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SOME POSITIVE CONCLUSIONS RELATED TO THE EMBRECHTS–GOLDIE CONJECTURE

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Abstract: We give some conditions under which if an infinitely divisible distribution supported on $[0,\infty)$ belongs to the intersection of the distribution class $\mathfrak{L}(\gamma)$ for some $\gamma \geq 0$ and the distribution class $\mathfrak{OS}$, then so does the corresponding Lévy distribution or its convolution with itself. To this end, we discuss the closure under compound convolution roots for the class and provide some types of distributions satisfying the above conditions. Therefore, this leads to some positive conclusions related to the Embrechts–Goldie conjecture in contrast to the fact that all corresponding previous results for the distribution class $\mathfrak{L}(\gamma) \cap \mathfrak{OS}$ were negative.

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1. Introduction

Let H be an infinitely divisible distribution supported on $[0,\infty)$ with the Laplace transform

$$\int_0^\infty \exp\{-\lambda y\}H(dy) = \exp\left\{-a\lambda - \int_0^\infty (1-e^{\lambda y})\upsilon(dy)\right\}, \quad \operatorname{Re}\lambda \geq 0, \tag{1.1}$$

where $\operatorname{Re}\lambda$ is the real part of λ, $a \geq 0$ is a constant, and υ is a Borel measurable function on $(0,\infty)$ satisfying $\mu := \upsilon((1,\infty)) < \infty$ and

$$\int_0^1 y\upsilon(dy) < \infty.$$

This υ is called the *Lévy measure*. Let

$$F(x) := \upsilon(x)\mathbf{1}_{(1,\infty)}(x)/\mu := \upsilon(0,x]\mathbf{1}_{(1,\infty)}(x)/\mu \quad \text{for all } x \in (-\infty,\infty)$$

be the Lévy distribution generated by υ. The distribution H admits the representation $H = H_1 * H_2$ which is reserved for convolution of two distributions H_1 and H_2 satisfying $1 - H_1(x) = O(e^{-\beta x})$ for all $\beta > 0$ and

$$H_2(x) = e^{-\mu}\sum_{k=0}^\infty F^{*k}(x)\mu^k/k! \quad \text{for all } x \in (-\infty,\infty), \tag{1.2}$$

where F^{*k} is the k-fold convolution of F with itself for all $k \geq 2$, while $F^{*1} = F$ and F^{*0} is the distribution degenerate at zero. See, for instance, Feller [1].

We might also say that F is an "input" and H is an "output" in a system. Usually, we use the "input" F to infer the "output" H. However, sometimes F is in a "black box," and we need H to infer F. The main topic of this paper is the search of some conditions under which a Lévy distribution F or its convolution with itself belongs to certain distribution class providing the corresponding infinitely

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divisible distribution H in the same class. In this way, we first need to recall the concepts and notations of some distribution classes.

Throughout the paper all limits are taken as x tends to infinity unless stated otherwise. If f and g are two positive functions then $f(x) = O(g(x))$ means that $\limsup f(x)/g(x) < \infty$, while $f(x) \lesssim g(x)$ means that $\limsup f(x)/g(x) = 1$. Also, $f(x) \asymp g(x)$ means that $f(x) = O(g(x))$ and $g(x) = O(f(x))$, while $f(x) \sim g(x)$ means that $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$. Furthermore, $f(x) = o(g(x))$ means that $\lim f(x)/g(x) = 0$. Additionally, given a distribution V, let $\overline{V} = 1 - V$ be the tail distribution of V.

For some constant $\gamma \geq 0$, a distribution V on $[0, \infty)$ or $(-\infty, \infty)$ belongs to the distribution class $\mathfrak{L}(\gamma)$, if $\overline{V}(x) > 0$ for all x and $\overline{V}(x - t) \sim \overline{V}(x)e^{\gamma t}$ for all $t > 0$. If $\gamma > 0$ and V is a lattice, then x and t should be restricted to the values of the lattice span of V; see Bertoin and Doney [2]. It is known that if $V \in \mathfrak{L}(\gamma)$, then

$$\mathfrak{H}(V, \gamma) = \{h : 0 < h(x) \uparrow \infty, h(x) = o(x), \overline{V}(x - t) \sim \overline{V}(x)e^{\gamma t} \text{ uniformly for all } |t| \leq h(x)\} \neq \phi.$$

A distribution V belongs to the distribution class $\mathfrak{S}(\gamma)$, if $V \in \mathfrak{L}(\gamma)$,

$$M(V, \gamma) := \int_0^\infty e^{\gamma y} V(dy) < \infty,$$

and

$$\overline{V^{*2}}(x) \sim 2M(V, \gamma)\overline{V}(x).$$

In particular, the classes $\mathfrak{L}(0)$ and $\mathfrak{S}(0)$ are respectively called the *long-tailed distribution class* and the *subexponential distribution class*, respectively denoted by $\mathfrak{L}$ and $\mathfrak{S}$. Moreover, the requirement $V \in \mathfrak{L}$ is not needed in the definition of $\mathfrak{S}$ when V is supported on $[0, \infty)$.

The class $\mathfrak{S}$ was introduced by Chistyakov [3], while the class $\mathfrak{S}(\gamma)$ for some $\gamma > 0$ by Chover et al. [4, 5] for the support $[0, \infty)$ and Tang and Tsitsiashvili [6] or Pakes [7] for the support $(-\infty, \infty)$. For the work on the class $\mathfrak{S}(\gamma)$, see Zachary and Foss [8], etc. The classes $\bigcup_{\gamma \geq 0} \mathfrak{S}(\gamma)$ and $\bigcup_{\gamma \geq 0} \mathfrak{L}(\gamma)$ are properly included in the following two distribution classes.

A distribution V on $[0, \infty)$ or $(-\infty, \infty)$ belongs to the distribution class $\mathfrak{OS}$ introduced by Klüppelberg [9] for the support $[0, \infty)$ and Shimura and Watanabe [10] for the support $(-\infty, \infty)$, if

$$C^*(V) := \limsup \overline{V^{*2}}(x)/\overline{V}(x) < \infty.$$

A distribution V on $[0, \infty)$ or $(-\infty, \infty)$ belongs to the distribution class $\mathfrak{OL}$ introduced by Shimura and Watanabe [10], if

$$C^*(V, t) := \limsup \overline{V}(x - t)/\overline{V}(x) < \infty$$

for all $t > 0$. Further, Shimura and Watanabe [10] show that the inclusion $\mathfrak{OS} \subset \mathfrak{OL}$ is proper.

