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LARGE DEVIATION PRINCIPLES FOR THE PROCESSES ADMITTING EMBEDDED COMPOUND RENEWAL PROCESSES

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Abstract: We obtain limit theorems in the domain of large and moderate deviations for the processes admitting embedded compound renewal processes. We justify the large and moderate deviation principles for the trajectories of periodic compound renewal processes with delay and find a moderate deviation principle for the trajectories of semi-Markov compound renewal processes.

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1. Introduction

Denote by $\mathbb{D} = \mathbb{D}[0,\infty)$ the space of functions without discontinuities of the second kind and right-continuous; and by $\mathscr{B}$, the σ-algebra of subsets of $\mathbb{D}$ generated by cylindrical sets. Regard a random process $X(t)$ for $t \geq 0$ with trajectories in $\mathbb{D}$ as defined on the probability space $(\mathbb{D}, \mathscr{B}, \mathbf{P})$.

Suppose that some random time moments

$$0 = T_0 \leq T_1 < T_2 < \cdots$$

divide the process into the cycles identical in the following sense: The random elements

$$\begin{aligned} &((\tau_j, \zeta_j), \{X_j(t),\ 0 \leq t \leq \tau_j\}) \\ &:= ((T_j - T_{j-1}, X(T_j) - X(T_{j-1})), \{X(T_{j-1}+t) - X(T_{j-1}),\ 0 \leq t \leq \tau_j\}) \end{aligned} \tag{1.1}$$

are independent for $j \geq 1$ and identically distributed for $j \geq 2$.

Denote by (τ, ζ) the vector with the same distribution as (τ_j, ζ_j) for $j \geq 2$. We can consider the sequence $\{(\tau_j, \zeta_j)\}$ as the controlling sequence of the compound renewal process (CRP)

$$Z(t) := Z_{\nu(t)}, \quad \text{where } \nu(t) := \max\{k : T_k \leq t\}, \quad t \geq 0,$$

where $Z_0 := 0$ and $Z_k := \sum_{j=1}^{k} \zeta_j$ for $k \geq 1$.

The processes $X(t)$ and $Z(t)$ are obviously related as

$$X(T_j) = Z(T_j), \quad j = 1, 2, \dots,$$

and it is natural to call the CRP $Z(t)$ *embedded* into $X(t)$ with the initial vector of jumps $(\tau_1, \zeta_1) = (\tau_1, X(\tau_1))$.

The classical examples of processes with discrete time $t = n$, admitting the embedded CRP, are Markov additive processes (MAPs), i.e., the sequences of sums of random variables defined on the states of a Harris Markov chain (a chain with a positive atom, possibly, artificial). Such a chain is subdivided into some independent identical cycles generated by the chain returning to the positive atom. Denoting

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the cycle lengths by $\tau_1, \tau_2, \dots$ and the sums of random variables in these cycles by $\zeta_1, \zeta_2, \dots$, we obtain the controlling sequence (1.1) for the embedded CRP. There are also other examples.

It is shown in [1, § 1.8, § 2.5, and § 5.7; 2, 3] that under some natural and rather mild assumptions many limit laws for a MAP on the increasing time interval $[0, n]$ assume the same form as the corresponding limit laws for the embedded CRP.

In order to simplify exposition, in each section we assume firstly that the process $Z(t)$ is homogeneous. In this case it is natural to call the corresponding process $X(t)$ *homogeneous*. We extend the results to the inhomogeneous case at the ends of the sections.

Assume henceforth that the controlling vector (τ, ζ) is nondegenerate; i.e., the distribution of (τ, ζ) in the plane of variables τ and ζ does not lie on any straight line or, which is the same, $\mathbf{D}(\zeta - c\tau) > 0$ for all c.

Following the approach of [1, § 1.8] in studying MAPs, as the main characteristic describing the difference between the processes $X(t)$ and $Z(t)$ we choose the oscillation $\widehat{\zeta}$ of $X(t)$ on the cycles:

$$\widehat{\zeta} := \max_{0 \le t < \tau} X(t) - \min_{0 \le t < \tau} X(t),$$

where $X(t)$ is a homogeneous process, so that $\widehat{\zeta}_j$ means the oscillations on a cycle j.

In many sections we assume that the random variables in question satisfy Cramér's moment condition. Assume that the random variable ξ satisfies Cramér's condition $[\mathbf{C}]$, written as $\xi \Subset [\mathbf{C}]$, whenever $\mathbf{E}e^{\lambda|\xi|} < \infty$ for some $\lambda > 0$. Similarly, we write $\xi \Subset [\mathbf{C}_\infty]$, implying the strong Cramér's condition, whenever $\mathbf{E}e^{\lambda|\xi|} < \infty$ for all $\lambda < \infty$. If all coordinates of $(\tau, \zeta, \widehat{\zeta})$ satisfy condition $[\mathbf{C}]$ then we write $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$.

The article has the following structure: Section 2 considers the large deviation principle (LDP) for the normalized process $\frac{X(T)}{T}$ in the phase space under the condition $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$. This problem is simpler than the similar problem for the trajectories of the process $\frac{X(t)}{T}$ on the increasing time interval $[0, T]$, but it illustrates the next circumstance: If the tails of the distributions of ζ and $\widehat{\zeta}$ have "comparable" asymptotics in the framework of condition $[\mathbf{C}]$ then the LDPs for

$$\mathrm{x}_T := \frac{X(T)}{T} \quad \text{and} \quad \mathrm{z}_T := \frac{Z(T)}{T}$$

coincide in some neighborhood of the point $a := \frac{\mathbf{E}\zeta}{\mathbf{E}\tau}$ and the size of the neighborhood depends on the relation between the tails of these distributions. Outside this neighborhood, the LDP for x_T will in general differ from that for z_T. In the subsequent sections the conditions on $\widehat{\zeta}$ are as a rule stronger than on ζ, which yields the coincidence of the LDP for the trajectories of

$$\mathbf{x}_T = \mathbf{x}_T(t) := \frac{X(Tt)}{T} \quad \text{and} \quad \mathbf{z}_T = \mathbf{z}_T(t) := \frac{Z(Tt)}{T}, \quad t \in [0, 1].$$

Henceforth $\mathbb{D}[0, 1]$ stands for the space of functions without discontinuities of the second kind and right-continuous on $[0, 1]$. Section 3 justifies the individual trajectory LDP for the process $\mathbf{x}_T$ in which for a fixed measurable set $B \subset \mathbb{D}[0, 1]$ we study the asymptotics

$$\log \mathbf{P}(\mathbf{x}_T \in B) \quad \text{as } T \to \infty$$

in the case that $(\tau, \zeta) \Subset [\mathbf{C}]$ and $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$. In particular, we obtain an LDP in the first and the second boundary crossing problems—on the trajectory of $\mathbf{x}_T$ intersecting and not intersecting a prescribed boundary. Section 4 is devoted to the trajectory LDP for $\mathbf{x}_T$ in the case that $\tau \Subset [\mathbf{C}]$ and $(\zeta, \widehat{\zeta}) \Subset [\mathbf{C}_\infty]$. In Section 5 we obtain a trajectory moderate deviation principle for $\mathbf{x}_T$ in the Cramér case when $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$, as well as in the so-called semiexponential case, when the moments $\mathbf{E}e^{\tau^\beta}$, $\mathbf{E}e^{|\zeta|^\beta}$, and $\mathbf{E}e^{|\widehat{\zeta}|^\beta}$ are finite for some $\beta \in (0, 1)$. In Sections 6 and 7 we use the theorems of the previous sections to study large deviations for the trajectories of a *periodic* CRP *with delay* (see also [4]) and a *semi-Markov* CRP (see also [2, 3]).

2. A Large Deviation Principle in the Phase Space

In this section we assume that $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$. Verify that in this case in some domain G including a neighborhood of the point $\alpha = a$ the normalized random variables $\mathrm{x}_T = \frac{X(T)}{T}$ satisfy the same local large deviation principle as the normalized random variables $\mathrm{z}_T = \frac{Z(T)}{T}$; see Theorem 2.2 below.

In order to state the main result more precisely, we need the following notation (cf. [1, 5–18]): Given $(\lambda, \mu) \in \mathbb{R}^2$, put

$$\psi(\lambda, \mu) := \mathbf{E}e^{\lambda\tau+\mu\zeta}, \quad A(\lambda, \mu) := \log \psi(\lambda, \mu)$$

and

$$\mathscr{A} := \{(\lambda, \mu) : A(\lambda, \mu) < \infty\}, \quad \mathscr{A}^{\leq 0} := \{(\lambda, \mu) : A(\lambda, \mu) \leq 0\}.$$

Clearly, the sets $\mathscr{A}$ and $\mathscr{A}^{\leq 0}$ are convex, $\mathscr{A}^{\leq 0} \subset \mathscr{A}$, and the interior $(\mathscr{A})$ of $\mathscr{A}$ is the domain of analyticity of the function $A(\lambda, \mu)$.

