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# ON THE $t$-EQUIVALENCE OF GENERALIZED ORDERED SETS

**N. N. Trofimenko and T. E. Khmyleva**

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**Abstract:** We consider the generalized ordered spaces $X$ and $Y$ such that the tightness $t(X)$ coincides with the tightness $t(Y)$ but $T(X) = \{x \in X : t(x, X) = t(X)\}$ and $T(Y) = \{y \in Y : t(y, Y) = t(Y)\}$ have different cardinalities. Some sufficient conditions are found under which such spaces $X$ and $Y$ are not $t$-equivalent.

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## 1. Introduction

We consider the homeomorphism of the spaces of continuous functions on generalized ordered spaces. A space $X$ is a *generalized ordered space* if $X$ is a subspace of some linearly ordered topological space $Y$ (see [1]). In fact, these are the linearly ordered spaces in which a neighborhood base of $x \in X$ is given by intervals $(y, z)$, with $x \in (y, z)$, or by half-intervals $[x, y)$ or $(y, x]$ or by the singleton $\{x\}$. Among the examples of these spaces are the Sorgenfrey line, the Michael line, the Hattori spaces, etc.

In generalized ordered spaces, the tightness $t(X)$ of a space $X$ coincides with the functional tightness of $(X)$. Therefore, if $t(X) \neq t(Y)$ for generalized ordered spaces $X$ and $Y$ then the spaces $C_p(X)$ and $C_p(Y)$ are nonhomeomorphic [2]. In the article, considering generalized ordered spaces $X$ and $Y$ of the same tightness but having the nonequipollent sets of the points of maximum tightness, we find the sufficient conditions for $X$ and $Y$ to be not $t$-equivalent, which means that $C_p(Y)$ and $C_p(X)$ are nonhomeomorphic.

As a consequence, using the homeomorphism theorems for $C_p[0, \alpha]$, we obtain the topological classification of $C_p(S_\alpha)$ on the "long Sorgenfrey lines $S_\alpha$." Note that the topological and linear classifications of $C_p[0, \alpha]$ depend on the $t$-equivalence of the ordinal segments $[0, \alpha]$. The theorems in this article imply that the topological classification of the spaces $C_p(S_\alpha)$ coincides with the linear homeomorphic classification of $C_p(S_\alpha)$ which was obtained in [3].

## 2. The Main Notations and Definitions

Throughout the sequel, $\mathbb{R}$ and $\mathbb{N}$ are the sets of reals and naturals respectively; $\aleph_0$ is a countable cardinal, $\aleph_1$ is the first uncountable cardinal; and $|A|$ is the cardinality of a set $A$. Given a generalized linear ordered space $X$, as usual, we put

$$(x, \rightarrow) = \{y \in X : y > x\} \quad \text{and} \quad (\leftarrow, x) = \{y \in X : y < x\}.$$

The spaces $[x, \rightarrow)$, $(\leftarrow, x]$, $[y, x)$, and $(x, y]$ are defined likewise. Observe that every linear ordered topological space $X$ is normal (see [4, 1.7.5]) and so is every subspace $Y \subset X$ (see [4, 2.7.5]). Therefore, all generalized ordered spaces are normal.

Definition 1. Let $X$ be a generalized ordered space. If $x$ is a limit point of $(\leftarrow, x)$ then the *cofinality of* $x$ is the cardinal

$$\operatorname{cf} x = \min\{|A| : A \text{ is a cofinal subset of } (\leftarrow, x)\}.$$

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If $x$ is an isolated point in $(\leftarrow, x] \subset X$ then we put $\operatorname{cf} x = 0$. The coinitiality of $x \in X$ is defined similarly: $\operatorname{cn} x = 0$ if $x$ is an isolated point in $[x, \rightarrow)$ and

$$\operatorname{cn} x = \min\{|A| : A \text{ is a coinitial subset of } (x, \rightarrow)\}$$

if $x$ is a limit point of $(x, \rightarrow)$.

