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CONFORMAL ENVELOPES OF NOVIKOV–POISSON ALGEBRAS

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Abstract: We prove that every Novikov–Poisson algebra over a field of zero characteristic can be embedded into a commutative conformal algebra with a derivation. As a corollary, we show that every commutator Gelfand–Dorfman algebra obtained from a Novikov–Poisson algebra is special, i.e., embeddable into a differential Poisson algebra.

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Introduction

The class of nonassociative algebras called "Novikov algebras" emerged in the papers [1] and [2] related with Hamiltonian operators in formal variational calculus and in the study of one-dimensional systems of differential equations of hydrodynamic type. Novikov–Poisson algebras were introduced in [3] as a tool in the theory of Novikov algebras. In the sequel, Novikov and Novikov–Poisson algebras were found to be related with Jordan superalgebras [4, 5] and the combinatorial properties of the varieties constructed from differential algebras of various types [6]. In the same context, the systems called Gelfand–Dorfman algebras (as appeared in [1]) were found to be useful. These algebras were studied in [7, 8].

The notion of conformal algebra was proposed in [9] as a formal language describing the algebraic properties of the coefficients of the singular part of operator product expansion (OPE) in a 2-dimensional conformal quantum field theory. As algebraic objects, these systems turn out useful for solving a series of abstract problems on "ordinary" nonassociative algebras (see, e.g., [10]). In this paper, we establish one more similar relation between conformal and Novikov–Poisson algebras. As an application, we prove that the commutator Gelfand–Dorfman algebra, obtained from a Novikov–Poisson algebra, is special.

Recall that a Novikov algebra [2] is a vector space V equipped with a bilinear operation $\circ$ satisfying the following identities:

$$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y), \quad x \circ (y \circ z) = y \circ (x \circ z). \tag{1}$$

Examples of Novikov algebras can be obtained from commutative algebras with a derivation. Namely, if A is a commutative (and associative) algebra with a linear operator $d : A \to A$ such that $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$ then the space A with the new operation $a \circ b = d(a)b$ is a Novikov algebra denoted by $A^{(d)}$. It was shown in [11] (see also [12]) that for every Novikov algebra $(V, \circ)$ there exists a commutative algebra A with a derivation d such that V is isomorphic to a subalgebra in the Novikov algebra $A^{(d)}$. Among all such differential envelopes there exists a universal system whose construction was described in [11].

A Novikov–Poisson algebra (see [3]) is a vector space V equipped with two bilinear operations $\circ$ and $*$ such that $(V, \circ)$ is a Novikov algebra, $(V, *)$ is a commutative (and associative) algebra, and the following identities hold:

$$(x \circ y) * z = x \circ (y * z), \quad (x * y) \circ z - x * (y \circ z) = (x * z) \circ y - x * (z \circ y). \tag{2}$$

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REMARK 1. In this paper we consider the “right” version of the definition of Novikov and Novikov–Poisson algebras which can be obtained from the commoner “left” version by permutation of the arguments of $\circ$.

Let us also recall (see [3]) that for two Novikov–Poisson algebras U and V the space $U \otimes V$ with operations

$$(u_1 \otimes v_1) * (u_2 \otimes v_2) = (u_1 * u_2) \otimes (v_1 * v_2), \tag{3}$$

$$(u_1 \otimes v_1) \circ (u_2 \otimes v_2) = (u_1 \circ u_2) \otimes (v_1 * v_2) + (u_1 * u_2) \otimes (v_1 \circ v_2) \tag{4}$$

is a Novikov–Poisson algebra too.

A Novikov–Poisson algebra can also be obtained from a commutative algebra A with a derivation d by means of the same operation $\circ$ as in $A^{(d)}$ on assuming that $*$ is the multiplication in the initial algebra A. However, this construction is far from being generic: There exist Novikov–Poisson algebras that cannot be embedded into any algebras of type $A^{(d)}$ [5]. The purpose of the present work is to prove that for every Novikov–Poisson algebra $(V, \circ, *)$ there exists a commutative conformal algebra C with a derivation D such that V embeds into C relative to the operations

$$D(u)\,{}_{(0)}\,v = u \circ v, \quad D(u)\,{}_{(1)}\,v = u * v$$

for all $u, v \in V$. Hence, we may construct differential enveloping algebras for Novikov–Poisson algebras of zero characteristic in the class of commutative conformal algebras.

This construction allows us to study the embedding of transpose Poisson algebras (t-Poisson, for short) into “ordinary” Poisson algebras with a derivation. The notion of t-Poisson algebra was introduced in [13]: This is a vector space V with two bilinear operations $[\cdot,\cdot]$ and $*$ such that V is a Lie algebra with respect to $[\cdot,\cdot]$, $(V, *)$ is an (associative) commutative algebra, and the identity

$$2x * [y, z] = [x * y, z] - [x * z, y]$$

holds for all $x, y, z \in V$.

For instance, if $(V, \circ, *)$ is a Novikov–Poisson algebra then the new operation $[a, b] = a \circ b - b \circ a$, with $a, b \in V$, turns V into a t-Poisson algebra denoted by $V^{(-)}$ [13].

On the other hand, every t-Poisson algebra satisfies the identity

$$[x * y, z] - x * [y, z] - [z * y, x] + z * [y, x] = [x, z] * y;$$

i.e., this is a Gelfand–Dorfman algebra in the sense of [7].

Among Gelfand–Dorfman algebras, a special place is occupied by special ones; i.e., those that can be embedded into differential Poisson algebras (see [6]). Namely, a Gelfand–Dorfman algebra V with operations $[\cdot,\cdot]$ and $*$ is *special* if there exists an (associative) commutative algebra P with a Poisson bracket $\{\cdot,\cdot\}$ and derivation d such that $V \subseteq P$ and $[u, v] = \{u, v\}$, with $u * v = d(u)v$ for all $u, v \in V$.

It was earlier shown in [10] that a Gelfand–Dorfman algebra obtained from a Novikov algebra $(V, \circ)$, considered as a system with the new operations $[a, b] = a \circ b - b \circ a$, $a * b = a \circ b$, and $a, b \in V$, is special. In this paper, we will show that for every Novikov–Poisson algebra V the t-Poisson algebra $V^{(-)}$ is special.

As proved in [12], special Gelfand–Dorfman algebras form a variety, and so there exists some set of defining identities. The entire set of these identities is unknown, but we may use the embedding of a Novikov–Poisson algebra into a differential conformal one for proving the speciality of the Gelfand–Dorfman algebra $V^{(-)}$ constructed from a Novikov–Poisson algebra V.

