ISSN 0037–4466, Siberian Mathematical Journal, 2022, Vol. 63, No. 4, pp. 735–742. © Pleiades Publishing, Ltd., 2022.
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NONLINEAR MIXED JORDAN TRIPLE $*$-DERIVATIONS ON $*$-ALGEBRAS

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UDC 512.57

Abstract: Let $\mathcal{A}$ be a unital $*$-algebra containing a nontrivial projection. Under some mild conditions on $\mathcal{A}$, it is shown that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation. In particular, we apply the above result to prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras, and standard operator algebras.

DOI: 10.1134/S0037446622040140

Keywords: mixed Jordan triple $*$-derivation, $*$-derivation, von Neumann algebra

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. Given $A, B \in \mathcal{A}$, we call the product $[A, B]_* = AB - BA^*$ the *skew Lie product* and $A \bullet B = AB + BA^*$ the *Jordan $*$-product.* The two new products are fairly meaningful and important and have been studied by many authors (see [1–15]). Recall that an additive map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be an *additive derivation* if

$$\Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{A}$. Furthermore, Φ is said to be an *additive $*$-derivation* if Φ is an additive derivation and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. A map (without the additivity assumption) $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a *nonlinear Jordan $*$-derivation* or a *nonlinear skew Lie derivation* if $\Phi(A \bullet B) = \Phi(A) \bullet B + A \bullet \Phi(B)$ or $\Phi([A, B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$ for all $A, B \in \mathcal{A}$. Many authors have paid more attentions on the problem about Jordan $*$-derivations, skew Lie derivations and triple derivations, such as Jordan triple $*$-derivations and skew Lie triple derivations (see [15–24]). For example, Taghavi et al. [24] investigated a nonlinear λ-Jordan triple $*$-derivation on prime $*$-algebras; i.e., for all $A, B, C \in \mathcal{A}$,

$$\Phi(A\diamondsuit_\lambda B\diamondsuit_\lambda C) = \Phi(A)\diamondsuit_\lambda B\diamondsuit_\lambda C + A\diamondsuit_\lambda \Phi(B)\diamondsuit_\lambda C + A\diamondsuit_\lambda B\diamondsuit_\lambda \Phi(C)$$

where $A\diamondsuit_\lambda B = AB + \lambda BA^*$ such that a complex scalar $|\lambda| \neq 0, 1$, and Φ is additive. Moreover, if $\Phi(I)$ is self-adjoint, then Φ is a $*$-derivation.

Recently, many authors have studied the isomorphisms and derivations corresponding to the new products of the mixture of Lie product and skew Lie product. For example, Yang and Zhang [25, 26] studied the nonlinear maps that preserve the mixed skew Lie triple product $[[A, B]_*, C]$ and $[[A, B], C]_*$ on factor von Neumann algebras. Zhou, Yang, and Zhang [27] studied the structure of the nonlinear mixed Lie triple derivations on prime $*$-algebras. In this paper, we consider the derivations corresponding to the new product of the mixture of the skew Lie product and the Jordan $*$-product. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear mixed Jordan triple $*$-derivation if

$$\Phi([A \bullet B, C]_*) = [\Phi(A) \bullet B, C]_* + [A \bullet \Phi(B), C]_* + [A \bullet B, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$. Under some mild conditions on a $*$-algebra $\mathcal{A}$, we prove that a map $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation. In particular, we apply the above result to prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras, and standard operator algebras.

Original article submitted April 30, 2021; revised February 2, 2022; accepted February 10, 2022.

2. The Main Result and Its Proof

Our main result in this paper reads as follows:

Theorem 2.1. *Let $\mathcal{A}$ be a unital $*$-algebra with the unit I. Assume that $\mathcal{A}$ contains a nontrivial projection P that satisfies*

$$X\mathcal{A}P = 0 \quad \text{implies} \quad X = 0 \tag{$\spadesuit$}$$

and

$$X\mathcal{A}(I-P) = 0 \quad \text{implies} \quad X = 0. \tag{$\clubsuit$}$$

Then a map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\Phi([A \bullet B, C]_*) = [\Phi(A) \bullet B, C]_* + [A \bullet \Phi(B), C]_* + [A \bullet B, \Phi(C)]_*$$

for all $A, B, C \in \mathcal{A}$ if and only if Φ is an additive $*$-derivation.