Denote by $t(X)$ the tightness of a space $X$ and designate as $t(x, X)$ the tightness of $X$ at a point $x$. If $X$ is a generalized ordered space, then clearly $t(x, X) = \max\{\operatorname{cf} x, \operatorname{cn} x\}$. Let $t(X) = \tau$. Consider the set

$$T(X) = \{x \in X : t(x, X) = \tau\}$$

and its subsets

$$T_l(X) = \{x \in T(X) : \operatorname{cf} x = \tau, \operatorname{cn} x < \tau\};$$
$$T_r(X) = \{x \in T(X) : \operatorname{cn} x = \tau, \operatorname{cf} x < \tau\};$$
$$T_{lr}(X) = \{x \in T(X) : \operatorname{cn} x = \operatorname{cf} x = \tau\}.$$

Clearly,

$$T(X) = T_l(X) \sqcup T_r(X) \sqcup T_{lr}(X).$$

Definition 2. Let $Z \subset X$ and let $\lambda$ be a cardinal. A point $x \in Z$ is *$\lambda$-inaccessible* in $Z$ if from $x \notin A \subset Z$ and $|A| \le \lambda$ it follows that $x \notin \overline{A}$.

Definition 3. A function $f : X \to \mathbb{R}$ is *$\lambda$-continuous* if, for every $A \subset X$, with $|A| \le \lambda$, the function $f|_A$ is continuous on $A$. If for every set $A \subset X$ such that $|A| \le \lambda$, there exists a continuous function $h : X \to \mathbb{R}$ such that $f|_A = h|_A$ then $f$ is *strictly $\lambda$-continuous.*

Note that for a normal space $X$, every $\lambda$-continuous function is strictly $\lambda$-continuous (see [5]).

It is not hard to see that the characteristic function $\chi_{\{x\}} : X \to \mathbb{R}$ is $\lambda$-continuous if and only if $x$ is $\lambda$-accessible in $X$. As usual, $C_p(X)$ means the space of all real-valued continuous functions on $X$ which is endowed with the topology of pointwise convergence, while $C_\omega(X)$ is the space of all $\omega$-continuous functions on $X$ which is also endowed with the topology of pointwise convergence. The notation $C_p(X) \sim C_p(Y)$ means that $C_p(X)$ and $C_p(Y)$ are homeomorphic.

Given a Tychonoff space $X$, denote the Hewitt completion of $X$ by $\nu X$. The following are well known:

**Theorem 1** [4]. *If $X$ and $Y$ are Tychonoff spaces and $\varphi : X \to Y$ is a homeomorphism then there exists a homeomorphism $\tilde{\varphi} : \nu X \to \nu Y$ such that $\tilde{\varphi}|_X = \varphi$.* □

**Theorem 2** [5]. *If $X$ is a normal space then*

$$\nu C_p(X) = \{f \in \mathbb{R}^X : f \text{ is } \omega\text{-continuous on } X\}. \quad \square$$

## 3. The Main Results

**Lemma 1.** *Let $X$ be a generalized ordered space, $t(X) > \aleph_0$, and let $P_l \subset T_l(X)$ be a right discrete space in $T_l(X)$; i.e., every $x \in P_l$ is isolated in $T_l(X) \cap [x, \rightarrow)$. Then there exists $A(P_l) \subset C_\omega(X) \setminus C_p(X)$, with $A(P_l) = \{f_x : x \in P_l\}$, such that for every $x \in P_l$ the function $f_x$ is continuous on $X \setminus \{x\}$, right continuous at $x \in X$, and*

$$f_{x'}^{-1}(\mathbb{R} \setminus \{0\}) \cap f_{x''}^{-1}(\mathbb{R} \setminus \{0\}) = \varnothing$$