In the description of the LDP for the CRP $Z(t)$, the following two functions are crucial. The first is

$$A(\mu) := -\sup\{\lambda : (\lambda, \mu) \in \mathscr{A}^{\leq 0}\},$$

where by definition we assume that the least upper bound over the empty set equals $-\infty$, so that $\Gamma(\mu) := (-A(\mu), \mu)$ for $\mu \in \mathbb{R}$ is a parametric definition of the boundary $\partial\mathscr{A}^{\leq 0}$ of $\mathscr{A}^{\leq 0}$ in $\mathbb{R}^2$. The second function is

$$D(\alpha) := \sup_{\mu}\{\mu\alpha - A(\mu)\}. \tag{2.1}$$

The function $A(\mu)$ is convex and, therefore, differentiable almost everywhere. From (2.1) we infer that the point $\mu(\alpha)$ at which sup in (2.1) is attained is an a.e. unique solution of the equation $A'(\mu) = \alpha$, so that

$$D(\alpha) = \mu(\alpha)\alpha - A(\mu(\alpha)).$$

This implies that $D'(\alpha) = \mu(\alpha)$, the function $\mu(\alpha)$ is nondecreasing, and $\mu(a) = 0$ for $a := \frac{\mathbf{E}\zeta}{\mathbf{E}\tau}$. Put $\lambda(\alpha) := -A(\mu(\alpha))$. Then $\Gamma(\mu(\alpha)) = (\lambda(\alpha), \mu(\alpha))$ for $\alpha \in \mathbb{R}$ is also a parametric definition of $\partial\mathscr{A}^{\leq 0}$ in $\mathbb{R}^2$; furthermore,

$$(\lambda(a), \mu(a)) = (0, 0), \quad A(0) = 0 = D(a), \quad A'(0) = a, \quad D'(a) = 0.$$

The functions $A(\mu)$ and $D(\alpha)$ are called the *basic function* and the *deviation function* for the CRP $Z(t)$ respectively; see [1, Chapter 3]. They are convex and lower continuous; furthermore,

$$A(\mu) = \sup_{\alpha}\{\mu\alpha - D(\alpha)\}.$$

Thus, $A(\mu)$ and $D(\alpha)$ constitute a pair of conjugate functions with respect to the Legendre transform. For more details on the properties of $A(\mu)$ and $D(\alpha)$; see [1, Chapter 3] and the bibliography therein.

Consider the function

$$E_X(\alpha) := \sup_{t>0} e^{\lambda(\alpha)t}\mathbf{E}(e^{\mu(\alpha)(X(t)-X(0))};\ \tau > t)$$

related to the evolution of the homogeneous process $X(t)$ on the typical cycle

$$\{X(t) - X(0);\ 0 \leq t < \tau\}.$$

Using this function, put

$$G_X := \{\alpha \in \mathbb{R} : E_X(\alpha) < \infty\}.$$

In the case that $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$, we have $(\lambda(a), \mu(a)) = (0, 0)$ for $a = \frac{\mathbf{E}\zeta}{\mathbf{E}\tau}$, Thus, G_X contains the point $\alpha = a$ together with some neighborhood.

Observe that $Z(t) - Z(0) = 0$ for all $t \in [0, \tau)$. Therefore,

$$E_Z(\alpha) := \sup_{t>0} e^{\lambda(\alpha)t}\mathbf{E}(e^{\mu(\alpha)(Z(t)-Z(0))};\ \tau > t) = \sup_{t>0} e^{\lambda(\alpha)t}\mathbf{P}(\tau > t). \tag{2.2}$$

(2.2) implies that $G_Z := \{\alpha \in \mathbb{R} : E_Z(\alpha) < \infty\}$ satisfies

$$\{\alpha \in \mathbb{R} : \lambda(\alpha) < \lambda_+\} \subseteq G_Z \subseteq \{\alpha \in \mathbb{R} : \lambda(\alpha) \leq \lambda_+\},$$

where $\lambda_+ := \sup\{\lambda : \mathbf{E}e^{\lambda\tau} < \infty\}$.

Lemma 2.1. *On assuming that* $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$, *the random sequences*

$$\mathrm{x}_T := \frac{X(T)}{T}, \quad \mathrm{z}_T := \frac{Z(T)}{T}$$

satisfy the following:

(i) *If* $\alpha \in G_X$ *then*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathrm{x}_T \in (\alpha)_\varepsilon) \le -D(\alpha) \tag{2.3}$$

for every sequence $\varepsilon = \varepsilon_T = o(1)$ *as* $T \to \infty$;

(ii) *if* $\alpha \in G_Z$ *then*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathrm{z}_T \in (\alpha)_\varepsilon) \le -D(\alpha) \tag{2.4}$$

for every sequence $\varepsilon = \varepsilon_T = o(1)$ *as* $T \to \infty$;

(iii) *for all* $\varepsilon > 0$ *and* $\alpha \in \mathbb{R}$ *we have*

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathrm{x}_T \in (\alpha)_\varepsilon) \ge -D(\alpha); \tag{2.5}$$

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathrm{z}_T \in (\alpha)_\varepsilon) \ge -D(\alpha). \tag{2.6}$$

Say that the *local large deviation principle* (LLDP) *holds* for a sequence $\{\mathrm{y}_T\}_{T>0}$ of random variables *in a domain* $G \subset \mathbb{R}$ *with the deviation function* (DF) $I = I(\alpha)$ as $G \to [0, \infty]$ if

$$\lim_{n\to\infty} \frac{1}{T} \log \mathbf{P}(\mathrm{y}_T \in (\alpha)_\varepsilon) = -I(\alpha)$$

for every $\alpha \in G$ where $\varepsilon = \varepsilon_T \to 0$ vanishes sufficiently slowly as $T \to \infty$.

Theorem 2.2. (i) *On assuming that* $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$, *the* LLDP *holds for two random sequences* $\{\mathrm{x}_T\}_{T>0}$ *and* $\{\mathrm{z}_T\}_{T>0}$ *in the domain* $G_X \cap G_Z$ *with the common* DF $D(\alpha)$.

(ii) *On assuming that* $(\tau, \zeta) \Subset [\mathbf{C}]$ *and* $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$, *the* LLDP *holds for two random sequences* x_T *and* z_T *in the domain* G_Z *with the common* DF $D(\alpha)$.

Proof of Theorem 2.2. Claim (i) is straightforward from Lemma 2.1.

Let us prove claim (ii). By Lemma 2.1, the sequence z_T satisfies the LLDP in G_Z with DF $D(\alpha)$. Condition $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$ implies (see Lemma 4.2) that

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(|\mathrm{x}_T - \mathrm{z}_T| > \varepsilon) = -\infty \quad \text{for all } \varepsilon > 0. \tag{2.7}$$

It is easy to see that (2.7) and the property that z_T satisfies the LLDP in G_Z with DF $D(\alpha)$ implies claim (ii). The proof of Theorem 2.2 is complete.

Let us prove Lemma 2.1(i),(ii). For an arbitrary $\varepsilon > 0$ we have

$$P(T) := \mathbf{P}(\mathrm{x}_T \in (\alpha)_\varepsilon) = \sum_{n=0}^{\infty} P_n(T),$$

where

$$P_n(T) := \mathbf{P}(\mathrm{x}_T \in (\alpha)_\varepsilon,\ T_n \le T < T_n + \tau_{n+1}).$$

Thus,

$$P(T) = \sum_{n\le T^2} P_n(T) + \sum_{n>T^2} P_n(T) =: \sum\nolimits_1 + \sum\nolimits_2 . \tag{2.8}$$

Firstly estimate the sum $\sum_2$ using Chebyshev's inequalities:

$$\sum\nolimits_2 \le \sum_{n>T^2} \mathbf{P}(T_n \le T) = \sum_{n>T^2} \mathbf{P}\Big(\sum_{k=1}^{n} \tau_k \le T\Big)$$
$$= \sum_{n>T^2} \mathbf{P}(e^{-\sum_{k=1}^{n}\tau_k} \ge e^{-T}) \le \sum_{n>T^2} \frac{(\mathbf{E}e^{-\tau})^n}{e^{-T}}.$$

Since $\mathbf{P}(\tau > 0) = 1$, we have $c := \mathbf{E}e^{-\tau} < 1$, and so

$$\sum\nolimits_2 \le \sum_{n>T^2} e^T c^n \le e^{\frac{T^2}{2}\log c + T} \frac{1}{1-c} \le e^{\frac{T^2}{3}\log c} \tag{2.9}$$

for all sufficiently large T. Estimate each term $P_n(T)$ in $\sum_1$ as $n \ge 0$. For $(\lambda,\mu) = (\lambda(\alpha),\mu(\alpha))$ and every $n \ge 0$ we have

$$P_n(T) = \mathbf{E}(e^{\pm\lambda T_n + \pm\mu Z_m};\ X(T) \in (T\alpha)_{T\varepsilon},\ T_n \le T < T_n + \tau_{n+1}). \tag{2.10}$$

The inequality $0 \le \mu X(T) - \mu\alpha T + |\mu|\varepsilon T$ holds on the event

$$H_n(T) := \{X(T) \in (T\alpha)_{T\varepsilon},\ T_n \le T < T_n + \tau_{n+1}\},$$

and for $\gamma := T - T_n < \tau_{n+1}$ we have

$$-\lambda T_n - \mu Z_n \le -\lambda T - \mu\alpha T + |\mu|\varepsilon T + \lambda\gamma + \mu(X(T_n+\gamma) - Z_n).$$

Therefore, on the event $H_n(T)$ by $Z_n = X(T_n)$ we have

$$e^{-\lambda T_n - \mu Z_n} \le e^{-\lambda T - \mu\alpha T + |\mu|\varepsilon T} e^{\lambda\gamma + \mu(X(T_n+\gamma) - X(T_n))}. \tag{2.11}$$

From (2.10), (2.11), and the equalities

$$-\lambda T - \mu\alpha(T) = -\lambda(\alpha)T - \mu(\alpha)\alpha(T) = -TD(\alpha)$$

we deduce that

$$P_n(T) \le e^{-TD(\alpha) + |\mu|\varepsilon} E_n(T), \tag{2.12}$$

where

$$E_n(T) := \mathbf{E}(e^{\lambda T_n + \mu Z_n} e^{\lambda\gamma + \mu(X(T_n+\gamma) - X(T_n))};\ T_n \le T < T_n + \tau_{n+1})$$
$$\le \int_0^\infty \mathbf{E}(e^{\lambda T_n + \mu Z_n};\ T - T_n \in dt)\mathbf{E}(e^{\lambda t + \mu(X(t) - X(0))};\ \tau > t).$$

Since the parameter α lies in G_X, the second factor in the integral is bounded uniformly in $t \ge 0$ by some constant $C < \infty$, so that (2.12) yields

$$P_n(T) \le e^{-TD(\alpha) - |\mu|\varepsilon} C\psi^n(\lambda,\mu) \le e^{-TD(\alpha) - |\mu|\varepsilon} C. \tag{2.13}$$

Using then (2.8), (2.9), and (2.13), we find that

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(T) \le -D(\alpha) + |\mu|\varepsilon,$$

which implies (2.3).