Proof. Let $P_1 = P$ and $P_2 = I - P$. Put $\mathcal{A}_{jk} = P_j\mathcal{A}P_k$, $j, k = 1, 2$. Then

$$\mathcal{A} = \sum_{j,k=1}^{2} \mathcal{A}_{jk}.$$

In the sequel A_{jk} indicates that $A_{jk} \in \mathcal{A}_{jk}$. Clearly, we only need to prove the necessity. We will complete the proof by several claims:

Claim 1. $\Phi(0) = 0$.

Indeed,

$$\Phi(0) = \Phi([0 \bullet 0, 0]_*) = [\Phi(0) \bullet 0, 0]_* + [0 \bullet \Phi(0), 0]_* + [0 \bullet 0, \Phi(0)]_* = 0.$$

Claim 2. *Φ is additive.*

We will complete the proof of Claim 2 in several steps.

Step 2.1. Given $A_{12} \in \mathcal{A}_{12}$ and $B_{21} \in \mathcal{A}_{21}$, we have $\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21})$.

We only need show that $T = \Phi(A_{12} + B_{21}) - \Phi(A_{12}) - \Phi(B_{21}) = 0$. Since

$$[I \bullet (i(P_2 - P_1)), A_{12}]_* = [I \bullet (i(P_2 - P_1)), B_{21}]_* = 0,$$

where i is the imaginary unit; it follows from Claim 1 that

$$\begin{aligned}
&[\Phi(I) \bullet (i(P_2 - P_1)), A_{12} + B_{21}]_* + [I \bullet \Phi(i(P_2 - P_1)), A_{12} + B_{21}]_* \\
&+[I \bullet (i(P_2 - P_1)), \Phi(A_{12} + B_{21})]_* = \Phi([I \bullet (i(P_2 - P_1)), A_{12} + B_{21}]_*) \\
&= \Phi([I \bullet (i(P_2 - P_1)), A_{12}]_*) + \Phi([I \bullet (i(P_2 - P_1)), B_{21}]_*) \\
&= [\Phi(I) \bullet (i(P_2 - P_1)), A_{12} + B_{21}]_* + [I \bullet \Phi(i(P_2 - P_1)), A_{12} + B_{21}]_* \\
&+[I \bullet (i(P_2 - P_1)), \Phi(A_{12}) + \Phi(B_{21})]_*.
\end{aligned}$$

From this we get $[I \bullet (i(P_2 - P_1)), T]_* = 0$. So $T_{11} = T_{22} = 0$.

Since $[I \bullet A_{12}, P_1]_* = 0$, it follows that

$$\begin{aligned}
&[\Phi(I) \bullet (A_{12} + B_{21}), P_1]_* + [I \bullet \Phi(A_{12} + B_{21}), P_1]_* + [I \bullet (A_{12} + B_{21}), \Phi(P_1)]_* \\
&= \Phi([I \bullet (A_{12} + B_{21}), P_1]_*) = \Phi([I \bullet A_{12}, P_1]_*) + \Phi([I \bullet B_{21}, P_1]_*) \\
&= [\Phi(I) \bullet (A_{12} + B_{21}), P_1]_* + [I \bullet (\Phi(A_{12}) + \Phi(B_{21})), P_1]_* + [I \bullet (A_{12} + B_{21}), \Phi(P_1)]_*.
\end{aligned}$$

Hence $[I \bullet T, P_1]_* = 0$, from which we get that $T_{21} = 0$. Similarly, we can show that $T_{12} = 0$, proving the step.

STEP 2.2. For all $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$ we have

$$\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})$$

and

$$\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Let $T = \Phi(A_{11} + B_{12} + C_{21}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21})$.

It follows from Step 2.1 that

$$\begin{gathered}
[\Phi(I) \bullet (iP_2), A_{11} + B_{12} + C_{21}]_* + [I \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21}]_* \\
+[I \bullet (iP_2), \Phi(A_{11} + B_{12} + C_{21})]_* = \Phi([I \bullet (iP_2), A_{11} + B_{12} + C_{21}]_*) \\
= \Phi([I \bullet (iP_2), A_{11}]_*) + \Phi([I \bullet (iP_2), B_{12} + C_{21}]_*) \\
= \Phi([I \bullet (iP_2), A_{11}]_*) + \Phi([I \bullet (iP_2), B_{12}]_*) + \Phi([I \bullet (iP_2), C_{21}]_*) \\
= [\Phi(I) \bullet (iP_2), A_{11} + B_{12} + C_{21}]_* + [I \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21}]_* \\
+[I \bullet (iP_2), \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})]_*.
\end{gathered}$$

From this we get $[I \bullet (iP_2), T]_* = 0$. So $T_{12} = T_{21} = T_{22} = 0$.