As regards our main topic, Embrechts et al. [11], Sgibnev [12], Pakes [7], and Watanabe [13] already gave some positive results for $\mathfrak{S}(\gamma)$. Their works show that the Lévy distribution F of an infinitely divisible distribution H belongs to $\mathfrak{S}(\gamma)$ when $H \in \mathfrak{S}(\gamma)$ combined with some conditions. For other classes, however, we only have a few negative results; i.e., there exists an infinitely divisible distribution H belonging to the class such that its Lévy distribution F does not belong to the same class; see Theorem 1.1(iii) of Shimura and Watanabe [10] for $\mathfrak{OS}$, Theorem 1.2(3) of Xu et al. [14] for $(\mathfrak{L} \cap \mathfrak{OS}) \setminus \mathfrak{S}$, and Theorem 1.1 of Xu et al. [15] for $(\mathfrak{L}(\gamma) \cap \mathfrak{OS}) \setminus \mathfrak{S}(\gamma)$ with some $\gamma > 0$. Therefore, for $\mathfrak{L}(\gamma) \cap \mathfrak{OS}$, more precisely, for $(\mathfrak{L}(\gamma) \cap \mathfrak{OS}) \setminus \mathfrak{S}(\gamma)$ with some $\gamma \geq 0$, this paper will discuss the following

Problem 1.1. *Under what conditions does a Lévy distribution F or the convolution of F with itself belong to $\mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for some $\gamma \geq 0$, if the corresponding infinitely divisible distribution H belongs to the class?*

We give the following positive answer to Problem 1.1.

Theorem 1.1. *Let H be an infinitely divisible distribution on* $[0,\infty)$ *with Laplace transform* (1.1) *and Lévy distribution F. Given some $\gamma \geq 0$, assume that $H \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ and*

$$\liminf \overline{F^{*k}}(x-t)/\overline{F^{*k}}(x) \geq e^{\gamma t} \quad \text{for all } \ t>0 \ \text{ and } \ k \geq 1. \tag{1.3}$$

Then

(i) *$H_2 \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ and $\overline{H}(x) \sim M(H_1,\gamma)\overline{H_2}(x)$;*

(ii) *there exists an integer $l_0 \geq 1$ such that $F^{*n} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $n \geq l_0$ and $F^{*n} \notin \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $1 \leq n \leq l_0 - 1$. In particular, if $F \in \mathfrak{OS}$ then $F^{*n} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $n \geq 1$.*

Remark 1.1. Condition (1.3) for $k = 1$ was used in Lemma 7 and Theorem 7 of Foss and Korshunov [16]. Clearly, if F is a heavy-tailed distribution or $F \in \mathfrak{L}(\gamma)$ for some $\gamma \geq 0$, then (1.3) holds for all $k \geq 1$ automatically. We also point out that many distributions satisfy (1.3) for some $\gamma > 0$ and all $k \geq 1$, but they do not belong to $\mathfrak{L}(\gamma)$; see Remark 1.3 below. The condition that (1.3) holds for all $k \geq 1$ can be replaced by some simpler and more convenient conditions; see Corollaries 1.1 and 1.2 below.

Corollary 1.1. *Let H be an infinitely divisible distribution on* $[0,\infty)$ *with Laplace transform* (1.1) *and Lévy distribution F. Assume that $H \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for some $\gamma \geq 0$, while $F \in \mathfrak{OL}$ and*

$$\lim_{t\to\infty} \overline{F}(t) C^*(F,t) = 0. \tag{1.4}$$

Then $F^{*k} \in \mathfrak{OL}$ and (1.3) *holds for all $k \geq 1$. Thus all conclusions of Theorem* 1.1 *hold.*

Remark 1.2. For the distribution F in $\mathfrak{L}$, we have $C^*(F,t) = 1$ for each $t > 0$, which means that (1.4) is true. In the class $\mathfrak{S}(\gamma)$, we have $C^*(F,t) = e^{\gamma t}$ for each $t > 0$, while $M(F,\gamma) < \infty$ means that (1.4) is also true. Thus, for the distribution F in $\mathfrak{OL}$, condition (1.4) means that the distribution F is "not far" from the class $\mathfrak{L}$ or $\mathfrak{S}(\gamma)$. Clearly, (1.4) as a sufficient condition is more natural and achievable than (1.3) for all $k \geq 1$.

In particular, we introduce some kind of distribution with more specific representation. Given a constant $\gamma > 0$ and an arbitrary distribution F_0, we define the distribution F as follows:

$$\overline{F}(x) = \mathbf{1}_{\{x<0\}}(x) + e^{-\gamma x}\overline{F_0}(x)\mathbf{1}_{\{x\geq 0\}}(x), \quad x \in (-\infty,\infty). \tag{1.5}$$

Clearly, F is light-tailed and (1.3) holds for $k = 1$. Further, we have the following

Corollary 1.2. *Let H be an infinitely divisible distribution on* $[0,\infty)$ *with Laplace transform* (1.1) *and Lévy distribution F. Assume that $H \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for some $\gamma > 0$, $F_0 \in \mathfrak{OL}$ in* (1.5) *and*

$$\lim_{t\to\infty} \overline{F_0}(t) C^*(F_0,t) = 0. \tag{1.6}$$

Then all conclusions of Corollary 1.1 *hold.*

Remark 1.3. In Example 4.1 below, we give a concrete type of distribution F_0 belonging to the class $\mathfrak{OS}\setminus\mathfrak{L} \subset \mathfrak{OL}\setminus\mathfrak{L}$. Let H be an infinitely divisible distribution with Lévy distribution F in (1.5) with some $\gamma > 0$. Proposition 5.1 of Xu et al. [15] shows that $H \in (\mathfrak{L}(\gamma)\cap\mathfrak{OS})\setminus\mathfrak{S}(\gamma)$ and $F^{*k} \in (\mathfrak{L}(\gamma)\cap\mathfrak{OS})\setminus\mathfrak{S}(\gamma)$ for all $k \geq 2$, while $F \notin \mathfrak{L}(\gamma)$. Also, we prove that F_0 satisfies (1.6). Thus, $F^{*k} \in \mathfrak{OL}$ and (1.3) holds for all $k \geq 1$ by Corollary 1.2. In fact, we can construct many distributions F_0 that satisfy (1.6). Let F_1 be a continuous distribution belonging to $\mathfrak{L}$ and let $y_1 > 0$ and $a > 1$ be two constants such that $a\overline{F_1}(y_1) \leq 1$. Define the distribution F_0 by

$$\overline{F_0}(x) = \overline{F_1}(x)\mathbf{1}_{\{x<x_1\}}(x) + \sum_{i=1}^{\infty}(\overline{F_1}(x_i)\mathbf{1}_{\{x_i\leq x<y_i\}}(x)$$
$$+\overline{F_1}(x)\mathbf{1}_{\{y_i\leq x<x_{i+1}\}}(x)), \quad x \in (-\infty,\infty),$$

where $\{x_i : i \geq 1\}$ and $\{y_i : i \geq 1\}$ are two sequences of positive constants satisfying $x_i < y_i < x_{i+1}$ and $\overline{F_1}(x_i) = a\overline{F}_1(y_i)$, $i \geq 1$. Then $F_0 \in \mathfrak{OL} \setminus \mathfrak{L}$ with $C^*(F_0,t) = a$ for each $t > 0$. Thus, (1.6) holds.

