplansky–Hilbert module was introduced by Kaplansky in 1953 under the name AW^*-module. He wrote: "...the new idea is to generalize Hilbert space by allowing the inner product to take values in a more general ring than the complex numbers. After the appropriate preliminary theory of these AW^*-modules has been developed, one can operate with a general AW^*-algebra of type I in almost the same manner as with the factor." In other words, the central elements of an AW^*-algebra can be taken as complex numbers and we can work with factors rather than general AW^*-algebras. Needles to say, this is a version of Kantorovich's heuristic principle who stated in his 1935 definitive article on vector lattices: "In this note, I define the new type of space that I call a semiordered linear space. The introduction of such space allows us to study linear operations of one abstract class (those with values in the space) as linear functionals."

9.3. Boolean dimension and the Kaplansky problem. The definition of the Boolean dimension of a Kaplansky–Hilbert module in 3.5 belongs to Kusraev [23] and presents the external deciphering of the internal definition given by Ozawa in [25] of the dimension of the Kaplansky–Hilbert module presenting some internal object of a Boolean valued universe which is the dimension of the Hilbert space that appears as the Boolean valued realization of the original module. The concepts of homogeneous and strictly homogeneous Kaplansky–Hilbert modules in 3.1 belong to Kaplansky [23] and Kusraev [24] respectively. Theorems 3.2 and 3.3 were proved respectively in [25] and [24]. The latter article contains also the results on the functional representation of Kaplansky–Hilbert modules (Theorem 4.5) and AW^*-algebras (Theorem 5.5).

Boolean valued analysis of AW^*-algebras yields a negative solution to the Kaplansky problem of the unique decomposition of a type I AW^*-algebra into the direct sum of homogeneous bands. Ozawa gave this solution in [22]. The lack of uniqueness is tied with the effect of the cardinal shift that may happens on ascending into a Boolean valued model $V^{(\mathbb{B})}$. The cardinal shift is impossible in the case that the Boolean algebra of central idempotents $\mathbb{B}$ under study satisfies the countable chain condition, and so the decomposition in question is unique. Kaplansky established uniqueness of the decomposition on assuming that $\mathbb{B}$ satisfies the countable chain condition and conjectured that uniqueness fails in general; see [22, Theorem 4].

9.4. Embeddable C^*-algebras and Tomita–Takesaki modular theory. Observe that the C^*-algebras realizable as bicommutants of a type I AW^*-algebra has attracted researchers since long ago; for instance see the Berberian book [6] where these algebras are called embeddable. The theory of embeddable algebras runs parallel to the theory of von Neumann algebras. The transfer principle for embeddable algebras, i.e., Theorem 6.2 established by Ozawa [25], explain the nature of this parallelism as well as provides some method for translating all theorems on non Neumann algebras to the corresponding theorems about embeddable C^*-algebras. Theorems 6.4–6.6, 6.7, 6.10, and 6.11, belonging to Ozawa, exhibit examples of the above translation; see [1] for more details and further references.

We will cite another Ozawa's result in [25]. A mapping $J : H \to H$, acting in a Kaplansky–Hilbert module H over Λ, is Λ-*conjugation* provided that

(a) $J(u+v) = J(u) + J(v) \quad (u, v \in H)$;
(b) $J(\lambda u) = \lambda^* J(u) \quad (\lambda \in \Lambda,\, u \in H)$;
(c) $\langle Ju|v\rangle = \langle u|Jv\rangle \quad (u, v \in H)$;
(d) $J = J^{-1}$.

A subset A of $\mathscr{L}_\Lambda(H)$ is a Λ-*factor* whenever A is an AW^*-subalgebra of $\mathscr{L}_\Lambda(H)$ and the center of A coincides with the center of $\mathscr{L}_\Lambda(H)$. If Λ-factor A admits a Λ-conjugation J on H such that $x \mapsto J \circ x^* \circ J$ is a $*$-anti-isomorphism of A to A', then A is *standard*. In case $\Lambda = \mathbb{R}$ we simply speak of conjugations and a Λ-factor is actually a von Neumann factor.

Tomita–Takesaki theory states that each von Neumann algebra is $*$-isomorphic to some standard von Neumann algebra; see, for instance, the book by Kadison and Ringrose [4]. Using the Boolean valued transfer principle for embeddable algebras, Ozawa establish that each embeddable AW^*-algebra with center Λ is $*$-isomorphic to a standard Λ-factor on some Kaplansky–Hilbert module over Λ.