Since

$$[I \bullet (i(P_2 - P_1)), B_{12}]_* = [I \bullet (i(P_2 - P_1)), C_{21}]_* = 0,$$

it follows that

$$\begin{gathered}
[\Phi(I) \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21}]_* + [I \bullet \Phi(i(P_2 - P_1)), A_{11} + B_{12} + C_{21}]_* \\
+[I \bullet (i(P_2 - P_1)), \Phi(A_{11} + B_{12} + C_{21})]_* = \Phi([I \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21}]_*) \\
= \Phi([I \bullet (i(P_2 - P_1)), A_{11}]_*) + \Phi([I \bullet (i(P_2 - P_1)), B_{12}]_*) + \Phi([I \bullet (i(P_2 - P_1)), C_{21}]_*) \\
= [\Phi(I) \bullet (i(P_2 - P_1)), A_{11} + B_{12} + C_{21}]_* + [I \bullet \Phi(i(P_2 - P_1)), A_{11} + B_{12} + C_{21}]_* \\
+[I \bullet (i(P_2 - P_1)), \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21})]_*,
\end{gathered}$$

from which we get $[I \bullet (i(P_2 - P_1)), T]_* = 0$. So $T_{11} = 0$, and then $T = 0$. Similarly, $\Phi(B_{12} + C_{21} + D_{22}) = \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})$.

STEP 2.3. For all $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, and $D_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Let $T = \Phi(A_{11} + B_{12} + C_{21} + D_{22}) - \Phi(A_{11}) - \Phi(B_{12}) - \Phi(C_{21}) - \Phi(D_{22})$. It follows from Step 2.2 that

$$\begin{gathered}
[\Phi(I) \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22}]_* + [I \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
+[I \bullet (iP_2), \Phi(A_{11} + B_{12} + C_{21} + D_{22})]_* = \Phi([I \bullet (iP_2), A_{11} + B_{12} + C_{21} + D_{22}]_*) \\
= \Phi([I \bullet (iP_2), A_{11}]_*) + \Phi([I \bullet (iP_2), B_{12} + C_{21} + D_{22}) \\
= \Phi([I \bullet (iP_2), A_{11}]_*) + \Phi([I \bullet (iP_2), B_{12}]_*) + \Phi([I \bullet (iP_2), C_{21}]_*) + \Phi([I \bullet (iP_2), D_{22}]_*) \\
= [\Phi(I) \bullet (iP_2), A_{11} + B_{12} + C_{21} + +D_{22}]_* + [I \bullet \Phi(iP_2), A_{11} + B_{12} + C_{21} + D_{22}]_* \\
+[I \bullet (iP_2), \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})]_*.
\end{gathered}$$

From this we get $[I \bullet (iP_2), T]_* = 0$. So $T_{12} = T_{21} = T_{22} = 0$. Similarly, we can show that $T_{11} = 0$, proving Step 2.3.

Step 2.4. Given $A_{jk}, B_{jk} \in \mathcal{A}_{jk}$, with $1 \leq j \neq k \leq 2$, we have $\Phi(A_{jk}+B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk})$. Since

$$\left[\frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_* = (A_{jk}+B_{jk}) - A_{jk}^* - B_{jk}A_{jk}^*,$$