Therefore, for F in (1.5), we have $F^{*k} \in \mathfrak{OL}$ and (1.3) holds for all $k \geq 1$ by Corollary 1.2. Moreover, when $\gamma = 0$, Theorem 2.2 of Xu et al. [14] shows that there is a type of infinitely divisible distribution H with Lévy distribution F such that H, H_2, and F^{*n} for all $n \geq 2$ belong to $(\mathfrak{L} \cap \mathfrak{OS}) \setminus \mathfrak{S}$ and (1.3) holds for all $k \geq 1$, while $F \in \mathfrak{OL} \setminus (\mathfrak{L} \cap \mathfrak{OS})$. Here (1.3) for all $k \geq 1$ is automatically established but (1.6) does not hold.

Note that $l_0 = 2$ in the above examples of the two cases: $\gamma > 0$ or $\gamma = 0$.

Remark 1.4. In the above results, the main object is the class $\mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for some $\gamma \geq 0$. We note that there are many distributions in $(\mathfrak{L}(\gamma) \cap \mathfrak{OS}) \setminus \mathfrak{S}(\gamma)$; see, for instance, Leslie [17], Klüppelberg and Villasenor [18], Shimura and Watanabe [10], Lin and Wang [19], Wang et al. [20], Xu et al. [14], and Xu et al. [15]. Alongside the above-mentioned literatures, the reader can refer the research on $\mathfrak{OS}$ to Watanabe and Yamamura [21], Yu and Wang [22], Beck et al. [23], Xu et al. [24], Xu et al. [25], and so on.

According to the decomposition of an infinitely divisible distribution and (1.2), we can find that, in order to prove Theorem 1.1, we first need to solve the following Problem 1.2 involving a compound distribution which is also called a *compound convolution*. Let V be a distribution on $[0, \infty)$ and let τ be a nonnegative integer-valued random variable with masses $p_k = \mathbf{P}(\tau = k)$ for all integers $k \geq 0$ satisfying

$$\sum_{k=0}^{\infty} p_k = 1.$$

Denote the corresponding compound convolution by $V^{*\tau}$; i.e.,

$$V^{*\tau} := \sum_{k=0}^{\infty} p_k V^{*k}. \tag{1.7}$$

For convenience, we set up $p_k > 0$ for all integers $k \geq 0$. In fact, if τ is a nonnegative integer-valued random variable with masses $p_{k_m} > 0$ for all integers $m \geq 1$ satisfying

$$\sum_{m=0}^{\infty} p_{k_m} = 1,$$

where $k_1 = 1$, then all conclusions of the paper still hold.

Problem 1.2. *Under what conditions does the distribution V on $[0, \infty)$ or the convolution of V with itself belong to $\mathfrak{L}(\gamma) \cap \mathfrak{OS}$ provided that $V^{*\tau} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$?*

Usually, Problem 1.2 is a topic on closure under compound convolution roots for a distribution class.

It is known that the compound convolution, as well as its convolution with other distribution, has extensive and important applications in various fields such as risk theory, queuing systems, branching processes, infinitely divisible distributions, and so on. See, for instance, Embrechts et al. [26], Borovkov and Borovkov [27], and Foss et al. [28].

The topic in Problem 1.2 is a natural extension of the well-known Embrechts and Goldie conjecture on the class $\mathfrak{L}(\gamma)$ with some $\gamma \geq 0$; see Embrechts and Goldie [29, 30]. Some of the latest results on the conjecture and the related problems can be found in Xu et al. [14], Xu et al. [15], Watanabe [31], Watanabe and Yamamuro [32], and so on.

In the references above, Theorem 2.2 of Xu et al. [14] and Proposition 5.1 of Xu et al. [15] show that the class $\mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for some $\gamma \geq 0$ is not closed under compound convolution roots. However, it is very interesting to find a positive answer to Problem 1.2.

In Section 3, we prove Theorem 1.1. To this end, we give a positive answer to Problem 1.2 in Section 2. In Section 4, we first prove Corollaries 1.1 and 1.2; next, we provide some example that satisfies all conditions of Corollary 1.2; finally, we give a general Kesten inequality to discuss condition (2.1) in Theorem 2.1 below.

2. On Compound Convolution

In what follows, we assume that all distributions are supported on $[0,\infty)$. Let V be a distribution and recall the random variable τ and the compound convolution $V^{*\tau}$ that are defined in Section 1.

Theorem 2.1. *Given some $\gamma \ge 0$, assume that $V^{*\tau} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$, condition* (1.3) *is satisfied for all $k \ge 1$, and for each $0 < \varepsilon < 1$ there is an integer $n_0 = n_0(V,\tau,\varepsilon) \ge 1$ such that*

$$\sum_{k=n_0+1}^{\infty} p_k \overline{V^{*(k-1)}}(x) \le \varepsilon \overline{V^{*\tau}}(x) \quad \text{for all} \;\; x \ge 0. \tag{2.1}$$

Then there is an integer $l_0 \ge 1$ such that $V^{*n} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $n \ge l_0$ and $V^{*n} \notin \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $1 \le n \le l_0 - 1$. Further, if $V \in \mathfrak{OS}$, then $V^{*n} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $n \ge 1$.

Remark 2.1. (i) Condition (2.1) was used in Watanabe and Yamamuro [21], Yu and Wang [22], Xu et al. [24], and Xu et al. [15]. As Watanabe and Yamamuro [21] pointed out that if $p_{k+1}/p_k \to 0$ as $k \to \infty$, for instance, $p_k = e^{-\lambda}\lambda^k/k!$ for all $k \ge 0$; then (2.1) is satisfied for all distributions. In fact, since $p_{k+1}/p_k \to 0$ as $k \to \infty$, there is an integer $n_0 \ge 1$ such that $p_{k+1}/p_k < \varepsilon$ for all $k \ge n_0$ and any $0 < \varepsilon < 1$. Thus,

$$\begin{aligned}\sum_{k=n_0}^{\infty} p_k \overline{V^{*(k-1)}}(x) &= \sum_{k=n_0}^{\infty} (p_k/p_{k-1}) p_{k-1} \overline{V^{*(k-1)}}(x) \\ &< \varepsilon \sum_{k=n_0}^{\infty} p_{k-1} \overline{V^{*(k-1)}}(x) < \varepsilon \overline{V^{*\tau}}(x).\end{aligned}$$

Moreover, some examples in Remark 4.1 below show that (2.1) holds but p_{k+1}/p_k does not vanish as $k \to \infty$.