1. Conformal Algebras

Let $\Bbbk$ be a field of characteristic zero, and let $\mathbb{Z}_+$ stand for the set of nonnegative integers.

Recall that (see [9, 14]) a conformal algebra is a vector space C equipped with a linear operator $\partial : C \to C$ and a bilinear operation $(\cdot\,{}_{(\lambda)}\,\cdot)$ with range in the space $\Bbbk[\partial, \lambda] \otimes_{\Bbbk[\partial]} C \simeq C[\lambda]$.

In other words, for all $a, b \in C$ there is a polynomial $(a_{(\lambda)} b) \in C[\lambda]$ whose coefficients at $\lambda^n/n!$ are denoted by $(a_{(n)} b) \in C$, with $n \in \mathbb{Z}_+$. Hence, we may define a conformal algebra as a system with an infinite family of the operations $(a, b) \mapsto (a_{(n)} b)$, with $n \in \mathbb{Z}_+$, satisfying the *locality axiom*: For all $a, b \in C$ there exists $N \in \mathbb{Z}_+$ such that $(a_{(n)} b) = 0$ for all $n \geq N$. The minimal $N \geq 0$ with this property is denoted by $N_C(a, b)$. Note that N_C is called the *locality function* on the conformal algebra C.

The remaining axioms of conformal algebras relate the operator ∂ and λ-product:

$$(\partial a_{(\lambda)} b) = -\lambda(a_{(\lambda)} b), \quad (a_{(\lambda)} \partial b) = (\partial + \lambda)(a_{(\lambda)} b). \tag{5}$$

Let us denote by H the algebra of polynomials $\Bbbk[\partial]$ in the formal variable ∂. Define the (right) action of H on the space of Laurent polynomials $\Bbbk[t, t^{-1}]$ by the rule $f(t)\partial = -f'(t)$, $f \in \Bbbk[t, t^{-1}]$.

Given a conformal algebra C, define the *coefficient algebra* $\mathscr{A}(C)$ of C (see [14, 15]) as follows: Consider the vector space $\mathscr{A}(C) = \Bbbk[t, t^{-1}] \otimes_H C$ and put $a(n) = t^n \otimes_H a$ for $a \in C$ and $n \in \mathbb{Z}$. The product rule

$$a(n)b(m) = \sum_{s \geq 0} \binom{n}{s} (a_{(s)} b)(n + m - s), \quad a, b \in C,\ n, m \in \mathbb{Z}, \tag{6}$$

is a well-defined bilinear operation on $\mathscr{A}(C)$. In particular, the subspace $\mathscr{A}_+(C)$ spanned by the elements $a(n)$, with $a \in C$ and $n \in \mathbb{Z}_+$, is a subalgebra of $\mathscr{A}(C)$ which is known as the *positive part* of $\mathscr{A}(C)$.

The coefficient algebra $\mathscr{A}(C)$ has the following property: The initial conformal algebra C embeds into the space of formal power series $\mathscr{A}(C)[[z, z^{-1}]]$ so that the operation ∂ on C turns into the formal derivation d/dz and the λ-product is expressed via the Fourier transform of the product of formal series [14]:

$$a(z)_{(\lambda)} b(z) = \mathrm{Res}_{w=0}\, a(w)b(z) \exp\{\lambda(w - z)\}.$$

A conformal algebra C is said to be associative (commutative, Lie, etc.) if so is its (ordinary) coefficient algebra $\mathscr{A}(C)$. In particular, C is a *Novikov conformal algebra* if $\mathscr{A}(C)$ satisfies identities (1).

An identity on the coefficient algebra $\mathscr{A}(C)$ corresponds to a certain relation on C (see [14]). For instance, $\mathscr{A}(C)$ is commutative if and only if

$$(a_{(\lambda)} b) = (b_{(-\partial-\lambda)} a) \tag{7}$$

for all $a, b \in C$. The associativity identity turns into

$$a_{(\lambda)} (b_{(\mu)} c) = (a_{(\lambda)} b)_{(\lambda+\mu)} c, \tag{8}$$

this equation should be considered as a coefficientwise equality between two polynomials in two variables λ and μ.

In what follows, a commutative conformal algebra will stand for an associative and commutative one. In every algebra of this kind, the following identity holds:

$$a_{(\lambda)} (b_{(\mu)} c) = b_{(\mu)} (a_{(\lambda)} c); \tag{9}$$

as can be easily deduced from (5), (7), and (8) (see [16]).

The following statement is a standard exercise on the identities of conformal algebras [17]. Its "left" version was obtained in [18] by means of the calculus of formal series [14].

Lemma 1. *A conformal algebra C with an operation $(\cdot \circ_{(\lambda)} \cdot)$ is a Novikov one if and only if C satisfies the identities*

$$(x \circ_{(\lambda)} y) \circ_{(\lambda+\mu)} z - x \circ_{(\lambda)} (y \circ_{(\mu)} z) = (x \circ_{(\lambda)} z) \circ_{(-\partial-\mu)} y - x \circ_{(\lambda)} (z \circ_{(-\partial-\mu)} y), \tag{10}$$

$$x \circ_{(\lambda)} (y \circ_{(\mu)} z) = y \circ_{(\mu)} (x \circ_{(\lambda)} z). \tag{11}$$

A *derivation* of a conformal algebra C is a linear map $D : C \to C$ such that $D\partial = \partial D$ and $D(a_{(\lambda)} b) = (D(a)_{(\lambda)} b) + (a_{(\lambda)} D(b))$ for all $a, b \in C$. For instance, $D = \partial$ is a derivation of every conformal algebra.

Various functors between categories of ordinary algebras have conformal analogues. For instance, the functor $(-)$ from the category As of associative algebras into the category Lie of Lie algebras which turns $A \in$ As into its commutator Lie algebra $A^{(-)} \in$ Lie also acts on the class of associative conformal algebras. Namely, each associative conformal algebra C relative to the new operation

$$[a_{(\lambda)} b] = (a_{(\lambda)} b) - (b_{(-\partial-\lambda)} a), \quad a, b \in C,$$

is a Lie conformal algebra. In contrast to the ordinary algebras, there exist Lie conformal algebras that cannot be embedded into associative ones. This is unclear even for the narrow class of quadratic Lie conformal algebras which was introduced in [7].