we get from Step 2.3 that

$$\begin{aligned}
\Phi(A_{jk}+B_{jk}) + \Phi(-A_{jk}^*) + \Phi(-B_{jk}A_{jk}^*) &= \Phi\left(\left[\frac{I}{2} \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_*\right) \\
= \left[\Phi(\frac{I}{2}) \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_* &+ \left[\frac{I}{2} \bullet \Phi(P_j + A_{jk}), P_k + B_{jk}\right]_* \\
+\left[\frac{I}{2} \bullet (P_j + A_{jk}), \Phi(P_k + B_{jk})\right]_* &= \left[\Phi(\frac{I}{2}) \bullet (P_j + A_{jk}), P_k + B_{jk}\right]_* \\
+\left[\frac{I}{2} \bullet \Big(\Phi(P_j) + \Phi(A_{jk})\Big), P_k + B_{jk}\right]_* &+ \left[\frac{I}{2} \bullet (P_j + A_{jk}), (\Phi(P_k) + \Phi(B_{jk}))\right]_* \\
= \Phi\left(\left[\frac{I}{2} \bullet P_j, P_k\right]_*\right) + \Phi\left(\left[\frac{I}{2} \bullet P_j, B_{jk}\right]_*\right) &+ \Phi\left(\left[\frac{I}{2} \bullet A_{jk}, P_k\right]_*\right) + \Phi\left(\left[\frac{I}{2} \bullet A_{jk}, B_{jk}\right]_*\right) \\
&= \Phi(B_{jk}) + \Phi(A_{jk} - A_{jk}^*) + \Phi(-B_{jk}A_{jk}^*) \\
&= \Phi(B_{jk}) + \Phi(A_{jk}) + \Phi(-A_{jk}^*) + \Phi(-B_{jk}A_{jk}^*).
\end{aligned}$$

Then $\Phi(A_{jk}+B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk})$.

Step 2.5. Given $A_{jj}, B_{jj} \in \mathcal{A}_{jj}$, with $1 \leq j \leq 2$, we have

$$\Phi(A_{jj}+B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

Let $T = \Phi(A_{jj}+B_{jj}) - \Phi(A_{jj}) - \Phi(B_{jj})$. For $1 \leq j \neq k \leq 2$, it follows that

$$\begin{aligned}
&[\Phi(I) \bullet (iP_k), A_{jj}+B_{jj}]_* + [I \bullet \Phi(iP_k), A_{jj}+B_{jj}]_* \\
&+[I \bullet (iP_k), \Phi(A_{jj}+B_{jj})]_* = \Phi([I \bullet (iP_k), A_{jj}+B_{jj}]_*) = \Phi([I \bullet (iP_k), A_{jj}]_*) + \Phi([I \bullet (iP_k), B_{jj}]_*) \\
&= [\Phi(I) \bullet (iP_k), A_{jj}+B_{jj}]_* + [I \bullet \Phi(iP_k), A_{jj}+B_{jj}]_* \\
&+[I \bullet (iP_k), \Phi(A_{jj}) + \Phi(B_{jj})]_*.
\end{aligned}$$

From this we get $[I \bullet (iP_k), T]_* = 0$. So $T_{jk} = T_{kj} = T_{kk} = 0$. Now $T = T_{jj}$.

For all $C_{jk} \in \mathcal{A}_{jk}$, $j \neq k$, it follows from Step 2.4 that

$$\begin{aligned}
&[\Phi(I) \bullet (A_{jj}+B_{jj}), C_{jk}]_* + [I \bullet \Phi(A_{jj}+B_{jj}), C_{jk}]_* \\
&+[I \bullet (A_{jj}+B_{jj}), \Phi(C_{jk})]_* = \Phi([I \bullet (A_{jj}+B_{jj}), C_{jk}]_*) \\
&= \Phi([I \bullet A_{jj}, C_{jk}]_*) + \Phi([I \bullet B_{jj}, C_{jk}]_*) \\
&= [\Phi(I) \bullet (A_{jj}+B_{jj}), C_{jk}]_* + [I \bullet \Phi(A_{jj}) + \Phi(B_{jj}), C_{jk}]_* \\
&+[I \bullet (A_{jj}+B_{jj}), \Phi(C_{jk})]_*.
\end{aligned}$$

Hence $[I \bullet T_{jj}, C_{jk}]_* = 0$ for all $C_{jk} \in \mathcal{A}_{jk}$; i.e., $T_{jj}CP_k = 0$ for all $C \in \mathcal{A}$. It follows from (♠) and (♣) that $T = T_{jj} = 0$, proving the step.

Now, it follows from Steps 2.3–2.5 that Φ is additive, proving Claim 2.

Claim 3. $\Phi(I)$ *is a self-adjoint central element in* $\mathcal{A}$.

On the one hand,

$$\begin{aligned}
0 = \Phi([I \bullet I, I]_*) &= [\Phi(I) \bullet I, I]_* + [I \bullet \Phi(I), I]_* + [I \bullet I, \Phi(I)]_* \\
&= [2\Phi(I), I]_* = 2\Phi(I) - 2\Phi(I)^*,
\end{aligned}$$

which implies that $\Phi(I)$ is a self-adjoint element in $\mathcal{A}$.