(ii) In the theorem, l_0 is not necessarily equal to 1; see also Remark 1.3 of this paper.

Proof. We first give an equivalent form of (2.1) in the case that $V^{*\tau} \in \mathfrak{OS}$.

Lemma 2.1. *If $V^{*\tau} \in \mathfrak{OS}$, then the following two propositions are equivalent to each other:*
(i) *For each $0 < \varepsilon < 1$, there is an integer $n_0 = n_0(V,\tau,\varepsilon) \ge 1$ such that*

$$\sum_{k=n_0+1}^{\infty} p_k \overline{V^{*k}}(x) \le \varepsilon \overline{V^{*\tau}}(x) \quad \text{for all } x \ge 0. \tag{2.2}$$

(ii) *For each $0 < \varepsilon < 1$, there is an integer $n_0 = n_0(V,\tau,\varepsilon) \ge 1$ such that* (2.1) *holds.*

Proof. We only need to prove (ii) $\Longrightarrow$ (i). To this end, we put

$$D^*(V^{*\tau}) = \sup_{x \ge 0} \overline{(V^{*\tau})^{*2}}(x)/\overline{V^{*\tau}}(x).$$

Clearly, $1 \le D^*(V^{*\tau}) < \infty$ by $V^{*\tau} \in \mathfrak{OS}$. For any $0 < \varepsilon < 1$, we take $\varepsilon_1 = \varepsilon p_1/D^*(V^{*\tau})$, then $0 < \varepsilon_1 < p_1/D^*(V^{*\tau}) < 1$. For the above ε_1, by (ii), there is an integer $n_0 = n_0(V,\tau,\varepsilon_1) \ge 1$ such that (2.1) holds and

$$a_{n_0} = \sum_{k=n_0+1}^{\infty} p_k < \varepsilon_1. \tag{2.3}$$

Define the distribution G_{n_0} by $G_{n_0}(x) = \sum_{k=n_0+1}^{\infty} p_k V^{*(k-1)}(x)/a_{n_0}$ for all x. Then for all $x \ge 0$, by (2.1), $a_{n_0}\overline{G_{n_0}}(x) \le \varepsilon_1 \overline{V^{*\tau}}(x)$. Further, by (2.3),

$$\sum_{k=n_0+1}^{\infty} p_k \overline{V^{*k}}(x) = \overline{V * \sum_{n_0+1 \le k < \infty} p_k V^{*(k-1)}}(x)$$
$$= a_{n_0} \int_{0-}^{x} \overline{G_{n_0}}(x-y)V(dy) + a_{n_0}\overline{V}(x) \le \varepsilon_1 \Big(\int_{0-}^{x} \overline{V^{*\tau}}(x-y)V(dy) + \overline{V}(x) \Big)$$
$$= \varepsilon_1 \overline{V * V^{*\tau}}(x) \le \varepsilon_1 D^*(V^{*\tau}) \overline{V^{*\tau}}(x)/p_1 = \varepsilon \overline{V^{*\tau}}(x),$$

which is just (i). □

Now, we prove the theorem. For some $0 < \varepsilon_0 < 1$, by (2.1) with $\varepsilon = \varepsilon_0$, $V^{*\tau} \in \mathfrak{OS}$ and Lemma 2.1, there is an integer $n_0 = n_0(V, \tau, \varepsilon_0) \ge 1$ such that

$$\sum_{k=1}^{n_0} p_k \overline{V^{*k}}(x) \ge (1 - \varepsilon_0) \overline{V^{*\tau}}(x) \text{ for all } x \ge 0,$$

which means that $\overline{V^{*\tau}}(x) \asymp \overline{V^{*n_0}}(x)$. Then from $V^{*\tau} \in \mathfrak{OS}$ it is immediate that $V^{*n_0} \in \mathfrak{OS}$. Consequently, there is an integer $l_0 = \min\{1 \le n \le n_0 : V^{*n} \in \mathfrak{OS}\}$ such that $1 \le l_0 \le n_0$ and $V^{*l_0} \in \mathfrak{OS}$. According to Proposition 2.6 of Shimura and Watanabe [10], $V^{*n} \in \mathfrak{OS}$ and $\overline{V^{*\tau}}(x) \asymp \overline{V^{*n}}(x)$ for all $n \ge l_0$. Thus, for each $n \ge l_0$, there is a constant $D_n = D_n(V, \tau) > 0$ such that

$$\limsup \overline{V^{*\tau}}(x)/\overline{V^{*n}}(x) = D_n < \infty.$$

Next, we continue to prove $V^{*n} \in \mathfrak{L}(\gamma)$ for each $n \ge l_0$. By Lemma 2.1 and (2.1), for every $0 < \varepsilon < \varepsilon_0$ small enough, there exists an integer $m_0 = m_0(V, \tau, \varepsilon)$ such that $m_0 > n$ and

$$\sum_{k=m_0+1}^{\infty} p_k \overline{V^{*k}}(x) \le \varepsilon \overline{V^{*\tau}}(x) \quad \text{for all } x \ge 0. \tag{2.4}$$

Further, since $V^{*\tau} \in \mathfrak{L}(\gamma)$, by (2.4) and (1.3) for all $k \ge 1$ and each $t > 0$, there is a constant $x_0 = x_0(V, \tau, \varepsilon, t) > t$ such that for all $x > x_0$,

$$\varepsilon \overline{V^{*\tau}}(x) \ge \overline{V^{*\tau}}(x-t) - e^{\gamma t}\overline{V^{*\tau}}(x)$$
$$= \Big(\sum_{1 \le k \ne n \le m_0} + \sum_{k=n} + \sum_{k \ge m_0+1} \Big) p_k (\overline{V^{*k}}(x-t) - e^{\gamma t}\overline{V^{*k}}(x))$$
$$\ge -\varepsilon e^{\gamma t} \sum_{1 \le k \ne n \le m_0} p_k \overline{V^{*k}}(x) + p_n(\overline{V^{*n}}(x-t) - e^{\gamma t}\overline{V^{*n}}(x)) - e^{\gamma t} \sum_{k \ge m_0+1} p_k \overline{V^{*k}}(x)$$
$$\ge p_n(\overline{V^{*n}}(x-t) - e^{\gamma t}\overline{V^{*n}}(x)) - 2\varepsilon e^{\gamma t}\overline{V^{*\tau}}(x),$$

which implies that for all $x > x_0$

$$\overline{V^{*n}}(x-t) \le e^{\gamma t}\overline{V^{*n}}(x) + (1 + 2e^{\gamma t})\varepsilon \overline{V^{*\tau}}(x)/p_n.$$

Hence,

$$\limsup \overline{V^{*n}}(x-t)/\overline{V^{*n}}(x) \le e^{\gamma t} + (1 + 2e^{\gamma t})\varepsilon D_n/p_n. \tag{2.5}$$

Clearly, the fixed integer n is independent of ε.