*for all $x', x'' \in P_l$, $x' \ne x''$.*

Proof. Since $P_l$ is a right discrete set in $T_l(X)$, for every $x \in P_l$ there exists a neighborhood $U(x)$ such that $U(x) \cap [x, \rightarrow) \cap T_l(X) = \{x\}$. Putting $f_x(x) = 1$ and $f_x(t) = 0$ if $t < x$ or $t \in [x, \rightarrow) \setminus U(x)$, we extend $f_x$ by continuity to $[x, \rightarrow)$. It is not hard to see that $f_x$ is left discontinuous at $x$, continuous on $[x, \rightarrow)$, and $\omega$-continuous since $\operatorname{cf} x = \tau > \aleph_0$. The set $A(P_l) = \{f_x : x \in P_l\}$ satisfies the hypotheses of the lemma. □

Obviously, if $T_r(X)$ contains a left discrete set $P_r$; then, as in Lemma 1, we can construct $A(P_r) \subset C_\omega(X) \setminus C(X)$.

**Lemma 2.** *Suppose that $Y$ is a generalized ordered space, $y_0 \in Y$, $\operatorname{cf} y_0 > \aleph_0$, and there exists a cofinal countably compact set $A \subset (\leftarrow, y_0)$. Then for every function $f \in C_\omega(Y)$ there exists $y_f \in Y$ such that $y_f < y_0$ and $f(y) = f(y_f)$ for every $y \in [y_f, y_0)$. The same holds for $y_0 \in Y$ such that $\operatorname{cn} y_0 > \aleph_0$.*

PROOF. Show that

$(*)$ for every $\varepsilon > 0$ there is $y_\varepsilon < y_0$ such that $|f(y'') - f(y')| < \varepsilon$ if $y', y'' \in (y_\varepsilon, y_0)$.

Indeed, if $(*)$ fails then there is $\varepsilon_0 > 0$ such that for every $y < y_0$ we have points $y', y'' > y$ for which $|f(y') - f(y'')| \geq \varepsilon_0$. By induction, we can choose some increasing sequence of points

$$y_1 < y_1' < y_1'' < \cdots < y_n < y_n' < y_n'' < \cdots$$

such that $\{y_n\}_{n=1}^\infty \subset A$ and $|f(y_n'') - f(y_n')| \geq \varepsilon_0$ for each $n \in \mathbb{N}$. Since $A$ is countably compact, there exists $a = \sup\{y_n : n \in \mathbb{N}\} \in A$. Clearly, $a$ is also a limit point of $\{y_n' : n \in \mathbb{N}\}$ and $\{y_n'' : n \in \mathbb{N}\}$. We come to a contradiction to the continuity of $f$ on the countable set $\{y_n' : n \in \mathbb{N}\} \cup \{y_n'' : n \in \mathbb{N}\} \cup \{a\}$. Consequently, $(*)$ holds.

Applying $(*)$ to $\varepsilon = \frac{1}{n}$, we obtain some sequence $y_1 < y_2 < \cdots$ such that $y_n \in A$ and $|f(y'') - f(y')| < \frac{1}{n}$ for $y', y'' \in (y_n, y_0)$. Then for $y_f = \sup\{y_n : n \in \mathbb{N}\}$ we have $f(y) = f(y_f)$ for all $y \in [y_f, y_0)$. □

**Lemma 3.** *Suppose that $X$ and $Y$ be generalized ordered sets, $t(X) = t(Y) = \tau > \aleph_0$, and $\Phi : C_\omega(X) \to C_\omega(Y)$ is a homeomorphism such that $\Phi(C_p(X)) = C_p(Y)$. If $f_x \in A(P_l)$ (or $f_x \in A(P_r)$) then $\Phi(f_x)$ is $\lambda$-continuous for all $\lambda < \tau$.*

