ISSN 0037–4466, Siberian Mathematical Journal, 2023, Vol. 64, No. 4, pp. 968–981. © Pleiades Publishing, Ltd., 2023.
Russian Text © The Author(s), 2022, published in Vladikavkazskii Matematicheskii Zhurnal, 2022, Vol. 24, No. 3, pp. 62–77.

ON THE THEORY OF SPACES OF GENERALIZED BESSEL POTENTIALS

A. L. Dzhabrailov and E. L. Shishkina

UDC 517.98

Abstract: We define the weighted Dirichlet integral and show that this integral can be represented by a multidimensional generalized shift. The corresponding norm does not allow us to define the function spaces of arbitrary fractional order of smoothness, and so we introduce the new norm that is related to a generalized Bessel potential. Potential theory originates from the theory of electrostatic and gravitational potentials and the study of the Laplace, wave, Helmholtz, and Poisson equations. The celebrated Riesz potentials are the realizations of the real negative powers of the Laplace and wave operators. In the meantime, much attention in potential theory is paid to the Bessel potential generating the spaces of fractional smoothness. We progress in generalization by considering the Laplace–Bessel operator constructed from the singular Bessel differential operator. The theory of singular differential equations with the Bessel operator as well as the theory of the corresponding weighted function spaces are closely connected and belong to the areas of mathematics whose theoretical and applied significance can hardly be overestimated.

DOI: 10.1134/S0037446623040183

Keywords: Bessel operator, generalized Bessel potentials, weighted Dirichlet integral

1. Introduction

Well known is the fundamental role of the classical Bessel potentials in the general theory of function spaces of fractional smoothness as well as applications to the theory of partial equations (see [1, 2]). The classical Bessel potentials are constructed on using the Fourier transform.

The goal of this article is to develop the theory of the space of generalized Bessel potentials $\mathbf{B}_\gamma^\alpha$ constructed with the Hankel transform. This space was first introduced by Lyakhov in [3] who based on the Stein–Lizorkin approach. In [3], the B-hypersingular integrals and B-Riesz potentials, introduced earlier by Lyakhov in [4, 5], were applied the constructing the norm on $\mathbf{B}_\gamma^\alpha$. In the present article, we use another approach to $\mathbf{B}_\gamma^\alpha$ which is based on the works [6–8] by Aronszajn and Smith. The approach consists in introducing some norm on $\mathbf{B}_\gamma^\alpha$ with the help of weighted Dirichlet integrals.

The spaces of generalized Bessel potentials of arbitrary order α are necessary for defining the classes of solutions to the boundary value problem

$$Au = f \text{ in } D, \quad B_i u = 0 \text{ on } \partial D,$$

where A is an elliptic operator containing the Bessel differential operators $B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$; in particular, A can be the Laplace–Bessel operator $\Delta_\gamma = \sum_{i=1}^n B_{\gamma_i}$.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, while

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ x_1 > 0, \dots, x_n > 0\},$$
$$\overline{\mathbb{R}}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ x_1 \geq 0, \dots, x_n \geq 0\}.$$

The first author was supported by the Ministry of Science and Higher Education of the Russian Federation (Project FEGS–2020–0001).

Original article submitted January 8, 2022; accepted January 18, 2023.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the multi-index consisting of fixed positive reals γ_i, with $i = 1, \ldots, n$, and $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

Let Ω be a finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each of the hyperplanes $x_i = 0$, with $i = 1, \ldots, n$, $\Omega_+ = \Omega \cap \mathbb{R}^n_+$, and $\overline{\Omega}_+ = \Omega \cap \overline{\mathbb{R}}^n_+$. We will work with the function class $C^m(\Omega_+)$ consisting of m times differentiable functions on Ω_+. Denote by $C^m(\overline{\Omega}_+)$ the subset of the functions in $C^m(\Omega_+)$ whose all derivatives with respect to x_i extend continuously to $x_i = 0$ for all $i = 1, \ldots, n$. Let $C^m_{ev}(\overline{\Omega}_+)$ consist of the functions in $C^m(\overline{\Omega}_+)$ such that

$$\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\Big|_{x_i=0} = 0$$

for all nonnegative integers $k \le \frac{m-1}{2}$ (see [9, p. 21]). In what follows, we use the abbreviation C^m_{ev} for $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ and put

$$C^\infty_{ev}(\overline{\Omega}_+) = \bigcap C^m_{ev}(\overline{\Omega}_+),$$

where the intersection is taken over all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$.

Let $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ be the space of compactly supported functions $f \in C^\infty_{ev}(\overline{\Omega}_+)$. Put

$$\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+) = \mathscr{D}_+(\overline{\Omega}_+) \quad \text{and} \quad \overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}_+) = \overset{\circ}{C}{}^\infty_{ev}.$$

Let $L^\gamma_p(\mathbb{R}^n_+) = L^\gamma_p$, with $1 \le p < \infty$, consists of the measurable functions on $\mathbb{R}^n_+$ even in each of the variables x_i, with $i = 1, \ldots, n$, such that

$$\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx < \infty.$$

Here and in the sequel $x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}$. For every $p \ge 1$, the norm of $f \in L^\gamma_p$ is defined as

$$\|f\|_{L^\gamma_p(\mathbb{R}^n_+)} = \|f\|_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx \right)^{\frac{1}{p}}.$$

It is know that L^γ_p is a Banach space (see [9]).

The multidimensional Hankel transform of $f \in L^\gamma_1(\mathbb{R}^n_+)$ is defined as

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}^n_+} f(x)\, \mathbf{j}_\gamma(x;\xi) x^\gamma dx,$$

where

$$\mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^n j_{\frac{\gamma_i - 1}{2}}(x_i \xi_i).$$

Denote by j_ν the normalized Bessel function of the first kind $j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x)$, where J_ν is a Bessel function of the first kind (see [10]).

Let $f \in L^\gamma_1(\mathbb{R}_+)$ be a function of bounded variation in a neighborhood of a continuity point x of f. Then for $\gamma > 0$ the inversion formula for the transformation looks as

$$\mathbf{F}^{-1}_\gamma\big[\widehat{f}(\xi)\big](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \mathbf{j}_\gamma(x,\xi) \widehat{f}(\xi)\, \xi^\gamma d\xi.$$

The Hankel transform reduces the Bessel operator to the multiplication of the square of the corresponding argument with the minus sign (see [9]):

$$F_{\gamma_i}\big[(B_{\gamma_i})_{x_i} f\big](\xi) = -|\xi_i|^2 F_{\gamma_i}[f](\xi), \tag{1}$$

where $(B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ is the Bessel operator, $i = 1, \dots, n$.

