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IDENTITIES AND QUASI-IDENTITIES OF POINTED ALGEBRAS

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Abstract: Each pointed enrichment of an algebra can be regarded as the same algebra with an additional finite set of constant operations. An algebra is pointed whenever it is a pointed enrichment of some algebra. We show that each pointed enrichment of a finite algebra in a finitely axiomatizable residually very finite variety admits a finite basis of identities. We also prove that every pointed enrichment of a finite algebra in a directly representable quasivariety admits a finite basis of quasi-identities. We give some applications of these results and examples.

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1. Introduction

Each pointed enrichment of an algebra can be regarded as the same algebra with an additional finite set of constant operations. Formally, a *pointed enrichment* of a given algebra $\mathbf{A}$ of signature σ is the algebra $\mathbf{A}^c$ of signature $\sigma \cup C$ such that C is a finite set of constant operations and the σ-reduct of $\mathbf{A}^c$ coincides with $\mathbf{A}$. An algebra is called *pointed* whenever it is a pointed enrichment of some algebra. For instance, a pointed group is an algebra of signature $\sigma \cup C$, where C is a finite set of constants and the σ-reduct of this algebra is a group.

Say that an algebra admits a *finite basis of* (*quasi-*)*identities* whenever the (quasi)variety generated by this algebra is finitely axiomatizable. The equational and quasiequational theories of an algebra and its pointed enrichment can be rather different. Indeed, by the Oates–Powell Theorem [1], each finite group admits a finite basis of identities. However, according to Bryant [2], there is a finite pointed group lacking finite bases of identities. A similar result holds for quasiequational theories; namely, there exists a finite lattice with a finite basis of quasi-identities whose one pointed enrichment lacks finite bases of quasi-identities [3]. At the same time, there is a finite lattice lacking finite bases of quasi-identities and having a pointed enrichment with a finite basis of quasi-identities [3]. The following question is natural: *Which pointed enrichments of finite algebras preserve the property of admitting finite bases of identities and* (*or*) *quasi-identities?* Note that this problem for groups is considered in [2, 4, 5] and still remains open; see Section 5.

The main goal of this article is to find conditions on a variety and quasivariety which enable us to find some satisfactory solutions to the problem. We show that each pointed enrichment of a finite algebra lying in a finitely axiomatizable residually very finite variety admits a finite basis of identities. This class of algebras includes, for instance, the finite groups in which all nilpotent subgroups are abelian. We prove also that each pointed enrichment of a finite algebra lying in a directly representable quasivariety admits a finite basis of quasi-identities. Among these algebras we find, for instance, finite abelian groups and finite boolean algebras. Also, we present some applications of these results.

2. Definitions and Auxiliary Results

Recall the basic definitions and results on varieties and quasivarieties which we will need. See [6, 7] for additional information about the main concepts of universal algebra to be introduced below and used in this article.

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We assume that all algebras and classes of algebras under study are of the same signature σ unless stated otherwise. We assume also that all classes of algebras are closed under isomorphic copies.

Given a class $\mathscr{K}$ of algebras, denote by $\mathbf{S}(\mathscr{K})$ ($\mathbf{H}(\mathscr{K})$) the class of all subalgebras (respectively homomorphic images) of the algebras in $\mathscr{K}$, while by $\mathbf{P}(\mathscr{K})$ (as well $\mathbf{P}_U(\mathscr{K})$ and $\mathbf{P}_s(\mathscr{K})$), the class of all direct products (respectively ultraproducts and subdirect products) of algebras in $\mathscr{K}$.

A *quasivariety* of algebras is a class of algebras $\mathscr{K}$ closed under subalgebras, direct products, and ultraproducts; i.e.,

$$\mathscr{K} = \mathbf{S}(\mathscr{K}) = \mathbf{P}(\mathscr{K}) = \mathbf{P}_U(\mathscr{K}).$$

In other words, a quasivariety is a class of algebras axiomatizable by a set of quasi-identities. A *quasi-identity* is a universal Horn sentence with nonempty positive part, i.e., a sentence of the form

$$(\forall \overline{x})[p_1(\overline{x}) \approx q_1(\overline{x}) \wedge \dots \wedge p_n(\overline{x}) \approx q_n(\overline{x})] \to p(\overline{x}) \approx q(\overline{x}),$$

where $p, q, p_1, q_1, \dots, p_n, q_n$ are terms of signature σ. The smallest quasivariety $\mathscr{K}$ including a given class $\mathscr{G}$ is the class $\mathbf{SPP}_U(\mathscr{G})$. If $\mathscr{G}$ is a finite family of finite algebras then $\mathscr{K} = \mathbf{SP}(\mathscr{G})$ and the quasivariety $\mathscr{K}$ is called *finitely generated*.