On the other hand, for all $A \in \mathcal{A}$ we get

$$\begin{aligned}0 = \Phi([I \bullet I, A]_*) &= [\Phi(I) \bullet I, A]_* + [I \bullet \Phi(I), A]_* + [I \bullet I, \Phi(A)]_* \\ &= 2[2\Phi(I), A]_* = 4(\Phi(I)A - A\Phi(I)),\end{aligned}$$

which implies that $\Phi(I)$ is a central element in $\mathcal{A}$.

Claim 4. *$P_1\Phi(P_1)P_2 = -P_1\Phi(P_2)P_2$, $P_2\Phi(P_1)P_1 = -P_2\Phi(P_2)P_1$, and $P_1\Phi(P_2)P_1 = P_2\Phi(P_1)P_2 = 0$.*

On the one hand, for $1 \leq j \neq k \leq 2$, it follows from Claim 3 that

$$\begin{aligned}0 &= \Phi([I \bullet P_j, P_k]_*) = [\Phi(I) \bullet P_j, P_k]_* + [I \bullet \Phi(P_j), P_k]_* + [I \bullet P_j, \Phi(P_k)]_* \\ &= [2\Phi(P_j), P_k]_* + [2P_j, \Phi(P_k)]_* = 2\Phi(P_j)P_k - 2P_k\Phi(P_j)^* + 2P_j\Phi(P_k) - 2\Phi(P_k)P_j.\end{aligned}$$

Multiplying both sides of the above equation by P_j and P_k from the left and right, respectively, we infer that $P_1\Phi(P_1)P_2 = -P_1\Phi(P_2)P_2$ and $P_2\Phi(P_1)P_1 = -P_2\Phi(P_2)P_1$.

On the other hand, we get

$$\begin{aligned}0 = \Phi([I \bullet (iP_j), P_k]_*) &= [\Phi(I) \bullet (iP_j), P_k]_* + [I \bullet \Phi(iP_j), P_k]_* + [I \bullet (iP_j), \Phi(P_k)]_* \\ &= [2\Phi(iP_j), P_k]_* + [2iP_j, \Phi(P_k)]_* \\ &= 2\Phi(iP_j)P_k - 2P_k\Phi(iP_j)^* + 2i(P_j\Phi(P_k) + \Phi(P_k)P_j).\end{aligned}$$

Multiplying both sides of the above equation by P_j from the left and right, respectively, we obtain that $P_1\Phi(P_2)P_1 = P_2\Phi(P_1)P_2 = 0$.

Claim 5. $P_1\Phi(P_1)P_1 = P_2\Phi(P_2)P_2 = 0$.

For every $A_{12} \in \mathcal{A}_{12}$, on the one hand, it follows from Claims 2 and 3 that

$$\begin{aligned}2\Phi(A_{12}) = \Phi([I \bullet P_1, A_{12}]_*) &= [\Phi(I) \bullet P_1, A_{12}]_* + [I \bullet \Phi(P_1), A_{12}]_* + [I \bullet P_1, \Phi(A_{12})]_* \\ &= [2\Phi(I)P_1, A_{12}]_* + [2\Phi(P_1), A_{12}]_* + [2P_1, \Phi(A_{12})]_* \\ &= 2\Phi(I)A_{12} + 2\Phi(P_1)A_{12} - 2A_{12}\Phi(P_1)^* + 2P_1\Phi(A_{12}) - 2\Phi(A_{12})P_1.\end{aligned}$$

Multiplying both sides of the above equation by P_1 and P_2 from the left and right, respectively, by Claim 4, we get that

$$P_1\Phi(P_1)A_{12} + \Phi(I)A_{12} = 0. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned}2\Phi(A_{12}) &= \Phi([P_1 \bullet P_1, A_{12}]_*) \\ &= [\Phi(P_1) \bullet P_1, A_{12}]_* + [P_1 \bullet \Phi(P_1), A_{12}]_* + [P_1 \bullet P_1, \Phi(A_{12})]_* \\ &= [\Phi(P_1)P_1 + P_1\Phi(P_1)^*, A_{12}]_* + [P_1\Phi(P_1) + \Phi(P_1)P_1, A_{12}]_* + [2P_1, \Phi(A_{12})]_* \\ &= \Phi(P_1)A_{12} + P_1\Phi(P_1)^*A_{12} - A_{12}\Phi(P_1)P_1 + P_1\Phi(P_1)A_{12} \\ &\quad + \Phi(P_1)A_{12} - A_{12}\Phi(P_1)^*P_1 + 2P_1\Phi(A_{12}) - 2\Phi(A_{12})P_1.\end{aligned}$$