Parseval's identity for the one-dimensional Hankel transform (see [9, p. 20]) and the fact that $f \in L_2^\gamma(\mathbb{R}_+^n)$ yield $\mathbf{F}_\gamma f \in L_2^\gamma(\mathbb{R}_+^n)$ and

$$\int\limits_{\mathbb{R}_+^n} \big|\mathbf{F}_\gamma[f](\xi)\big|^2 \xi^\gamma d\xi = 2^{|\gamma|-n} \prod_{j=1}^n \Gamma^2\Big(\frac{\gamma_j+1}{2}\Big) \int\limits_{\mathbb{R}_+^n} \big|f(x)\big|^2 x^\gamma dx. \tag{2}$$

The multidimensional generalized shift is defined as

$$({}^\gamma\mathbf{T}_x^y f)(x) = {}^\gamma\mathbf{T}_x^y f(x) = ({}^{\gamma_1}T_{x_1}^{y_1} \dots {}^{\gamma_n}T_{x_n}^{y_n} f)(x), \tag{3}$$

where each one-dimensional generalized shift ${}^{\gamma_i}T_{x_i}^{y_i}$, $i = 1, \dots, n$, acts by the formula

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x) = \frac{\Gamma\big(\frac{\gamma_i+1}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{\gamma_i}{2}\big)} \int\limits_0^\pi f\Big(x_1, \dots, x_{i-1}, \sqrt{x_i^2 + \tau_i^2 - 2x_i y_i \cos\varphi_i}, x_{i+1}, \dots, x_n\Big) \sin^{\gamma_i - 1}\varphi_i \, d\varphi_i.$$

In what follows, put

$$C(\gamma) = \pi^{-\frac{n}{2}} \prod_{i=1}^n \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})}.$$

The generalized convolution generated by the multidimensional generalized shift ${}^\gamma\mathbf{T}_x^y$ has the form

$$(f * g)_\gamma(x) = (f * g)_\gamma = \int\limits_{\mathbb{R}_+^n} f(y)({}^\gamma\mathbf{T}_x^y g)(x)\, y^\gamma dy. \tag{4}$$

The multidimensional Poisson operator $\mathbf{P}_x^\gamma$ acts an integrable function f by the formula

$$\mathbf{P}_x^\gamma f(x) = C(\gamma) \int\limits_0^\pi \cdots \int\limits_0^\pi f(x_1 \cos\alpha_1, \dots, x_n \cos\alpha_n) \prod_{i=1}^n \sin^{\gamma_i - 1}\alpha_i \, d\alpha_i. \tag{5}$$

3. The Weighted Dirichlet Integral

Let $i = (i_1, \dots, i_m)$ be a multi-index consisting of the integers from 1 to n, with $|i| = i_1 + \dots + i_n$; while $\xi^i = \prod_{k=1}^m \xi_{ik}$, $\xi = (\xi_1, \dots, \xi_n)$, and

$$\mathbb{B}_i = (B_{\gamma_{i_m}})_{x_{i_m}} \dots (B_{\gamma_{i_1}})_{x_{i_1}},$$

where $(B_{\gamma_{i_k}})_{x_{i_k}} = \frac{\partial^2}{\partial x_{i_k}^2} + \frac{\gamma_{i_k}}{x_{i_k}}\frac{\partial}{\partial x_{i_k}}$ is a Bessel operator for all $k = 1, \dots, m$. Given an integer $\alpha \geq 0$, define the weighted Dirichlet integral of order α as

$$d_{\alpha,\gamma}(u) = \sum_{|i|=\alpha} \int\limits_{\mathbb{R}_+^n} |\mathbb{B}_i u|^2 x^\gamma dx.$$

In the image of (1), we obtain

$$d_{\alpha,\gamma}(u) = \int\limits_{\mathbb{R}_+^n} |\xi|^{4\alpha} \big|\mathbf{F}_\gamma[u](\xi)\big|^2 \xi^\gamma d\xi. \tag{6}$$

Formula (6) can be used for defining the weighted Dirichlet integral d_α^γ for an arbitrary $\alpha \geq 0$.

Lemma 1. *Given $0<\alpha<1/2$ and $u\in \overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$, we have*

$$d_{\alpha,\gamma}(u)=\int\limits_{\mathbb{R}^n_+}|\xi|^{4\alpha}\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi$$

$$=\frac{1}{C(n,\gamma,\alpha)}\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\left|u(y)-u(x)\right|^2\Big({}^\gamma\mathbf{T}^x_y\,\frac{1}{|y|^{n+|\gamma|+4\alpha}}\Big)x^\gamma y^\gamma dxdy, \tag{7}$$

where

$$C(n,\gamma,\alpha)=\frac{2^{1-|\gamma|-4\alpha}\pi}{\sin(2\alpha\pi)\,\Gamma(2\alpha+1)\,\Gamma\big(\frac{n+|\gamma|}{2}+2\alpha\big)\prod\limits_{i=1}^n\Gamma\big(\frac{\gamma_i+1}{2}\big)}.$$

Proof. If

$$\mathbf{I}=\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{\left|{}^\gamma\mathbf{T}^y_x u(x)-u(x)\right|^2}{|y|^{n+|\gamma|+4\alpha}}\,x^\gamma y^\gamma dxdy,$$

then

$$\mathbf{I}=C(\gamma)\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\int\limits_0^\pi\cdots\int\limits_0^\pi\frac{\left|u\big(\sqrt{x_1^2-2x_1y_1\cos\beta_1+y_1^2},\ldots,\sqrt{x_n^2-2x_ny_n\cos\beta_1+y_n^2}\,\big)-u(x)\right|^2}{|y|^{n+|\gamma|+4\alpha}}$$

$$\times\prod_{i=1}^n\sin^{\gamma_i-1}\beta_i\,d\beta_i\,x^\gamma y^\gamma dxdy.$$

Passing to the coordinates

$$\widetilde{y}_1=y_1\cos\beta_1,\quad \widetilde{y}_2=y_1\sin\beta_1,\quad \widetilde{y}_3=y_2\cos\beta_2,$$
$$\widetilde{y}_4=y_2\sin\beta_2,\ldots,\widetilde{y}_{2n-1}=y_n\cos\beta_n,\quad \widetilde{y}_{2n}=y_n\sin\beta_n,$$

we infer that

$$\mathbf{I}=C(\gamma)\int\limits_{\mathbb{R}^n_+}\int\limits_{\widetilde{\mathbb{R}}^{2n}_+}\frac{\left|u\big(\sqrt{(x_1-\widetilde{y}_1)^2+\widetilde{y}_2^2},\ldots,\sqrt{(x_n-\widetilde{y}_{2n-1})^2+\widetilde{y}_{2n}^2}\,\big)-u(x)\right|^2}{|\widetilde{y}\,|^{n+|\gamma|+4\alpha}}\prod_{i=1}^n y_{2i}^{\gamma_i-1}x^\gamma d\widetilde{y}dx$$

$$=C(\gamma)\int\limits_{\mathbb{R}^n_+}\int\limits_{\widetilde{\mathbb{R}}^{2n}_+}\frac{\left|u\Big(\sqrt{z_1^2+\widetilde{y}_2^2},\ldots,\sqrt{z_{2n-1}^2+\widetilde{y}_{2n}^2}\,\Big)-u(x)\right|^2}{\big((x_1-z_1)^2+\widetilde{y}_2^2+\cdots+(x_n-z_{2n-1})^2+\widetilde{y}_{2n}^2\big)^{\frac{n+|\gamma|+4\alpha}{2}}}$$