A *variety* is a quasivariety closed under homomorphic images. According to Birkhoff's Theorem [8], a variety is a class of algebras axiomatizable by a set of identities, where by an identity we understand a sentence of the form $(\forall \overline{x})[s(\overline{x}) \approx t(\overline{x})]$ for some terms $s(\overline{x})$ and $t(\overline{x})$. The smallest variety $\mathbf{V}(\mathscr{K})$ including a class $\mathscr{K}$ of algebras is the class $\mathbf{HSP}(\mathscr{K})$ [8]. A variety $\mathscr{K}$ is finitely generated whenever there exists a finite set $\mathscr{G}$ of finite algebras with $\mathscr{K} = \mathbf{V}(\mathscr{G})$.

A quasivariety (or variety) $\mathscr{K}$ admits a finite basis of quasi-identities (respectively identities) whenever $\mathscr{K}$ is finitely axiomatizable; i.e., by the Compactness Theorem there exists a finite set of quasi-identities (identities) Σ such that

$$\mathscr{K} = \operatorname{Mod}(\Sigma) = \{\mathbf{A} \mid \mathbf{A} \models \varphi \quad \text{for all } \varphi \in \Sigma\}.$$

Suppose that $\mathscr{K}$ is a quasivariety. A congruence α of some algebra $\mathbf{A}$ is called a *$\mathscr{K}$-congruence* whenever $\mathbf{A}/\alpha \in \mathscr{K}$. Observe that $\mathbf{A} \in \mathscr{K}$ if and only if the smallest congruence 0_A on $\mathbf{A}$ is a $\mathscr{K}$-congruence. The set $\operatorname{Con}_{\mathscr{K}} \mathbf{A}$ of all $\mathscr{K}$-congruences on $\mathbf{A}$ forms an algebraic lattice that is a lower subsemilattice of the lattice of congruences $\operatorname{Con} \mathbf{A}$.

An algebra $\mathbf{A}$ is called *trivial* whenever it consists of one element.

For a quasivariety $\mathscr{K}$ a nontrivial algebra $\mathbf{A} \in \mathscr{K}$ is called *subdirectly $\mathscr{K}$-irreducible* whenever the smallest congruence 0_A is completely meet irreducible in $\operatorname{Con}_{\mathscr{K}} \mathbf{A}$. By Birkhoff's Theorem for quasivarieties, each algebra in $\mathscr{K}$ is a subdirect product of subdirectly $\mathscr{K}$-irreducible algebras [9]; see also [7]. In particular, for a finitely generated quasivariety $\mathbf{Q}(\mathbf{A})$ each subdirectly $\mathbf{Q}(\mathbf{A})$-irreducible algebra is isomorphic to some subalgebra of $\mathbf{A}$. Denote the class of all $\mathscr{K}$-subdirectly irreducible algebras in a quasivariety $\mathscr{K}$ by $\mathscr{K}_{RSI}$. Say also that $\mathscr{K}_{RSI}$ is the class of all relatively subdirectly irreducible algebras. For a variety $\mathscr{V}$ the class $\mathscr{V}_{SI}$ is the class of all subdirectly irreducible algebras in $\mathscr{V}$.

Take some algebra $\mathbf{A}$ and $a, b \in \mathbf{A}$. The smallest $\mathscr{K}$-congruence including the pair (a, b) is called a *principal $\mathscr{K}$-congruence* or a *relative principal congruence* and denoted by $\theta_{\mathscr{K}}(a, b)$.

Say that a quasivariety $\mathscr{R}$ has *definable relative principal congruences* whenever there exists an existentially positive formula $\Gamma(x, y, u, v)$ with free variables x, y, u, and v such that

$$\theta_{\mathscr{R}}(a, b) = \{(x, y) \in A^2 \mid \mathbf{A} \models \Gamma(x, y, a, b)\}$$

for all $\mathbf{A} \in \mathscr{R}$ and $a, b \in \mathbf{A}$. For varieties this coincides with the definition of "having definable principal congruences."

A locally finite quasivariety $\mathscr{R}$ is *directly representable* whenever there exists a finite subset $\mathscr{C} \subset \mathscr{R}$ of finite algebras such that each finite algebra in $\mathscr{R}$ is isomorphic to a direct product of algebras in $\mathscr{C}$.

Theorem 2.1 [10]. *Each subquasivariety of a directly representable quasivariety has definable relative principal congruences.*

Theorem 2.2 [10, 11]. *Suppose that the quasivariety $\mathscr{K}$ has definable relative principal congruences and the class $\mathscr{K}_{RSI}$ of relatively subdirectly irreducible algebras in $\mathscr{K}$ is finitely axiomatizable. Then $\mathscr{K}$ admits a finite basis of quasi-identities.*

These theorems yield the following corollary.

Corollary 2.3. *Each finite algebra in a directly representable quasivariety admits a finite basis of quasi-identities.*

Theorem 2.2 for varieties and the property that each directly representable variety has a definable principal congruence are established in [12].

3. Pointed Enrichments of Algebras in a Residually Very Finite Variety

Here we provide the conditions for a finite basis of identities of a finite algebra to be preserved under its every point enrichment.

Say that a variety $\mathscr{V}$ is *residually at most* n whenever all subdirectly irreducible algebras in $\mathscr{V}$ are of cardinality at most n. A variety $\mathscr{V}$ is *residually very finite* or *has a finite residual bound* whenever $\mathscr{V}$ is residually at most n for some $0 < n < \omega$.