The functor $(d) : A \mapsto A^{(d)}$, described above, acts from the category of commutative algebras with a derivation into the category of Novikov algebras. Quite expectedly, (d) also has a conformal analog.

Lemma 2. *Let C be a commutative conformal algebra with a binary operation $(\cdot_{(\lambda)} \cdot)$ and a derivation D. Then the new operation $(u \circ_{(\lambda)} v) = (Du_{(\lambda)} v)$, with $u, v \in C$, turns the same H-module C into the Novikov conformal algebra $C^{(D)}$.*

Proof. On the one hand, it suffices to note that $D : C \to C$ induces a derivation d on the coefficient algebra $\mathscr{A}(C)$ by the rule $d(a(n)) = (Da)(n)$, with $n \in \mathbb{Z}$ and $a \in C$. Since $\mathscr{A}(C)^{(d)} = \mathscr{A}(C^{(D)})$, the conformal algebra $C^{(D)}$ satisfies (10) and (11).

On the other hand, it is easy to check that the conditions of Lemma 1 hold. Indeed, let us express the right-hand side of (10) in terms of the commutative conformal operation:

$$\begin{gathered}(x \circ_{(\lambda)} z) \circ_{(-\partial-\mu)} y - x \circ_{(\lambda)} (z \circ_{(-\partial-\mu)} y) = D(Dx_{(\lambda)} z)_{(-\partial-\mu)} y - Dx_{(\lambda)} (Dz_{(-\partial-\mu)} y) \\ = (D^2 x_{(\lambda)} z)_{(-\partial-\mu)} y + (Dx_{(\lambda)} Dz)_{(-\partial-\mu)} y - Dx_{(\lambda)} (Dz_{(-\partial-\mu)} y) = (D^2 x_{(\lambda)} z)_{(-\partial-\mu)} y \\ = y_{(\mu)} (D^2 x_{(\lambda)} z) = D^2 x_{(\lambda)} (y_{(\mu)} z).\end{gathered}$$

The latter coincides with the expression obtained similarly from the left-hand side of (10). Equation (11) is immediate from (9). $\square$

Commutative conformal algebras also allow us to construct (ordinary) Novikov–Poisson algebras.

Theorem 1. *Let C be a commutative conformal algebra with a derivation D, and let $X \subseteq C$ be a subset of C such that $N_C(x, y) \le 1$ and $N_C(D(x), y) \le 2$ for all $x, y \in X$. Then the subalgebra generated in C by X relative to the operations*

$$a \circ b = D(a)_{(0)} b, \quad a * b = D(a)_{(1)} b \tag{12}$$

is a Novikov–Poisson algebra.

Proof. We will use the following obvious observation: $N_C(x, y) \le N$ for $x, y \in C$ if and only if the degree of the polynomial $(x_{(\lambda)} y)$ in λ is strictly less than N. Given $a, b, c \in C$, assume that $N_C(a, b), N_C(a, c), N_C(b, c) \le 1$. Then $N_C(a \circ b, c), N_C(a * b, c) \le 1$. Indeed, the polynomial $(b_{(\lambda)} c)$ does not depend on λ. Since

$$(a \circ b + \mu(a * b))_{(\lambda)} c = (D(a)_{(\mu)} b)_{(\lambda)} c = D(a)_{(\mu)} (b_{(\lambda-\mu)} c)$$

by (8), the left-hand side of the last equation does not depend on λ, as desired. (Note that $N_C(a, b) = N_C(b, a)$ in a commutative conformal algebra.)

Hence, all elements of the algebra $V \subseteq C$ generated by X by means of the operations $\circ$ and $*$ have mutual locality ≤ 1. Moreover, if $u, v \in V$ then $N_C(D(u), v) \le 2$. Indeed, by induction reasoning, it suffices to consider the degree in λ of the following polynomials:

$$D(D(a)_{(\mu)} b)_{(\lambda)} c, \quad D(a)_{(\lambda)} (D(b)_{(\mu)} c)$$

for $a, b, c \in V$, $N_C(D(b), c), N_C(D(a), c) \le 2$. It follows from (8) and (9) that

$$D(D(a)\,_{(\mu)}\, b)\,_{(\lambda)}\, c = D^2(a)\,_{(\mu)}\,(b\,_{(\lambda-\mu)}\, c) + D(a)\,_{(\mu)}\,(D(b)\,_{(\lambda-\mu)}\, c),$$
$$D(a)\,_{(\lambda)}\,(D(b)\,_{(\mu)}\, c) = D(b)\,_{(\mu)}\,(D(a)\,_{(\lambda)}\, c).$$

The right-hand side degrees in λ do not exceed 1, as desired.

Let us check the axioms of a Novikov–Poisson algebra for elements of V with operations $\circ$ and $*$.

The commutativity of $*$ on V follows from the commutativity of the conformal algebra C:

$$a * b = D(a)\,_{(1)}\, b = -b\,_{(1)}\, D(a) = -D(b\,_{(1)}\, a) + Db\,_{(1)}\, a = b * a,$$

since $b\,_{(1)}\, a = 0$ for all $a, b \in V$. Further, consider the corollary of (9):

$$Da\,_{(\lambda)}\,(Db\,_{(\mu)}\, c) = Db\,_{(\mu)}\,(Da\,_{(\lambda)} c), \quad a, b, c \in V.$$

Let us calculate the left- and right-hand sides:

$$a \circ (b \circ c) + \lambda a * (b \circ c) + \mu a \circ (b * c) + \lambda\mu a * (b * c)$$
$$= b \circ (a \circ c) + \mu b * (a \circ c) + \lambda b \circ (a * c) + \lambda\mu b * (a * c).$$

Comparing the coefficients, we obtain the second equation of (1), the first one of (2), and the condition of left commutativity of $*$. The latter implies associativity of $*$.