$$\times\prod_{i=1}^n y_{2i}^{\gamma_i-1}x^\gamma dz_1d\widetilde{y}_2\ldots dz_{2n-1}\widetilde{y}_{2n}\,dx,$$

where $\{\widetilde{y}_{2i-1}-x_i=z_{2i-1},\ i=1,\ldots,n\}$ and $\widetilde{\mathbb{R}}^{2n}_+=\{\widetilde{y}\in\mathbb{R}^{2n}:\widetilde{y}_{2i}>0,\ i=1,\ldots,n\}$. Putting $z_{2i-1}=y_i\cos\beta_i$ and $\widetilde{y}_{2i}=y_i\sin\beta_i$, with $i=1,\ldots,n$, we obtain

$$\mathbf{I}=C(\gamma)\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\int\limits_0^\pi\cdots\int\limits_0^\pi\frac{\left|u(y)-u(x)\right|^2}{\big(x_1^2-2x_1y_1\cos\beta_1+y_1^2+\cdots+x_n^2-2x_ny_n\cos\beta_1+y_n^2\big)^{\frac{n+|\gamma|+4\alpha}{2}}}$$

$$\times\prod_{i=1}^n\sin^{\gamma_i-1}\beta_i\,d\beta_i\,x^\gamma y^\gamma dxdy=\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\left|u(y)-u(x)\right|^2\Big({}^\gamma\mathbf{T}^x_y\frac{1}{|y|^{n+|\gamma|+4\alpha}}\Big)\,x^\gamma y^\gamma dxdy.$$

Therefore,

$$\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} \frac{\left|{}^\gamma\mathbf{T}^y_x u(x)-u(x)\right|^2}{|y|^{n+|\gamma|+4\alpha}}\, x^\gamma y^\gamma dxdy$$

$$=\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} \left|u(y)-u(x)\right|^2 \left({}^\gamma\mathbf{T}^x_y \frac{1}{|y|^{n+|\gamma|+4\alpha}}\right) x^\gamma y^\gamma dxdy. \qquad (8)$$

If $0<\alpha<1/2$; then, using Parseval's identity for (2) and formula 3.170 in [11, p. 155], we get

$$\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+} \frac{\left|{}^\gamma\mathbf{T}^y_x u(x)-u(x)\right|^2}{|y|^{n+|\gamma|+4\alpha}}\, x^\gamma y^\gamma dxdy$$

$$=\int\limits_{\mathbb{R}^n_+} \frac{y^\gamma dy}{|y|^{n+|\gamma|+4\alpha}} \int\limits_{\mathbb{R}^n_+} \left|{}^\gamma\mathbf{T}^y_x u(x)-u(x)\right|^2 x^\gamma dx$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{y^\gamma dy}{|y|^{n+|\gamma|+4\alpha}} \int\limits_{\mathbb{R}^n_+} \left|(\mathbf{F}_\gamma)_x\left[{}^\gamma\mathbf{T}^y_x u(x)\right](\xi)-\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{y^\gamma dy}{|y|^{n+|\gamma|+4\alpha}} \int\limits_{\mathbb{R}^n_+} \left|\mathbf{j}_\gamma(y;\xi)-1\right|^2 \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi \int\limits_{\mathbb{R}^n_+} \frac{|\mathbf{j}_\gamma(y;\xi)-1|^2}{|y|^{n+|\gamma|+4\alpha}}\, y^\gamma dy$$

$$=\int\limits_{\mathbb{R}^n_+} \mathscr{A}(\xi)\left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi,$$

where

$$\mathscr{A}(\xi)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{|\mathbf{j}_\gamma(y;\xi)-1|^2}{|y|^{n+|\gamma|+4\alpha}}\, y^\gamma dy.$$

Since

$$\mathscr{A}(\xi)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{|\mathbf{j}_\gamma(y;\xi)-1|^2}{|y|^{n+|\gamma|+4\alpha}}\, y^\gamma dy=\left\{y=\frac{z}{|\xi|}\right\}$$

$$=|\xi|^{4\alpha}\,\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{\left|\mathbf{j}_\gamma\left(z;\frac{\xi}{|\xi|}\right)-1\right|^2}{|z|^{n+|\gamma|+4\alpha}}\, z^\gamma dz = C(n,\gamma,\alpha)|\xi|^{4\alpha},$$

where

$$C(n,\gamma,\alpha)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}^n_+} \frac{\left|\mathbf{j}_\gamma\left(z;\frac{\xi}{|\xi|}\right)-1\right|^2}{|z|^{n+|\gamma|+4\alpha}}\, z^\gamma dz, \qquad (9)$$

we see that $\mathscr{A}(\xi)$ is a homogeneous function of order 4α which is invariant under orthogonal transformations and

$$\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{\left|{}^\gamma\mathbf{T}^y_x u(x)-u(x)\right|^2}{|y|^{n+|\gamma|+4\alpha}}\,x^\gamma y^\gamma dxdy=C(n,\gamma,\alpha)\int\limits_{\mathbb{R}^n_+}|\xi|^{4\alpha}\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi. \tag{10}$$

Let us calculate $C(n,\gamma,\alpha)$. Using the representation for $\mathbf{j}_\gamma$ of the form $\mathbf{j}_\gamma(x,\xi)=\mathbf{P}^\gamma_\xi\left[e^{-i\langle x,\xi\rangle}\right]$ (see [11, p. 137, formula 3.138]), where $\mathbf{P}^\gamma_\xi$ is as in (5), we infer

$$\int\limits_{\mathbb{R}^n_+}\frac{\left|\mathbf{j}_\gamma\left(z;\frac{\xi}{|\xi|}\right)-1\right|^2}{|z|^{n+|\gamma|+4\alpha}}\,z^\gamma dz=\int\limits_{\mathbb{R}^n_+}\frac{\left|\mathbf{P}^\gamma_\xi\left[e^{-i\left\langle z;\frac{\xi}{|\xi|}\right\rangle}\right]-1\right|^2}{|z|^{n+|\gamma|+4\alpha}}\,z^\gamma dz.$$

Introduce the new coordinates in the last integral on the right-hand side as follows:

$$\widetilde{x}_1=z_1\cos\alpha_1,\quad \widetilde{x}_2=z_1\sin\alpha_1,\quad \widetilde{x}_3=z_2\cos\alpha_2,$$
$$\widetilde{x}_4=z_2\sin\alpha_2,\dots,\widetilde{x}_{2n-1}=z_n\cos\alpha_n,\quad \widetilde{x}_{2n}=z_n\sin\alpha_n.$$