Lemma 3.1. *If three algebras $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{B}$ satisfy $\mathbf{B} \leq_s \mathbf{A}_1 \times \mathbf{A}_2$ then $\mathbf{B}^c \leq_s \mathbf{A}_1^{c_1} \times \mathbf{A}_2^{c_2}$ for some pointed enrichments $\mathbf{A}_1^{c_1}$ of $\mathbf{A}_1$ and $\mathbf{A}_2^{c_2}$ of $\mathbf{A}_2$.*

Proof. The definition of congruence implies that a binary relation α on an algebra $\mathbf{A}$ is a congruence if and only if α is a congruence on some pointed enrichment of $\mathbf{A}$. Consequently, $\operatorname{Con}\mathbf{A} = \operatorname{Con}\mathbf{A}^c$ for every pointed enrichment $\mathbf{A}^c$ of $\mathbf{A}$. Since $\ker\pi_1 \cap \ker\pi_2 = \Delta_{\mathbf{B}}$; it follows that $\mathbf{B}^c \leq_s \mathbf{B}^c/\ker\pi_1 \times \mathbf{B}^c/\ker\pi_2$, where π_i is the projection of $\mathbf{B}$ onto $\mathbf{A}_i$ for $i = 1, 2$. □

Here is one of the main results of this article.

Theorem 3.2. *If a finite algebra $\mathbf{A}$ lies in a finitely axiomatizable residually very finite variety then each pointed enrichment of $\mathbf{A}$ admits a finite basis of identities.*

Proof. Consider a variety $\mathscr{U}$ with a finite basis of identities Σ and a finite residual bound $n > 0$. Assume that $\mathbf{A} \in \mathscr{U}$ and $\mathscr{V}^c = \mathbf{V}(\mathbf{A}^c)$. It is not difficult to see that $\mathbf{A}^c \models \Sigma$ and consequently $\mathscr{V}^c \models \Sigma$.

Let us prove firstly that the variety $\mathscr{V}^c$ is residually very finite. Take a subdirectly irreducible algebra $\mathbf{B}^c$ in $\mathscr{V}^c$ and the σ-reduct $\mathbf{B}$ of $\mathbf{B}^c$. Since $\operatorname{Con}\mathbf{B} = \operatorname{Con}\mathbf{B}^c$, it follows that $\mathbf{B}$ is a subdirectly irreducible algebra. Since $\mathbf{B}^c \in \mathscr{V}^c$; therefore, $\mathbf{B}^c \models \Sigma$. Consequently, $\mathbf{B} \models \Sigma$. Hence, $\mathbf{B} \in \mathscr{U}$. Since $\mathbf{B}$ is subdirectly irreducible and $\mathscr{U}$ is residually at most n, we obtain $|\mathbf{B}^c| \leq n$ for every subdirectly irreducible algebra $\mathbf{B}^c$ in $\mathscr{V}^c$. Since $\mathscr{V}^c$ is a locally finite variety, the number of algebras of cardinality at most n is finite. So, $\mathscr{V}^c$ has finitely many subdirectly irreducible algebras, each of which is of cardinality at most n, i.e., $\mathscr{V}^c$ is residually very finite.

To verify that $\mathscr{V}^c$ is finitely axiomatizable, assume the contrary. This means that there exist an infinite set of pointed algebras $\{\mathbf{A}_i^c \mid i \in I\}$ and an ultrafilter D over I such that $\mathbf{A}_i^c \notin \mathscr{V}^c$ and $\prod \mathbf{A}_i^c/D \in \mathscr{V}^c$. Since $\mathscr{V}^c \models \Sigma$, for every $\varphi \in \Sigma$ we have $\{i \in I \mid \mathbf{A}_i^c \models \varphi\} \in D$ by Los's Theorem. Since Σ is finite, we have

$$\bigcap \big\{ \{i \in I \mid \mathbf{A}_i^c \models \varphi\} \mid \varphi \in \Sigma \big\} = \{i \in I \mid \mathbf{A}_i^c \models \Sigma\} \in D.$$

Consequently, $\{i \in I \mid \mathbf{A}_i \models \Sigma\} \in D$. Thus, we may assume that $\mathbf{A}_i^c \models \Sigma$ for all $i \in I$ and, as a corollary, $\mathbf{A}_i \in \mathscr{U}$ for all $i \in I$. Since $\mathbf{A}_i^c \notin \mathscr{V}^c$, there exists a finitely generated subalgebra $\mathbf{B}_i^c$ of the algebra $\mathbf{A}_i^c$ such that $\mathbf{B}_i^c \notin \mathscr{V}^c$. Since $\mathscr{V}(A)$ is locally finite, $\mathbf{B}_i$ is a finite algebra, whence so is $\mathbf{B}_i^c$. Moreover, since $\prod \mathbf{B}_i^c/D$ is a subalgebra of $\prod \mathbf{A}_i^c/D$, we infer that $\prod \mathbf{B}_i^c/D \in \mathscr{V}^c$. Thus, we may assume that $\mathbf{A}_i^c$ is a finite algebra for all $i \in I$. Since $\mathscr{U}$ is residually very finite and $\mathbf{A}_i \in \mathscr{U}$, it follows that