Finally, consider the expression

$$D(Da\,_{(\lambda)}\, b)\,_{(\mu)}\, c - Da\,_{(\lambda)}\,(Db\,_{(\mu-\lambda)}\, c) = (D^2 a\,_{(\lambda)}\, b)\,_{(\mu)}\, c = D^2 a\,_{(\lambda)}\,(b\,_{(\mu-\lambda)}\, c).$$

For $a, b, c \in V$, the right-hand side of this equality does not depend on μ and is symmetric relative to the permutation of b and c (since $N_C(b, c) \le 1$). On the other hand,

$$D(Da_{(\lambda)}b)_{(\mu)}c - Da_{(\lambda)}(Db_{(\mu-\lambda)}c)$$
$$= (a \circ b) \circ c + \lambda(a * b) \circ c + \mu(a \circ b) * c + \lambda\mu(a * b) * c - a \circ (b \circ c)$$
$$-(\mu - \lambda) a \circ (b * c) - \lambda a * (b \circ c) - \lambda(\mu - \lambda) a * (b * c).$$

Indeed, the already-proved properties of $\circ$ and $*$ imply that all terms with μ cancel each other. The coefficients at λ^i, with $i = 0, 1$, are symmetric on b and c. So, $(a\circ b)\circ c - a\circ(b\circ c)$ and $(a*b)\circ c - a*(b\circ c)$ do not change under the transposition of b and c. This implies the remaining identities from (1) and (2). □

Novikov conformal algebras may be obtained from Novikov–Poisson algebras by the following construction similar to quadratic Lie conformal algebras in [7].

Lemma 3. *Let V be a Novikov–Poisson algebra with operations $\circ$ and $*$. Then the free H-module $\mathscr{N}(V) = H \otimes V$ equipped with the operation*

$$(u \circ_{(\lambda)} v) = u \circ v + \lambda(u * v), \quad u, v \in V,$$

is a Novikov conformal algebra.

This is a particular case of the general definition in [18].

Proof. On the one hand, it suffices to note that $\mathscr{A}(\mathscr{N}(V))$ as a vector space is isomorphic to the space of Laurent polynomials $V[t, t^{-1}] = V \otimes \Bbbk[t, t^{-1}]$, and the product $\circ$ on $\mathscr{A}(\mathscr{N}(V))$ given by (6) is calculated as follows:

$$(u \otimes t^n) \circ (v \otimes t^m) = (u \circ v) \otimes t^{n+m} + n(u * v) \otimes t^{n+m-1}.$$

It is easy to see that this operation is exactly (4) on the tensor product of V and the Novikov–Poisson algebra $(\Bbbk[t, t^{-1}])^{(d)}$ for $d = d/dt$. Hence, $\mathscr{A}(\mathscr{N}(V))$ is a Novikov algebra relative to $\circ$.

On the other hand, it is easy to check conditions (10) and (11). □

In the sequel, following [18], $\mathscr{N}(V)$ will be called the *quadratic* Novikov conformal algebra constructed from a Novikov–Poisson algebra V as described in Lemma 3.

Remark. It follows from the proof of Lemma 3 that $\mathscr{N}(V)$ is actually a Novikov–Poisson conformal algebra relative to the two operations $(\cdot \circ_{(\lambda)} \cdot)$ and $(\cdot\,_{(\lambda)}\, \cdot)$, where the latter is defined by

$$(u\,_{(\lambda)}\, v) = u * v, \quad u, v \in V.$$

2. Reconstruction of a Differential Conformal Algebra

In this section, we present some method of constructing a conformal algebra which is based on an ordinary algebra (analogous to the *reconstruction* in [19]). This method is adapted to the case of differential algebras.

Fix a nonempty set B and put

$$X = \{b^{(k)}(n) \mid b \in B,\ k, n \in \mathbb{Z}_+\}. \tag{13}$$

Here $b^{(k)}(n)$ is just a formal notation for a triple of the form $(k, n, b) \in \mathbb{Z}_+^2 \times B$.

Denote by $F = \Bbbk[X]$ the polynomial algebra on the variables X with a derivation d defined by $b^{(k)}(n) \mapsto b^{(k+1)}(n)$. Suppose that J is an ideal in F which is invariant under d (differential ideal). Assume that for all $a, b \in B$ and $k, p \in \mathbb{Z}_+$ there exists $N = N(a, k; b, p) \in \mathbb{Z}_+$ such that

$$\sum_{s=0}^{N} (-1)^s \binom{N}{s} a^{(k)}(n-s) b^{(p)}(m+s) \in J \tag{14}$$

for all $n, m \in \mathbb{Z}_+$, with $n \geq N$. Then $A = F/J$ is a commutative algebra with a derivation that is also denoted by d.

Remark 3. Condition (14) implies that $\sum_{s=0}^{M} (-1)^s \binom{M}{s} a^{(k)}(n-s) b^{(p)}(m+s) \in J$ for all $M > N$, with $n, m \in \mathbb{Z}_+$ and $n \geq M$. Hence, in particular,

$$\sum_{s=0}^{N+1} (-1)^s s \binom{N+1}{s} a^{(k)}(n-s) b^{(p)}(m+s) \in J$$

for all $n \geq N+1$.

Let $\mathscr{F}(A)$ stand for the set of all functions $\mathbb{Z}_+ \to A$. Define the following operations on $\mathscr{F}(A)$: a unary map ∂ and a family of binary operations $(\cdot\, _{(n)}\, \cdot)$, with $n \in \mathbb{Z}_+$, such that

$$\partial f : m \mapsto -m f(m-1), \quad m \in \mathbb{Z}_+,$$

$$(f\, _{(n)}\, g) : m \mapsto \sum_{s=0}^{n} (-1)^s \binom{n}{s} f(n-s) g(m+s), \quad m \in \mathbb{Z}_+,$$

for $f, g \in \mathscr{F}(A)$ and $n \in \mathbb{Z}_+$.

It is well known and easy to check (see, e.g., [15]) that these operations meet the following relations:

$$(\partial f\, _{(n)}\, g) = -n(f_{(n-1)} g), \quad (f\, _{(n)}\, \partial g) = \partial(f\, _{(n)}\, g) + n(f_{(n-1)} g),$$

and

$$(f\, _{(n)}\, g)_{(m)} h = \sum_{s=0}^{n} (-1)^s \binom{n}{s} f_{(n-s)}(g_{(m+s)} h)$$

for all $f, g, h \in \mathscr{F}(A)$ and $n, m \in \mathbb{Z}+$.