PROOF. Suppose that there exists a subset $Y_0 \subset Y$, with $|Y_0| < \tau$, and $\Phi(f_x)|_{Y_0}$ is discontinuous at some point $y_0 \in Y_0$. Given $y \in Y_0$ and $n \in N$, consider the standard neighborhoods of $\Phi(f_x)$; i.e.,

$$U(y,n) = U\left(\Phi(f_x), y, y_0, \frac{1}{n}\right)$$
$$= \left\{g \in C_\omega(Y) : |g(y) - \Phi(f_x)(y)| < \frac{1}{n},\ |g(y_0) - \Phi(f_x)(y_0)| < \frac{1}{n}\right\}.$$

Then for every $g \in \bigcap\{U(y,n) : y \in Y_0,\ n \in \mathbb{N}\}$, we have $g|_{Y_0} = \Phi(f_x)|_{Y_0}$. Hence,

$$\left(\bigcap\{U(y,n) : y \in Y_0,\ n \in \mathbb{N}\}\right) \cap C(Y) = \varnothing.$$

Since $\Phi(C_p(X)) = C_p(Y)$; therefore,

$$\left(\bigcap\{\Phi^{-1}U(y,n) : y \in Y_0,\ n \in \mathbb{N}\}\right) \cap C_p(X) = \varnothing. \tag{1}$$

On the other hand, for every neighborhood $\Phi^{-1}U(y,n)$ of $f_x$, there exists a neighborhood

$$V\left(f_x, F(y,n), \frac{1}{k(y,n)}\right) \subset \Phi^{-1}U(y,n),$$

where $F(y,n) \subset X$ is a finite subset and $k(y,n) \in \mathbb{N}$. Obviously, $|\bigcup\{(F(y,n) : y \in Y_0,\ n \in \mathbb{N}\}| < \tau$ and hence $F = \bigcup\{F(y,n) : y \in Y_0,\ n \in \mathbb{N}\}$ is not cofinal to $x$. Since $f_x \in A(P_l)$ is $\lambda$-continuous for all $\lambda < \tau$ and hence strictly $\lambda$-continuous (see [5]), there exists a continuous function $h \in C_p(X)$ such that $f_x|_F = h|_F$. Clearly,

$$h \in \bigcap\left\{V\left(f_x, F(y,n), \frac{1}{k(y,n)}\right) : y \in Y_0,\ n \in \mathbb{N}\right\} \subset \bigcap\{\Phi^{-1}U(y,n) : y \in Y_0,\ n \in \mathbb{N}\},$$

which contradicts (1). □

**Lemma 4.** *Suppose that $X$ and $Y$ are generalized ordered spaces, $t(X) = t(Y) = \tau > \aleph_0$, and $\Phi : C_\omega(X) \to C_\omega(Y)$ is a homeomorphism such that $\Phi(C_p(X)) = C_p(Y)$. If $x \in T_{lr}(X)$ then $\Phi(\chi_{\{x\}})$ is $\lambda$-continuous for all $\lambda < \tau$.*

Proof. Observe that the function $\chi_{\{x\}}$ is $\lambda$-continuous for $\lambda < \tau$ provided that $x \in T_{lr}(X)$. The rest of the proof is finished as in Lemma 3. $\square$

**Theorem 3.** *Suppose that $X$ and $Y$ are generalized ordered spaces, $t(X) = t(Y) = \tau > \aleph_0$, $|T(X)| > |T(Y)| \geq \aleph_0$, and the following are fulfilled:*

*(1) Either $|T_{lr}(X)| > |T(Y)|$ or there exists a right discrete subset $P_l \subset T_l(X)$ such that $|P_l| > |T(Y)|$ or there exists a left discrete set $P_r \subset T_r(X)$ such that $|P_r| > |T(Y)|$.*

*(2) If $y \in Y$ and $\operatorname{cf} y = \tau$ then there exists a cofinal countably compact subset in $(\leftarrow, y)$. If $\operatorname{cn} y = \tau$ then there exists a coinitial countably compact subset in $(y, \to)$.*