$$\mathbf{A}_i \leq_s \mathbf{B}_1^{i_1} \times \cdots \times \mathbf{B}_k^{i_k}$$

for some integers $i_1, \dots, i_k \geq 0$ and $\{\mathbf{B}_1, \dots, \mathbf{B}_k\} = \mathscr{U}_{SI}$. Denote the set of all pointed enrichments of algebras in $\mathscr{U}_{SI}$ by $\mathscr{P} = \{\mathbf{C}_1, \dots, \mathbf{C}_m\}$. Lemma 3.1 yields

$$\mathbf{A}_i^c \leq_s \mathbf{C}_1^{i_1} \times \dots \times \mathbf{C}_m^{i_m}$$

for some $i_1, \dots, i_m \geq 0$. This implies that for each $i \in I$ there exists a tuple of numbers $i_1, \dots, i_m \geq 0$ defining the algebra $\mathbf{A}_i^c$. Denote the index s element of tuple i by i_s.

Given $s \leq m$, put $I_s = \{i \in I \mid i_s \neq 0\}$. Verify that $S = \{s \leq m \mid I_s \in D\}$ is nonempty. Indeed, suppose that $I_s \notin D$ for all $s \leq m$. Then $I - I_s \in D$ for all $s \leq m$. Consequently, $\bigcap\{I - I_s \mid s \leq m\} \in D$; in particular, $\bigcap\{I - I_s \mid s \leq m\} \neq \varnothing$. This means that there exists $i \in \bigcap\{I - I_s \mid s \leq m\}$ such that $i_1 = \dots = i_m = 0$. Consequently, $\mathbf{A}_i \leq_s \mathbf{C}_1^0 \times \dots \times \mathbf{C}_m^0$, meaning that $\mathbf{A}_i$ is a trivial algebra. This contradicts the choice of $\mathbf{A}_i$. Thus, $\{s \leq m \mid I_s \in D\} = S \neq \varnothing$.

Put $I' = \bigcap\{I_s \in D \mid s \in S\}$. Since S is finite, we have $I' \in D$. The properties of ultraproducts yield $\prod_{i \in I} \mathbf{A}_i^c/D \cong \prod_{i \in I'} \mathbf{A}_i^c/D'$, where $D' = \{H \cap I' \mid H \in D\}$ is an ultrafilter over I'. As shown in [13], ultraproducts preserve finite subdirect irreducibility, and so for all algebras $\{\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i \mid I\}$ and ultrafilters D over I we can infer from $\mathbf{A}_i \leq_s \mathbf{B}_i \times \mathbf{C}_i$ that

$$\prod_{i \in I} \mathbf{A}_i/D \leq_s \prod_{i \in I} \mathbf{B}_i/D \times \prod_{i \in I} \mathbf{C}_i/D.$$

Hence,

$$\prod_{i \in I'} \mathbf{A}_i^c/D' \leq_s \prod_{i \in I'} \mathbf{C}_{s_1}^{i_{s_1}}/D' \times \dots \times \prod_{i \in I'} \mathbf{C}_{s_r}^{i_{s_r}}/D',$$

where $s_1, \dots, s_r \in S$ and $r = |S|$.

The construction of I_s and I' yields $i_{s_k} \neq 0$ for all $i \in I'$ and $1 \leq k \leq r$. Thus, $\prod_{i \in I'} \mathbf{C}_{s_k}^{i_{s_k}}/D'$ is a nontrivial algebra. Moreover, since the ultraproduct of homomorphic images is the homomorphic image of the ultraproduct, $\mathbf{C}_{s_k}$ is the homomorphic image of $\prod_{i \in I'} \mathbf{C}_{s_k}^{i_{s_k}}/D'$. Since $\prod_{i \in I'} \mathbf{C}_{s_k}^{i_{s_k}}/D'$ is the homomorphic image of $\prod_{i \in I'} \mathbf{A}_i^c/D'$ and $\prod_{i \in I'} \mathbf{A}_i^c/D' \in \mathscr{V}^c$, it follows that $\prod_{i \in I'} \mathbf{C}_{s_k}^{i_{s_k}}/D' \in \mathscr{V}^c$ for all $s_k \in S$. Thus, $\mathbf{C}_{s_k} \in \mathscr{V}^c$. Since $\mathbf{A}_i^c$ is a subdirect product of algebras of type $\mathbf{C}_{s_k}$, we have $\mathbf{A}_i^c \in \mathscr{V}^c$ for $i \in I_s$. This contradicts the choice of $\mathbf{A}_i^c$. Consequently, the variety $\mathscr{V}^c$ admits a finite basis of identities. □

Note that the requirement of residually very finite variety in Theorem 3.2 is a necessary condition. Indeed, the group reduct $\mathbf{B}$ of the finite pointed Bryant group $\mathbf{B}^c$ constructed in [2] includes a nonabelian nilpotent subgroup. According to [14] the variety $\mathbf{V}(\mathbf{B})$ is not residually very finite, and by the Oates–Powell Theorem $\mathbf{V}(\mathbf{B})$ is finitely axiomatizable. However, the finite Bryant group $\mathbf{B}^c$ lacks a finite basis of identities.