In these coordinates,

$$\int\limits_{\mathbb{R}^n_+}\frac{\left|\mathbf{j}_\gamma\left(z;\frac{\xi}{|\xi|}\right)-1\right|^2}{|z|^{n+|\gamma|+4\alpha}}\,z^\gamma dz=C(\gamma)\int\limits_{\widetilde{\mathbb{R}}^{2n}_+}\left|e^{-i\left\langle\widetilde{x},\frac{\widetilde{\xi}'}{|\xi|}\right\rangle}-1\right|^2\prod_{i=1}^n\widetilde{x}_{2i}^{\,\gamma_i-1}d\widetilde{x},$$

where $\widetilde{x}=(\widetilde{x}_1,\dots,\widetilde{x}_{2n})\in\mathbb{R}^{2n}$, $\widetilde{x}_{2i}>0$; $i=\overline{1,n}$, $|\widetilde{x}|=|z|$, $\widetilde{\xi}'=(\xi_1,0,\xi_2,0,\dots,\xi_n,0)\in\mathbb{R}^{2n}$, $|\widetilde{\xi}'|=|\xi|$, and $\widetilde{\mathbb{R}}^{2n}_+=\{x\in\mathbb{R}^{2n}:x_{2i}>0,\ i=1,\dots,n\}$. As the integrand in $C(n,\gamma,\alpha)$, we have some function of the type of a planar wave. Choosing integration over $\widetilde{x}_1=p$, noticing that $\left\langle\widetilde{x},\frac{\widetilde{\xi}'}{|\xi|}\right\rangle=p$, and putting $\widetilde{x}'=(\widetilde{x}_2,\dots,\widetilde{x}_{2n})$, we obtain

$$\int\limits_{\mathbb{R}^n_+}\frac{\left|\mathbf{j}_\gamma\left(z;\frac{\xi}{|\xi|}\right)-1\right|^2}{|z|^{n+|\gamma|+4\alpha}}\,z^\gamma dz=C(\gamma)\int\limits_{\widetilde{\mathbb{R}}^{2n}_+}\frac{|e^{-ip}-1|^2}{\left|p^2+|\widetilde{x}'|^2\right|^{\frac{n+|\gamma|+4\alpha}{2}}}\prod_{i=1}^n\widetilde{x}_{2i}^{\,\gamma_i-1}dpd\widetilde{x}'$$

$$=\left\{\widetilde{x}'=pt\right\}=C(\gamma)\int\limits_{-\infty}^{\infty}\frac{|e^{-ip}-1|^2}{|p|^{4\alpha+1}}\,dp\int\limits_{\substack{\mathbb{R}^{2n-1},\,t_{2i-1}>0\\ i=1,\dots,n}}\frac{\prod\limits_{i=1}^n t_{2i-1}^{\,\gamma_i-1}\,dt}{(1+|t|^2)^{\frac{n+|\gamma|+4\alpha}{2}}}.$$

Since

$$\int\limits_{-\infty}^{\infty}\frac{|e^{-ip}-1|^2}{|p|^{4\alpha+1}}\,dp=2^{3-4\alpha}\int\limits_0^\infty\frac{\sin^2 p}{p^{4\alpha+1}}\,dp=-4\cos(2\pi\alpha)\Gamma(-4\alpha),$$

we have

$$\int\limits_{\substack{\mathbb{R}^{2n-1},\,t_{2i-1}>0\\ i=1,\dots,n}}\frac{\prod\limits_{i=1}^n t_{2i-1}^{\,\gamma_i-1}\,dt}{(1+|t|^2)^{\frac{n+|\gamma|+4\alpha}{2}}}=\int\limits_0^\infty\frac{\rho^{n+|\gamma|-2}}{(1+\rho^2)^{\frac{n+|\gamma|+4\alpha}{2}}}\,d\rho\int\limits_{\widetilde{S}^+_1(2n-1)}\prod_{i=1}^n\theta_{2i-1}^{\,\gamma_i-1}\,dS$$

$$\times\frac{\Gamma\left(2\alpha+\frac{1}{2}\right)\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}{2\Gamma\left(\frac{n+|\gamma|}{2}+2\alpha\right)}\left|\widetilde{S}^+_1(2n-1)\right|_{|\gamma|-1}$$

for $t=\rho\theta$. Here $\widetilde{S}_1^+(2n-1)$ is the part of the unit sphere centered at the origin in $\mathbb{R}^{2n-1}$ for $\widetilde{x}_{2i}>0$, with $i=1,\dots,n$. Applying formula (107) in [11, p. 49], we set

$$\int\limits_{\widetilde{S}_1^+(2n-1)} \prod_{i=1}^n \theta_{2i-1}^{\gamma_i-1}\, dS = \frac{\Gamma^{n-1}\big(\frac12\big)\prod\limits_{i=1}^n \Gamma\big(\frac{\gamma_i}{2}\big)}{2^{n-1}\Gamma\big(\frac{n+|\gamma|-1}{2}\big)} = \frac{\pi^{\frac{n-1}{2}}\prod\limits_{i=1}^n \Gamma\big(\frac{\gamma_i}{2}\big)}{2^{n-1}\Gamma\big(\frac{n+|\gamma|-1}{2}\big)} = \big|\widetilde{S}_1^+(2n-1)\big|_{|\gamma|-1}.$$

Consequently,

$$\int\limits_{\mathbb{R}_+^n} \frac{\big|\mathbf{j}_\gamma\big(z;\frac{\xi}{|\xi|}\big)-1\big|^2}{|z|^{n+|\gamma|+4\alpha}}\, z^\gamma dz = C(\gamma)\int\limits_{-\infty}^{\infty}\frac{|e^{-ip}-1|^2}{|p|^{4\alpha+1}}\,dp \int\limits_{\substack{\mathbb{R}^{2n-1},\, t_{2i-1}>0\\ i=1,\dots,n}} \frac{\prod\limits_{i=1}^n t_{2i-1}^{\gamma_i-1}\,dt}{(1+|t|^2)^{\frac{n+|\gamma|+4\alpha}{2}}}.$$

Finally, using Legendre's duplication formula and Euler's reflection formula, simplify $C(n,\gamma,\alpha)$ to see that

$$C(n,\gamma,\alpha) = \frac{2^{1-|\gamma|-4\alpha}\pi}{\sin(2\alpha\pi)\,\Gamma(2\alpha+1)\,\Gamma\big(\frac{n+|\gamma|}{2}+2\alpha\big)\prod\limits_{i=1}^n \Gamma\big(\frac{\gamma_i+1}{2}\big)}.$$

Taking into account (8), (10), and the form of the constant $C(n,\gamma,\alpha)$, we complete the proof of the lemma. □

The constant $C(n,\gamma,\alpha)$ possesses the properties

$$\lim_{\alpha\to 0+}\frac{1}{C(n,\gamma,\alpha)}=0,\qquad \lim_{\alpha\to\frac12-0}\frac{1}{C(n,\gamma,\alpha)}=0.$$

Therefore,

$$d_{\alpha,\gamma}(u) = \sum_{|i|=\alpha}\int\limits_{\mathbb{R}_+^n} |\mathbb{B}_i u|^2\, x^\gamma dx \quad \text{if } \alpha \text{ is an integer;}$$

in the remaining cases,

$$d_{\alpha,\gamma}(u) = \frac{1}{C(n,\gamma,\alpha-l)}\sum_{|i|=l}\int\limits_{\mathbb{R}_+^n}\int\limits_{\mathbb{R}_+^n}\big|\mathbb{B}_i u(y)-\mathbb{B}_i u(x)\big|^2\Big({}^\gamma\mathbf{T}_y^x\frac{1}{|y|^{n+|\gamma|+4\alpha}}\Big)\,x^\gamma y^\gamma,$$

where $l=[\alpha]$. The weighted integral $d_{\alpha,\gamma}$ is continuous in α and is independent of the orthogonal coordinates in $\mathbb{R}_+^n$.