Moreover, for all $f, g \in \mathscr{F}(A)$, with $n \in \mathbb{Z}_+$, the formally infinite sum

$$\{g\, _{(n)}\, f\} := \sum_{s \geq 0} (-1)^{n+s} \frac{1}{s!} \partial^s (g_{(n+s)} f)$$

also defines a function $\mathbb{Z}_+ \to A$ since for every $m \in \mathbb{Z}_+$ we have

$$\{g\, _{(n)}\, f\}(m) = \sum_{s \geq 0} (-1)^{n+s} \frac{1}{s!} \partial^s (g_{(n+s)} f)(m) = \sum_{s=0}^{m} (-1)^n \binom{m}{s} (g_{(n+s)} f)(m-s).$$

Further, let us calculate

$$\{g_{(n)} f\}(m) = \sum_{s,t\geq 0} (-1)^{n+t} \binom{m}{s}\binom{n+s}{t} g(n+s-t) f(m-s+t)$$
$$= \sum_{u\in\mathbb{Z}} (-1)^u \left(\sum_{s\geq 0} (-1)^{m+s} \binom{m}{s}\binom{n+s}{m+u} \right) g(m+u) f(n-u).$$

To get the last equation, we introduce the new summation index $u = n - m + s - t$ instead of t. The expression in brackets is equal to $\binom{n}{u}$ by the classical combinatorial identity. Therefore,

$$\{g_{(n)} f\} = (f_{(n)} g)$$

for all $f, g \in \mathscr{F}(A)$ and $n \in \mathbb{Z}_+$. Hence, the space $\mathscr{F}(A)$ with operations ∂ and $(\cdot_{(n)}\cdot)$, with $n \geq 0$, satisfies all axioms of a commutative conformal algebra apart from locality.

Consider the subset

$$\mathscr{B} = \{b^{(k)} : m \mapsto b^{(k)}(m) \mid b \in B,\, k \in \mathbb{Z}_+\} \subset \mathscr{F}(A).$$

It follows from (14) on the ideal J that for every $f, g \in \mathscr{B}$ the locality axiom holds: There exists $N \in \mathbb{Z}_+$ such that $(f_{(n)} g) = 0$ for all $n \geq N$. The associative analog of the Dong Lemma (see [15]) implies that $\mathscr{B}$ generates the subalgebra that is denoted by $\mathscr{R}(A)$ in the system $\mathscr{F}(A)$ with the operations $\partial(\cdot)$ and $(\cdot_{(n)}\cdot)$, where $n \in \mathbb{Z}_+$, which is a commutative conformal algebra relative to the λ-product

$$(f_{(\lambda)} g) = \sum_{n\geq 0} \frac{\lambda^n}{n!} (f_{(n)} g), \quad f, g \in \mathscr{R}(A).$$

Define one more operation $D(\cdot)$ on $\mathscr{F}(A)$ by the following rule:

$$D(f) : m \mapsto d(f(m)), \quad f \in \mathscr{F}(A),\ m \in \mathbb{Z}_+.$$

It is easy to see that

$$D\partial = \partial D$$

and

$$D(f_{(n)} g) = D(f)_{(n)} g + f_{(n)} D(g), \quad n \in \mathbb{Z}_+,\ f, g \in \mathscr{F}(A),$$

since d is a derivation on A. Moreover, $D(b^{(k)}) = b^{(k+1)} \in \mathscr{B}$ for all $b \in B$ and $k \in \mathbb{Z}_+$, and so $\mathscr{R}(A)$ is invariant relative to D.

3. Differential Enveloping Conformal Algebras

Let $\mathscr{N}(V)$ be the quadratic Novikov conformal algebra constructed from a Novikov–Poisson algebra $(V, \circ, *)$ as described in Lemma 3.

The positive part of the coefficient algebra of $\mathscr{N}(V)$ is isomorphic to the space of polynomials $V[t]$. The isomorphism identifies $u(n)$ with $u \otimes t^n$ for $u \in V$ and $n \in \mathbb{Z}_+$. Therefore, $V[t]$ is a Novikov algebra relative to the multiplication

$$u(n) \circ v(m) = (u \circ v)(n+m) + n(u * v)(n+m-1). \tag{15}$$

Suppose that B is a basis for the space V. Construct the set X and the differential algebra $F = \Bbbk[X]$ as in Section 2. Define the integer-valued *weight* function $\mathrm{wt}(\cdot)$ on the set of monomials in F assuming

$$\mathrm{wt}(a_1^{(k_1)}(n_1) \cdots a_l^{(k_l)}(n_l)) = k_1 + \cdots + k_l - l.$$

It is easy to see that $\mathrm{wt}(uv) = \mathrm{wt}(u) + \mathrm{wt}(v)$ for all monomials $u, v \in F$, and

$$F = \bigoplus_{w\in\mathbb{Z}} F_w,$$

where F_w is the linear span of all monomials in F of weight w. It is clear that $d(F_w) \subseteq F_{w+1}$. We will identify the space $V[t]$ with a subspace in F spanned by the variables $a(n) = a^{(0)}(n)$, with $a \in B$ and $n \in \mathbb{Z}_+$. Given $v \in V$, we will denote by $v(n)$ the image of vt^n in F: If $v = \sum_i \alpha_i b_i$, $\alpha_i \in \Bbbk$, $v_i \in B$, then

$$v(n) = \sum_i \alpha_i b_i(n).$$

Let us choose the following families of polynomials in F:

$$S_0 = \left\{ \sum_{s=0}^{k+p+1} (-1)^s \binom{k+p+1}{s} a^{(k)}(n-s) b^{(p)}(m+s) \mid a, b \in B,\ k, p, n, m \in \mathbb{Z}_+,\ n \geq k+p+1 \right\},$$

$$S_1 = \{a'(n)b(m) - a(n) \circ b(m) \mid a, b \in B,\ n, m \in \mathbb{Z}_+\}.$$

Define the ideal J of‘F as the sum of two ideals J_0 and J_1, where each J_i is generated as a differential ideal by the corresponding S_i, with $i = 0, 1$. In other words, these J_i as ideals of F are generated by all derivatives of the polynomials from S_i.

Note that each S_i consists of wt-homogeneous polynomials, and so the ideals J_i are wt-homogeneous too:

$$J_i = \bigoplus_{w\in\mathbb{Z}} (J_i \cap F_w), \quad i = 0, 1.$$

It is also useful to note that F/J_1 is exactly the universal differential envelope of the Novikov algebra $V[t]$ with multiplication (15). The mapping $u(n) \mapsto u^{(0)}(n) + J_1 \in F/J_1$ is injective by [11] and [12], and hence $V[t] \cap J_1 = 0$.