The following example demonstrates a finite algebra $\mathbf{M}$ with a pointed enrichment $\mathbf{M}^c$ such that the variety $\mathbf{V}(\mathbf{M})$ is not residually very finite and lacks finite bases of identities. At the same time $\mathbf{V}(\mathbf{M}^c)$ is residually very finite and admits a finite basis of identities.

Example 3.3. Consider the four-element algebra $\mathbf{M} = \langle\{0, a, b, c\}; \cdot, m\rangle$ with one binary operation $\cdot$ and one quaternary operation m defined as

$$x \cdot y = \begin{cases} c, & \text{if } x = c \text{ or } y = c, \\ a, & \text{if } x = a \text{ and } y = b, \\ b, & \text{if } x = b \text{ and } y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$m(x, y, z, u) = \begin{cases} u, & \text{if } u \neq c, \\ f(x, y, z), & \text{if } u = c, \end{cases}$$

where $f(x, y, z)$ is a majority function on $\{0, a, b, c\}$; i.e.,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x$$

for all $x \in \{0, a, b, c\}$. Consider the pointed enrichment $\mathbf{M}^c = \langle \{0, a, b, c\}; \cdot, m, \overline{c} \rangle$ of $\mathbf{M}$, where $\nu(\overline{c}) = c$. Then the variety $\mathbf{V}(\mathbf{M})$ is not residually very finite and lacks a finite basis of identities. At the same time $\mathbf{V}(\mathbf{M}^c)$ is residually very finite and has a finite basis of identities.

PROOF. Verify firstly that $\mathbf{V}(\mathbf{M}^c)$ is residually very finite and admits a finite basis of identities. Since $m(x, y, z, \overline{c})$ is a term of signature $(\cdot, m, \overline{c})$, the definition of m implies that $m(x, y, z, \overline{c})$ is a majority term of this signature for the variety $\mathbf{V}(\mathbf{M}^c)$. Consequently, $\mathscr{V}(\mathbf{M}^c)$ is a congruence-distributive variety [15]; see also [6]. Johnson's Lemma on subdirectly irreducible algebras in a congruence-distributive variety [15] yields $\mathscr{V}(\mathbf{M}^c)_{SI} \subseteq \mathbf{HS}(\mathbf{M}^c)$. Since $\mathbf{M}^c$ is a finite algebra, $\mathbf{V}(\mathbf{M}^c)$ is residually very finite. Since $\mathscr{V}(\mathbf{M}^c)$ is congruence-distributive, Baker's Theorem [16] shows that $\mathscr{V}(\mathbf{M}^c)$ is finitely axiomatizable.

Verify that the variety $\mathscr{V}(\mathbf{M})$ is not residually very finite. It is not difficult to see that $\mathbf{G} = \langle \{0, a, b\}; \cdot, m \rangle$ is a proper subalgebra of $\mathbf{M}$. Hence,

$$\mathscr{V}(\mathbf{G}) \subseteq \mathscr{V}(\mathbf{M}).$$

The definition of m implies that the identity $(\forall xyzu)[m(x, y, z, u) \approx u]$ is true on $\mathbf{G}$ and false on $\mathbf{M}$. Consequently,

$$\mathscr{V}(\mathbf{G}) \subset \mathscr{V}(\mathbf{M}) \text{ and } \mathscr{V}(\mathbf{G}) \models (\forall xyzu)[m(x, y, z, u) \approx u].$$

The identity $(\forall xyzu)[m(x, y, z, u) \approx u]$ also implies the following property of the variety $\mathscr{V}(\mathbf{G})$: For every term $t(\overline{x})$ of signature $(\cdot, m)$ there exists a term $t'(\overline{x})$ of signature $(\cdot)$ such that

$$\mathscr{V}(\mathbf{G}) \models (\forall \overline{x})[t(\overline{x}) \approx t'(\overline{x})].$$

Indeed, we can construct the term t' from t by replacing all subterms of the form $m(t_1, t_2, t_3, t_4)$ in the expression for t with t_4. Since $\approx$ is a transitive relation; therefore, each identity of signature $(\cdot, m)$ is logically equivalent to some identity of signature $(\cdot)$ relative to the identity $(\forall xyzu)[m(x, y, z, u) \approx u]$. As a corollary, we infer that the set Σ of all identities true on $\mathbf{G}$ is logically equivalent to the set of identities $\Sigma' \cup \{(\forall xyzu)[m(x, y, z, u) \approx u]\}$, where Σ' is the set of all identities true on the $(\cdot)$-reduct of the algebra $\mathbf{G}$. In other words, $\mathbf{A} \in \mathbf{V}(\mathbf{G})$ if and only if $\mathbf{A}^{(\cdot)} \in \mathbf{V}(\mathbf{G}^{(\cdot)})$, where $\mathbf{A}^{(\cdot)}$ and $\mathbf{G}^{(\cdot)}$ are the $(\cdot)$-reducts of the algebras $\mathbf{A}$ and $\mathbf{G}$ respectively.