4. Finding a Suitable Norm for $\overset{\circ}{C}{}_{ev}^{\infty}(\mathbb{R}_+^n)$

From the practical and theoretical standpoints, it is important to find out in what space and with what norm $\overset{\circ}{C}{}_{ev}^{\infty}(\mathbb{R}_+^n)$ is dense.

In [3], there was introduced some function space that is connected with multiplication by $|x|^{-\alpha}$ in the images of the Hankel transform. This was called the *space of Riesz B-potentials*. Recall that in the theory of B-potentials a Riesz B-potential has the form (see [4])

$$(U_\gamma^\alpha f)(x) = u(x) = C_{n,\gamma}\int\limits_{\mathbb{R}_+^n} f(y)\big({}^\gamma\mathbf{T}_x^y|x|^{\alpha-n-|\gamma|}\big)\,y^\gamma dy,\quad \alpha>0.$$

An analog of Sobolev's Theorem holds for U_γ^α (see [4, Theorem 1]). Namely, for $0<\alpha<\frac{n+|\gamma|}{p}$, $p>1$, the operator U_γ^α with density $f\in L_p^\gamma$ is bounded from L_p^γ into L_q^γ, where $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n+|\gamma|}$. For $\alpha\geq\frac{n+|\gamma|}{p}$, the potential U_γ^α can be defined in the sense of weighted distributions. As a consequence of this fact, in [3, Theorem 5], $\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$ was shown to be dense in the space of Riesz B-potentials only for $0<\alpha<\frac{n+|\gamma|}{p}$. Therefore, it is inconvenient to use the norm that bases on a Riesz B-potential in differential problems since these problems need potentials of arbitrarily large order.

We will prove that since norm convergence does not imply the pointwise convergence of a sequence in the spaces with the norm $\sqrt{d_{\alpha,\gamma}}$; therefore, $\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$ is not a function space for $\alpha\geq\frac{n+|\gamma|}{4}$.

Theorem 1. *If $\alpha\geq\frac{n+|\gamma|}{4}$ then the space $\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$ normalized by $\sqrt{d_{\alpha,\gamma}}$ is not a function space with respect to any exceptional class.*

Proof. Let $u\in\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$ and let u be identically 1 in a neighborhood of zero in $\mathbb{R}^n_+$. Put $u_\rho=u(x/\rho)$. We have

$$d_{\alpha,\gamma}(u_\rho)=\int\limits_{\mathbb{R}^n_+}|\xi|^{4\alpha}\left|\mathbf{F}_\gamma[u](\xi/\rho)\right|^2\xi^\gamma d\xi=\{\xi/\rho=y\}$$

$$=\rho^{n+|\gamma|-4\alpha}\int\limits_{\mathbb{R}^n_+}|y|^{4\alpha}\left|\mathbf{F}_\gamma[u](y)\right|^2 y^\gamma dy=\rho^{n+|\gamma|-4\alpha}\,d_{\alpha,\gamma}(u).$$

Consequently, for $\alpha>\frac{n+|\gamma|}{4}$, we see that

$$\lim_{\rho\to\infty}d_{\alpha,\gamma}(u_\rho)=0,\quad\text{but}\quad\lim_{\rho\to\infty}u_\rho(x)=1.$$

This shows that the space under consideration cannot be a function space (since otherwise the whole of $\mathbb{R}^n_+$ must be an exceptional set).

Now, consider the case of $\alpha=\frac{n+|\gamma|}{4}$. Choosing $\varepsilon\in(0,\alpha)$ and $v\in\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$, for the bilinear norm corresponding to the quadratic form $d_{\alpha,\gamma}(u)$, we see that

$$d_{\alpha,\gamma}(u_\rho,v)$$

$$=\int\limits_{\mathbb{R}^n_+}|\xi|^{4\alpha}\,\mathbf{F}_\gamma[u_\rho](\xi)\,\overline{\mathbf{F}_\gamma[v](\xi)}\,\xi^\gamma d\xi=\int\limits_{\mathbb{R}^n_+}|\xi|^{2\alpha+2\varepsilon}\,\mathbf{F}_\gamma[u_\rho](\xi)\,|\xi|^{2\alpha-2\varepsilon}\,\overline{\mathbf{F}_\gamma[v](\xi)}\,\xi^\gamma d\xi$$

$$\leq\left(\int\limits_{\mathbb{R}^n_+}|\xi|^{4\alpha+4\varepsilon}\left|\mathbf{F}_\gamma[u_\rho](\xi)\right|^2\xi^\gamma d\xi\right)^{\frac12}\left(\int\limits_{\mathbb{R}^n_+}|\xi|^{4\alpha-4\varepsilon}\left|\mathbf{F}_\gamma[v](\xi)\right|^2\xi^\gamma d\xi\right)^{\frac12}$$

$$=\sqrt{d_{\alpha+\varepsilon,\gamma}(u_\rho)}\,\sqrt{d_{\alpha-\varepsilon,\gamma}(v)}.$$

Here we used the Cauchy–Bunyakovsky inequality. Since

$$\lim_{\rho\to\infty}d_{\alpha+\varepsilon,\gamma}(u_\rho)=0,$$

the equality $\lim_{\rho\to\infty}d_{\alpha,\gamma}(u_\rho,v)=0$ is fulfilled for every $v\in\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$. We consider the Hilbert space that is a complete metric space with respect to the distance function induced by the inner product $d_{\alpha,\gamma}(u,v)$. This space is the abstract completion of $\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$ under the norm $\sqrt{d_{\alpha,\gamma}}$. Thus, u_ρ converges weakly to 0 as $\rho \to \infty$ in this Hilbert space because $d_{\alpha,\gamma}(u_\rho)$ is bounded. Then there exists a sequence $\rho_k \to \infty$ such that the arithmetic means of $\{u_{\rho_k}\}$ converges strongly to 0 (see [12]). But the equality $\lim_{\rho\to\infty} u_\rho(x) = 1$ implies that the sequence of the arithmetic means pointwise converges everywhere to 1. Thus, $\overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$ normalized by $\sqrt{d_{\alpha,\gamma}}$ cannot be a function space. $\square$

One of the simplest norms on $\overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$ equivalent to $\sqrt{d_{\alpha,\gamma}}$ looks as

$$\|u\|^2_{\alpha,\gamma} = \int\limits_{\mathbb{R}^n_+} \left(1+|\xi|^2\right)^\alpha \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi. \tag{11}$$

Below we prove that (11) is representable by the convolution kernels generating a generalized Bessel potential.

5. The Class of Generalized Bessel Potentials

In this section, we obtain a special representation of (11) which is most convenient for classes of generalized Bessel potentials. The generalization of the spaces of Bessel potentials has a rich history. In [13], for functions in the space of generalized Bessel potentials which is constructed from rearrangement invariant spaces, some equivalent description was obtained for the cone of decreasing rearrangements. In [14], some equivalent characterizations were established for the cones of decreasing rearrangements for the spaces of generalized Riesz and Bessel potentials.