*Then $C_p(Y)$ and $C_p(X)$ are not homeomorphic.*

Proof. Suppose that $P_l \subset T_l(X)$ is right discrete in $T_l(X)$ and $|P_l| > |T(Y)|$. Consider

$$A(P_l) \subset C_\omega(X) \setminus C_p(X)$$

and suppose that there exists a homeomorphism $\Phi : C_p(X) \to C_p(Y)$. Without loss of generality, we may assume that $\Phi(0) = 0$. Using Theorems 1 and 2, we can extend $\Phi$ to a homeomorphism $\Phi : C_\omega(X) \to C_\omega(Y)$. Since the function $f \equiv 0$ is a limit point of the nonstationary sequence $\{f_{x_n}\}_{n=1}^{\infty} \subset A(P_l)$, we see that

$$|\{x \in P_l : \Phi(f_x)(y) \neq 0\}| \leq \aleph_0 \quad \text{for all } y \in Y.$$

Since $|T(Y)| < |P_l|$, there exists $P_{l1} \subset P_l$ such that $|P_{l1}| = |P_l|$ and $\Phi(f_x)(y) = 0$ for all $x \in P_l$ and $y \in T(Y)$.

Since all $\Phi(f_x)$ do not belong to $C_p(Y)$ and are $\lambda$-continuous for $\lambda < \tau$ by Lemma 3, for every $x \in P_{l1}$ there exists $y \in T(Y)$ at which $\Phi(f_x)$ is left discontinuous if $y \in T_l(Y)$ and right discontinuous if $y \in T_r(Y)$.

Let $y \in T(Y)$ and

$$P_{ly} = \{x \in P_{l1} : \Phi(f_x) \text{ is discontinuous at } y\}.$$

Since $\bigcup\{P_{ly} : y \in T(Y)\} = P_{l1}$ and $|P_{l1}| = |P_l| > T(Y)$, there exists $y_0 \in T(Y)$ such that $|P_{ly_0}| > \aleph_0$. Without loss of generality, we may assume that all functions $\{\Phi(f_x) : x \in P_{ly_0}\}$ are left discontinuous at $y_0$. By Lemma 2, for every $\Phi(f_x)$ there exists a point $y_x \in Y$ such that $\Phi(f_x)(y) = \Phi(f_x)(y_x)$ for all $y \in [y_x, y_0)$; moreover, $\Phi(f_x)(y_x) \neq 0$ because $\Phi(f_x)$ is discontinuous at $y_0$ and $\Phi(f_x)(y_0) = 0$. Since $|P_{ly_0}| > \aleph_0$, for some $n \in \mathbb{N}$ there is an uncountable set $P_{ly_0n} = \big\{x \in P_{ly_0} : \Phi(f_x)(y_x) \geq \frac{1}{n}\big\}$.

Consider an arbitrary countable subset $B \subset P_{ly_0n}$. Since $\{y_x : x \in B\}$ is not cofinal to $(\leftarrow, y_0)$, there exists a point $y_1$ such that $y_x < y_1 < y_0$ for every $x \in B$; i.e., the identically zero function is not limit point of $\{\Phi(f_x) : x \in B\}$. This contradicts the fact that the function $f \equiv 0$ is a limit point of $\{f_x : x \in B\}$.

For the case of the existence of a left discrete subset $P_r$ in $T_r(X)$ such that $|P_r| > |T(Y)|$, the proof is similar. If $|T_{lr}(X)| > |T(Y)|$; then, instead of $A(P_l)$, we must consider the set $\{\chi_{\{x\}} : x \in T_{lr}(x)\}$ and use Lemma 4. $\square$

