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THE ROOT-CLASS RESIDUALITY OF THE FUNDAMENTAL GROUPS OF CERTAIN GRAPHS OF GROUPS WITH CENTRAL EDGE SUBGROUPS

E. V. Sokolov

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Abstract: Consider a class $\mathcal{C}$ of groups which contains at least one nontrivial group and is closed under subgroups, extensions, and direct products of the form $\prod_{y\in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$. Suppose that G is either a tree product of finitely many groups with central edge subgroups or the fundamental group of an arbitrary graph of groups with trivially intersecting central edge subgroups. We establish some sufficient conditions for G to be residually a $\mathcal{C}$-group.

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1. Introduction

This article continues [1–3] and aims mainly at studying the root-class residuality of the fundamental groups of graphs of groups with trivially intersecting central edge subgroups. See, for instance, [2] for details on the root classes and the root-class residuality of free constructions of groups. Here we only recall that a class $\mathcal{C}$ of groups is a root class if and only if $\mathcal{C}$ contains at least one nontrivial group and is closed under subgroups, extensions, and direct products of the form $\prod_{y\in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$.

Most statements on the residuality of free constructions of groups are proved for the generalized free products of two groups and HNN-extensions with one stable letter, whereas fewer for tree products. In almost all of those statements the restrictions on the groups in the construction do not enable us to use other free constructions as the groups. Therefore, even though the fundamental groups of arbitrary graphs of groups amount to HNN-extensions of tree products, their study does not reduce to considering the above particular cases. So, rather few results on the residuality of such groups are available; namely,

criteria for residual finiteness [4] and residual nilpotence [5] in the case that all vertex groups are finite;

criteria for residual finiteness [6] and residual nilpotence [7], residuality with respect to finite ρ-groups [8], where ρ is a nonempty set of primes, arbitrary root-class residuality [7] on assuming that the graph is finite and all its vertex groups and edge subgroups are infinite cyclic;

a criterion for residual finiteness in the case that the graph is finite and all its vertex groups are finitely generated, nilpotent, and torsion-free [9];

a criterion for arbitrary root-class residuality for the fundamental group of a graph of isomorphic groups [10];

certain sufficient conditions for the residual finiteness of a finite graph of groups with either all vertex groups virtually free [11] or all edge subgroups cyclic [12].

Thus, the results of the article make an advance in this area despite rather strong restrictions on the graph of groups.

In [3] there appeared the general necessary and sufficient conditions for the root-class residuality of the fundamental group of an arbitrary graph of groups. However, to apply them it is necessary in particular to construct homomorphisms of the fundamental group onto groups of the approximating class

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which are injective on all vertex groups. In the case of central edge subgroups some general approach to constructing such homomorphisms is proposed in [1]. Here we apply it to the graphs of groups with trivially intersecting central edge subgroups (Theorem 1). The same method enables us to prove, under certain additional assumptions, that we can drop the requirement of trivial intersection of edge subgroups whenever the graph has at most one simple cycle (Theorem 2). Together with that, the available results on the residuality of HNN-extensions with central associated subgroups; see [13–15] for instance, show that the appearance of cycles and nontrivially intersecting edge subgroups in the graph of groups considerably complicates the residuality conditions for its fundamental group, and the full analog of Theorem 1 in this case cannot be obtained.

Theorems 1 and 2 in particular yield some sufficient conditions for the residuality of the fundamental group of a graph of groups. However, they assume implicitly that all edge subgroups of the graph belong to the approximating class. Using the results of [3], we can overcome this restriction for the graphs of groups with trivially intersecting edge subgroups and the tree products of finitely many groups satisfying certain additional assumptions (Theorems 3 and 4).

Even if some group under study is finitely presented, its structure might have description in terms of the fundamental groups of infinite graphs of groups: For instance, the normal closure of the base of an HNN-extension amounts to the tree product of infinitely many isomorphic copies of the base group. Thus, in this article we try to avoid wherever possible the finiteness requirement for the graph under consideration despite some complications in the statements and proofs of a series of propositions.

2. Statements of the Results

Concerning the graphs of groups, we stick to the notation and assumptions of [3]. Namely, assume throughout that Γ is a nonempty undirected connected graph with vertex set V and edge set E which may have loops and multiple edges. Also, assume that T is a maximal subtree in Γ with edge set E_T.