A generalized Bessel potential is defined by the relation (see [15, 16])

$$u = (\mathbf{G}^\alpha_\gamma \varphi)(x) = \int\limits_{\mathbb{R}^n_+} G^\gamma_\alpha(y)\left({}^\gamma\mathbf{T}^y_x \varphi(x)\right) y^\gamma dy, \tag{12}$$

where

$$G^\gamma_\alpha(x) = \mathbf{F}^{-1}_\gamma\left[\left(1+|\xi|^2\right)^{-\frac{\alpha}{2}}\right](x) \tag{13}$$

is a generalization of the Bessel kernel. The two forms of the inverse operator of (12) were constructed in [16].

In [3], the space $\mathbf{B}^\alpha_\gamma(L^\gamma_p) = \{u : u = \mathbf{G}^\alpha_\gamma\varphi, \varphi \in L^\gamma_p\}$ with the norm $\|u\|_{\mathbf{B}^\alpha_\gamma(L^\gamma_p)} = \|\varphi\|_{L^\gamma_p}$ was introduced by using B-hypersingular integrals.

It was shown in [15] that

$$G^\gamma_\alpha(x) = \frac{2^{\frac{n-|\gamma|-\alpha}{2}+1}}{|x|^{\frac{n+|\gamma|-\alpha}{2}}\,\Gamma\left(\frac{\alpha}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)} K_{\frac{n+|\gamma|-\alpha}{2}}(|x|), \tag{14}$$

where $K_{\frac{n+|\gamma|-\alpha}{2}}$ is a modified Bessel function of the second kind (see [10]).

Since G^γ_α is integrable with the weight x^γ (see [15, 16]), its Hankel transform exists for every ξ. The kernel G^γ_α is an analytic function of α for $\alpha > 0$. Thus, from (13), by analytic continuation, we conclude that the Hankel transform of the generalized Bessel kernel for $\alpha > 0$ is equal to

$$\mathbf{F}_\gamma[G^\gamma_\alpha](\xi) = \left(1+|\xi|^2\right)^{-\frac{\alpha}{2}}. \tag{15}$$

Moreover, the kernel G^γ_α satisfies the properties

$$\int\limits_{\mathbb{R}^n_+} G^\gamma_\alpha(x)x^\gamma dx = 1 \quad \text{and} \quad (G^\gamma_\alpha * G^\gamma_\beta)_\gamma = G^\gamma_{\alpha+\beta},\ \alpha > 0,\ \beta > 0$$

(see [15, 16]), where $(G^\gamma_\alpha * G^\gamma_\beta)_\gamma$ is the generalized convolution; see (4).

To obtain (11) for $0<\alpha<1/2$, we first introduce the function

$$\omega_{\alpha,\gamma}(|x|)=\frac{2^{\frac{n-|\gamma|-\alpha}{2}+1}}{\Gamma\left(\frac{\alpha}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,|x|^{\frac{n+|\gamma|-\alpha}{2}}\,K_{\frac{n+|\gamma|-\alpha}{2}}(|x|)$$

$$=\frac{2^{n-\alpha+2}}{\Gamma\left(\frac{\alpha}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\int\limits_0^\infty t^{\frac{n+|\gamma|-\alpha}{2}-1}e^{-t-\frac{|x|^2}{4t}}\,dt. \tag{16}$$

Then the generalized Bessel potential in (12) is representable as the generalized convolution operator of (4):

$$\left(\mathbf{G}^\alpha_\gamma\varphi\right)(x)=\left(\frac{\omega_{\alpha,\gamma}(|x|)}{|x|^{n+|\gamma|-\alpha}}*\varphi\right)_\gamma,\quad \alpha>0.$$

Below we will need $\omega_{-4\alpha,\gamma}(|x|)$ for $0<\alpha<1/2$. The asymptotic properties of the modified Bessel function K_ν guarantee that the kernel function $\omega_{-4\alpha,\gamma}(|x|)$ decreases exponentially at infinity and turns into a constant at the origin:

$$\lim_{|x|\to\infty}\omega_{-4\alpha,\gamma}(|x|)=\frac{2^{\frac{n-|\gamma|+4\alpha}{2}+1}}{\Gamma\left(-2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\lim_{|x|\to\infty}|x|^{\frac{n+|\gamma|+4\alpha}{2}}\,K_{\frac{n+|\gamma|+4\alpha}{2}}(|x|)$$

$$=\frac{2^{\frac{n-|\gamma|+4\alpha}{2}+1}}{\Gamma\left(-2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\sqrt{\frac{\pi}{2}}\lim_{|x|\to\infty}|x|^{\frac{n+|\gamma|+4\alpha}{2}-\frac{1}{2}}\,e^{-|x|}=0,$$

$$\lim_{|x|\to0}\omega_{-4\alpha,\gamma}(|x|)=\frac{2^{\frac{n-|\gamma|+4\alpha}{2}+1}}{\Gamma\left(-2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\lim_{|x|\to0}|x|^{\frac{n+|\gamma|+4\alpha}{2}}\,K_{\frac{n+|\gamma|+4\alpha}{2}}(|x|)$$

$$=\frac{2^{\frac{n-|\gamma|+4\alpha}{2}+1}}{\Gamma\left(-2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\lim_{|x|\to0}|x|^{\frac{n+|\gamma|+4\alpha}{2}}\,\frac{\Gamma\left(\frac{n+|\gamma|+4\alpha}{2}\right)}{2^{1-\frac{n+|\gamma|+4\alpha}{2}}}\,|x|^{-\frac{n+|\gamma|+4\alpha}{2}}$$

$$=\frac{2^{n+4\alpha}\,\Gamma\left(\frac{n+|\gamma|+4\alpha}{2}\right)}{\Gamma\left(-2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

Consequently,

$$\lim_{|x|\to\infty}\omega_{-4\alpha,\gamma}(|x|)=0,\qquad \lim_{|x|\to0}\omega_{-4\alpha,\gamma}(|x|)=\frac{2^{n+4\alpha}\,\Gamma\left(\frac{n+|\gamma|+4\alpha}{2}\right)}{\Gamma\left(-2\alpha\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}. \tag{17}$$

Theorem 2. *The norm $\|u\|_{\alpha,\gamma}$ admits the representation*

$$\|u\|^2_{\alpha,\gamma}=2^{|\gamma|-n+1}\prod_{i=1}^{n}\Gamma^2\left(\frac{\gamma_i+1}{2}\right)$$

$$\times\left(\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{\left|{}^\gamma\mathbf{T}^y_x u(x)+u(y)\right|^2}{|x|^{n+|\gamma|+4\alpha}}\left(\omega_{-4\alpha,\gamma}(|x|)-\omega_{-4\alpha,\gamma}(0)\right)x^\gamma dx\,y^\gamma dy\right.$$

$$\left.-\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{\left|{}^\gamma\mathbf{T}^y_x u(x)-u(y)\right|^2}{|x|^{n+|\gamma|+4\alpha}}\left(\omega_{-4\alpha,\gamma}(|x|)+\omega_{-4\alpha,\gamma}(0)\right)x^\gamma dx\,y^\gamma dy\right). \tag{18}$$