Given a groupoid $\mathbf{A} = \langle A; \cdot \rangle$, denote by $\mathbf{A}^+$ the algebra of signature $(\cdot, m)$ in which the $(\cdot)$-reduct coincides with $\mathbf{A}$ and $m(a, b, c, d) = d$ for all $a, b, c, d \in A$. It is not difficult to see that $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}^+$ and $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ if and only if $\mathbf{B}^+$ is a subalgebra of $\mathbf{A}^+$.

Observe that $\mathbf{G}^{(\cdot)} = \langle \{0, a, b\}; \cdot \rangle$ is Murskii's groupoid. It is shown in [17] (see also [18]) that the variety $\mathbf{V}(\mathbf{G}^{(\cdot)})$ generated by Murskii's groupoid is not residually very finite. Since $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}^+$ for $\mathbf{A} \in \mathbf{V}(\mathbf{G}^{(\cdot)})$, it follows that $\mathbf{A}^+$ is subdirectly irreducible for every subdirectly irreducible groupoid $\mathbf{A} \in \mathbf{V}(\mathbf{G}^{(\cdot)})$. Hence, $\mathbf{V}(\mathbf{G})$ is not residually very finite. Since $\mathbf{V}(\mathbf{G}) \subset \mathbf{V}(\mathbf{M})$, the variety $\mathbf{V}(\mathbf{M})$ is not residually very finite either.

It remains to verify that $\mathbf{V}(\mathbf{M})$ is not finitely based. Recall that a locally finite variety $\mathscr{V}$ is called *inherently nonfinitely based* whenever every locally finite variety including $\mathscr{V}$ is not finitely based. According to [18], the variety $\mathscr{V}$ is inherently nonfinitely based if and only if for every $n > 0$ there exists an $(n+1)$-generated infinite algebra $\mathbf{A}$ such that every n-generated subalgebra of it belongs to $\mathscr{V}$. As [19] established (see also [18]) the variety $\mathbf{V}(\mathbf{G}^{(\cdot)})$ generated by Murskii's groupoid is inherently nonfinitely based. This means that for every $n > 0$ there exists an $(n+1)$-generated infinite groupoid $\mathbf{A}_n$ such that every n-generated subgroupoid $\mathbf{G}_n$ of it belongs to $\mathbf{V}(\mathbf{G}^{(\cdot)})$. As we observed above, $\mathbf{G}_n \in \mathbf{V}(\mathbf{G}^{(\cdot)})$ if and only if $\mathbf{G}_n^+ \in \mathbf{V}((\mathbf{G}^{(\cdot)})^+)$. Since $\mathbf{A}_n^+$ is not locally finite, it follows that $\mathbf{A}_n^+ \notin \mathbf{V}((\mathbf{G}^{(\cdot)})^+))$.

Since $(\mathbf{G}^{(\cdot)})^+ = \mathbf{G}$, for every $n > 0$ there exists an $(n+1)$-generated infinite algebra $\mathbf{A}_n^+$ such that every n-generated subalgebra $\mathbf{G}_n^+$ of it belongs to $\mathbf{V}(\mathbf{G})$. Thus, by definition $\mathbf{V}(\mathbf{G})$ is inherently nonfinitely based. Therefore, $\mathbf{V}(\mathbf{M})$ is not finitely based, so that $\mathbf{G} \leq \mathbf{M}$. $\square$

Johnson posed the following problem (see [20]): *Is it true that every residually very finite variety of algebras of finite signature is finitely axiomatizable?* The problem still remains open, but it is solved for many classes of algebras. One of the most general results belongs to Kearnes, Szendrei, and Willard [20].

Theorem 3.4 [20]. *Suppose that a residually very finite variety $\mathscr{V}$ of algebras has the difference term. Then $\mathscr{V}$ is finitely axiomatizable.*

For the definition of the difference term and its properties see [20]. Theorem 3.4 generalizes McKenzie's Theorem [21] on the finite basedness of congruence-modular varieties and Willard's Theorem on the finite basedness of congruence meet-semidistributive varieties [22]. Adding McKenzie's Theorem on the finite basedness of a variety with definable principal congruences [12] (see also [10, 11]) we obtain the following corollary:

Corollary 3.5. *Suppose that a residually very finite variety $\mathscr{V}$ of algebras satisfies one of the following conditions:*

- *$\mathscr{V}$ has the difference term, in particular, it is a congruence-modular variety or a congruence meet-semidistributive variety;*
- *$\mathscr{V}$ has definable principal congruences.*

Then each pointed enrichment of a finite algebra of $\mathscr{V}$ admits a finite basis of identities.

We present some applications of Corollary 3.5 in Section 5.

4. Pointed Enrichments of Algebras in a Directly Representable Quasivariety

This section gives some conditions for a finite basis of quasi-identities to be preserved under pointed enrichments.