Choose the directions for all edges of Γ arbitrarily, and denote the endpoints of an edge $e \in E$ by $e(1)$ and $e(-1)$. Associate to each vertex $v \in V$ some group G_v, while to each edge $e \in E$ some group H_e and injective homomorphisms $\varphi_{+e}\colon H_e \to G_{e(1)}$ and $\varphi_{-e}\colon H_e \to G_{e(-1)}$. So, we obtain the directed *graph of groups*

$$\mathcal{G}(\Gamma) = (\Gamma,\ G_v\ (v \in V),\ H_e\ (e \in E),\ \varphi_{\varepsilon e}\ (e \in E,\ \varepsilon = \pm 1)).$$

Refer to G_v for $v \in V$ as a *vertex group*, and to the subgroups $H_{+e} = H_e\varphi_{+e}$ and $H_{-e} = H_e\varphi_{-e}$ as *edge subgroups*. Given $v \in V$, put

$$\Theta_v = \{(e,\varepsilon) \mid e \in E,\ \varepsilon = \pm 1,\ v = e(\varepsilon)\}, \quad H_v = \operatorname{sgp}\{H_{\varepsilon e} \mid (e,\varepsilon) \in \Theta_v\}.$$

Recall that the presentation of the fundamental group of a graph of groups depends in general on the choice of a maximal subtree. In this article we assume that the fundamental group $\pi_1(\mathcal{G}(\Gamma))$ of the graph of groups $\mathcal{G}(\Gamma)$ has the presentation corresponding to a tree T fixed in advance:

$$\pi_1(\mathcal{G}(\Gamma)) = \left\langle \begin{matrix} G_v\ (v \in V),\ t_e\ (e \in E \setminus E_T); \\ H_{+e} = H_{-e}\ (e \in E_T),\ t_e^{-1}H_{+e}t_e = H_{-e}\ (e \in E \setminus E_T) \end{matrix} \right\rangle.$$

Consider the following collection of properties of the graph $\mathcal{G}(\Gamma)$:

(1) for each $v \in V$ the subgroup H_v amounts to the direct product of the subgroups $H_{\varepsilon e}$ for $(e,\varepsilon) \in \Theta_v$;

(2) Γ is a tree;

(3) Γ has precisely one simple cycle.

Say that a graph of groups $\mathcal{G}(\Gamma)$ *is of type* (k), where $1 \le k \le 3$, if $\mathcal{G}(\Gamma)$ enjoys property (k) and H_v lies in the center of G_v for each $v \in V$.

Observe that if (3) holds then $\pi_1(\mathcal{G}(\Gamma))$ amounts to the HNN-extension with one stable letter of the tree product of G_v for $v \in V$. Considering such groups below, we always require that the approximating class $\mathcal{C}$ contain nonperiodic groups. Although this additional restriction has no relation to the group $\pi_1(\mathcal{G}(\Gamma))$, in order to simplify statements it is convenient to include the restriction in the description of the type of a graph of groups. Thus, we use the expression "the graph of groups is of type $(3)_{+\mathcal{C}}$" as a brief version of the conjunction "the graph of groups is of type (3) and the class $\mathcal{C}$ contains at least one nonperiodic group."

Theorem 1. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients. Consider a graph $\mathcal{G}(\Gamma)$ of groups of type (1) such that for each $v \in V$ the group G_v admits a homomorphism σ_v onto a $\mathcal{C}$-group injective on the subgroup H_v. Then the following hold:*

(1) *If the direct product D of $G_v\sigma_v$ for $v \in V$ lies in $\mathcal{C}$ then there exists a homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a $\mathcal{C}$-group extending the homomorphisms σ_v for $v \in V$.*

(2) *If all G_v for $v \in V$ are $\mathcal{C}$-residual then so is $\pi_1(\mathcal{G}(\Gamma))$.*

(3) *Denote by $\mathcal{C}_{\mathrm{tf}}$ the class of all torsion-free $\mathcal{C}$-groups and suppose that the group $G_v\sigma_v$ is torsion-free and the subgroup $H_v\sigma_v$ is isolated in $G_v\sigma_v$ for each $v \in V$. Then*

(a) *if $D \in \mathcal{C}$ then we can choose the homomorphism σ so that $\pi_1(\mathcal{G}(\Gamma))\sigma \in \mathcal{C}_{\mathrm{tf}}$;*

(b) *if all G_v for $v \in V$ are $\mathcal{C}_{\mathrm{tf}}$-residual then so is $\pi_1(\mathcal{G}(\Gamma))$.*