Lemma 4. *Let $g \in F$ be a* wt*-homogeneous polynomial of weight* 1*. Then the class $g + J_1$ contains a linear combination of monomials of the form $c_1'(l_1) \cdots c_r'(l_r) c''(l)$.*

Proof. If g contains a monomial with a letter of type $b(m)$, with $m \in \mathbb{Z}_+$; then in the same monomial should be a letter of type $a^{(k+1)}(n)$, with $k \geq 0$ and $n \in \mathbb{Z}_+$. We can use the relation $d^k(a'(n)b(m) - a(n) \circ b(m)) \in J_1$ to replace an occurrence of a factor $a^{(k+1)}(n)b(m)$ into a linear combination of words like $a^{(i)}(n)b^{(j)}(m)$, $i + j = k + 1$, $i, j > 0$, and $(a \circ b)^{(k)}(n + m)$, $(a * b)^{(k)}(n + m - 1)$. Let us iterate such a procedure to reduce g modulo J_1 to such a form that neither of monomials contain a letter of type $b(m)$. All monomials of weight 1 that do not contain letters of type $b(m)$ must be of the desired form. □

Lemma 5. $F_{-1} \cap J_0 \subseteq J_1$.

Proof. Put

$$f_{n,m}^{(k,p)}(a, b) = \sum_{s=0}^{k+p+1} (-1)^s \binom{k+p+1}{s} a^{(k)}(n-s) b^{(p)}(m+s),$$

$$\mathrm{wt}\, f_{n,m}^{(k,p)}(a, b) = k + p - 2 \in \{-2, -1, 0, 1, 2, \dots\}.$$

It suffices to show that $\mathrm{wt}\, ud^r(f) = -1$ implies $ud^r(f) \in J_1$, where $f = f_{n,m}^{(k,p)}(a, b) \in S_0$, with $r \geq 0$, while u is a monomial from F. Proceed by induction on r starting from $r = 0$.

Assume that $\mathrm{wt}\, uf = -1$ for a monomial $u \in F$. We need to prove $uf \in J_1$.

CASE 1: $\operatorname{wt} f = -2$. It is possible only for $k = p = 0$, i.e., $f = a(n)b(m) - a(n-1)b(m+1)$. The weight of u equals 1. So by Lemma 4 we may suppose that u contains a letter of type $c''(l)$, with $l \in \mathbb{Z}_+$. Then

$$uf = vc''(l)f = v(c''(l)a(n)b(m) - c''(l)a(n-1)b(m+1))$$
$$\equiv v((c(l), a(n), b(m))_\circ - (c(l), a(n-1), b(m+1))_\circ) \pmod{J_1},$$

where $(\alpha, \beta, \gamma)_\circ$ is the associator of elements from F relative to the operation $\circ$. Straightforward computation via (15) leads to

$$(c(l), a(n), b(m))_\circ = ((c \circ a) \circ b)(n+m+l) - (c \circ (a \circ b))(n+m+l)$$
$$+l((c \circ a) * b + (c * a) \circ b - c * (a \circ b))(n+m+l-1) + l(l-1)(a * b * c)(n+m+l-2).$$

All terms of the last expression depend only on l and on the sum $n+m$, and so $uf \in J_1$.

CASE 2: $\operatorname{wt} f = -1$. Here we have either $k = 1$ and $p = 0$, or $k = 0$ and $p = 1$. If $f = f_{n,m}^{(1,0)}(a, b)$ then

$$f \equiv a(n) \circ b(m) - 2a(n-1) \circ b(m+1) + a(n-2) \circ b(m+2) \pmod{J_1},$$

and (15) implies $f \in J_1$. In a similar way, $f_{n,m}^{(0,1)}(a, b) \in J_1$.

CASE 3: $\operatorname{wt} f \geq 0$. Proceed by induction on $N = k + p + 1 \geq 3$. The values of k and p considered above correspond to $N = 1$ and $N = 2$, which is the base of induction. Suppose that $f = f_{n,m}^{(k,p)}(a, b)$, $k + p \geq 2$. Without loss of generality, assume that $k > 0$. If $\operatorname{wt}(uf) = -1$ for a monomial u then $\operatorname{wt} u \leq -1$ and u contains a letter of type $c(l)$, $c \in B$, $l \in \mathbb{Z}_+$. Then

$$uf = vc(l)f = v\left(\sum_{s=0}^{N}(-1)^s\binom{N}{s}c(l)a^{(k)}(n-s)b^{(p)}(m+s)\right)$$
$$\equiv v\left(\sum_{s=0}^{N}(-1)^s\binom{N}{s}\left((a(n-s) \circ c(l))^{(k-1)} - \sum_{i=1}^{k-1}\binom{k-1}{i}c^{(i)}(l)a^{(k-i)}(n-s)\right)b^{(p)}(m+s)\right)$$

modulo J_1. Let us distribute the brackets and switch the order of summation to get

$$uf \equiv \sum_{s=0}^{N}(-1)^s\binom{N}{s}v(a \circ c)^{(k-1)}(n+l-s)b^{(p)}(m+s)$$
$$+\sum_{s=0}^{N}(-1)^s(n-s)\binom{N}{s}v(a * c)^{(k-1)}(n+l-1-s)b^{(p)}(m+s)$$
$$-\sum_{i=1}^{k-1}\binom{k-1}{i}vc^{(i)}(l)\sum_{s=0}^{N}(-1)^s\binom{N}{s}a^{(k-i)}(n-s)b^{(p)}(m+s) \pmod{J_1}.$$

All summands of this expression belong to the ideal J_0, as they come from the polynomials $f_{n+l,m}^{(k-1,p)}(a \circ c, b)$, $f_{n+l-1,m}^{(k-1,p)}(a * c, b)$, and $f_{n,m}^{(k-i,p)}(a, b)$, $i = 1, \ldots, k-1$ (see Remark 3). Hence, by the induction hypothesis, each summand belongs to J_1.

Next, suppose we already proved for some $r \geq 0$ that $\operatorname{wt} ud^r(f) = -1$ implies $ud^r(f) \in J_1$. Let $\operatorname{wt} ud^{r+1}(f) = -1$. Then $\operatorname{wt} ud^r(f) = -2$, and $\operatorname{wt} d^r(f) = \operatorname{wt} f + r \geq -2$.