Thus, combined with the arbitrariness of ε, (2.5) and (1.3) lead to $V^{*n} \in \mathfrak{L}(\gamma)$. In particular, if $V \in \mathfrak{OS}$, then by the same method we can get $V^{*n} \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for all $n \ge 1$. □

3. Proof of Theorem 1.1

We first prove the following lemma with a more general form than the first part of Theorem 1.1, which is a key to the proof of the theorem and is of interest in its own right.

Let G_1 be a distribution. Write $G = G_1 * G_2$, where $G_2 = V^{*\tau}$ is a compound convolution. Recall that all distributions are supported on $[0, \infty)$.

Lemma 3.1. *Assume that $G \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ for $\gamma \geq 0$ and $\overline{G_1}(x) = o(\overline{G_2}(x))$. Further, suppose that (2.1) for all $0 < \varepsilon < 1$ and (1.3) for the distribution V and all $k \geq 1$ are satisfied. Then $G_2 \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ and $\overline{G}(x) \sim M(G_1, \gamma)\overline{G_2}(x)$.*

Proof. First, we prove $\overline{G}(x) \asymp \overline{G_2}(x)$ and $G_2 \in \mathfrak{OS}$. For each $0 < \varepsilon < 1/(2C^*(G))$, since $G \in \mathfrak{L}(\gamma)$ and $\overline{G_1}(x) = o(\overline{G_2}(x))$, there is a constant $A > 0$ large enough such that, when $x \geq A$, we have

$$\begin{aligned}\overline{G}(x) &= \int_{0-}^{x-A} \overline{G_1}(x-y)G_2(dy) + \int_{0-}^{A} \overline{G_2}(x-y)G_1(dy) + \overline{G_1}(A)\overline{G_2}(x-A)\\ &\leq \varepsilon \int_{0-}^{x-A} \overline{G}(x-y)G_2(dy) + G_1(A)\overline{G_2}(x-A) + \overline{G_1}(A)\overline{G_2}(x-A)\\ &\leq \varepsilon\overline{G^{*2}}(x) + \overline{G_2}(x-A) \leq 2\varepsilon C^*(G)\overline{G}(x) + \overline{G_2}(x-A).\end{aligned}$$

In the above expression, we replace x with $x + A$, then

$$(1 - 2\varepsilon C^*(G))\overline{G}(x+A) \leq \overline{G_2}(x).$$

Therefore, by $G \in \mathfrak{OS} \subset \mathfrak{OL}$, for x large enough, we have

$$(1 - 2\varepsilon C^*(G))\overline{G}(x)/(2C^*(G, A)) \leq (1 - 2\varepsilon C^*(G))\overline{G}(x+A) \leq \overline{G_2}(x);$$

i.e., $\overline{G}(x) \asymp \overline{G_2}(x)$, and so $G_2 \in \mathfrak{OS}$.

Next, we prove that $G_2 \in \mathfrak{L}(\gamma)$. On the one hand, by (2.1), $G_2 \in \mathfrak{OS}$ and Lemma 2.1, there is an integer $n_0 = n_0(V, \tau, \varepsilon) \geq 1$, such that (2.2) holds. For each $0 < \varepsilon < 1$ and every $t > 0$, by (1.3) for all $k \geq 1$, there is a constant $x_0 = x_0(F, \varepsilon, t)$ such that

$$e^{\gamma t}\overline{V^{*k}}(x)/\overline{V^{*k}}(x-t) \leq 1 + \varepsilon, \quad \text{for all } 1 \leq k \leq n_0 \text{ and } x \geq x_0. \tag{3.1}$$

By (3.1) and (2.2), we have

$$(e^{\gamma t}\overline{G_2}(x) - \overline{G_2}(x-t))/\overline{G_2}(x-t) \leq \sum_{k=1}^{n_0}\left(\frac{e^{\gamma t}\overline{V^{*k}}(x)}{\overline{V^{*k}}(x-t)} - 1\right) + \varepsilon e^{\gamma t} \leq \varepsilon(n_0 + e^{\gamma t}),$$

which implies that

$$\limsup(e^{\gamma t}\overline{G_2}(x) - \overline{G_2}(x-t))/\overline{G_2}(x-t) \leq 0. \tag{3.2}$$

On the other hand, for each $0 < \varepsilon < 1$ and every $t > 0$, we take a constant B large enough such that $B > 2t$. When $x \geq 3B$, using integration by parts, by (3.2), $\overline{G_1}(x) = o(\overline{G_2}(x))$ and $G \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$, we obtain

$$e^{2\gamma t}\overline{G}(x) - \overline{G}(x-2t) \leq e^{2\gamma t}\left(\int_{0-}^{x-B} + \int_{x-B}^{x}\right)\overline{G_1}(x-y)G_2(dy) + e^{2\gamma t}\overline{G_2}(x)$$

$$-\int\limits_{x-2t-B}^{x-2t} \overline{G_1}(x-2t-y)G_2(dy) - \overline{G_2}(x-2t)$$

$$\leq \varepsilon e^{2\gamma t}\overline{G^{*2}}(x) + \int\limits_{0-}^{B}(e^{2\gamma t}\overline{G_2}(x-y) - \overline{G_2}(x-2t-y))G_1(dy)$$

$$+\overline{G_1}(B)(e^{2\gamma t}\overline{G_2}(x-B) - \overline{G_2}(x-2t-B))$$

$$\leq 2\varepsilon(1+\varepsilon)e^{2\gamma t}C^*(G)\overline{G}(x) + \left(\int\limits_{0-}^{t} + \int\limits_{t}^{2t} + \int\limits_{2t}^{B}\right)(e^{2\gamma t}\overline{G_2}(x-y) - \overline{G_2}(x-2t-y))G_1(dy)$$

$$\leq 3\varepsilon(1+\varepsilon)e^{2\gamma t}C^*(G)\overline{G}(x) + \int\limits_{t}^{2t}(e^{2\gamma t}\overline{G_2}(x-y) - e^{\gamma y}\overline{G_2}(x-2t))G_1(dy)$$

$$+\int\limits_{t}^{2t}(e^{\gamma y}\overline{G_2}(x-2t) - e^{\gamma(y-t)}\overline{G_2}(x-3t))G_1(dy)$$

$$+\int\limits_{t}^{2t}(e^{\gamma(y-t)}\overline{G_2}(x-3t) - \overline{G_2}(x-3t-(y-t)))G_1(dy). \tag{3.3}$$