Theorem 2. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients. Consider a graph $\mathcal{G}(\Gamma)$ of groups of type (2) or $(3)_{+\mathcal{C}}$. Suppose also that, for each $v \in V$, the group G_v admits a homomorphism σ_v onto a $\mathcal{C}$-group injective on the subgroup H_v and the direct product D of $G_v\sigma_v$ for $v \in V$ lies in $\mathcal{C}$. Then the following hold:*

(1) *There exists a homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of class $\mathcal{C}$ extending the homomorphisms σ_v for $v \in V$.*

(2) *If all G_v for $v \in V$ are $\mathcal{C}$-residual, then so is $\pi_1(\mathcal{G}(\Gamma))$.*

(3) *Denote by $\mathcal{C}_{\mathrm{tf}}$ the class of all torsion-free $\mathcal{C}$-groups. Suppose that for each $v \in V$ the group $G_v\sigma_v$ is torsion-free and all subgroups $H_{\varepsilon e}\sigma_v$ for $((e,\varepsilon) \in \Theta_v)$ are isolated in $G_v\sigma_v$. Then*

(a) *we can choose a homomorphism σ so that $\pi_1(\mathcal{G}(\Gamma))\sigma \in \mathcal{C}_{\mathrm{tf}}$;*

(b) *if all G_v for $v \in V$ are $\mathcal{C}_{\mathrm{tf}}$-residual then so is $\pi_1(\mathcal{G}(\Gamma))$.*

In the assumptions of Theorem 2 the requirement that $\mathcal{C}$ contains the direct product D and, unless Γ is a tree, at least one nonperiodic group, in general is essential for the $\mathcal{C}$-residuality of $\pi_1(\mathcal{G}(\Gamma))$, as follows from the main results of [16] and [15] respectively. The following generalize Corollaries 2 and 3 of [1] and Corollaries 1 and 2 of [2].

Corollary 1. *Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type (1), (2), or (3), for each $v \in V$ the group G_v is solvable, and the solvability lengths of all groups G_v are jointly bounded. Then the following hold:*

(1) *$\pi_1(\mathcal{G}(\Gamma))$ is residually solvable.*

(2) *Suppose that all G_v for $v \in V$ are torsion-free and at least one of the following holds:*

(a) *the graph $\mathcal{G}(\Gamma)$ of groups is of type (1) and the subgroup H_v is isolated in G_v for all $v \in V$;*

(b) *the graph $\mathcal{G}(\Gamma)$ of groups is of type (2) or (3) and for all $e \in E$ and $\varepsilon = \pm 1$ the subgroup $H_{\varepsilon e}$ is isolated in $G_{e(\varepsilon)}$.*

Then $\pi_1(\mathcal{G}(\Gamma))$ is residually torsion-free solvable.

(3) *If the graph $\mathcal{G}(\Gamma)$ of groups is of type (1) or (2) and for some set ρ of primes all G_v for $v \in V$ are periodic ρ-groups with jointly bounded exponents then $\pi_1(\mathcal{G}(\Gamma))$ is residually a periodic solvable ρ-group of finite exponent.*

Corollary 2. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients and that $\mathcal{G}(\Gamma)$ is a graph of groups of type (1) or a finite graph of groups of type (2) or $(3)_{+\mathcal{C}}$. The group $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}$-residual provided that for every $v \in V$ at least one of the following holds:*

(1) *G_v lies in $\mathcal{C}$;*

(2) *G_v is $\mathcal{C}$-residual and the subgroup H_v is finite;*

(3) *G_v is residually a torsion-free $\mathcal{C}$-group and the subgroup H_v is of finite rank.*

Henceforth, given a class $\mathcal{C}$ of groups and some group X, denote by $\mathcal{C}^*(X)$ the family of all normal subgroups of X the quotients by which lie in $\mathcal{C}$.

Recall [17] that a subgroup Y of a group X is called *$\mathcal{C}$-separable* in X whenever

$$\bigcap_{Z\in\mathcal{C}^*(X)} YZ = Y.$$

Therefore, a group X is $\mathcal{C}$-residual if and only if the trivial subgroup of X is $\mathcal{C}$-separable. Say also that a group X is *$\mathcal{C}$-regular* with respect to a subgroup Y if each subgroup $M \in \mathcal{C}^*(Y)$ coincides with some subgroup of the form $N \cap Y$ with $N \in \mathcal{C}^*(X)$.