The weight of $d^r(f)$ reaches its minimal value only for $k = p = r = 0$, i.e., $f = a(n)b(m) - a(n-1)$ $b(m+1)$. In this case, (15) implies

$$\begin{aligned} d^{r+1}(f) = d(f) &= a'(n)b(m) + b'(m)a(n) - a'(n-1)b(m+1) - b'(m+1)a(n-1) \\ &\equiv a(n) \circ b(m) + b(m) \circ a(n) - a(n-1) \circ b(m+1) - b(m+1) \circ a(n-1) \\ &\equiv (a * b)(n+m-1) - (b * a)(n+m-1) = 0 \pmod{J_1}. \end{aligned}$$

Now, let $\operatorname{wt} d^r(f) \geq -1$. Note that

$$ud^{r+1}(f) = d(ud^r(f)) - d(u)d^r(f).$$

The weights of both summands are equal to -1, and so $d(u)d^r(f) \in J_1$ by the induction hypothesis. Moreover, $\operatorname{wt} ud^r(f) = -2$, i.e., $\operatorname{wt} u < 0$. Hence, the monomial u contains a letter $c(l) \in X$ of weight -1. Then $u = c(l)v$, $ud^r(f) = c(l)vd^r(f)$, and $\operatorname{wt} vd^r(f) = -1$, and so $vd^r(f) \in J_1$ by the induction hypothesis. Therefore, $ud^r(f) \in J_1$. Finally, $ud^{r+1}(f) \in J_1$. □

Theorem 2. *For every quadratic Novikov conformal algebra $\mathscr{N}(V)$ there exists a commutative conformal algebra C with a derivation D such that $\mathscr{N}(V) \subseteq C^{(D)}$ while $N_C(u, v) \leq 1$ and $N_C(Du, v) \leq 2$ for all $u, v \in V$.*

Proof. Consider the commutative algebra $A = F/(J_0 + J_1)$ as above. The ideal $J = J_0 + J_1$ is invariant under d and meets (14). Construct the commutative conformal algebra $\mathscr{R}(A)$ with a derivation D as described in Section 2.

The mapping $B \to \mathscr{R}(A)$ sending a basis element $b \in B$ to the function $b^{(0)} \in \mathscr{B}$ can be extended to some H-linear map $\theta : \mathscr{N}(V) = H \otimes V \to \mathscr{R}(A)$.

Note that the values of $D(\theta(a))_{(n)}\theta(b) \in \mathscr{R}(A)$ at each $m \in \mathbb{Z}_+$ coincide with the corresponding values of the function $\theta(a \circ_{(n)} b)$ for all $a, b \in B$ and $n \geq 0$. Indeed,

$$\begin{aligned} D(\theta(a))_{(n)}\theta(b) : m \mapsto & \sum_{s=0}^{n} (-1)^s \binom{n}{s} a'(n-s)b(m+s) \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} a(n-s) \circ b(m+s) \\ &= \sum_{s=0}^{n} (-1)^s \binom{n}{s} \big((a \circ b)(n+m) + (n-s)(a * b)(n+m-1)\big) \\ &= \delta_{n,0}(a \circ b)(m) + \delta_{n,1}(a * b)(m). \end{aligned}$$

On the other hand, $a \circ_{(n)} b = \delta_{n,0}(a \circ b) + \delta_{n,1}(a * b)$ by the definition of $\mathscr{N}(V)$, and so $\theta(a \circ_{(n)} b) : m \mapsto \delta_{n,0}(a \circ b)(m) + \delta_{n,1}(a * b)(m)$, as desired.

Hence, $\theta : \mathscr{N}(V) \to \mathscr{R}(A)^{(D)}$ is a homomorphism of Novikov conformal algebras.

It remains to prove that θ is injective. Assume that

$$w = \sum_{s=0}^{M} \frac{1}{s!}(-\partial)^s \otimes u_s \in \ker\theta$$

for some $M \geq 0$ and $u_s \in V$. If not all u_s are zero then the polynomials

$$w(m) = \sum_{s=0}^{M} \binom{m}{s} u_s(m-s) \in V[t] \subset \Bbbk X$$

are nonzero for all $m \geq M$. (Here we identify $a(n) \in X$ and $at^n \in V[t]$, as above.) On the other hand, $\mathrm{wt}(w(m)) = -1$ and

$$\theta(w) : m \mapsto w(m) + J = 0 \in A;$$

i.e., $w(m) \in J \cap F_{-1} = (J_0 + J_1) \cap F_{-1} \subset J_1$ by Lemma 5 due to wt-homogeneity of the ideals J_i, $i = 0, 1$. Recall that F/J_1 is the universal differential envelope of the Novikov algebra $V[t]$, and so $V[t] \cap J_1 = 0$; a contradiction.

Therefore, $C = \mathscr{R}(A)$ is the desired commutative conformal algebra with a derivation D, while $\theta : \mathscr{N}(V) \to \mathscr{R}(A)^{(D)}$ is an injective homomorphism of conformal algebras. Moreover, $\theta(a) {}_{(n)} \theta(b) = a^{(n)} b^{(0)} = 0$ for $n \geq 1$, and $D(\theta(a)) {}_{(n)} \theta(b) = a^{(1)} {}_{(n)} b^{(0)} = 0$ for $n \geq 2$ due to the defining relations S_0 that generate J_0. □

Corollary 1. *For every Novikov–Poisson algebra V there exists a commutative conformal algebra C with a derivation D such that V embeds into C as described in Theorem 1.*

Indeed, in the conformal algebra C from the proof of Theorem 2 the space $1 \otimes V \subset \mathscr{N}(V)$ forms a subalgebra relative to the operations (12) which is isomorphic to V.

4. Commutator Novikov–Poisson Algebras Are Special

Let V be a Novikov–Poisson algebra with operations $\circ$ and $*$. We continue using the notations from Sections 2 and 3: B is a basis for V, while X is constructed by the rule (13), and $F = \Bbbk[X]$. Let A stand for the algebra in the proof of Theorem 2. There are two commuting derivations ∂ and d on A, defined on the generators as

$$\partial : a^{(k)}(n) \mapsto n a^{(k)}(n-1), \quad d : a^{(k)}(n) \mapsto a^{(k+1)}(n),$$

with $a \in B$ and $k, n \in \mathbb{Z}_+$. Indeed, the ideal J in Section 3 is invariant under the corresponding derivation ∂ of F.

Every commutative algebra A with two commuting derivations d and ∂ is a Poisson algebra relative to the bracket

$$\{f, g\} = d(f)\partial(g) - \partial(f)d(g), \quad f, g \in A;$$

and, moreover, d is a derivation with respect to this bracket, i.e.,

$$d\{f, g\} = \{d(f), g\} + \{f, d(g)\}$$

for all $f, g \in A$. In particular, the algebra $A = F/J$ considered above is a differential Poisson algebra.

Recall that, given a Novikov–Poisson algebra V with operations $\circ$ and $*$, we denote by $V^{(-)}$ the Gelfand–Dorfman algebra constructed on the same space V with the operations $*$ and $[a, b] = a \circ b - b \circ a$, where $a, b \in V$.