Replacing in the above argument the process $X(t)$ with the process $Z(t)$, we obtain a proof of (2.4).
(iii) Choose and fix for this proof some constants $c \in (0,\infty)$, $R \in (0,\infty)$, and $q \in (0,1]$ so that

$$\mathbf{P}(\tau \geq c,\ |\zeta| \leq R,\ \widehat{\zeta} \leq R) \geq q. \tag{2.14}$$

Consider for $r > 0$, $T > \frac{1}{r}$, $\varepsilon > 0$, and $\delta \in (0,1)$ for $n := [rT]$ the event

$$C_T(\varepsilon,\delta) := \left\{ \frac{T_n}{n} \in \frac{T}{n}(1-\delta)_\delta,\ \frac{Z_n}{n} \in \frac{T}{n}(\alpha)_{\frac{\varepsilon}{2}} \right\}.$$

On this event the sum T_n lies to the left of the value of T, and the distance $\mathscr{T} := T - T_n$ lies within the limits

$$\mathscr{T} \in (0, 2T\delta) \quad \text{with probability 1.} \tag{2.15}$$

On the event $C_T(\varepsilon,\delta)$ consider the new process

$$X'(t) := X(T_n + t) - X(T_n), \quad t \geq 0,$$

independent of the process $\{X(t),\ 0 \leq t \leq T_n\}$, where we emphasize that $n = [rT]$. The new process $X'(t)$ is obviously homogeneous and admits the embedded CRP which we will denote by

$$Z'(t) := Z(t + T_n) - Z(T_n), \quad t \geq 0.$$

All notation related to the new process $X'(t)$ includes the superscript $'$; for instance, $T'_k := T_{n+k} - T_n$ for $k \geq 0$, and so on. Then on the event $C_T(\varepsilon,\delta)$ we have

$$X(T) = X(T_n) + X'(\mathscr{T}) = Z_n + Z'_{\nu'(\mathscr{T})} + X'(\mathscr{T}) - Z'_{\nu'(\mathscr{T})},$$

where the random variable $\mathscr{T}$ satisfies (2.15). For

$$k(T) := \left[\frac{2T\delta}{c}\right] + 1$$

consider the event

$$B_T := \{\tau'_m \geq c,\ |\zeta'_m| \leq R,\ \widehat{\zeta}'_m \leq R,\ m \in \{1,\ldots,k(T)\}\}.$$

Then by (2.15) the event $C_T(\varepsilon,\delta) \cap B_T$ satisfies

$$|Z'_{\nu'(\mathscr{T})}| \leq k(T)R, \quad |X'(\mathscr{T}) - Z'_{\nu'(\mathscr{T})}| \leq R,$$

so that

$$\left|\frac{X(T)}{T} - \frac{Z_n}{T}\right| \leq \frac{2R\delta}{c} + \frac{2R}{T}$$

on this event. Choosing $\delta > 0$ sufficiently small and $T < \infty$ sufficiently large so that

$$\frac{2R\delta}{c} + \frac{2R}{T} \leq \frac{\varepsilon}{2},$$

we see that

$$\left|\frac{X(T)}{T} - \alpha\right| < \varepsilon$$

on $C_T(\varepsilon,\delta) \cap B_T$. Hence,

$$\mathbf{P}\left(\frac{X(T)}{T} \in (\alpha)_\varepsilon\right) \geq \mathbf{P}(C_T(\varepsilon,\delta) \cap B_T) = \mathbf{P}(C_T(\varepsilon,\delta))\mathbf{P}(B_T) \geq \mathbf{P}(C_T(\varepsilon,\delta))q^{k(T)},$$

where the last inequality follows from (2.14). Thus,

$$L_-(\alpha,\varepsilon) := \liminf_{T\to\infty} \frac{1}{T}\log \mathbf{P}\left(\frac{X(T)}{T}\in(\alpha)_\varepsilon\right) \ge \liminf_{T\to\infty}\frac{1}{T}\log\mathbf{P}(C_T(\varepsilon,\delta)) + \frac{2\delta}{c}\log q.$$

The available lower bound in the LLDP for the random walk (T_n, Z_n), see Theorem 1.2.1 of [12] for instance, implies that

$$\liminf_{T\to\infty}\frac{1}{T}\log\mathbf{P}(C_T(\varepsilon,\delta)) \ge -r\Lambda\left(\frac{1-\delta}{r},\frac{\alpha}{r}\right),$$

where $\Lambda(\theta,\alpha) := \sup_{\lambda,\mu}\{\lambda\theta+\mu\alpha - A(\lambda,\mu)\}$ is the (first) deviation function for the random vector (τ,ζ). Thus,

$$L_-(\alpha,\varepsilon) \ge -r\Lambda\left(\frac{1-\delta}{r},\frac{\alpha}{r}\right) + \frac{2\delta}{c}\log q.$$

Minimizing the right-hand side of the last inequality over $r>0$, for $\beta := \frac{1}{1-\delta}\alpha$ we obtain

$$L_-(\alpha,\varepsilon) \ge -(1-\delta)D_\Lambda(1,\beta) + \frac{2\delta}{c}\log q,$$

where $D_\Lambda(\theta,\beta) := \inf_{r>0} r\Lambda\big(\frac{\theta}{r},\frac{\beta}{r}\big)$ is the second deviation function for the random vector (τ,ζ). Choosing furthermore $\alpha'\in(\alpha)_\varepsilon$ and $\varepsilon'>0$ so that $(\alpha')_{\varepsilon'}\subset(\alpha)_\varepsilon$, for $\beta' := \frac{\alpha'}{1-\delta}$ we find that

$$L_-(\alpha,\varepsilon) \ge L_-(\alpha',\varepsilon') \ge -(1-\delta)D_\Lambda(1,\beta') + \frac{2\delta}{c}\log q.$$

Maximizing the right-hand side of the last inequalities over $\alpha'\in(\alpha)_\varepsilon$, we find that

$$L_-(\alpha,\varepsilon) \ge -(1-\delta)\inf_{\alpha'=(1-\delta)\beta'\in(\alpha)_\varepsilon} D_\Lambda(1,\beta') + \frac{2\delta}{c}\log q.$$

Since for every $\varepsilon'\in(1,\varepsilon)$ and all sufficiently small $\delta>0$ the right-hand side of the last inequality is at most

$$-(1-\delta)\inf_{\alpha'\in(\alpha)_{\varepsilon'}} D_\Lambda(1,\alpha') + \frac{2\delta}{c}\log q;$$

passing to the limit as $\delta\downarrow 0$, we see that

$$L_-(\alpha,\varepsilon) \ge -\inf_{\alpha'\in(\alpha)_{\varepsilon'}} D_\Lambda(1,\alpha').$$

Passing furthermore on the right-hand side here to the limit as $\varepsilon'\downarrow 0$ and using Lemma 3.1 of [10], we arrive at the required relation (2.5)

$$L_-(\alpha,\varepsilon) \ge -D(1,\alpha) = -D(\alpha).$$

Since (2.6) can be justified similarly, the proof of Lemma 2.1 is complete.

Remark 2.3. The results of Theorem 2.2 can be carried over to the inhomogeneous case. To this end, in the hypotheses of claim (i) of Theorem 2.2 it suffices to require in addition that $(\tau_1,\zeta_1,\widehat{\zeta}_1)\Subset[\mathbf{C}]$; furthermore, G_Z and G_X should be replaced in general by some smaller sets G'_Z and G'_X containing the point a with some neighborhood. In the hypotheses of claim (ii) of Theorem 2.2 it suffices to require in addition that $(\tau_1,\zeta_1)\Subset[\mathbf{C}]$ and $\widehat{\zeta}_1\Subset[\mathbf{C}_\infty]$; furthermore, G_Z should be replaced in general by a smaller set G'_Z containing the point a with some neighborhood.

3. Individual Trajectory Large Deviation Principles

While discussing LDP for the trajectories of random walks or CRP, we usually mean the fulfillment of appropriate relations for all measurable sets $B \subset \mathbb{D}$. These propositions require, as a rule, rather restrictive conditions on the jumps of the processes (condition $[\mathbf{C}_\infty]$). At the same time, there may exist quite large classes of sets $B \subset \mathbb{D}$ for which these relations hold on only assuming condition $[\mathbf{C}]$. As [1, Chapter 4] shows, that includes sets related to a trajectory of the process intersecting or not intersecting a receding curvilinear boundary.