Proof. Let $\gamma_{n+1}\geq 0$ be arbitrary and

$$\mathbf{J}=\int\limits_0^\infty\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)u(x)-u(y)\right|^2}{\left[{}^\gamma\mathbf{T}^y_x\,|x|^2+z_0^2\right]^{\frac{n+|\gamma'|+1+4\alpha}{2}}}\,x^\gamma dx\,y^\gamma dy\,z_0^{\gamma_{n+1}}dz_0,$$

where

$$j_\nu(x)=\frac{2^\nu\Gamma(\nu+1)}{x^\nu}\,J_\nu(x),$$

and J_ν is the Bessel function of the first kind. Acting as in proving (8) and applying (2) to the integral over y, we infer

$$\mathbf{J}=\int\limits_0^\infty\int\limits_{\mathbb{R}^n_+}\int\limits_{\mathbb{R}^n_+}\frac{\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)\,{}^\gamma\mathbf{T}^y_x u(x)-u(y)\right|^2}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma'|+1+4\alpha}{2}}}\,x^\gamma dx\,y^\gamma dy\,z_0^{\gamma_{n+1}}dz_0$$

$$=\int\limits_0^\infty z_0^{\gamma_{n+1}}dz_0\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma dx}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}}\int\limits_{\mathbb{R}^n_+}\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)\,{}^\gamma\mathbf{T}^y_x u(x)-u(y)\right|^2 y^\gamma dy$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_0^\infty dz_0\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma dx}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}}\int\limits_{\mathbb{R}^n_+}\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)\,\mathbf{j}_\gamma(x;\xi)-1\right|^2\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi.$$

Putting $(x,z_0)=\widetilde{z}$, $\widetilde{\xi}=(\xi,1)$, and $\gamma'=(\gamma,\gamma_{n+1})$, we have

$$\mathbf{J}=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_0^\infty z_0^{\gamma_{n+1}}dz_0$$

$$\times\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma dx}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma'|+1+4\alpha}{2}}}\int\limits_{\mathbb{R}^n_+}\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)\mathbf{j}_\gamma(x;\xi)-1\right|^2\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi\int\limits_{\mathbb{R}^{n+1}_+}\frac{\left|\mathbf{j}_{\gamma'}\left(\widetilde{z};\widetilde{\xi}\right)-1\right|^2}{|\widetilde{z}\,|^{n+|\gamma'|+1+4\alpha}}\,\widetilde{z}^{\,\gamma'}d\widetilde{z}$$

$$=\int\limits_{\mathbb{R}^n_+}\mathscr{B}(\xi)\left|\mathbf{F}_\gamma[u](\xi)\right|^2\xi^\gamma d\xi.$$

Performing the change of variables $\widetilde{z}=z/|\widetilde{\xi}|$, we obtain

$$\mathscr{B}(\xi)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^{n+1}_+}\frac{|\mathbf{j}_{\gamma'}(\widetilde{z};\widetilde{\xi})-1|^2}{|\widetilde{z}\,|^{n+|\gamma'|+1+4\alpha}}\,\widetilde{z}^{\,\gamma'}d\widetilde{z}$$

$$=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left(1+|\xi|\right)^{4\alpha}\int\limits_{\mathbb{R}^{n+1}_+}\frac{\left|\mathbf{j}_{\gamma'}\left(z;\frac{\widetilde{\xi}}{|\widetilde{\xi}\,|}\right)-1\right|^2}{|z|^{n+|\gamma'|+1+4\alpha}}\,z^{\gamma'}dz$$

and

$$\mathbf{J} = D(n,\gamma',\alpha) \int\limits_{\mathbb{R}^n_+} \big(1+|\xi|\big)^{4\alpha} \big|\mathbf{F}_\gamma[u](\xi)\big|^2 \xi^\gamma d\xi.$$

It follows from (9) that

$$\int\limits_{\mathbb{R}^{n+1}_+} \frac{\left|\mathbf{j}_{\gamma'}\left(z;\frac{\widetilde{\xi}}{|\widetilde{\xi}|}\right)-1\right|^2}{|z|^{n+|\gamma'|+1+4\alpha}} z^{\gamma'} dz = \frac{\prod\limits_{j=1}^{n+1}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}{2^{n+1-|\gamma'|}} C(n+1,\gamma',\alpha)$$

and

$$D(n,\gamma',\alpha) = \frac{\Gamma^2\left(\frac{\gamma_{n+1}+1}{2}\right)}{2^{1-\gamma_{n+1}}} C(n+1,\gamma',\alpha),$$

where

$$C(n+1,\gamma',\alpha) = \frac{2^{1-|\gamma'|-4\alpha}\pi}{\sin(2\alpha\pi)\,\Gamma(2\alpha+1)\,\Gamma\left(\frac{n+|\gamma'|+1}{2}+2\alpha\right)\prod\limits_{i=1}^{n+1}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

Hence, for every $\gamma_{n+1} \geq 0$, we infer

$$\|u\|^2_{\alpha,\gamma} = \frac{1}{D(n,\gamma',\alpha)} \int\limits_0^\infty \int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{\left|j_{\frac{\gamma_{n+1}-1}{2}}(z_0)u(x)-u(y)\right|^2}{\left[{}^\gamma\mathbf{T}^y_x\,|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx\, y^\gamma dy\, z_0^{\gamma_{n+1}} dz_0.$$

Passing to $\gamma_{n+1}=0$ and putting

$$E(n,\gamma,\alpha) = D(n,\gamma',\alpha)\big|_{\gamma_{n+1}=0} = \frac{\pi\sqrt{\pi}\,2^{-|\gamma|-4\alpha}}{\sin(2\alpha\pi)\,\Gamma(2\alpha+1)\,\Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)},$$

we write

$$\begin{aligned}
\|u\|^2_{\alpha,\gamma} &= \frac{1}{E(n,\gamma,\alpha)} \int\limits_{-\infty}^\infty \int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{\left|e^{iz_0}u(x)-u(y)\right|^2}{\left[{}^\gamma\mathbf{T}^y_x\,|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx\, y^\gamma dy dz_0 \\
&= \frac{1}{E(n,\gamma,\alpha)} \int\limits_{-\infty}^\infty \int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{\left|e^{\frac{iz_0}{2}}u(x)-e^{-\frac{iz_0}{2}}u(y)\right|^2}{\left[{}^\gamma\mathbf{T}^y_x\,|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx\, y^\gamma dy dz_0 \\
&= \frac{1}{E(n,\gamma,\alpha)} \int\limits_{-\infty}^\infty \int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{\left|{}^\gamma\mathbf{T}^y_x u(x)-u(y)\right|^2 \cos^2\frac{z_0}{2}}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx\, y^\gamma dy dz_0 \\
&+ \frac{1}{E(n,\gamma,\alpha)} \int\limits_{-\infty}^\infty \int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{\left|{}^\gamma\mathbf{T}^y_x u(x)+u(y)\right|^2 \sin^2\frac{z_0}{2}}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} x^\gamma dx\, y^\gamma dy dz_0.
\end{aligned}$$