Theorem 3. *Let V be a Novikov–Poisson algebra. Then the commutator Gelfand–Dorfman algebra $V^{(-)}$ is special.*

Proof. It suffices to show that $V^{(-)}$ satisfies every defining identity of the variety of special Gelfand–Dorfman algebras.

Assume that $f(x_1, \dots, x_n) = 0$ on all special Gelfand–Dorfman algebras, where f belongs to the free Gelfand–Dorfman algebra $\mathrm{GD}\langle x_1, \dots, x_n\rangle$ freely generated by $x_1, \dots, x_n$. In particular, this identity holds on the differential Poisson algebra A constructed above with operations $[\cdot, \cdot]$ and $*$. We need to prove that $f(x_1, \dots, x_n) = 0$ on $V^{(-)}$.

The space $V[t]$ is embedded into A in such a way that bt^n corresponds to $b(n)$ for $b \in B$, $n \in \mathbb{Z}_+$.

Denote the free Novikov–Poisson algebra generated by $\{x_1, \dots, x_n\}$ by $\mathrm{NP}\langle x_1, \dots, x_n\rangle$. Let us fix the homomorphism

$$\pi : \mathrm{GD}\langle x_1, \dots, x_n\rangle \to (\mathrm{NP}\langle x_1, \dots, x_n\rangle)^{(-)}$$

that sends x_i to x_i. In particular,

$$\pi([f, g]) = \pi(f) \circ \pi(g) - \pi(g) \circ \pi(f), \quad \pi(f * g) = \pi(f) * \pi(g).$$

Lemma 6. *For all $f \in \mathrm{GD}\langle x_1, \ldots, x_n \rangle$ and $v_1, \ldots, v_n \in V$ there exist $f_2, \ldots, f_n \in V$ such that*

$$f(v_1(1), \ldots, v_n(1)) = (\pi(f)(v_1, \ldots, v_n))(1) + f_2(2) + \cdots + f_n(n).$$

Proof. By definition, the following holds in A:

$$\begin{aligned} \{v(n), u(m)\} &= mv'(n)u(m-1) - nv(n-1)u'(m) \\ &= m(v'\,_{(0)}\,u)(n+m-1) + mn(v'\,_{(1)}\,u)(n+m-2) \\ &\quad -n(u'\,_{(0)}\,v)(n+m-1) - nm(u'\,_{(1)}\,v)(n+m-2) \\ &= (mv \circ u - nu \circ v)(n+m-1), \quad u, v \in V, \end{aligned} \tag{16}$$

since

$$v'\,_{(1)}\,u = v * u = u * v = u'\,_{(1)}\,v.$$

Moreover,

$$\begin{aligned} u(n) * v(m) = u'(n)v(m) &= (u'_{(0)}v)(n+m) + n(u'\,_{(1)}\,v)(n+m-1) \\ &= n(u*v)(n+m-1) + (u \circ v)(n+m) \end{aligned} \tag{17}$$

for all $u, v \in V$.

It suffices to prove the claim for monomials in the free Gelfand–Dorfman algebra. Proceed by induction on the degree of f. For $f = x_1$, the claim is obvious. Assume that $f = [g, h]$ or $f = g * h$, where $g, h \in \mathrm{GD}\langle x_1, \ldots, x_n \rangle$ are of degree smaller than n. Then, by the induction hypothesis,

$$g(v_1(1), \ldots, v_n(1)) = g_1(1) + g_2(2) + g_3(3) + \cdots, \quad g_1 = \pi(g)(v_1, \ldots, v_n),$$

$$h(v_1(1), \ldots, v_n(1)) = h_1(1) + h_2(2) + h_3(3) + \cdots, \quad h_1 = \pi(h)(v_1, \ldots, v_n)$$

for appropriate $g_i, h_i \in V$, with $i \geq 2$. Let us calculate $f(v_1(1), \ldots, v_n(1))$ with (16) and (17). If $f = [g, h]$ then

$$\begin{aligned} f(v_1(1), \ldots, v_n(1)) &= \{g(v_1(1), \ldots, v_n(1)),\ h(v_1(1), \ldots, v_n(1))\} \\ &= \{g_1(1), h_1(1)\} + f_2(2) + f_3(3) + \cdots \\ &= (g_1 \circ h_1)(1) - (h_1 \circ g_1)(1) + f_2(2) + f_3(3) + \cdots \\ &= \pi([g, h])(1) + f_2(2) + f_3(3) + \cdots, \end{aligned}$$

as desired. If $f = g * h$ then

$$\begin{aligned} f(v_1(1), \ldots, v_n(1)) &= g(v_1(1), \ldots, v_n(1)) * h(v_1(1), \ldots, v_n(1)) \\ &= (g_1 * h_1)(1) + (g_1 \circ h_1)(2) + f_2(2) + f_3(3) + \cdots \\ &= \pi(g * h)(1) + (g_1 \circ h_1)(2) + f_2(2) + f_3(3) + \cdots, \end{aligned}$$

and so the lemma is proved. □

Now we may finish the proof of the theorem. If some identity $f(x_1, \ldots, x_n) = 0$ holds on the algebra A then $f(v_1(1), \ldots, v_n(1)) = 0$ for all $v_1, \ldots, v_n \in V$. On the other hand,

$$0 = f(v_1(1), \ldots, v_n(1)) = f_1(1) + f_2(2) + \cdots + f_n(n), \quad f_1 = \pi(f)(v_1, \ldots, v_n),$$

by Lemma 6. Since $f_i(i)$ are the images of linearly independent elements $f_i t^i \in V[t]$ under the embedding of $V[t]$ into A, we have $f_i = 0$ for all $i = 1, \ldots, n$. In particular,

$$f_1 = \pi(f)(v_1, \ldots, v_n) = 0.$$

By the definition of π, the element $\pi(f)(v_1, \dots, v_n) \in V$ is exactly the value of f at the elements $v_1, \dots, v_n \in V$ in the Gelfand–Dorfman algebra $V^{(-)}$. Hence, $V^{(-)}$ satisfies all those identities that hold on all special Gelfand–Dorfman algebras. Since the class of special Gelfand–Dorfman algebras forms a variety [12], $V^{(-)}$ itself is special. $\square$

Remark 4. This is an open question whether each t-Poisson algebra is a special Gelfand–Dorfman algebra or not. The results of this paper answer in the affirmative for the commutator t-Poisson algebras obtained from Novikov–Poisson algebras over a field of characteristic zero.

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