9.5. Jordan–Banach algebras. The JB-algebras are nonassociative real analogs of C^*-algebras and von Neumann operator algebras. The theory of these algebras stems from the classification by Jordan, von Neumann, and Wigner of all finite-dimensional formally real Jordan algebras: These are only the algebras of selfadjoint $(n \times n)$-matrices, with $3 \le n < \infty$, over the reals, or complexes, or quaternions as well as the algebra of selfadjoint (3×3)-matrices over the Cayley numbers. In the mid-1960s Topping [26] and Størmer [27] had started the study of the nonassociative real analogs of von Neumann algebras, the JW-algebras presenting weakly closed Jordan algebras of bounded selfadjoint operators in a Hilbert space. Then Alfsen, Shultz, and Størmer introduced JB-algebras in [28] and Shultz [19] distinguished the class of predual JB-algebras called JBW-algebras. Kusraev outlined the Boolean valued approach to JB-algebras in [29] which contains 7.5–7.8 as well as the results of Section 8 on $\mathbb{B}$-JBW-algebras. The class of $\mathbb{B}$-JBW-algebras of [30] is wider than the class of JBW-algebra. Their fundamental distinctions consist in the fact that each $\mathbb{B}$-JBW-algebra has a faithful representation in the algebra of selfadjoint operators on some AW^*-module rather than on a Hilbert space as in the case of JBW-algebras.

9.6. Noncommutative and nonassociative integration. One of the most fruitful ideas of the theory of involutive and Jordan algebras with important applications to quantum mechanics consists in considering a von Neumann algebra or a JW-algebra as a noncommutative analog of the classical L^∞ space as a base for developing the theory of *noncommutative* or *nonassociative integration*. Segal proposed the foundations of some noncommutative integration theory invoking the notion of measurability of an unbounded operator affiliated to a von Neumann algebra $\mathscr{M}$ with respect to a faithful normal semifinite trace τ. He defined the noncommutative $L_1(\mathscr{M}, \tau)$, $L_2(\mathscr{M}, \tau)$, and $L_\infty(\mathscr{M}, \tau)$ spaces. Extension of the noncommutative integration theory to an arbitrary von Neumann algebra equipped with an arbitrary normal semifinite weight ω became possible only after appearance of Tomita–Takesaki modular theory. The scale of noncommutative $L_p(\mathscr{M}, \omega)$ spaces for $1 \le p \le \infty$ was firstly constructed by Haagerup. Among the contributors to this theory, we mention only Dixmier, Nelson, and Yeadon. Kostecki gave a detailed overview of the noncommutative integration theory and extensive bibliography in [30].

If integration bases over a JW-algebra or an AJW-algebra instead of a von Neumann algebra then we arrive at the so-called *nonassociative integration*. Ayupov started the study of measurable operators over JW-algebras. The L^p spaces over JBW-algebras are studied by Abdullaev, Ayupov, Berdikulov, and Iochum.

Nonassociative L_p spaces provide an essentially larger class of Banach spaces than noncommutative L_p spaces. Ayupov and Abdullaev proved that $L_p(A, \tau)$ is isometrically isomorphic to the selfadjoint part of a noncommutative $L_p(\mathscr{M}, \nu)$ associated with a von Neumann algebra $\mathscr{M}$ and a semifinite trace ν if and only if the JBW-algebra A is special and isomorphic to the selfadjoint part of $\mathscr{M}$.

Clearly, Boolean valued analysis must have nontrivial applications to the study of measurable operators associated with an AW^*-algebra. Unfortunately, we are aware of the only paper on this topic which belongs to Korol' and Chilin [31].

9.7. Derivations. A *derivation* in an algebra A is a linear operator $D : A \to A$ satisfying the Leibniz rule $D(xy) = D(x)y + xD(y)$. If $a \in A$ then the multiplication operator $x \mapsto xa - ax$, with $x \in A$ is a derivation which is called *inner*. Quantum mechanics is among the main motivations for studying derivations since the latter arise as the generators of the one-parameter groups describing the symmetries and dynamical evolutions of quantum mechanical systems; see [32]. Denote by $S(M)$ or $LS(M)$ the set of measurable or locally measurable operators associated with M. If M is a finite algebra then $S(M) = LS(M)$ and $S(N)$ is the *Murray–von Neumann algebra* associated with M.

The theory of bounded derivations of C^*-algebras and AW^*-algebras started from the Kaplansky paper [23] and the definitive result by Singer which was an answer to a question by Kaplansky. Singer proved that every derivation of the algebra of continuous functions on a Hausdorff compact space is the zero mapping; see [32]. Since the 2000s much popularity has gained by the *Ayupov problem*: Is each derivation inner on the algebra of measurable operators associated with a von Neumann algebra?