Multiplying both sides of the above equation by P_1 and P_2 from the left and right, respectively, we get that

$$3P_1\Phi(P_1)A_{12} + P_1\Phi(P_1)^*A_{12} = 0. \tag{2.2}$$

Finally,

$$
\begin{aligned}
2\Phi(A_{12}) &= \Phi([P_1 \bullet I, A_{12}]_*) \\
&= [\Phi(P_1) \bullet I, A_{12}]_* + [P_1 \bullet \Phi(I), A_{12}]_* + [P_1 \bullet I, \Phi(A_{12})]_* \\
&= [\Phi(P_1) + \Phi(P_1)^*, A_{12}]_* + [2P_1\Phi(I), A_{12}]_* + [2P_1, \Phi(A_{12})]_* \\
&= (\Phi(P_1) + \Phi(P_1)^*)A_{12} - A_{12}(\Phi(P_1) + \Phi(P_1)^*) + 2\Phi(I)A_{12} + 2P_1\Phi(A_{12}) - 2\Phi(A_{12})P_1.
\end{aligned}
$$

Multiplying both sides of the above equation by P_1 and P_2 from the left and right, respectively, by Claim 4, we get that

$$
P_1\Phi(P_1)A_{12} + P_1\Phi(P_1)^*A_{12} + 2\Phi(I)A_{12} = 0. \tag{2.3}
$$

It follows from (2.2) and (2.3) that

$$
P_1\Phi(P_1)A_{12} - \Phi(I)A_{12} = 0. \tag{2.4}
$$

Now, by (2.1) and (2.4), we have $P_1\Phi(P_1)A_{12} = 0$; i.e., $P_1\Phi(P_1)P_1AP_2 = 0$ for all $A \in \mathcal{A}$. It follows from (♣) that $P_1\Phi(P_1)P_1 = 0$. Similarly, we can prove that $P_2\Phi(P_2)P_2 = 0$.

Claim 6. $\Phi(I) = 0$.

By Claims 2, 4, and 5, we can get that

$$
\Phi(I) = \Phi(P_1) + \Phi(P_2) = P_1\Phi(P_1)P_2 + P_2\Phi(P_1)P_1 + P_1\Phi(P_2)P_2 + P_2\Phi(P_2)P_1 = 0.
$$

Claim 7. $\Phi([A,B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$ *for all* $A, B \in \mathcal{A}$.

It follows from Claims 2 and 6 that

$$
\begin{aligned}
2\Phi([A,B]_*) = \Phi([I \bullet A, B]_*) &= [\Phi(I) \bullet A, B]_* + [I \bullet \Phi(A), B]_* + [I \bullet A, \Phi(B)]_* \\
&= [2\Phi(A), B]_* + [2A, \Phi(B)]_* = 2([\Phi(A), B]_* + [A, \Phi(B)]_*),
\end{aligned}
$$

which implies that $\Phi([A,B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_*$.

Claim 8. $\Phi(A^*) = \Phi(A)^*$ *for all* $A \in \mathcal{A}$.

For every $A \in \mathcal{A}$, by Claims 2, 6, and 7, we have

$$
\Phi(A) - \Phi(A^*) = \Phi([A, I]_*) = [\Phi(A), I]_* = \Phi(A) - \Phi(A)^*.
$$

Hence $\Phi(A^*) = \Phi(A)^*$.

Claim 9. $\Phi(iI) = 0$.

By Claims 2 and 8, we can get $\Phi(iI)^* = -\Phi(iI)$. So

$$
0 = -2\Phi(I) = \Phi([iI, iI]_*) = [\Phi(iI), iI]_* + [iI, \Phi(iI)]_* = 4i\Phi(iI),
$$

which implies that $\Phi(iI) = 0$.

Claim 10. $\Phi(iA) = i\Phi(A)$ *for all* $A \in \mathcal{A}$.