The concept of regularity generalizes the classical concept of a *potent element* [18]: If $\mathcal{F}$ is the class of all finite groups then an element $x \in X$ is potent if and only if the group X is $\mathcal{F}$-regular with respect to the cyclic subgroup $\langle x\rangle$. Like potency, regularity is used in studying the residuality of free constructions of groups to construct the subgroups of the vertex groups with prescribed intersections with the edge subgroups; for more details, see [19, § 2.3].

Theorem 3. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients while $\mathcal{G}(\Gamma)$ is a graph of groups of type* (1)*. Suppose also that G_v is $\mathcal{C}$-residual and $\mathcal{C}$-regular with respect to the subgroup H_v which is $\mathcal{C}$-separable in G_v for all $v \in V$. Then $\pi_1(\mathcal{G}(\Gamma))$ is a $\mathcal{C}$-residual group.*

Before stating the next theorem, observe that if $\mathcal{G}(\Gamma)$ is a finite graph of groups of type (1) or (2) then without loss of generality we may assume that $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$. Indeed, if Γ is a tree and $H_{\varepsilon e} = G_{e(\varepsilon)}$ for some $e \in E$ and $\varepsilon = \pm 1$ then $G_{e(\varepsilon)} = H_{-\varepsilon e} \leq G_{e(-\varepsilon)}$ in $\pi_1(\mathcal{G}(\Gamma))$; hence, we can eliminate from its presentation the generators of $G_{e(\varepsilon)}$. The result of this operation is the fundamental group of the graph of groups which is obtained from $\mathcal{G}(\Gamma)$ by contracting the edge e and replacing for each pair $(f, \delta) \in \Theta_{e(\varepsilon)} \setminus \{(e, \varepsilon)\}$ the homomorphism $\varphi_{\delta f}$ by the composition $\varphi_{\delta f}\varphi_{\varepsilon e}^{-1}\varphi_{-\varepsilon e}$. It is clear that this graph of groups is also a tree.

If $\mathcal{G}(\Gamma)$ is of type (1) and $H_{\varepsilon e} = G_{e(\varepsilon)}$; then, choosing a tree T containing the edge e, we find that $H_{\varepsilon e} = H_{-\varepsilon e}$ in $\pi_1(\mathcal{G}(\Gamma))$. Consequently, we can modify its presentation and the graph of groups as above. Since $H_{\varepsilon e}$ is a unique nontrivial edge subgroup of $G_{e(\varepsilon)}$, the resulting graph of groups is still of type (1).

If at least one among the groups G_v for $v \in V$ is nontrivial then finitely many transformations described above reduce the graph of groups $\mathcal{G}(\Gamma)$ to the required form. Otherwise, $\pi_1(\mathcal{G}(\Gamma))$ is a free group with basis $\{t_e \mid e \in E \setminus E_T\}$ known to be residual with respect to each root class of groups [20, Theorem 1].

Theorem 4. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients, while $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in E$, $\varepsilon = \pm 1$, and at least one of the following holds:*

(1) *$\mathcal{G}(\Gamma)$ is a finite graph of groups of type* (1) *and G_v is $\mathcal{C}$-regular with respect to the subgroup H_v for every $v \in V$;*

(2) *$\mathcal{G}(\Gamma)$ is a finite graph of groups of type* (2) *and $G_{e(\varepsilon)}$ is $\mathcal{C}$-regular with respect to the subgroup $H_{\varepsilon e}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

The group $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}$-residual if and only if G_v is $\mathcal{C}$-residual for all $v \in V$ and the subgroup $H_{\varepsilon e}$ is $\mathcal{C}$-separable in $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$.

It is easy to see that if $\mathcal{C}$ is a root class of groups and Y is a central subgroup of some group X with $X/Y \in \mathcal{C}$ then X is a $\mathcal{C}$-regular group with respect to Y and the latter is $\mathcal{C}$-separable in X. Corollaries 3 and 4 provide more meaningful examples of the situation when the separability and regularity requirements of Theorems 3 and 4 hold.

Suppose that ρ is a nonempty set of primes. Following [21], call an abelian group *ρ-bounded* whenever in each of its quotients all primary components of the periodic part corresponding to the numbers in ρ are finite. A solvable (nilpotent) group is called *ρ-bounded* whenever it possesses a finite subnormal (respectively central) series with ρ-bounded abelian factors. Note that if ρ includes all primes then every ρ-bounded solvable group is a *bounded solvable* group in the sense of Maltsev [17].