In this section we study the cases in which for some specified classes of sets $B \subset \mathbb{D}[0,1]$ the trajectories of the process

$$\mathbf{z}_T = \mathbf{z}_T(t) := \frac{Z(tT)}{T}, \quad 0 \le t \le 1, \tag{3.1}$$

satisfy an "individual" LDP; we find conditions under which the same individual LDP remains valid for all trajectories of

$$\mathbf{x}_T = \mathbf{x}_T(t) := \frac{X(tT)}{T}, \quad 0 \le t \le 1. \tag{3.2}$$

Equip the space $\mathbb{D}[0,1]$ with the uniform metric, i.e., given $f, g \in \mathbb{D}[0,1]$, put

$$\rho(f,g) := \sup_{t\in[0,1]} |t(t) - g(t)|.$$

Denote the resulting metric space by $\mathbb{D}_U[0,1]$.

Define in $\mathbb{D}_U[0,1]$ the deviation functional (integral)

$$I = I(f) := \begin{cases} \int\limits_0^1 D(f'(t))dt & \text{if} \quad f \in \mathbb{C}_a, \\ \infty & \text{if } f \in \mathbb{D}_U[0,1] \setminus \mathbb{C}_a, \end{cases} \tag{3.3}$$

where $\mathbb{C}_a$ is the class of absolutely continuous functions $f \in \mathbb{D}_U[0,1]$ with $f(0) = 0$.

Denote by $(B)_\varepsilon$ the ε-neighborhood of a measurable set $B \subset \mathbb{D}_U[0,1]$, i.e.,

$$B_\varepsilon := \{g \in \mathbb{D}_U[0,1] : \inf_{f\in B} \rho(f,g) < \varepsilon\}.$$

Say that a measurable set $B \subset \mathbb{D}_U[0,1]$ *is of class* $\mathscr{L}D_+$ whenever for some $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$ we have

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{z}_T \in (B)_\varepsilon) \le -I([(B)_\varepsilon]), \quad \liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{z}_T \in (B)) \ge -I((B)); \tag{3.4}$$

and, furthermore,

$$I((B)) = I(B+) := \lim_{\varepsilon\downarrow 0} I([(B)_\varepsilon]). \tag{3.5}$$

If (3.4) and (3.5) hold for the normalized process $\mathbf{x}_T$ (see (3.2)) and a fixed measurable set $B \subset \mathbb{D}_U[0,1]$ then

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{x}_T \in B) = -I(B), \tag{3.6}$$

and we say that *the trajectories of* $\mathbf{x}_T$ *satisfy the individual* LDP *in* $\mathbb{D}_U[0,1]$.

To construct simple examples of sets B of class $\mathscr{L}D_+$, consider for $\alpha \in \mathbb{R}$ the family of sets $B(\alpha) := \{f \in \mathbb{D}_U[0,1] : f(1) \ge \alpha\}$. If $(\tau, \lambda) \Subset [\mathbf{C}]$ and $\lambda_+ := \sup\{\lambda : \mathbf{E}e^{\lambda\tau} < \infty\} \ge D(0)$, while the real α is such that the convex deviation function $D(t)$ (see (2.1)) is finite in some neighborhood of the point $t = \alpha$; then by the local LDP in the phase space for the CRP $Z(t)$ (see [10]) the relations (3.4) and (3.5) for the set $B(\alpha)$ are satisfied and, consequently, $B(\alpha) \in \mathscr{L}D_+$. Other examples of sets B of class $\mathscr{L}D_+$ will appear at the end of this section.

Theorem 3.1. *Suppose that* $(\tau,\zeta) \Subset [\mathbf{C}]$, $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$, *and* $\lambda_+ \ge D(0)$. *Then*

(i) *for all* $f \in \mathbb{C}_a$, $\varepsilon > 0$, *and* $\delta = \delta_T \downarrow 0$ *as* $T \to \infty$ *we have the inequalities* (*i.e., the partial* LLDP *for* $\mathbf{x}_T$)

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{x}_T \in (f)_\delta) \le -I(f), \tag{3.7}$$

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{x}_T \in (f)_\varepsilon) \ge -I(f); \tag{3.8}$$

(ii) *if* $B \in \mathscr{L}D_+$ *then the trajectories of* $\mathbf{x}_T$ *satisfy the individual* LDP (3.6).

Proof. (i): By hypotheses, the CRP $\mathbf{z}_T$ satisfies the so-called first partial LLDP, [1, Theorem 4.2.1]:

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{z}_T \in (f)_\delta) \le -I(f), \tag{3.9}$$

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{z}_T \in (f)_\varepsilon) \ge -I(f) \tag{3.10}$$

for all $f \in \mathbb{C}_a$, $\varepsilon > 0$, and $\delta = \delta_T \downarrow 0$ as $T \to \infty$. Use the obvious inequalities

$$\mathbf{P}(\mathbf{z}_T \in (f)_{2\gamma}) \ge \mathbf{P}(\mathbf{x}_T \in (f)_\gamma,\ \rho(\mathbf{x}_T, \mathbf{z}_T) < \gamma) \ge \mathbf{P}(\mathbf{x}_T \in (f)_\gamma) - \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma).$$

Thus,

$$\mathbf{P}(\mathbf{z}_T \in (f)_{2\gamma(T)}) \ge \mathbf{P}(\mathbf{x}_T \in (f)_{\gamma(T)}) - \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma(T)) \tag{3.11}$$

for all $T > 0$ and $\gamma(T) > 0$.

Let us verify that for every fixed $\gamma > 0$ we have

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma) = -\infty. \tag{3.12}$$

If $\gamma > 0$ then

$$\mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma) \le P_1 + P_2, \tag{3.13}$$

where $P_1 := \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma,\ \nu(T) \le T^2)$ and $P_2 := \mathbf{P}(\nu(T) > T^2)$. Let us estimate P_1 from above. Since $Z(T_n) = X(T_n)$ for $n \ge 0$; therefore,

$$\begin{aligned} P_1 &= \mathbf{P}\Big(\max_{1\le n\le [T^2]+1} \sup_{t\in(T_{n-1},T_n)} |X(t) - X(T_{n-1})| > T\gamma\Big) \\ &= \mathbf{P}\Big(\max_{1\le n\le [T^2]+1} \widehat{\zeta}_n > T\gamma\Big) \le (T^2+1)\mathbf{P}(\widehat{\zeta} > T\gamma). \end{aligned} \tag{3.14}$$

Since $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$, using (3.14) and Chebyshev's inequality, for every $\lambda > 0$ we obtain

$$\limsup_{T\to\infty} \frac{1}{T} \log P_1 \le \limsup_{T\to\infty} \frac{1}{T} \log \left(\frac{\mathbf{E} e^{\lambda\widehat{\zeta}}}{e^{\lambda\gamma T}} \right) = -\lambda\gamma.$$

Letting $\lambda \to \infty$, we find that

$$\limsup_{T\to\infty} \frac{1}{T} \log P_1 = -\infty. \tag{3.15}$$

Now, estimate P_2 from above. Chebyshev's inequality yields

$$\begin{aligned} P_2 &= \mathbf{P}(\nu(T) > T^2) \le \mathbf{P}(T_{[T^2]} < T) \\ &= \mathbf{P}\bigg(\exp\Big\{ -\sum_{k=1}^{[T^2]} \tau_k \Big\} > \exp\{-T\} \bigg) \le \frac{(\mathbf{E} e^{-\tau})^{[T^2]}}{e^{-T}}. \end{aligned}$$

Since $\mathbf{P}(\tau > 0) = 1$, it follows that $c := \mathbf{E}e^{-\tau} < 1$. Hence,

$$\limsup_{T\to\infty} \frac{1}{T} \log P_2 \le \limsup_{T\to\infty} \frac{1}{T}([T^2] \log c + T) = -\infty. \tag{3.16}$$

Using (3.13), (3.15), and (3.16), we establish (3.12).

From (3.12) we infer that there is a function $\gamma_0 = \gamma_0(T) \downarrow 0$ as $T \to \infty$ such that

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma(T)) = -\infty \tag{3.17}$$

for all $\gamma(T) \ge \gamma_0(T)$. Using (3.9), (3.11), and (3.17), we establish (3.7).

Swapping in (3.11) the processes $\mathbf{z}_T$ and $\mathbf{x}_T$, we see that

$$\mathbf{P}(\mathbf{x}_T \in (f)_{2\gamma}) \ge \mathbf{P}(\mathbf{z}_T \in (f)_\gamma) - \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \gamma) \tag{3.18}$$

for every fixed $\gamma > 0$. So, (3.10), (3.12), and (3.18) yield (3.8). Claim (i) of the theorem is justified.

(ii): Use the obvious inequalities

$$\mathbf{P}(\mathbf{z}_T \in (B)_\varepsilon) \ge \mathbf{P}(\mathbf{x}_T \in B,\ \rho(\mathbf{x}_T, \mathbf{z}_T) < \varepsilon) \ge \mathbf{P}(\mathbf{x}_T \in B) - \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) \ge \varepsilon).$$

Furthermore, from (3.12) and the property that B belongs to $\mathscr{L}D_+$, we find that

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{x}_T \in B) \le -I([(B)_\varepsilon]) \le -I((B)_{2\varepsilon}) \tag{3.19}$$

for all sufficiently small $\varepsilon > 0$.

The relation

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{x}_T \in B) \ge -I((B)) \tag{3.20}$$

is straightforward from (3.8). Since (3.6) follows from (3.19) and (3.20), the proof of Theorem 3.1 is complete.