It follows from Claims 2 and 9 that

$$
2\Phi(iA) = \Phi(2iA) = \Phi([iI, A]_*) = [\Phi(iI), A]_* + [iI, \Phi(A)]_* = 2i\Phi(A),
$$

and then $\Phi(iA) = i\Phi(A)$.

Claim 11. Φ *is a derivation.*

On the one hand, by Claim 7, we have

$$
\begin{aligned}
\Phi(AB) - \Phi(BA^*) = \Phi(AB - BA^*) &= \Phi([A,B]_*) = [\Phi(A), B]_* + [A, \Phi(B)]_* \\
&= \Phi(A)B - B\Phi(A)^* + A\Phi(B) - \Phi(B)A^*.
\end{aligned} \tag{2.5}
$$

On the other hand, by Claims 2, 7, and 10, we also have

$$
\begin{aligned}
-\Phi(AB) - \Phi(BA^*) &= \Phi([iA, iB]_*) = [i\Phi(A), iB]_* + [iA, i\Phi(B)]_* \\
&= -\Phi(A)B - B\Phi(A)^* - A\Phi(B) - \Phi(B)A^*.
\end{aligned} \tag{2.6}
$$

From (2.5) and (2.6) we obtain $\Phi(AB) = \Phi(A)B + A\Phi(B)$.

Now, by Claims 2, 8, and 11, we have proved that Φ is an additive $*$-derivation. This completes the proof of Theorem 2.1. □

3. Corollaries

In this section, we present some corollaries of the main result. An algebra $\mathcal{A}$ is called *prime* if $A\mathcal{A}B = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. It is easy to see that prime $*$-algebras satisfy (♠) and (♣). So we have the following corollary.

Corollary 3.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and let P be a nontrivial projection in $\mathcal{A}$. Then Φ is a nonlinear mixed Jordan triple $*$-derivation on $\mathcal{A}$ if and only if Φ is an additive $*$-derivation.*

We denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ and by $\mathcal{F}(H) \subseteq B(\mathcal{H})$, the subalgebra of all bounded finite rank operators. A subalgebra $\mathcal{A} \subseteq B(\mathcal{H})$ is called a *standard operator algebra* if $\mathcal{A}$ includes $\mathcal{F}(H)$. Now we have the following corollary.

Corollary 3.2. *Let $\mathcal{A}$ be a standard operator algebra on an infinite-dimensional complex Hilbert space $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then $\Phi : \mathcal{A} \to \mathcal{A}$ is a nonlinear mixed Jordan triple $*$-derivation if and only if Φ is a linear $*$-derivation. Moreover, there exists an operator $T \in B(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\Phi(A) = AT - TA$ for all $A \in A$, i.e., Φ is inner.*

Proof. Since $\mathcal{A}$ is prime, we have that Φ is an additive $*$-derivation. It follows from [28] that Φ is a linear inner derivation, i.e., there exists an operator $S \in B(\mathcal{H})$ such that $\Phi(A) = AS - SA$. Since $\Phi(A^*) = \Phi(A)^*$, we have

$$A^*S - SA^* = \Phi(A^*) = \Phi(A)^* = -A^*S^* + S^*A^*$$

for all $A \in A$. Hence

$$A^*(S + S^*) = (S + S^*)A^*,$$

and then $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Let

$$T = S - \frac{1}{2}\lambda I.$$

It is easy to see that $T + T^* = 0$ such that $\Phi(A) = AT - TA$. □

A von Neumann algebra $\mathcal{M}$ is a weakly closed self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. Note that $\mathcal{M}$ is a *factor von Neumann algebra* if its center only contains the scalar operators. It is well known that a factor von Neumann algebra is prime. So we have the following corollary:

Corollary 3.3. *Let $\mathcal{M}$ be a factor von Neumann algebra with $\dim \mathcal{M} \geq 2$. Then $\Phi : \mathcal{M} \to \mathcal{M}$ is a nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation.*

It is shown in [2] and [18] that if a von Neumann algebra has no central summands of type I_1, then $\mathcal{M}$ satisfies (♠) and (♣). Now we have the following corollary:

Corollary 3.4. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. Then $\Phi : \mathcal{M} \to \mathcal{M}$ is a nonlinear mixed Jordan triple $*$-derivation if and only if Φ is an additive $*$-derivation.*

Acknowledgment. The authors thank the referee for the very thorough reading of the paper and many helpful comments.

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