## 4. On the $t$-Equivalence of the Spaces $X_\alpha$

Let $X$ be a separable generalized ordered spaces and let $\alpha$ be an ordinal. Endow the product $[0, \alpha) \times X$ with the order relation $(\gamma_1, x_1) \leq (\gamma_2, x_2)$ if $\gamma_1 < \gamma_2$ or $\gamma_1 = \gamma_2$ and $x_1 \leq x_2$. Let $B(x)$ be a neighborhood base of $x \in X$ and let $x_0$ be the first element in $X$ (if existent). Define the neighborhood base of $(\gamma, x) \in [0, \alpha) \times X$ as follows:

$$\mathscr{B}(\gamma, x) = \{\{\gamma\} \times O(x) : O(x) \in \mathscr{B}(x)\}$$

if $x \neq x_0$

$$\mathscr{B}(\gamma, x_0) = \{(\beta, \gamma) \times X \sqcup \{\gamma\} \times O(x_0) : \beta < \gamma,\ O(x_0) \in \mathscr{B}(x_0)\},$$

if $\gamma$ is a limit ordinal and

$$\mathscr{B}(\gamma, x_0) = \{\{\gamma - 1\} \times (x, \rightarrow) \sqcup \{\gamma\} \times O(x_0) : x \in X,\ O(x_0) \in \mathscr{B}(x_0)\}$$

if $\gamma$ is a nonlimit ordinal.

Denote the space $[0, \alpha) \times X$ with the topology base $\{B(\gamma, x) : \gamma \in [0, \alpha),\ x \in X\}$ by $X_\alpha$.

Considering the segment $[0, \alpha]$, we obtain the space $X_{\alpha+1}$. In particular, if $X = [0, 1) \subset \mathbb{R}$ and $\alpha = \omega_1$ then we get a "long line," and if $S$ is a Sorgenfrey line with neighborhood base $\mathscr{B}(x) = \{(a, x] : a < x\}$ and $X = [0, 1) \subset S$ then $[0, \alpha) \times X = S_\alpha$ is a "long Sorgenfrey line." As $X$, we can take the "two arrows" or $[0, 1) \subset H(A)$, where $H(A)$ is the Hattori space (see [6]). Note that the existence of the first element $x_0 \in X$ makes it possible to define some mapping $\varphi(\gamma) = (\gamma, x_0)$ that is a homeomorphic embedding of the ordinal interval $[0, \alpha)$ onto the closed subspace $\{(\gamma, x_0) : \gamma \in [0, \alpha)\} \subset [0, \alpha) \times X$. If $\tau$ and $\sigma$ are initial ordinals, $\omega \le \sigma \le \tau$, and $\tau$ is a regular ordinal; then $t(X_{\tau\sigma+1}) = \tau$ and $|T(X_{\tau\sigma+1})| = |\sigma|$. It is not hard to see that $T(X_{\tau\sigma+1})$ has the form

$$\{(\tau(\gamma + 1), x_0) : 1 \le \gamma < \sigma\},$$

is right discrete, and $T(X_{\tau\sigma+1}) = T_l$. (In case $\sigma = \tau$, we must add the point $(\tau \cdot \tau, x_0)$.)

The set $\{(\tau \cdot \gamma + \beta, x_0) : 1 \le \beta < \tau\}$ is homeomorphic to the ordinal interval $[1, \tau)$ and, hence, it is countably compact. Moreover, this is a cofinal subset of $(\leftarrow, (\tau(\gamma + 1)), x_0)$. Thus, $X_{\tau\sigma+1}$ satisfy conditions (1) and (2) of Theorem 3. Consequently, we have the following

**Theorem 4.** *Suppose that $X$ is a separable generalized ordered space with the first element; $\tau$, $\lambda$, and $\sigma$ are initial ordinals, and $\tau$ is a regular ordinal. If $\omega \le \sigma < \lambda \le \tau$ then $C_p(X_{\tau\sigma+1})$ and $C_p(X_{\tau\lambda+1})$ are nonhomeomorphic.* □

**Theorem 5.** *Let $\alpha$ and $\beta$ be infinite ordinals and let $X$ be a separable generalized ordered space with the first element. The space $C_p(X_{\alpha+1})$ is homeomorphic to $C_p(X_{\beta+1})$ if and only if $C_p[0, \alpha]$ is homeomorphic to $C_p[0, \beta]$.*