Example 3.2 (an individual LDP in the first boundary crossing problem for the process $\mathbf{x}_T$). This problem is related to the asymptotics of the probability $\mathbf{P}(\mathbf{x}_T \in B_g)$, where for a prescribed function $g = g(t) \in \mathbb{D}_U[0,1]$ with $g(t) - at > 0$ for all $t \in [0,1]$, the set B_g is of the form

$$B_g := \{f \in \mathbb{D}_U[0,1] : f(0) = 0, \quad \sup_{t\in[0,1]} \{f(t) - g(t)\} \ge 0\}.$$

In other words, B_g consists of $f \in \mathbb{D}_U[0,1]$ starting at zero and exceeding or reaching the level $g(t)$ on the segment $[0,1]$.

Theorem 4.6.2 of [1] proposed some mild conditions under which for some $t_g \in (0,1]$ the function

$$f_g(t) := \begin{cases} t\frac{g(t_g)}{t_g} & \text{for } 0 \le t \le t_g, \\ g(t_g) + a(t - t_g) & \text{for } t_g \le t \le 1, \end{cases}$$

is the most probable trajectory in B_g for the process $\mathbf{z}_T$. In other words,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{z}_T \in B_g) = -I((B_g)) = -I([B_g]) = -I(f_g).$$

Furthermore, since $B_g \in \mathscr{L}D_+$ Theorem 3.1 implies, under a suitable condition on $\widehat{\zeta}$, an individual LDP for the process $\mathbf{x}_T$ in the first boundary crossing problem.

Example 3.3 (an individual LDP in the second boundary crossing problem for the process $\mathbf{x}_T$). This problem is related to the asymptotics of the probability $\mathbf{P}(\mathbf{x}_T \in B_{g_-,g_+})$, where for the prescribed functions $g_- = g_-(t)$ and $g_+ = g_+(t) \in \mathbb{D}_U[0,1]$ with $g_-(0) < 0 < g_+(0)$ and $g_+(t) - g_-(t) > 0$ for all $t \in [0,1]$ the set B_{g_-,g_+} is of the form

$$B_{g_-,g_+} := \{f \in \mathbb{D}_U[0,1] : f(0) = 0,\ g_-(t) \le f(t) \le g_+(t),\ t \in [0,1]\}.$$

In other words, B_{g_-,g_+} consists of the functions $f \in \mathbb{D}_U[0,1]$ starting at zero and on the segment $[0,1]$ remaining in the curvilinear strip with the boundaries $g_-(t)$ and $g_+(t)$.

Theorem 4.7.2 of [1] proposed conditions under which the most probable trajectory f_{g_-,g_+} in B_{g_-,g_+} for the process $\mathbf{z}_T$ is constructed. In other words,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\mathbf{z}_T \in B_{g_-,g_+}) = -I(B_{g_-,g_+}) = -I([B_{g_-,g_+}]) = -I(f_{g_-,g_+}).$$

Furthermore, since $B_{g_-,g_+} \in \mathscr{L}D_+$ Theorem 3.1 implies, under a suitable condition on $\widehat{\zeta}$, an individual LDP for the process $\mathbf{x}_T$ in the second boundary crossing problem.

Remark 3.4. The result of Theorem 3.1 can be carried over to the inhomogeneous case. To this end, in its hypotheses it suffices to require in addition that $\widehat{\zeta}_1 \Subset [\mathbf{C}_\infty]$ and $\mathscr{A}^{\le 0} \subset [\mathscr{A}_1]$, where $\mathscr{A}_1 := \{(\lambda,\mu) : \mathbf{E}e^{\lambda\tau_1+\mu\zeta_1} < \infty\}$.

4. Trajectory Large Deviation Principles

Given a family $\{\mathbf{y}_T(t);\ 0 \le t \le 1\}_{T>0}$ of processes in $\mathbb{D}_U[0,1]$, a sequence $\psi(T) \to \infty$ as $T \to \infty$, a functional $J(f) : \mathbb{D}[0,1] \to [0,\infty]$ such that for every $v \ge 0$ the set $\{f : J(f) \le v\}$ is compact in $\mathbb{D}_U[0,1]$, say that $\mathbf{y}_T(t)$ *satisfies* an LDP *with the deviation functional J and normalizing sequence $\psi(T)$* (briefly we denote this as $(J,\psi(T))$-LDP) if

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \log \mathbf{P}(\mathbf{y}_T \in B) \le -J([B]), \quad \liminf_{T\to\infty} \frac{1}{\psi(T)} \log \mathbf{P}(\mathbf{y}_T \in B) \ge -J((B)),$$

for every measurable set $B \subset \mathbb{D}_U[0,1]$, where (B) and $[B]$ are the interior and closure of B, while $J(B) := \inf_{f\in B} J(f)$.

In this section we use the notation of Sections 2 and 3. We consider the case that the trajectories of the normalized process $\mathbf{z}_T$ (see (3.1)) satisfy the (J,T)-LDP and study the conditions on $\widehat{\zeta}$ under which a similar LDP holds for $\mathbf{x}_T$ (see (3.2)).

As the deviation functional $J = J(f)$ below we use the functional $I = I(f)$ defined in (3.3).

Theorem 4.1. *Assume that $\tau \Subset [\mathbf{C}]$, $(\zeta,\widehat{\zeta}) \Subset [\mathbf{C}_\infty]$, and $\lambda_+ \ge D(0)$. Then $\mathbf{x}_T$ satisfies the* (I,T)-LDP.

Proof. Theorem 4.5.1 of [1] (see also [5, 6]) implies that under the hypotheses of our theorem the family $\mathbf{z}_T$ satisfies the (I,T)-LDP. Use the following statement:

Lemma 4.2. *Assume that $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$. Then for every $\varepsilon > 0$ we have*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbf{P}(\rho(\mathbf{z}_T,\mathbf{x}_T) > \varepsilon) = -\infty. \tag{4.1}$$

It is known (see [19, Theorem 4.2.13] for instance) that if the process $\mathbf{z}_T$ satisfies the (I,T)-LDP and (4.1) holds then the process $\mathbf{x}_T$ satisfies the same LDP. Thus, Theorem 4.1 is justified, and it remains to prove the lemma.

Proof of Lemma 4.2. If $\varepsilon > 0$ then

$$\mathbf{P}(\rho(\mathbf{z}_T,\mathbf{x}_T) > \varepsilon) \le P_1 + P_2 := \mathbf{P}(\rho(\mathbf{z}_T,\mathbf{x}_T) > \varepsilon,\ \nu(T) \le T^2) + \mathbf{P}(\nu(T) > T^2). \tag{4.2}$$

Estimate P_1 from above. Since $X(T_n) = Z(T_n)$ for $n \ge 0$, it follows that

$$\mathbf{P}(\max_{1\le n\le [T^2]+1} \sup_{t\in(T_{n-1},T_n)} |X(t) - X(T_{n-1})| > \varepsilon T)$$
$$\le \mathbf{P}(\max_{1\le n\le [T^2]+1} \widehat{\zeta}_n > \varepsilon T) \le (T^2+1)\mathbf{P}(\widehat{\zeta} > \varepsilon T). \tag{4.3}$$

Since $\widehat{\zeta} \Subset [\mathbf{C}_\infty]$, using the exponential Chebyshev's inequality, by (4.3) for all $\varepsilon > 0$ and $\lambda < \infty$ we obtain

$$\limsup_{T\to\infty} \frac{1}{T} \log P_1 \le \lim_{T\to\infty} \frac{1}{T} \log\left(\frac{\mathbf{E}e^{\lambda\widehat{\zeta}}}{e^{\lambda\varepsilon T}}\right) = -\lambda\varepsilon.$$

Letting $\lambda \to \infty$, for every fixed $\varepsilon > 0$ we arrive at

$$\limsup_{T\to\infty} \frac{1}{T} \log P_1 \le -\infty. \tag{4.4}$$

To estimate P_2, applying Chebyshev's inequality, we find that

$$P_2 = \mathbf{P}(\nu(T) > T^2) \le \mathbf{P}(T_{[T^2]} < T) \le \mathbf{P}(\exp\{-T_{[T^2]}\} > \exp\{-T\}) \le \frac{(\mathbf{E}e^{-\tau})^{[T^2]}}{e^{-T}}.$$

Since $c := \mathbf{E}e^{-\tau} < 1$, the last inequality yields

$$\limsup_{T\to\infty} \frac{1}{T} \log P_2 \le 1 + \lim_{T\to\infty} \frac{[T^2]\log c}{T} = -\infty. \tag{4.5}$$

Now (4.2), (4.4), and (4.5) imply the claim of Lemma 4.2.

Remark 4.3. The result of Theorem 4.1 can be carried over to the inhomogeneous case. In the conditions of the theorem it suffices to require in addition that $(\zeta_1, \widehat{\zeta}_1) \Subset [\mathbf{C}_\infty]$ and

$$\lambda_+^1 := \sup\{\lambda : \mathbf{E}e^{\lambda\tau_1} < \infty\} \ge D(0).$$

5. A Trajectory Moderate Deviation Principle

In this section, as before, we study a process $X(t)$ that admits an embedded homogeneous CRP $Z(t)$. In the case that the trajectories of $Z(t)$ satisfy a moderate deviation principle (MDP), we find the conditions under which the process $X(t)$ satisfies the same MDP. More precisely, for a fixed sequence $x = x(T) \to \infty$ as $T \to \infty$ we consider the two families of processes

$$\mathbf{x}_T = \mathbf{x}_T(t) := \frac{X(tT) - atT}{\sqrt{T}x}, \quad \mathbf{z}_T = \mathbf{z}_T(t) := \frac{Z(tT) - atT}{\sqrt{T}x}, \tag{5.1}$$

where

$$a = \frac{\mathbf{E}\zeta}{\mathbf{E}\tau}, \quad \sigma^2 := \frac{\mathbf{E}(\zeta - a\tau)^2}{\mathbf{E}\tau},$$

and the random vector (τ, ζ) controls the embedded CRP $Z(t)$.