The problem was solved in the case of a commutative von Neumann algebra M independently by Ber, Chilin, and Sukochev in [33] as well as by Kusraev in [34]: The algebra $S(M)$ has a nontrivial derivations if and only if the projection lattice $\mathfrak{P}(M)$ is atomic.[14)] The noncommutative version required more efforts: Only the case of Murray–von Neumann algebras remained unsettled by 2014. This was stated as open by Kadison and Liu in [32]. The complete solution of the Ayupov problem was announced by Ber, Kudaybergenov, and Sukochev in [35] and published in [36]: Given a von Neumann algebra M, every derivation of $LS(M)$ or $S(M)$ is inner if and only if a type I_{fin} direct summand of M is atomic. The overview of some other results about derivations on various algebras of measurable operators associated with von Neumann algebras can be found in [37]. Of interest are the linear operators close to derivations, i.e., local and 2-local derivations; this area is surveyed in [38].

9.8. Boolean transcendence degree. Let $\mathscr{C}$ be the complexes and $\tau(\mathscr{C})$ the transcendence degree of $\mathscr{C}$ over $\mathbb{C}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. The Boolean valued cardinal $\tau(\mathscr{C})$ carries some information on the connection between the Boolean algebra $\mathbb{B}$ and the commutative AW^*-algebra $\mathscr{C}{\downarrow}$. Say that $\tau(\mathscr{C})$ is the *B-transcendence degree of* $\mathscr{C}{\downarrow}$. Given $\mathscr{E} \subset \mathscr{C}{\downarrow}$, denote by $\langle \mathscr{E} \rangle$ the set of elements of the form $e_1^{n_1} \cdot \ldots \cdot e_k^{n_k}$ with $e_1, \dots, e_k \in \mathscr{E}$ and $k, n_1, \dots, n_k \in \mathbb{N}$. A set $\mathscr{E} \subset G$ is *locally algebraically independent* provided that $\langle \mathscr{E} \rangle$ is locally linearly independent in the sense of [2, 4.5]. It was conjectured in [39, Comment 5.3.7] that the notions of B-transcendence degree and local algebraic independence which are Boolean valued interpretations of conventional field extension concepts of algebraic independence and transcendence degree may be useful in studying the descents of fields or general regular rings. This prediction was confirmed by Ayupov, Karimov, and Kudaybergenov in [40]: If (Ω, Σ, μ) is a Maharam homogeneous measure space, then two homogeneous unital regular subalgebras of $S(\Omega, \Sigma, \mu)$ are isomorphic if and only if their Boolean algebras of idempotents are isomorphic and their transcendence degrees coincide (also see [41]).

9.9. Noncommutative and nonassociative geometry. *Noncommutative geometry* by Connes develops the ideas of Descartes *calcul géométrique*. The harbinger of noncommutative geometry is the Gelfand transform that establishes the equivalence of the categories of commutative C^*-algebras and Hausdorff compact topological spaces. Therefore, each property of a locally compact space has an algebraic formulation in terms of an appropriate commutative C^*-algebras; and, conversely, each statement about a commutative C^*-algebra admits translation into topological-geometrical terms. Noncommutative geometry extends this correlation between "space geometry" and "function algebra" to noncommutative objects; see Khalkhali [42] for a general introduction to noncommutative geometry. Recently, Boyle and Farnsworth started to generalize Connes' ideas of noncommutative geometry to nonassociative geometry; see [43]. We see that the tool kit of noncommutative geometry extends essentially by invoking various nonassociative objects like JB-algebras, JB^*-algebras, O^*-algebras, etc. We opine that Boolean valued analysis will be of use in this trend of research.

9.10. Takeuti's quantum mathematics. Usually, an order complete orthomodular lattice is taken as a general model of *quantum logic*. The *orthomodularity* of a complete lattice Q means that if $p, q \in Q$ and $p \leq q$, then there exists a Boolean subalgebra of Q containing p and q. A *standard quantum logic* is a lattice consisting of the projections on a Hilbert space.

Takeuti introduced the universe $\mathbb{V}^{(Q)}$ of sets which bases on a standard quantum logic Q on a Hilbert space H in his seminal paper [44]. Takeuti pointed out the remarkable fact of quantum set theory:

[14)] Recall that $\mathfrak{P}(M)$ is a Boolean algebra in the commutative case.

The reals within $\mathbb{V}^{(Q)}$ corresponds bijectively to the selfadjoint operators on H, as is straightforward by Boolean valued analysis. He wrote in [44, p. 303]: "Since quantum logic is an intrinsic logic, i.e., the logic of the quantum world [...], it is an important problem to develop mathematics based on quantum logic, more specifically set theory based on quantum logic. It is also a challenging problem for logicians since quantum logic is drastically different from the classical logic or the intuitionistic logic and consequently mathematics based on quantum logic is extremely difficult. On the other hand, mathematics based on quantum logic has a very rich mathematical content."

Ozawa made a deep and welcome analysis of Takeuti's mathematical views as well as his contributions to the foundations of mathematics and the development of the concept of set in a many-valued logic in [11].

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