When $t \leq y \leq 2t$, we have

$$|e^{2\gamma t}\overline{G_2}(x-y) - e^{\gamma y}\overline{G_2}(x-2t)|/\overline{G}(x) \leq (e^{2\gamma t}/\overline{G_1}(y)) + (e^{\gamma y}/\overline{G_1}(2t))$$

and

$$|e^{\gamma(y-t)}\overline{G_2}(x-3t) - \overline{G_2}(x-3t-(y-t))|/\overline{G}(x) \leq (e^{\gamma(y-t)}/\overline{G_1}(3t)) + (1/\overline{G_1}(2t+y)).$$

Then,

$$\int\limits_{t}^{2t}(e^{2\gamma t}/\overline{G_1}(y)) + (e^{\gamma y}/\overline{G_1}(2t))G_1(dy) \leq 2e^{2\gamma t}/\overline{G_1}(2t) < \infty$$

and

$$\int\limits_{t}^{2t}((e^{\gamma(y-t)}/\overline{G_1}(3t)) + (1/\overline{G_1}(2t+y))G_1(dy) \leq (e^{\gamma t}/\overline{G_1}(3t)) + (1/\overline{G_1}(4t)) < \infty.$$

Thus, by Fatou's Lemma and (3.2), we obtain

$$\limsup \int\limits_{t}^{2t}(e^{2\gamma t}\overline{G_2}(x-y) - e^{\gamma y}\overline{G_2}(x-2t))G_1(dy)/\overline{G}(x)$$

$$\leq \int\limits_{t}^{2t}\limsup\left(e^{2\gamma t}\frac{\overline{G_2}(x-y)}{\overline{G_2}(x-2t)} - e^{\gamma y}\right)G_1(dy)/\overline{G}_1(2t) \leq 0 \tag{3.4}$$

and

$$\limsup \int\limits_{t}^{2t}(e^{\gamma(y-t)}\overline{G_2}(x-3t) - \overline{G_2}(x-3t-(y-t)))G_1(dy)/\overline{G}(x)$$

$$\leq \int\limits_{t}^{2t}\left(\limsup\frac{e^{\gamma(y-t)}\overline{G_2}(x-3t)}{\overline{G_2}(x-3t-(y-t))} - 1\right)G_1(dy)/\overline{G}_1(4t) \leq 0. \tag{3.5}$$

According to (3.3)–(3.5), we know that

$$(\overline{G_2}(x-2t) - e^{-\gamma t}\overline{G_2}(x-3t)) \int\limits_t^{2t} e^{\gamma y} G_1(dy) = \int\limits_t^{2t} (e^{\gamma y}\overline{G_2}(x-2t) - e^{\gamma(y-t)}\overline{G_2}(x-3t)) G_1(dy)$$
$$\geq -3\varepsilon(1+\varepsilon)e^{\gamma t} C^*(G)\overline{G}(x) - 3\varepsilon\overline{G}(x).$$

Further, by $\overline{G}(x) \asymp \overline{G_2}(x-t)$ and the arbitrariness of ε, we deduce that

$$\liminf (e^{\gamma t}\overline{G_2}(x) - \overline{G_2}(x-t))/\overline{G_2}(x-t) \geq 0,$$

which leads to $G_2 \in \mathfrak{L}(\gamma)$ combined with (3.2).

Finally, since $G_2 \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ and $\overline{G}_1(x) = o(\overline{G}_1(x))$, using Lemma 2.1 of Pakes [7], we have $\overline{G}(x) \sim M(G_1, \gamma)\overline{G_2}(x)$. $\square$

Now, we prove Theorem 1.1.

(i) In Lemma 3.1, we take $V = F$, $G_1 = H_1$, $G_2 = H_2 = F^{*\tau}$, and $G = H$. According to Remark 2.1(i), condition (2.1) is satisfied for the Poisson compound convolution H_2. Therefore, by Lemma 3.1 and $H \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$, we have $H_2 \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ and $\overline{H_2}(x) \asymp \overline{H}(x)$.

Further, using Lemma 3.1 again, we have $\overline{H}(x) \sim M(H_1, \gamma)\overline{H_2}(x)$ because $H_2 \in \mathfrak{L}(\gamma) \cap \mathfrak{OS}$ and $\overline{H}_1(x) = O(e^{-\beta x})$ for all $\beta > 0$.

(ii) According to (i), Theorem 2.1, and Remark 2.1(i), we get the corresponding results directly.

4. Proofs of Corollaries 1.1 and 1.2

In this section, we follow the notations of Section 1. In order to prove Corollaries 1.1 and 1.2, we first give two lemmas.

Lemma 4.1. *If $F \in \mathfrak{OL}$, then for all $k \geq 1$, $F^{*k} \in \mathfrak{OL}$ and*

$$C^*(F^{*k}, t) \leq C^*(F, t) \quad \text{for each } t \geq 0. \tag{4.1}$$

Proof. We proceed by induction. Clearly, (4.1) holds for $k = 1$. Assume that $F^{*k} \in \mathfrak{OL}$ and (4.1) holds for some integer $k \geq 1$, then for any $0 < \varepsilon < 1$ and each $t > 0$, there is a constant $x_0 = x_0(F^{*k}, \varepsilon, t)$ such that when $x \geq x_0$,

$$\overline{F^{*k}}(x-t) \leq (1+\varepsilon)C^*(F^{*k}, t)\overline{F^{*k}}(x) \leq (1+\varepsilon)C^*(F,t)\overline{F^{*k}}(x) < \infty.$$

Further, according to the induction hypothesis, for above $0 < \varepsilon < 1$ and $t > 0$, we have

$$\overline{F^{*(k+1)}}(x-t) = \Bigg(\int\limits_{0-}^{x-t-x_0} + \int\limits_{x-t-x_0}^{x-t}\Bigg)\overline{F^{*k}}(x-t-y)F(dy) + \overline{F}(x-t)$$
$$\leq (1+\varepsilon)C^*(F,t)\Bigg(\int\limits_{0-}^{x-x_0} \overline{F^{*k}}(x-y)F(dy) + \int\limits_{0-}^{x_0} \overline{F}(x-y)F^{*k}(dy) + \overline{F}(x-x_0)\overline{F^{*k}}(x_0)\Bigg)$$
$$= (1+\varepsilon)C^*(F,t)\overline{F^{*(k+1)}}(x).$$

Thus, $F^{*(k+1)} \in \mathfrak{OL}$ and (4.1) holds for $k+1$ by the arbitrariness of ε. $\square$

Lemma 4.2. *For $i = 1, 2$, let F_i be a distribution satisfying*

$$\liminf \overline{F_i}(x-t)/\overline{F_i}(x) \geq e^{\gamma t} \quad \textit{for each } t > 0. \tag{4.2}$$

If $F_2 \in \mathfrak{OL}$ and

$$\lim_{t\to\infty} \overline{F_1}(t) C^*(F_2, t) = 0, \tag{4.3}$$

then

$$\liminf \overline{F_1 * F_2}(x-t)/\overline{F_1 * F_2}(x) \geq e^{\gamma t} \quad \textit{for each } t > 0. \tag{4.4}$$