We study the conditions under which the LDP for trajectories of the processes $\mathbf{x}_T$ and $\mathbf{z}_T$ are the same (see the definition of a $(J, \psi(T))$-LDP in Section 4).

As $J = J(f)$ in this section we use the deviation functional (integral)

$$I_0 = I_0(f) := \begin{cases} \frac{1}{2\sigma^2}\int\limits_0^1 (f'(t))^2 dt & \text{if } f \in \mathbb{C}_a, \\ \infty & \text{if } f \in \mathbb{D}[0,1] \setminus \mathbb{C}_a. \end{cases} \tag{5.2}$$

The properties of I_0 are thoroughly studied; see [12, Chapter 5] for instance. In particular, I_0 is a lower semicontinuous convex functional on $\mathbb{D}_U[0,1]$ and the sets $\{f : I_0(f) \le c\}$ are compact in the uniform metric for every $c \ge 0$.

To state the main theorem of this section, along with condition $[\mathbf{C}]$ we need the milder moment condition $[\mathbf{C}^V]$. Denote by $\mathscr{L}_\beta$, for $\beta \in (0,1)$, the class of functions $V = V(t) : (0,\infty) \to (0,\infty)$ satisfying the following:

(1) $V(t) = t^\beta l(t)$ is a regularly varying function (r.v.f.), and so $l(t)$ is a slowly varying function (s.v.f.) as $t \to \infty$;

(2) if $t \to \infty$ and $u = o(t)$ then

$$V(t+u) - V(t) = \beta u \frac{V(t)}{t}(1 + o(1)) + o(1).$$

Say that a random variable ξ *satisfies condition* $[\mathbf{C}^V]$, whenever for $V \in \mathscr{L}_\beta$ the inequality $\mathbf{P}(|\xi| > t) \le e^{-V(t)}$ holds for all $t > 0$. Denote the fulfillment of condition $[\mathbf{C}^V]$ as $\xi \Subset [\mathbf{C}^V]$. By analogy with the above, if $\tau \Subset [\mathbf{C}^V]$ and $\zeta \Subset [\mathbf{C}^V]$ simultaneously, then for the vector (τ,ζ) we write $(\tau,\zeta) \Subset [\mathbf{C}^V]$.

Addressing processes (5.1), consider the deviation $x = x(T)$ of the form

$$x \to \infty, \quad x = o(\widehat{x}) \quad \text{as } T \to \infty, \tag{5.3}$$

where the sequence $\widehat{x} = \widehat{x}(T)$ depends on whether the condition $[\mathbf{C}]$ or $[\mathbf{C}^V]$ being used and is defined as follows:

$$\widehat{x}(T) := \begin{cases} \sqrt{T} & \text{for} \quad [\mathbf{C}], \\ T^{-\frac{1}{2}} v^{(-1)}(1/T) & \text{for } [\mathbf{C}^V]. \end{cases}$$

Here

$$v^{(-1)}(1/T) := \sup\left\{ t \ge 0 : t^{-2}V(t) \ge \frac{1}{T} \right\},$$

i.e., $v^{(-1)}(u)$ is a generalized inverse function to $v(t) := t^{-2}V(t)$. Section 4.8.1 of [1] (see also [20]) contains the explicit form of the function $v^{(-1)}(1/T)$:

$$v^{(-1)}(1/T) = T^{\frac{1}{2-\beta}} l_1(T),$$

where $l_1(T)$ is an SVF as $T \to \infty$.

Thus, the smaller the parameter β within the interval $(0,1)$, the weaker condition $[\mathbf{C}^V]$ for $V \in \mathscr{L}_\beta$ and the smaller the zone of moderate deviations (5.3).

Theorem 5.1 (an MLDP for trajectories of $\mathbf{x}_T$). *Suppose that either* $(\tau,\zeta,\widehat{\zeta}) \Subset [\mathbf{C}]$ *or* $(\tau,\zeta,\widehat{\zeta}) \Subset [\mathbf{C}^V]$ *and the sequence* $x = x(T)$ *satisfies condition* (5.3). *Then the family* $\mathbf{x}_T$ *of processes, see* (5.1), *satisfy the* $(I_0, x^2(T))$*-LDP, where the deviation functional* $I_0 = I_0(f)$ *is defined in* (5.2).

Proof. Theorem 4.8.2 of [1], see also [20, 21], implies that under the hypotheses of the theorem the family of processes $\mathbf{z}_T$, see (5.1), satisfies the $(I_0, x^2(T))$-LDP. Thus, the claim of Theorem 5.1 follows from the next lemma.

Lemma 5.2. *Assume the hypotheses of Theorem* 3.1. *Then*

$$\lim_{T\to\infty} \frac{1}{x^2} \log \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) > \varepsilon) = -\infty$$

for every $\varepsilon > 0$.

It remains to prove this lemma.

Proof of Lemma 5.2. Given $\varepsilon > 0$ and $C > 0$, we have

$$\mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) > \varepsilon) \le P_1 + P_2, \tag{5.4}$$

where $P_1 := \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) > \varepsilon, \nu(T) \le CT)$ and $P_2 := \mathbf{P}(\nu(T) > CT)$. Estimate P_1 from above. Since $Z(T_n) = X(T_n)$ for $n \ge 0$; therefore,

$$P_1 = \mathbf{P}\Big(\max_{1\le n\le [CT]+1} \sup_{T_{n-1}<t<T_n} |X(t) - X(T_{n-1})| > \sqrt{T}x\varepsilon,\ \nu(T) \le CT\Big) \le \mathbf{P}\Big(\max_{1\le n\le [CT]+1} \widehat{\zeta}_n > \sqrt{T}x\varepsilon\Big) \le (1 + [CT])\mathbf{P}(\widehat{\zeta} > \sqrt{T}x\varepsilon). \tag{5.5}$$

If condition [**C**] holds then, applying to the right-hand side of (5.5) the exponential Chebyshev inequality and taking into account the zone of deviations (5.3), for all $\varepsilon > 0$, $C > 0$, and some $\lambda > 0$ we obtain the upper bound

$$\limsup_{T\to\infty} \frac{1}{x^2} \log P_1 \le \limsup_{T\to\infty} \frac{1}{x^2} \log \frac{\mathbf{E}e^{\lambda\widehat{\zeta}}}{e^{\lambda\sqrt{T}x\varepsilon}} = -\lim_{T\to\infty} \frac{\lambda\sqrt{T}x\varepsilon}{x^2} = -\infty. \tag{5.6}$$

If condition [$\mathbf{C}^V$] holds then, using condition (5.3) and the definition of the functions $v(t)$ and $v^{(-1)}(u)$, we see that

$$\limsup_{T\to\infty} \frac{1}{x^2} \log P_1 \le \limsup_{T\to\infty} \frac{1}{x^2} \log((1 + [CT])e^{-V(\sqrt{T}x\varepsilon)}) \le -\varepsilon^\beta \lim_{T\to\infty} \frac{V(\sqrt{T}x)}{x^2}$$
$$= -\varepsilon^\beta \lim_{T\to\infty} \frac{v(\sqrt{T}x)}{v(v^{(-1)}(1/T))} = -\varepsilon^\beta \lim_{T\to\infty} \frac{v(o(v^{(-1)}(1/T)))}{v(v^{(-1)}(1/T))} = -\infty \tag{5.7}$$

for every $\varepsilon > 0$. To estimate P_2 from above, applying Chebyshev's inequality, we infer that

$$P_2 = \mathbf{P}(\nu(T) > CT) \le \mathbf{P}(T_{[CT]} < T) = \mathbf{P}\Big(e^{\sum_{k=1}^{[CT]} \tau_k} < e^T\Big) \le \mathbf{P}\Big(e^{-\sum_{k=1}^{[CT]} \tau_k} > e^{-T}\Big) \le \frac{(\mathbf{E}e^{-\tau})^{[CT]}}{e^{-T}}.$$

Since $\mathbf{P}(\tau > 0) = 1$, we have $c := \mathbf{E}e^{-\tau} < 1$. Therefore, choosing the constant $C = C_0(c) := -\frac{2}{\log c} + 1$, for $T \ge 2$ we find that

$$P_2 \le \frac{c^{-\frac{2T}{\log c}}}{e^{-T}} = e^{-T}.$$

Using condition (5.3), we infer that

$$\limsup_{T\to\infty} \frac{1}{x^2} \log P_2 \le -\lim_{T\to\infty} \frac{T}{x^2} = -\infty. \tag{5.8}$$

Now (5.4)–(5.8) imply that

$$\limsup_{T\to\infty} \frac{1}{x^2} \log \mathbf{P}(\rho(\mathbf{x}_T, \mathbf{z}_T) > \varepsilon) \le \limsup_{T\to\infty} \frac{1}{x^2} \log(2\max(P_1, P_2)) = -\infty$$

for every $\varepsilon > 0$. The proof of Lemma 5.2 is complete.