Theorem 4.1. *If $\mathbf{A}$ is a finite algebra lying in a directly representable quasivariety then each pointed enrichment of $\mathbf{A}$ admits a finite basis of quasi-identities.*

Proof. Verify firstly that each pointed enrichment of a finite algebra in a directly representable quasivariety itself belongs to some directly representable quasivariety.

Take a directly representable quasivariety $\mathscr{K}$. By definition, there is a finite set $S = \{\mathbf{A}_1, \dots, \mathbf{A}_m\}$ of finite algebras in $\mathscr{K}$ such that each finite algebra in $\mathscr{K}$ is isomorphic to a direct product of algebras in S. Denote the set of all pointed enriched algebras lying in S by S^c. Since both S and the set of constant symbols are finite, so is S^c. Verify that the quasivariety $\mathscr{K}^c$ generated by S^c is directly representable.

Take a finite algebra $\mathbf{A}^c \in \mathscr{K}^c$. Since S^c is finite and $\mathscr{K}^c = \mathbf{SP}(S^c) = \mathbf{P}_s\mathbf{S}(S^c)$, it follows that

$$\mathbf{A}^c \leq_s \mathbf{B}_1^c \times \dots \times \mathbf{B}_n^c$$

for some $\mathbf{B}_i^c \in \mathbf{S}(S^c)$ for $i = 1, \dots, n$. Since $\operatorname{Con} \mathbf{A}^c = \operatorname{Con} \mathbf{A}$, we have

$$\mathbf{A} \leq_s \mathbf{B}_1 \times \dots \times \mathbf{B}_n.$$

It is easy to see that if $\mathbf{B}^c$ is a subalgebra of $\mathbf{D}^c$ then $\mathbf{B}$ is a subalgebra of $\mathbf{D}$. This yields $\mathbf{B}_1, \dots, \mathbf{B}_n \in \mathscr{K}$. Thus, $\mathbf{A} \in \mathscr{K}$. Since $\mathscr{K}$ is directly representable, we infer that

$$\mathbf{A} = \mathbf{C}_1 \times \dots \times \mathbf{C}_k$$

for some $\mathbf{C}_j \in S$ for $0 < j \leq k$. Lemma 3.1 yields

$$\mathbf{A}^c = \mathbf{C}_1^{c_1} \times \dots \times \mathbf{C}_k^{c_k}.$$

Since $\mathbf{C}_1^{c_i} \in S^c$ for all $i \leq k$, we find that $\mathbf{A}^c$ is a direct product of algebras of S^c. This means that $\mathscr{K}^c$ is a directly representable quasivariety.

Thus, a pointed enrichment $\mathbf{A}^c$ of the finite algebra $\mathbf{A}$ belongs to a directly representable quasivariety provided that so does $\mathbf{A}$. Corollary 2.3 implies that $\mathbf{A}^c$ admits a finite basis of quasi-identities. □

Note that every directly representable variety is congruence-permutable [12], therefore congruence-modular, and by definition residually very finite. Therefore, Theorems 3.2 and 4.1 yield the following statement.

Corollary 4.2. *If $\mathscr{V}$ is a directly representable variety then every pointed enrichment of a finite algebra $\mathbf{A} \in \mathscr{V}$ admits a finite basis of identities and quasi-identities.*

In contrast to directly representable quasivarieties, directly representable varieties are well understood; see [6, 12, 14]. We will exhibit an example of directly representable quasivariety which is not a variety.

Example 4.3. Consider the two-element pointed group $\mathbf{G}^c = \langle \{0, 1\}; +, -, \overline{0}, c\rangle$ with $\nu(c) = 1$. The quasivariety $\mathbf{Q}(\mathbf{G}^c)$ is directly representable but not a variety.

Suppose that

$$\mathbf{G} = (\mathbf{G}^c \times \mathbf{G}^c)/\theta((0, 0), (1, 1)).$$

It is not difficult to see that

$$\mathbf{G} \cong \langle \{0, 1\}; +, -, \overline{0}, c\rangle,$$

where $\nu(c) = 0$. Since the quasi-identity $(\forall xy)[\overline{0} \approx c \to x \approx y]$ is true on $\mathbf{G}^c$ but false on $\mathbf{G}$, it follows that