Proof. To prove (4.4), we only need to prove its equivalent:

$$\limsup (e^{\gamma t}\overline{F_1 * F_2}(x) - \overline{F_1 * F_2}(x-t))/\overline{F_1 * F_2}(x-t) \leq 0. \tag{4.5}$$

From (4.2), we know that, for all $0 < \varepsilon < 1$ and $i = 1, 2$, there exists a constant $x_0 = x_0(\varepsilon, \gamma, F_1, F_2) > 0$ such that

$$\overline{F_i}(x-t) - e^{\gamma t}\overline{F_i}(x) \geq -\varepsilon \overline{F_i}(x) \quad \text{for all } x \geq x_0. \tag{4.6}$$

When $x > t + 2x_0$, using integration by parts, by (4.6), we have

$$\begin{aligned}
& e^{\gamma t}\overline{F_1 * F_2}(x) - \overline{F_1 * F_2}(x-t) \\
&= e^{\gamma t} \int_{x-t-x_0}^{x-x_0} \overline{F_1}(x-y) F_2(dy) - \overline{F_1}(x_0)(\overline{F_2}(x-t-x_0) - e^{\gamma t}\overline{F_2}(x-x_0)) \\
&- \int_0^{x-t-x_0} (\overline{F_1}(x-t-y) - e^{\gamma t}\overline{F_1}(x-y))F_2(dy) - \int_0^{x_0} (\overline{F_2}(x-t-y) - e^{\gamma t}\overline{F_2}(x-y))F_1(dy) \\
&\leq e^{\gamma t}\overline{F_1}(x_0)\overline{F_2}(x-t-x_0) + \varepsilon \overline{F_1}(x_0)\overline{F_2}(x-x_0) \\
&+ \varepsilon \int_0^{x-t-x_0} \overline{F_1}(x-y)F_2(dy) + \varepsilon \int_0^{x_0} \overline{F_2}(x-y)F_1(dy) \\
&\leq e^{\gamma t}\overline{F_1}(x_0)\overline{F_2}(x-t-x_0) + \varepsilon \overline{F_1 * F_2}(x).
\end{aligned}$$

Then

$$\begin{aligned}
& \limsup (e^{\gamma t}\overline{F_1 * F_2}(x) - \overline{F_1 * F_2}(x-t))/\overline{F_1 * F_2}(x-t) \\
&\leq e^{\gamma t}\overline{F_1}(x_0) C^*(F_2, x_0) + \varepsilon.
\end{aligned}$$

Therefore, by (4.3) and the arbitrariness of ε, (4.5) holds. □

Proof of Corollary 1.1. By Lemma 4.1 and $F \in \mathfrak{OL}$, we have $F^{*k} \in \mathfrak{OL}$ for all $k \geq 1$. Then we only need to prove that (1.3) holds for all $k \geq 1$ according to Theorem 1.1.

We know already that (1.3) holds for $k = 1$. Assume that (1.3) holds for $k = n \geq 2$. In Lemma 4.2, we take $F_1 = F$ and $F_2 = F^{*n}$, then by (4.1) and (1.4), we have

$$\overline{F_1}(t) C^*(F_2, t) = \overline{F}(t) C^*(F^{*n}, t) \leq \overline{F}(t) C^*(F, t) \to 0 \quad \text{as } t \to \infty.$$

Thus, according to Lemma 4.2, we know that (1.3) holds for $k = n + 1$.

Therefore, (1.3) holds for all $k \geq 1$ by induction. □

Proof of Corollary 1.2. Clearly, it is easy to find that

$$C^*(F, t) = \limsup e^{\gamma t}\overline{F_0}(x-t)/\overline{F_0}(x) = e^{\gamma t} C^*(F_0, t).$$

Thus, $F \in \mathfrak{OL}$ follows from $F_0 \in \mathfrak{OL}$. Further, by (1.6),

$$\overline{F}(t)C^*(F,t) = \overline{F_0}(t)C^*(F_0,t) \to 0 \quad \text{as } t \to \infty.$$

Therefore, we complete the proof by Corollary 1.1. $\square$

Next, we give a distribution F in (1.5) that satisfies $F_0 \in \mathfrak{OL} \setminus \mathfrak{L}$ and (1.6). Thus, (1.3) holds for all $k \geq 1$ by Corollary 1.2.

EXAMPLE 4.1. Let $\alpha \in (3/2, (\sqrt{5}+1)/2)$ and $r = (\alpha+1)/\alpha$ be constants. Assume that $a > 1$ is so large that $a^r > 8a$. We define some distribution F_0 that is supported on $[0,\infty)$ and such that

$$\overline{F_0}(x) = \mathbf{1}_{\{x<a_0\}}(x) + C\sum_{n=0}^{\infty}\Bigg(\Bigg(\sum_{i=n}^{\infty} a_i^{-\alpha} - a_n^{-\alpha-1}(x-a_n)\Bigg)\mathbf{1}_{\{x\in[a_n,2a_n)\}}(x) + \sum_{i=n+1}^{\infty} a_i^{-\alpha}\mathbf{1}_{\{x\in[2a_n,a_{n+1})\}}(x)\Bigg), \tag{4.7}$$

where C is a regularization constant and $a_n = a^{r^n}$ for all nonnegative integers; see Definition 5.1 of Xu et al. [15]. By Proposition 5.1 in [15], $F_0 \in \mathfrak{OS} \setminus \mathfrak{L}$ and

$$\int_0^\infty \overline{F}_0(y)dy < \infty.$$

Now, we prove that (1.6) holds.

For each $t > 0$ and all enough large integer n such that $2a_n + t < a_{n+1}$, when $x \in [a_n, a_n+t)$,

$$1 \leq \overline{F_0}(x-t)/\overline{F_0}(x) \leq \overline{F_0}(a_n)/\overline{F_0}(a_n+t) \to 1;$$

when $x \in [a_n+t, 2a_n)$,

$$1 \leq \overline{F_0}(x-t)/\overline{F_0}(x) \leq (\overline{F_0}(a_n) - a_n^{-\alpha-1}t)/\overline{F_0}(a_n) \to 1;$$

when $x \in [2a_n, 2a_n+t)$, by $r = (\alpha+1)/\alpha$, we have

$$\overline{F_0}(x-t)/\overline{F_0}(x) \leq \overline{F_0}(2a_n-t)/\overline{F_0}(2a_n) \to 1+t;$$

and when $x \in [2a_n+t, a_{n+1})$,

$$\overline{F_0}(x-t)/\overline{F_0}(x) = 1.$$

This fact implies $C^*(F_0,t) = 1+t$, thus $F_0 \notin \mathfrak{L}$. Further, by $\int_0^\infty \overline{F}_0(y)dy < \infty$, we find