Remark 5.3. The result of Theorem 5.1 can be carried over to the inhomogeneous case. In the conditions of the theorem it suffices to require in addition that $(\tau_1, \zeta_1, \widehat{\zeta}_1) \Subset [\mathbf{C}]$ or $(\tau_1, \zeta_1, \widehat{\zeta}_1) \Subset [\mathbf{C}^V]$.

6. Large and Moderate Deviation Principles for Periodic Compound Renewal Processes with Delay

To define the object of study, consider the sequence

$$\xi_k^* = (\tau_k^*, \zeta_k^*), \quad \mathbf{P}(\tau_k^* \geq 0) = 1, \quad k \geq 1, \tag{6.1}$$

of random vectors and split (6.1) into disjoint independent groups $\mathscr{G}_1, \mathscr{G}_2, \dots, \mathscr{G}_k, \dots$. To this end, choose some integer parameter $s \geq 0$ corresponding to the delay, an integer parameter $m \geq 1$ corresponding to the period, and put

$$\mathscr{G}_1 := \{\xi_1^*, \dots, \xi_s^*\}, \quad \mathscr{G}_k := \{\xi_{s+(k-2)m+1}^*, \dots, \xi_{s+(k-1)m}^*\}, \quad k \geq 2.$$

Assume that the tuples of vectors in the distinct group $\mathscr{G}_k$ for $k \geq 1$ are jointly independent and for $k \geq 2$ the identically distributed. We emphasize incidentally that the vectors ξ_k^* in one of the groups $\mathscr{G}_1$, or $\mathscr{G}_2$ for instance, can be "however dependent."

Let us show how sequence (6.1) under our assumptions determines some periodic CRP with delay $X(t)$. To this end, construct a random walk $\{(T_n^*, Z_n^*)\}_{n\geq 0}$ generated by the sums of random vectors (6.1):

$$(T_0^*, Z_0^*) := (0,0), \quad (T_n^*, Z_n^*) := \left(\sum_{k=1}^n \tau_k^*, \sum_{k=1}^n \zeta_k^*\right), \quad n \geq 1.$$

Then $X(t) := Z_{\nu^*(t)}^*$, $t \geq 0$, where

$$\nu^*(t) := \begin{cases} 0 & \text{if } t = 0; \\ \max\{n \geq 0 : T_n^* \leq t\} & \text{if } \quad t > 0. \end{cases}$$

Thus, for $m = 1$ the periodic CRP with delay $X(t)$ becomes an homogeneous CRP if $s = 0$ and an inhomogeneous CRP if $s = 1$. Construct for a process $X(t)$ an embedded CRP $Z(t)$. For that, it suffices to construct for $Z(t)$ the controlling sequence

$$\xi_k = (\tau_k, \zeta_k), \quad k \geq 1. \tag{6.2}$$

Perform this construction by letting

$$\xi_k := \sum_{\{j:\xi_j^* \in \mathscr{G}_k\}} \xi_j^* = \Bigl(\sum_{\{j:\xi_j^* \in \mathscr{G}_k\}} \tau_j^*, \sum_{\{j:\xi_j^* \in \mathscr{G}_k\}} \zeta_j^*\Bigr).$$

We also require that the vector $\xi = (\tau, \zeta)$ having the same distribution with the vectors $\xi_k = (\tau_k, \zeta_k)$ for $k \geq 2$ is nondegenerate in $\mathbb{R}^2$ and $\mathbf{P}(\tau > 0) = 1$. It is obvious that the so-constructed sequence (6.2) is controlling for the CRP $Z(t)$; which, in turn, is embedded into the periodic CRP with delay $X(t)$. Furthermore, we have defined the characteristics (see Section 1) $(\tau_1, \zeta_1, \widehat{\zeta}_1)$ and $(\tau, \zeta, \widehat{\zeta})$, where

$$\widehat{\zeta}_1 := \max_{0 \leq t < \tau_1} X(t) - \min_{0 \leq t < \tau_1} X(t), \quad \widehat{\zeta} = \max_{\tau_1 \leq t < \tau_2} X(t) - \min_{\tau_1 \leq t < \tau_2} X(t).$$

In the following statement we use the notation $D(\alpha)$, $I(f)$, and so on as in Section 4.

Theorem 6.1. *Suppose that* $\tau_k^* \Subset [\mathbf{C}]$, $\zeta_k^* \Subset [\mathbf{C}_\infty]$ *for* $1 \leq k \leq s+m$ *and*

$$\lambda_+^{(1)} := \sup\{\lambda : \mathbf{E}e^{\lambda\tau_1} < \infty\} \geq D(0), \quad \lambda_+ := \sup\{\lambda : \mathbf{E}e^{\lambda\tau} < \infty\} \geq D(0).$$

Then the two families of processes

$$\mathbf{x} = \mathbf{x}(t) := \frac{1}{x} X(tT), \quad \mathbf{z} = \mathbf{z}(t) := \frac{1}{x} Z(tT), \quad 0 \leq t \leq 1,$$

where $x = x(T) \sim T$ *as* $t \to \infty$, *satisfy the* (I, T)-LDP *in* $\mathbb{D}_U[0,1]$.

Proof. Theorem 4.1 and Remark 4.3 imply that it suffices to show that $(\tau_1,\tau)\Subset[\mathbf{C}]$ and $(\zeta_1,\widehat{\zeta}_1,\zeta,\widehat{\zeta})\Subset[\mathbf{C}_\infty]$. Verify firstly that $(\tau_1,\tau)\Subset[\mathbf{C}]$. Clearly, it suffices to show that $\mathbf{E}e^{\lambda(\widetilde{\tau}+\widetilde{\zeta})}<\infty$ for some $\lambda>0$, where

$$\widetilde{\tau}:=\sum_{k=1}^{s+m}\tau_k^*.$$

Since $\tau_k^*\Subset[\mathbf{C}]$ for $1\le k\le s+m$, there is $\mu>0$ such that

$$\max_{1\le k\le s+m}\mathbf{E}e^{\mu\tau_k^*}<\infty.$$

Put $\lambda:=\frac{\mu}{s+m}$. Hölder's inequality yields

$$\mathbf{E}e^{\lambda\widetilde{\tau}}=\mathbf{E}\exp\left\{\lambda\sum_{k=1}^{s+m}\tau_k^*\right\}\le\left(\prod_{k=1}^{s+m}\mathbf{E}e^{\mu\tau_k^*}\right)^{\frac{1}{s+m}}<\infty.$$

Now the property $(\zeta_1,\widehat{\zeta}_1,\zeta,\widehat{\zeta})\Subset[\mathbf{C}_\infty]$, follows since $\zeta_k^*\Subset[\mathbf{C}_\infty]$ for $1\le k\le s+m$ and

$$\max(|\zeta_1|,|\widehat{\zeta}_1|,|\zeta|,|\widehat{\zeta}|)\le\sum_{k=1}^{s+m}|\zeta_k^*|.$$

The proof of Theorem 6.1 is complete.

In the next statement we use the notation a, σ^2, $I_0(f)$, and so on as in Section 5.

Theorem 6.2. *Suppose that $\tau_k^*\Subset[\mathbf{C}]$ and $\zeta_k^*\Subset[\mathbf{C}]$ for $1\le k\le s+m$. Then the two families of processes*

$$\mathbf{x}=\mathbf{x}(t):=\frac{1}{x\sqrt{T}}(X(tT)-atT),\quad \mathbf{z}=\mathbf{z}(t):=\frac{1}{x\sqrt{T}}(Z(tT)-atT),\quad 0\le t\le 1,$$

where $x=x(T)\to\infty$ and $\frac{x}{\sqrt{T}}\to 0$ as $t\to\infty$, satisfy the (I_0,T)-LDP in $\mathbb{D}_U[0,1]$.

Proof. By Theorem 5.1 and Remark 5.3, it suffices to show that $(\tau,\zeta,\widehat{\zeta})\Subset[\mathbf{C}]$ and $(\tau_1,\zeta_1,\widehat{\zeta}_1)\Subset[\mathbf{C}]$. Obviously, it suffices to verify that for some $\lambda>0$ we have

$$\mathbf{E}e^{\lambda(\widetilde{\tau}+\widetilde{\zeta})}<\infty,$$

where

$$\widetilde{\tau}+\widetilde{\zeta}:=\sum_{k=1}^{s+m}\tau_k^*+\sum_{k=1}^{s+m}|\zeta_k^*|.$$

Further arguments reduce to applying Hölder's inequality, as in the proof of Theorem 6.1, and we omit them. The proof of Theorem 6.2 is complete.

Remark 6.3. In [4], the law of large numbers (LLN) and the functional central limit theorem are obtained for periodic renewal processes with delay, as well as the LLN is justified for periodic CRPs with delay in the case that τ_k^* and ζ_k^* for $k\ge 1$ are independent. The reference also includes some survey of the results related to the applications of processes of this type. Thus, Theorem 6.2 refines the results of [4], extending the invariance principle into the domain of moderate deviations.

Remark 6.4. The results of this section are valid for a larger class of processes than the periodic CRP with delay. Actually, the random vectors appearing in the class $\mathscr{G}_k$, for $k\ge 1$, can be ordered arbitrarily.