Proof. Consider the closed subset $A = [0, \alpha] \times \{x_0\} \subset X_{\alpha+1}$ and show that there exists a continuous linear extension operator $\Psi : C_p(A) \overset{in}{\rightarrow} C_p(X_{\alpha+1})$. On every segment $I_\gamma = [(\gamma, x_0), (\gamma + 1, x_0)]$, $0 \le \gamma < \alpha$, fix $x_1 \in X$, with $x_1 > x_0$. By Urysohn's Lemma, there is a linear function $g_0 : I_\gamma \to [0, 1]$ such that $g_0([(\gamma, x_0), (\gamma, x_1)]) \subset \{1\}$ and $g_0(\gamma + 1, x_0) = 0$. In a similar fashion, define the function $g_1 : I_\gamma \to [0, 1]$, $g_1([(\gamma, x_1), (\gamma + 1, x_0)]) \subset \{1\}$, and $g_1(\gamma, x_0) = 0$. Putting $f_0 = \frac{g_0}{g_0 + g_1}$ and $f_1 = \frac{g_1}{g_0 + g_1}$, we obtain the partition of unity $\{f_0, f_1\}$. Consider the operator $\Psi : C_p(A) \to C_p(X_{\alpha+1})$ defined by the formula

$$\Psi(f)(\gamma, x) = f(\gamma, x_0) f_0(\gamma, x) + f(\gamma + 1, x_0) f_1(\gamma, x)$$

if $0 \le \gamma < \alpha$ and $\Psi(f)(\alpha, x) = f(\alpha, x_0)$. The function $\Psi(f)|_A$ is equal to $f$, whereas $\Psi(f)$ takes values between $f(\gamma, x_0)$ and $f(\gamma + 1, x_0)$ on $I_\gamma$ and hence $\Psi(f)$ is continuous on $X_{\alpha+1}$. It is easy to check that $\Psi$ is linear and continuous. In this event (see [5, 1.5]) $C_p(X_{\alpha+1})$ is linearly homeomorphic to $C_p(A) \times C_p^0(X_{\alpha+1}, A)$, where $C_p^0(X_{\alpha+1}, A) = \{f \in C_p(X_{\alpha+1}) : f(A) \subset \{0\}\}$.

By the compactness of the ordinal segment $[0, \alpha]$, the set $\{\gamma : \sup_{x \in X} |f(\gamma, x)| \ge \varepsilon\}$ is finite for all $f \in C_p^0(X_{\alpha+1}, A)$ and $\varepsilon > 0$. Therefore, $C_p^0(X_{\alpha+1}, A)$ is linearly homeomorphic to the space $\left(\prod_{0 \le \gamma \le \alpha} C_p^0(I_\gamma)\right)_{c_0}$ defined as follows:

$$\left(\prod_{0 \le \gamma \le \alpha} C_p^0(I_\gamma)\right)_{c_0}$$

$$= \left\{ f = \{f_\gamma\}_{\gamma \le \alpha} \in \prod_{0 \le \gamma \le \alpha} C_p^0(I_\gamma) : \{\gamma : \sup_{(\gamma, x) \in I_\gamma} |f_\gamma(\gamma, x)| \ge \varepsilon\} \text{ is finite for any } \varepsilon > 0 \right\},$$

where

$$C_p^0(I_\gamma) = \{f \in C_p(I_\gamma) : f(\gamma, x_0) = f(\gamma+1, x_0) = 0\}$$

if $0 \leq \gamma < \alpha$ and

$$C_p^0(I_\alpha) = \{f \in C_p(I_\alpha) : f(\alpha, x_0) = 0\}.$$

Since all $I_\gamma$'s, with $0 \leq \gamma < \alpha$, are homeomorphic to $I_0$ and $I_\alpha$ is homeomorphic to $X$, the space

$$\left(\prod_{0\leq\gamma\leq\alpha} C_p^0(I_\gamma)\right)_{c_0}$$

is linearly homeomorphic to

$$\left(\prod_{0\leq\gamma<\alpha} C_p^0(I_0)\right)_{c_0} \times C_p^0(I_\alpha) \sim \left(\prod_{|\alpha|} C_p^0(I_0)\right)_{c_0} \times C_p^0(X).$$