$$\lim_{t\to\infty} \overline{F_0}(t)C^*(F_0,t) = \lim_{t\to\infty} \overline{F_0}(t)(1+t) = 0;$$

i.e., (1.6) holds. $\square$

Finally, we give a general Kesten inequality which implies (2.1) under some conditions. To this end, write $A_n := \sup_{x\geq 0} \frac{\overline{V^{*n}}(x)}{\overline{G}(x)}$ for all $n \geq 1$ and

$$a := M(V,\gamma) + A_1(C^*(G) - 2M(G,\gamma)) =: M(V,\gamma) + b.$$

Proposition 4.1. *Let V and G be two distributions such that $G \in \mathfrak{L}(\gamma) \cap \mathfrak{O}\mathfrak{S}$ for some $0 \le \gamma < \infty$, $M(G,\gamma) < \infty$ and $\overline{V}(x) = O(\overline{G}(x))$. Then for every constant A satisfying*

$$a < A < 1 + a \tag{4.8}$$

and every constant $\varepsilon > 0$ satisfying

$$(1+\varepsilon)(a + (2 + A_1)\varepsilon) < A, \tag{4.9}$$

there exists a constant $K = K(V, G, \gamma, \varepsilon) > 0$ such that

$$\overline{V^{*k}}(x) \le KA^k\overline{G}(x) \quad \text{for all } \ x \ge 0 \ \text{ and } \ k \ge 1. \tag{4.10}$$

Remark 4.1. (i) Clearly, $C^*(G) \ge 2M(G,\gamma)$ and $M(V,\gamma) > 1$. Thus $a > 1$. If $C^*(G) = 2M(G,\gamma)$, i.e., $G \in \mathfrak{S}(\gamma)$; then we only require that $A > M(V,\gamma)$. This particular result is attributed to Lemma 2.1 of Yu et al. [33]. When $G = V \in \mathfrak{O}\mathfrak{S}$, i.e., $A_1 = 1$, the result is due to Lemma 6.3(ii) of Watanabe [13]. In the two results, the distribution V is supported on $(-\infty, \infty)$.

(ii) Clearly, in Theorem 2.1, if

$$\sum_{k=1}^{\infty} p_k A^{k-1} < \infty,$$

then condition (2.1) is satisfied by Proposition 4.1 with $G = V^{*\tau}$. For instance, we can take $p_k = pq^k$ for all nonnegative integers k, where $p, q > 0$ and $p + q = 1$, if q is so small that

$$qa = q(M(V,\gamma) + A_1(C^*(V^{*\tau}) - 2M(V^{*\tau},\gamma))) < 1,$$

then we can choose a constant A satisfying (4.10) for all $k \ge 1$, which means that (2.1) holds, while $p_{k+1}/p_k = q$ does not vanish as $k \to \infty$.

(iii) From (4.8), we know that $M(V,\gamma) < A - b$. Thus, for all $n \ge 1$ and any $K > 0$,

$$M^n(V,\gamma) < A^n((A-b)/A)^n \le KA^n((A-b)/(KA)). \tag{4.11}$$

Proof. Clearly, (4.10) holds for $k = 1$ and all $x \ge 0$. Further, we assume that (4.10) holds for $k = n$ and all $x \ge 0$. For the above mentioned $\varepsilon > 0$ and every $h \in \mathfrak{H}(G,\gamma)$, by $G \in \mathfrak{L}(\gamma) \cap \mathfrak{O}\mathfrak{S}$, there is a constant $x_0 > 0$ such that

$$\int_0^{h(x)} \overline{G}(x-y)V^{*n}(dy) \le (1+\varepsilon)M^n(V,\gamma)\overline{G}(x) \quad \text{for all } \ n \ge 1 \ \text{ and } \ x \ge x_0, \tag{4.12}$$

$$\int_{h(x)}^{x-h(x)} \overline{G}(x-y)G(dy) \le (1+\varepsilon)(C^*(G) - 2M(G,\gamma) + \varepsilon)\overline{G}(x), \tag{4.13}$$

and

$$\overline{V}(h(x))\overline{G}(x - h(x)) < \varepsilon\overline{G}(x). \tag{4.14}$$

For $\varepsilon > 0$, we take

$$K \ge \max\{A_1(A-b)/(A\varepsilon), 1/\overline{G}(x_0)\}.$$

Further, by (4.11), we have

$$A_1M^n(V,\gamma) \le \varepsilon KA^n \quad \text{for all } n \ge 1. \tag{4.15}$$

Now, we prove that (4.10) holds for $k = n + 1$.

For all $x \geq x_0$, using integration by parts and inductive hypothesis, by (4.8) and (4.12)–(4.15), we have

$$\overline{V^{*(n+1)}}(x) = \int\limits_0^{h(x)} \overline{V}(x-y)V^{*n}(dy) + \int\limits_0^{x-h(x)} \overline{V^{*n}}(x-y)V(dy) + \overline{V^{*n}}(h(x))\overline{V}(x-h(x))$$

$$\leq A_1 \int\limits_0^{h(x)} \overline{G}(x-y)V^{*n}(dy) + A_n\Bigg(\int\limits_0^{x-h(x)} \overline{G}(x-y)V(dy) + \overline{G}(h(x))\overline{V}(x-h(x))\Bigg)$$

$$= A_1 \int\limits_0^{h(x)} \overline{G}(x-y)V^{*n}(dy) + A_n\Bigg(\int\limits_0^{h(x)} \overline{G}(x-y)V(dy)$$

$$+ \int\limits_{h(x)}^{x-h(x)} \overline{V}(x-y)G(dy) + \overline{V}(h(x))\overline{G}(x-h(x))\Bigg)$$

$$\leq A_1(1+\varepsilon)M^n(V,\gamma)\overline{G}(x) + A_n\Bigg((1+\varepsilon)M(V,\gamma)\overline{G}(x)$$

$$+A_1 \int\limits_{h(x)}^{x-h(x)} \overline{G}(x-y)G(dy) + \varepsilon\overline{G}(x)\Bigg)$$

$$\leq (1+\varepsilon)\overline{G}(x)KA^n(M(V,\gamma) + A_1(C^*(G) - 2M(G,\gamma)) + (2+A_1)\varepsilon).$$

Further, by (4.9), we know that

$$A_{n+1} \leq KA^n(1+\varepsilon)(a+2\varepsilon) \leq KA^{n+1}.$$

Additionally, for all $0 \leq x \leq x_0$, by $A > 1$ and $K \geq 1/\overline{G}(x_0)$, we have

$$A_{n+1} \leq 1/\overline{G}(x_0) \leq KA^{n+1}.$$

Combining with the above two inequalities, we know immediately that (4.10) holds for $k = n+1$. □

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