7. The Moderate Deviation Principle for Semi-Markov Compound Renewal Processes

Let us define the object of study. We are given a time-homogeneous Markov chain $\kappa(n)$, $n \geq 0$, with finitely many $l \geq 1$ essential states $\mathbb{L} := \{1, \dots, l\}$ which is *indecomposable* and *nonperiodic*, i.e., *ergodic*; see [22, Section 13.4] for instance. Denote the matrix of transition probabilities by

$$\|p_{i,j}\| := \|\mathbf{P}(\kappa(1) = j \mid \kappa(0) = i)\|, \quad i, j \in \mathbb{L}.$$

Suppose also that we are given a sequence of independent tuples of random vectors

$$\mathscr{F}_k := \{\xi_k^{i,j} = (\tau_k^{i,j}, \zeta_k^{i,j}), i, j \in \mathbb{L}\}$$

and $k \geq 1$, independent of the Markov chain $\kappa(n)$ and having the same distributions as the tuple of random vectors

$$\mathscr{F} := \{\xi^{i,j} = (\tau^{i,j}, \zeta^{i,j}),\ i, j \in \mathbb{L}\},$$

where $\mathbf{P}(\tau^{i,j} > 0) = 1$. Assume the following nondegeneracy condition: *There exist $i, j \in \mathbb{L}$ such that $p_{i,j} > 0$ and*

$$\mathbf{P}(b\tau^{i,j} + c\zeta^{i,j} = d) < 1$$

for all $b, c, d \in \mathbb{R}$ with $|b| + |c| \neq 0$.

Fix the initial position of the chain $\kappa(0) = i_0 \in \mathbb{L}$ and define the walk (T_n^*, Z_n^*), for $n \geq 0$, by putting

$$T_0^* = Z_0^* := 0, \quad T_n^* := \sum_{k=1}^{n} \tau_k^{\kappa(k-1),\kappa(k)}, \quad Z_n^* := \sum_{k=1}^{n} \zeta_k^{\kappa(k-1),\kappa(k)}, \quad n \geq 1.$$

From the coordinate T_n^* construct the renewal process

$$\nu^*(t) := \begin{cases} 0 & \text{if} \quad t = 0, \\ \max\{n \geq 0 : T_n^* \leq t\} & \text{if} \quad t > 0. \end{cases}$$

Now we can define the semi-Markov CRP (for the initial position $\kappa(0) = i_0$ of the chain)

$$X(t) = X_{i_0}(t) := Z^*_{\nu^*(t)}, \quad t \geq 0.$$

This is the main process studied in this section.

Along with the initial state i_0 of the chain, fix an arbitrary state $i \in \mathbb{L}$, put $r(0) := 0$, and for $n \geq 1$ define the successive moments

$$r(n) := \inf\{k > n(n-1) : \kappa(n) = i\}$$

of the return of κ into the state i. It is obvious that the random variables

$$m_1 := r(1),\ \dots,\ m_n := r(n) - r(n-1),\ \dots$$

are jointly independent, and for $n \geq 2$ identically distributed. Furthermore, $m_1, m_2 \Subset [\mathbf{C}]$; see [22, Section 13.4] for instance.

Consider the sequence

$$(T_0, Z_0) := (0, 0), \dots, (T_n, Z_n) := (T^*_{r(n)}, Z^*_{r(n)})$$

and define the vectors $(\tau_j, \zeta_j) := (T_j - T_{j-1},\ Z_j - Z_{j-1},)$ for $j \geq 1$. It is easy to see that these vectors are independent and for $j \geq 2$ identically distributed; hence, they determine for the process $X(t)$ an embedded CRP $Z(t)$.

Define the two constants

$$a = a_i := \frac{\mathbf{E}_i\zeta_2}{\mathbf{E}_i\tau_2}, \quad \sigma^2 = \sigma_i^2 := \frac{\mathbf{E}_i(\zeta_2 - a_i\tau_2)^2}{\mathbf{E}_i\tau_2}, \tag{7.1}$$

where $\mathbf{E}_i Y$ means $\mathbf{E}(Y \mid \kappa(0) = i)$. In Theorem 7.1 we establish that constants (7.1) are independent of $i \in \mathbb{L}$. Thus, in the notation of Section 5, which we use below and which involves these parameters, they are independent of $i \in \mathbb{L}$.

Theorem 7.1. *Suppose that $(\tau^{i,j}, \zeta^{i,j}) \Subset [\mathbf{C}]$ for $i, j \in \mathbb{L}$. Then the constants a and σ^2 in (7.1) are independent of the parameter $i \in \mathbb{L}$, and the family of processes*

$$\mathbf{x}_T = \mathbf{x}_T(t) := \frac{1}{x\sqrt{T}}(X(tT) - atT), \quad 0 \le t \le 1,$$

where the sequence $x = x(T)$ is such that

$$\lim_{T\to\infty} x = \infty, \quad \lim_{T\to\infty} \frac{x}{\sqrt{T}} = 0,$$

for an arbitrary initial state of the Markov chain $\kappa(0) = i_0$ satisfies the $(I_0, x^2(T))$-LDP in $\mathbb{D}_U[0,1]$.

Proof. Fix some initial state i_0 of the Markov chain and an arbitrary state $i \in \mathbb{L}$, using which we construct an embedded CRP $Z(t)$. Theorem 5.1 and Remark 5.3 imply that the family of processes

$$\mathbf{x}_T(t) := \frac{1}{x\sqrt{T}}(X(tT) - a_i tT), \quad \mathbf{z}_T(t) := \frac{1}{x\sqrt{T}}(Z(tT) - a_i tT), \quad 0 \le t \le 1, \tag{7.2}$$

for the initial state i_0 satisfy the $\bigl(I_0^{(i)}, x^2(T)\bigr)$-LDP, where the deviation functional is defined for the constant σ_i^2 if $(\tau, \zeta, \widehat{\zeta}) \Subset [\mathbf{C}]$ and $(\tau_1, \zeta_1, \widehat{\zeta}_1) \Subset [\mathbf{C}]$. Obviously, it suffices to show that $\mathbf{E}e^{\lambda(\widetilde{\tau}+\widetilde{\zeta})} < \infty$ for some $\lambda > 0$, where

$$\widetilde{\tau} := \sum_{k=1}^{m_1+m_2} \tau_k^{\kappa(k-1),\kappa(k)} \quad \text{and} \quad \widetilde{\zeta} := \sum_{k=1}^{m_1+m_2} \bigl|\zeta_k^{\kappa(k-1),\kappa(k)}\bigr|.$$

The condition $(\tau^{i,j}, \zeta^{i,j}, m_1, m_2) \Subset [\mathbf{C}]$ for $i, j \in \mathbb{L}$ implies that there are $\lambda > 0$ and $\mu > 0$ such that

$$\max_{i,j\in\mathbb{L}} \mathbf{E}e^{\lambda(\tau^{i,j}+|\zeta^{i,j}|)} \le e^{\frac{\mu}{2}}, \quad \mathbf{E}e^{\mu(m_1+m_2)} < \infty. \tag{7.3}$$

Using (7.3) and Chebyshev's inequality, we obtain

$$\begin{aligned}
\mathbf{E}e^{\lambda(\widetilde{\tau}+\widetilde{\zeta})} &= \sum_{d=1}^{\infty} \mathbf{E}_{i_0}\left(\exp\left\{\lambda \sum_{k=1}^{d} \bigl(\tau_k^{\kappa(k-1),\kappa(k)} + \bigl|\zeta_k^{\kappa(k-1),\kappa(k)}\bigr|\bigr)\right\} \mid m_1 + m_2 = d\right)\mathbf{P}(m_1 + m_2 = d) \\
&\le \sum_{d=1}^{\infty} (\max_{i,j\in\mathbb{L}} \mathbf{E}e^{\lambda(\tau^{i,j}+|\zeta^{i,j}|)})^d \mathbf{P}(m_1 + m_2 = d) \\
\le \mathbf{E}e^{\mu(m_1+m_2)} &\sum_{d=1}^{\infty} \bigl(\max_{i,j\in\mathbb{L}} \mathbf{E}e^{\lambda(\tau^{i,j}+|\zeta^{i,j}|)}\bigr)^d e^{-\mu d} \le \mathbf{E}e^{\mu(m_1+m_2)} \sum_{d=1}^{\infty} e^{-\frac{\mu}{2}d} < \infty.
\end{aligned}$$

Therefore, we showed that the family $\mathbf{x}_T(t)$ satisfies the $\bigl(I_0^{(i)}, x^2(T)\bigr)$-LDP. This implies that

$$\lim_{T\to\infty} \mathbf{P}\left(\left|\frac{1}{T}X(T) - a_i\right| > \varepsilon\right) = 0 \tag{7.4}$$

for an arbitrary initial state i_0 and every $\varepsilon > 0$. Similarly we can prove (7.4), which instead of a_i involves a_j for every $j \in \mathbb{L}$. This circumstance shows that $a = a_1 = a_2 = \cdots = a_l$. The latter implies that the families of centered and normalized processes $\mathbf{x}_T$ defined in (7.2) for distinct $i \in \mathbb{L}$ coincide.

Apart from the initial state $i_0 \in \mathbb{L}$, choose two arbitrary distinct states $i, j \in L$ and denote by $I_0^{(i)} = I_0^{(i)}(f)$ and $I_0^{(j)} = I_0^{(j)}(f)$ the deviation functionals constructed for the parameters σ_i^2 and σ_j^2. The above implies that the family of processes $\mathbf{x}_T$ simultaneously satisfies the $\bigl(I_0^{(i)}, x^2\bigr)$-LDP and the $\bigl(I_0^{(j)}, x^2\bigr)$-LDP. Since this is possible only if σ_i^2 and σ_j^2 coincide, the parameters σ_i^2 are independent of $i \in \mathbb{L}$. The proof of Theorem 7.1 is complete.

Remark 7.2. In [18], local theorems are obtained for multidimensional arithmetic semi-Markov CRP in the domains of normal, moderately large, and partially large deviations.

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