Using the Wolfram Mathematica and (17), we get

$$\begin{aligned}
&\int\limits_{-\infty}^\infty \frac{\cos^2\frac{z_0}{2}\,dz_0}{\left[|x|^2+z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}} \\
&= \frac{\sqrt{\pi}\,2^{1-n-4\alpha}\,\Gamma(-2\alpha)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{|x|^{n+|\gamma|+4\alpha}\,\Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right)} \big(\omega_{-4\alpha,\gamma}(0)+\omega_{-4\alpha,\gamma}(|x|)\big),
\end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{\sin^2 \frac{z_0}{2} dz_0}{\left[|x|^2 + z_0^2\right]^{\frac{n+|\gamma|+1+2\alpha}{2}}}$$

$$= \frac{\sqrt{\pi}\, 2^{1-n-4\alpha}\, \Gamma(-2\alpha) \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{|x|^{n+|\gamma|+4\alpha}\, \Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right)} \left(\omega_{-4\alpha,\gamma}(0) - \omega_{-4\alpha,\gamma}(|x|)\right).$$

Since

$$\frac{1}{E(n,\gamma,\alpha)} \frac{\sqrt{\pi}\, 2^{1-n-4\alpha}\, \Gamma(-2\alpha) \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{|x|^{n+|\gamma|+4\alpha}\, \Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right)}$$

$$= \frac{\sin(2\alpha\pi)\, \Gamma(2\alpha+1)\, \Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right) \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\pi\sqrt{\pi}\, 2^{-|\gamma|-4\alpha}} \frac{\sqrt{\pi}\, 2^{1-n-4\alpha}\, \Gamma(-2\alpha) \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{|x|^{n+|\gamma|+4\alpha}\, \Gamma\left(\frac{n+|\gamma|+4\alpha+1}{2}\right)}$$

$$= -\frac{2^{|\gamma|-n+1}}{|x|^{n+|\gamma|+4\alpha}} \prod_{i=1}^{n} \Gamma^2\left(\frac{\gamma_i+1}{2}\right),$$

we have

$$\|u\|^2_{\alpha,\gamma} = 2^{|\gamma|-n+1} \prod_{i=1}^{n} \Gamma^2\left(\frac{\gamma_i+1}{2}\right)$$

$$\times \left(\int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{\left|{}^\gamma\mathbf{T}^y_x u(x) + u(y)\right|^2}{|x|^{n+|\gamma|+4\alpha}} \left(\omega_{-4\alpha,\gamma}(|x|) - \omega_{-4\alpha,\gamma}(0)\right) x^\gamma dx\, y^\gamma dy \right.$$

$$\left. - \int\limits_{\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+} \frac{|{}^\gamma\mathbf{T}^y_x u(x) - u(y)|^2}{|x|^{n+|\gamma|+4\alpha}} \left(\omega_{-4\alpha,\gamma}(|x|) + \omega_{-4\alpha,\gamma}(0)\right) x^\gamma dx\, y^\gamma dy \right).$$

Thus, we have validated (18). □

6. Conclusion

In conclusion, we observe that the two approaches are possible to defining the class of generalized Bessel potentials $\mathbf{B}^\alpha_\gamma(L^\gamma_p)$ of order α in $\mathbb{R}^n_+$. The first is that $u \in \mathbf{B}^\alpha_\gamma(L^\gamma_p)$ if u is a generalized convolution $(G^\gamma_\alpha * \varphi)_\gamma$ for some $\varphi \in L^\gamma_p(\mathbb{R}^n_+)$. This approach was presented in [3], which used B-hypersingular integrals. The second approach consists in endowing $\mathbf{B}^\alpha_\gamma(L^\gamma_p)$ with the norm

$$\|u\|^2_{\alpha,\gamma} = \int\limits_{\mathbb{R}^n_+} \left(1+|\xi|^2\right)^\alpha \left|\mathbf{F}_\gamma[u](\xi)\right|^2 \xi^\gamma d\xi,$$

which can be written down on using the convolution kernel that generates the generalized Bessel potential. This expression shows that the quadratic interpolation between $\|u\|_{\alpha,\gamma}$ and $\|u\|_{\beta,\gamma}$ gives $\|u\|_{\delta,\gamma}$, where δ is the interpolation order $\alpha(l-t)+\beta t$. The norm $\|u\|_{\alpha,\gamma}$ is most convenient for studying the class of generalized Bessel potentials in $\mathbb{R}^n_+$.

References

1. Stein E.M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University, Princeton (1970) (Princeton Math. Series; vol. 30).
2. Nikol'skii S.M., *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer, Berlin and Heidelberg (1975).
3. Lyakhov L.N. and Polovinkina M.V., "The space of weighted Bessel potentials," Proc. Steklov Inst. Math., vol. 250, 178–182 (2005).
4. Lyakhov L.N., "Inversion of Riesz B-potentials," Soviet Math. Dokl., vol. 44, no. 3, 717–720 (1992).
5. Lyakhov L.N., "On a class of hypersingular integrals," Soviet Math. Dokl., vol. 42, no. 3, 765–769 (1991).
6. Aronszajn N. and Smith K.T., "Functional spaces and functional completion," Ann. Inst. Fourier (Grenoble), vol. 6, 125–185 (1956).
7. Aronszajn N. and Smith K.T., "Characterization of positive reproducing kernels. Applications to Green's functions," Amer. J. Math., vol. 79, no. 3, 611–622 (1957).
8. Aronszajn N. and Smith K.T., "Theory of Bessel potentials," Ann. Inst. Fourier (Grenoble), vol. 11, 385–475 (1961).
9. Kipriyanov I.A., *Singular Elliptic Boundary-Value Problems*, Nauka and Fizmatlit, Moscow (1997) [Russian].
10. Watson G.N., *A Treatise on the Theory of Bessel Functions*. 2nd ed., Cambridge University, Cambridge (1996).
11. Shishkina E.L. and Sitnik S.M., *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics*, Academic, Cambridge (2020).
12. Banach S. and Saks S., "Sur la Convergence Forte dans les Champs L^p," Studia Math., vol. 2, 51–57 (1930).
13. Goldman M.L., "The cone of rearrangements for generalized Bessel potentials," Proc. Steklov Inst. Math., vol. 260, 144–156 (2008).
14. Goldman M.L., "Rearrangement invariant envelopes of generalized Bessel and Riesz potentials," Dokl. Math., vol. 78, no. 3, 814–818 (2008).
15. Ekincioglu I., Shishkina E.L., and Keskin C., "Generalized Bessel potential and its application to non-homogeneous singular screened Poisson equation," Integral Transforms Spec. Funct., vol. 32, no. 12, 932–947 (2021).
16. Dzhabrailov A., Luchko Y., and Shishkina E., "Two forms of an inverse operator to the generalized Bessel potential," Axioms, vol. 10, no. 3, 1–20 (2021).

A. L. Dzhabrailov
Kadyrov Chechen State University, Grozny, Russia
https://orcid.org/0000-0001-8208-3695
E-mail address: ahmed_0065@mail.ru

E. L. Shishkina
Voronezh State University, Voronezh, Russia
Belgorod State National Research University
Belgorod, Russia
https://orcid.org/0000-0003-4083-1207
E-mail address: ilina_dico@mail.ru