$$\mathbf{Q}(\mathbf{G}^c) \neq \mathbf{V}(\mathbf{G}^c).$$

Verify that the quasivariety $\mathbf{Q}(\mathbf{G}^c)$ is directly representable. Suppose that $\mathbf{A} \in \mathbf{Q}(\mathbf{G}^c)$ is a finite algebra. Then $\mathbf{A} \leq_s (\mathbf{G}^c)^n$ for some $n > 0$. Verify by induction on n that $\mathbf{A} \cong (\mathbf{G}^c)^m$ for some $m \leq n$. For $n = 1$ we have $\mathbf{A} = \mathbf{G}^c$. Assuming that $n > 1$, we obtain $\mathbf{A} \leq_s \mathbf{G}^c \times \mathbf{B}$, where $\mathbf{B} = (\mathbf{G}^c)^{n-1}$. The projection $\pi_B(\mathbf{A})$ of $\mathbf{A}$ onto $\mathbf{B}$ is a subalgebra of $\mathbf{B}$. By induction $\pi_B(\mathbf{A}) \cong (\mathbf{G}^c)^k$ for some $k < n$. If $k < n - 1$ then $\mathbf{A} \leq_s (\mathbf{G}^c)^{k+1}$. By the inductive assumption $\mathbf{A} \cong (\mathbf{G}^c)^m$ for some $0 < m < n$. If $k + 1 = n$ then $\mathbf{A} \leq_s \mathbf{G}^c \times (\mathbf{G}^c)^{n-1}$. Since $\mathbf{G}$ is an abelian group, $|A| = 2^n$ and $|(\mathbf{G}^c)^{n-1}| = 2^{n-1}$. Hence, $\mathbf{A} \cong (\mathbf{G}^c)^n$.

Remarks. As [12] shows, every free algebra of a directly representable variety has permutable congruences and definable principal congruences. Inspecting those proofs, we can see that the same properties hold for directly representable quasivarieties. In other words, each free algebra of a directly representable quasivariety has permutable congruences and definable principal congruences. In this case, according to [23, 10], the variety generated by a directly representable quasivariety is congruence-permutable and has definable principal congruences.

5. Some Applications

This section applies Corollary 3.5 and Theorem 4.1 to some well-known classes of algebras and exhibits a related example.

1. Congruence-modular varieties. Residually finite congruence-modular varieties are described in [14] in terms of the theory of commutators. It shows in particular that each finite algebra of a congruence-modular variety generates a residually finite variety if and only if all its subalgebras satisfy the commutator identity (C1): $x \wedge [y, y] = [x \wedge y, y]$. Let us apply this result to groups and rings.

Groups and rings. It is known that a locally finite variety of groups or rings satisfies the commutators identity (C1) if and only if every finite nilpotent group (ring) of the variety is abelian (see [14]) or respectively has zero multiplication [24]. McKenzie's Theorem on the finite basedness of congruence-modular varieties [21] and Corollary 3.5 yield the following assertions:

Proposition 5.1. *If every nilpotent subgroup of a finite group is abelian then each pointed enrichment of it generates a finitely axiomatizable variety.*

Proposition 5.2. *If every nilpotent subring of a finite ring has zero multiplication then each pointed enrichment of it generates a finitely axiomatizable variety.*

It is established in [5] that each one-pointed enrichment of a finite nilpotent group admits a finite basis of identities (here a one-pointed enrichment means a pointed enrichment whose set of constants C consists of one element). Moreover, as Bryant noted in [2], "One further fact worthy of note is that every finite pointed group (G, g) belongs to the variety generated by a finite pointed group (G^*, g^*) with a finite basis for its laws." Thus, the problem of *which pointed enrichment of a finite group or ring admits a finite basis of identities* faces many obstacles.

We should also note that, as [25] shows, a finite group admits a finite basis of quasi-identities if and only if all its nilpotent subgroups are abelian. We do not know whether a similar result holds for finite pointed groups.

Abelian groups. The fundamental structure theorem for finitely generated abelian groups asserts in particular that the variety generated by a finite abelian group is directly representable. Thus, Proposition 5.1 and Corollary 4.2 imply

Proposition 5.3. *Each finite pointed abelian group admits a finite basis of identities and quasi-identities.*

2. Boolean algebras. Since each finite boolean algebra is isomorphic to a direct product of two-element boolean algebras, Corollary 4.2 implies

Proposition 5.4. *Each finite pointed boolean algebra admits a finite basis of identities and quasi-identities.*

Conjecturally, a finite lattice admits a finite basis of quasi-identities if and only if it generates a relatively congruence-distributive quasivariety [7]. The following example shows that the conjecture is false for congruence-distributive varieties in general: there exists a finite algebra of a congruence-distributive variety with a finite basis of quasi-identities but the quasivariety generated by this algebra is not relatively congruence-distributive.

Example 5.5. The pointed two-atom boolean algebra $\mathbf{B}_2 = \langle 2^2; +, \cdot, ', 0, 1, a\rangle$, with $a \neq 0, 1$, admits a finite basis of quasi-identities and does not generate a relatively congruence-distributive quasivariety.

Proof. Proposition 5.4 shows that $\mathbf{B}_2$ admits a finite basis of quasi-identities. It is known that a relatively subdirectly irreducible algebra of a relatively congruence-distributive quasivariety is subdirectly irreducible [26–28]. Since every relatively subdirectly irreducible algebra in $\mathbf{Q}(\mathbf{B}_2)$ isomorphically embeds into $\mathbf{B}_2$ and the latter lacks proper subalgebras, $\mathbf{B}_2$ is relatively subdirectly irreducible. Moreover, $\mathbf{B}_2$ is not subdirectly irreducible because it is isomorphic to a direct product of two two-element pointed Boolean algebras. Consequently, $\mathbf{B}_2$ does not generate a relatively congruence-distributive quasivariety. □

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