Suppose that $C_p[0,\alpha]$ is homeomorphic to $C_p[0,\beta]$. Clearly, in this case $|\alpha| = |\beta|$. Since $A$ is homeomorphic to $[0,\alpha]$, we obtain

$$C_p(X_{\alpha+1}) \sim C_p(A) \times C_p^0(X_{\alpha+1}, A) \sim C_p[0,\alpha] \times \left(\prod_{|\alpha|} C_p^0(I_0)\right)_{c_0} \times C_p^0(X)$$
$$\sim C_p[0,\beta] \times \left(\prod_{|\beta|} C_p^0(I_0)\right)_{c_0} \times C_p^0(X) \sim C_p(X_{\beta+1}).$$

If $C_p[0,\alpha]$ is not homeomorphic to $C_p[0,\beta]$ then this means that (see [7,8]) either

(a) $|\alpha| \neq |\beta|$

or

(b) $|\alpha| = |\beta| = |\tau|$, where $\tau$ is an initial regular ordinal and there exist initial ordinals $\sigma, \lambda, \sigma < \lambda \leq \tau$ such that $\tau\sigma \leq \alpha < \tau\sigma^+$ and $\tau\lambda \leq \beta < \tau\lambda^+$.

In case (a), granted the separability of $X$, we obtain

$$d(X_{\alpha+1}) \neq d(X_{\beta+1}),$$

and so $C_p(X_{\alpha+1})$ and $C_p(X_{\beta+1})$ are nonhomeomorphic.

In case (b), $C_p[0,\alpha] \sim C_p[0,\tau\sigma]$ and $C_p[0,\beta] \sim C_p[0,\tau\lambda]$; therefore, by the above,

$$C_p(X_{\alpha+1}) \sim C_p(X_{\tau\sigma+1}),$$

and, respectively, $C_p(X_{\beta+1}) \sim C_p(X_{\tau\lambda+1})$. By Theorem 4, we conclude that $C_p(X_{\alpha+1})$ and $C_p(X_{\beta+1})$ are nonhomeomorphic. □

Remark. If $m, n \in \mathbb{N}$ and $m \neq n$; then, essentially repeating the proof in [9], we can prove that $C_p(X_{\tau n+1})$ and $C_p(X_{\tau m+1})$ are nonhomeomorphic.

If $\sigma = n \in \mathbb{N}$ and $\omega \leq \lambda < \tau$ then $C_p(X_{\tau\sigma+1})$ is nonhomeomorphic to its square, whereas $C_p(X_{\tau\lambda+1})$ is homeomorphic to its square by Theorem 5. Therefore, $C_p(X_{\tau\sigma+1})$ and $C_p(X_{\tau\lambda+1})$ are nonhomeomorphic.

**Corollary 6.** *Let $\alpha$ and $\beta$ be infinite ordinals and let $S_{\alpha+1}$ and $S_{\beta+1}$ be "long Sorgenfrey lines." The spaces $C_p(S_{\alpha+1})$ and $C_p(S_{\beta+1})$ are homeomorphic if and only if so are $C_p[0,\alpha]$ and $C_p[0,\beta]$.* □

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N. N. Trofimenko
Tomsk State University, Tomsk, Russia
*E-mail address*: trofnadezhda99@yandex.ru

T. E. Khmyleva
Tomsk State University, Tomsk, Russia
*E-mail address*: tex2150@yandex.ru