Given a class $\mathcal{C}$ of groups consisting of periodic groups, denote by $\rho(\mathcal{C})$ the set of all prime divisors of the orders of elements of the groups in $\mathcal{C}$. Recall also that a subgroup Y of a group X is called ρ'*-isolated* in X for some set of primes ρ whenever, given $x \in X$ and $q \notin \rho$, from $x^q \in Y$ it follows that $x \in Y$.

Corollary 3. *Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type (1) or a finite graph of groups of type (2). Suppose also that $\mathcal{C}$ is a root class of groups consisting of periodic groups and the set $\rho(\mathcal{C})$ includes all possible primes. If G_v are $\rho(\mathcal{C})$-bounded solvable groups for all $v \in V$ then $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}$-residual.*

Corollary 4. *Suppose that $\mathcal{C}$ is a root class of groups consisting of periodic groups and G_v are $\rho(\mathcal{C})$-bounded nilpotent for all $v \in V$.*

(1) *If $\mathcal{G}(\Gamma)$ is a graph of groups of type (1) and for each $v \in V$ the subgroups 1 and H_v are $\rho(\mathcal{C})'$-isolated in G_v then $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}$-residual.*

(2) *Suppose that $\mathcal{G}(\Gamma)$ is a finite graph of groups of type (1) or (2) and $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$. The group $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}$-residual if and only if the subgroups 1 and $H_{\varepsilon e}$ are $\rho(\mathcal{C})'$-isolated in $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

Observe that, in contrast to the previous assertions, Corollaries 3 and 4 do not require the approximating class to be closed under quotients. The rest of the article contains proofs of the above theorems and corollaries.

3. Some Properties of the Fundamental Groups of the Graphs of Groups under Study

Henceforth, given a nonempty connected subgraph Γ' of a graph Γ, denote by $\mathcal{G}(\Gamma')$ the graph of groups with the same groups and homomorphisms assigned to the vertices and edges as in the graph $\mathcal{G}(\Gamma)$. It is not difficult to show, see [22] for instance, that if $T' = \Gamma' \cap T$ is a tree and the presentation of the group $\pi_1(\mathcal{G}(\Gamma'))$ corresponds to T' then the identity mapping of the generators of $\pi_1(\mathcal{G}(\Gamma'))$ into $\pi_1(\mathcal{G}(\Gamma))$ determines an injective homomorphism, and so we may assume that $\pi_1(\mathcal{G}(\Gamma'))$ is a subgroup of $\pi_1(\mathcal{G}(\Gamma))$.

Suppose that in G_v a normal subgroup R_v is chosen for each $v \in V$ so that

$$(R_{e(1)} \cap H_{+e})\varphi_{+e}^{-1} = (R_{e(-1)} \cap H_{-e})\varphi_{-e}^{-1}$$

for every edge $e \in E$. Then we refer to the family $\mathcal{R} = \{R_v \mid v \in V\}$ as a *system of compatible normal subgroups* of $\pi_1(\mathcal{G}(\Gamma))$ and denote by $\mathcal{G}_{\mathcal{R}}(\Gamma)$ the graph of groups

$$(\Gamma,\ \overline{G}_v\ (v \in V),\ \overline{H}_e\ (e \in E),\ \overline{\varphi}_{\varepsilon e}\ (e \in E,\ \varepsilon = \pm 1)),$$

in which

$$\overline{G}_v = G_v/R_v, \quad \overline{H}_e = H_e/(R_{e(1)} \cap H_{+e})\varphi_{+e}^{-1} = H_e/(R_{e(-1)} \cap H_{-e})\varphi_{-e}^{-1}$$

and the homomorphism $\overline{\varphi}_{\varepsilon e}\colon \overline{H}_e \to \overline{G}_{e(\varepsilon)}$ for $e \in E$ and $\varepsilon = \pm 1$ carries the coset $\overline{h}$ for $h \in H_e$ into $(h\varphi_{\varepsilon e})R_{e(\varepsilon)}$. We assume that the presentation of $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ corresponds to the same tree T. It is easy to see that then the identity mapping of the generators of $\pi_1(\mathcal{G}(\Gamma))$ into $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ determines a surjective homomorphism whose kernel is the normal closure in $\pi_1(\mathcal{G}(\Gamma))$ of $\bigcup_{v \in V} R_v$. Denote this homomorphism by $\rho_{\mathcal{R}}$.

Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type (1), take $F \subseteq E$ and put

$$R_v = \prod_{(e,\varepsilon) \in \Theta_v,\, e \in F} H_{\varepsilon e}$$

for each $v \in V$, with the product of the empty set of subgroups equal to 1. Then we denote the family $\{R_v \mid v \in V\}$ by $\mathcal{R}(F)$. It is easy to see that it is a system of compatible normal subgroups of $\pi_1(\mathcal{G}(\Gamma))$ and the graph of groups $\mathcal{G}_{\mathcal{R}(F)}(\Gamma)$ is of type (1).

Proposition 1. *Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type* (1). *Then for each $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus \{1\}$ there is a finite set of edges F_g such that $g\rho_{\mathcal{R}(E \setminus F_g)} \neq 1$ and $g \notin H_{\varepsilon e}$ implies that $g\rho_{\mathcal{R}(E\setminus F_g)} \notin H_{\varepsilon e}\rho_{\mathcal{R}(E\setminus F_g)}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

PROOF. Suppose firstly that $g \in \pi_1(\mathcal{G}(T)) \setminus \{1\}$. Then $g \in \pi_1(\mathcal{G}(T'))$ for some finite subtree T' of T. Induct on the number k of vertices in T'.

Assume that $k = 1$, i.e., $g \in G_v$ for some $v \in V$. If $g \notin H_v$ then put $F_g = \varnothing$; otherwise, $g \in \operatorname{sgp}\{H_{\varepsilon e} \mid (e, \varepsilon) \in \vartheta\}$ for some finite subset $\vartheta \subseteq \Theta_v$ and $F_g = \{e \mid (e, \varepsilon) \in \vartheta\}$. It is easy to see that F_g defined is the required set.

Assume that $k > 1$ and that the claim holds for the elements of $\pi_1(\mathcal{G}(\Gamma))$ belonging to the subtrees with fewer vertices. Take an edge e of T' and view $\pi_1(\mathcal{G}(T'))$ as the free product of the groups $\pi_1(\mathcal{G}(T'_1))$ and $\pi_1(\mathcal{G}(T'_{-1}))$ with amalgamated subgroups H_{+e} and H_{-e}; here T'_ε for $\varepsilon = \pm 1$ is the connected component containing the vertex $e(\varepsilon)$ of the graph obtained from T' by removing e. Take a reduced form $g = g_1 \dots g_n$ for the element g in the generalized free product; for the definitions and properties of reduced forms for the elements of a generalized free product and HNN-extension, see [3] for instance. If $n = 1$ then the required set F_g exists by the inductive assumption. Assume that $n > 1$.

Then by the inductive assumption for each $i \in \{1, \dots, n\}$ there is a finite set $F_{g_i} \subseteq E$ such that if $g_i \notin H_{\varepsilon e}$ for $\varepsilon = \pm 1$ then $g_i\rho_{\mathcal{R}(E\setminus F_{g_i})} \notin H_{\varepsilon e}\rho_{\mathcal{R}(E\setminus F_{g_i})}$. Put $F_g = \bigcup_{i=1}^n F_{g_i}$. Then for every $i \in \{1, \dots, n\}$ we have $\ker \rho_{\mathcal{R}(E\setminus F_g)} \leq \ker \rho_{\mathcal{R}(E\setminus F_{g_i})}$; therefore, if $g_i \notin H_{\varepsilon e}$ for $\varepsilon = \pm 1$ then $g_i\rho_{\mathcal{R}(E\setminus F_g)} \notin H_{\varepsilon e}\rho_{\mathcal{R}(E\setminus F_g)}$. Consequently, in $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus F_g)}(T))$, regarded as the free product with amalgamated subgroups $H_{+e}\rho_{\mathcal{R}(E\setminus F_g)}$ and $H_{-e}\rho_{\mathcal{R}(E\setminus F_g)}$, the element $g\rho_{\mathcal{R}(E\setminus F_g)}$ has a reduced form of length $n > 1$ and so cannot lie in the free factors of this product which contain all edge subgroups of the group $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus F_g)}(\Gamma))$. Thus, F_g is the required set.

Take some nontrivial element g of $\pi_1(\mathcal{G}(\Gamma))$ regarded as the HNN-extension of $\pi_1(\mathcal{G}(T))$ with a reduced form $g = g_0 t_{e_1}^{\varepsilon_1} g_1 \dots t_{e_n}^{\varepsilon_n} g_n$. Induct on n. If $n = 0$ then $g \in \pi_1(\mathcal{G}(T))$ and the required claim is already proved. Assume that $n > 0$. By the inductive assumption, for each $i \in \{1, \dots, n\}$ if $g_i \notin H_{-\varepsilon_i e_i}$ then there exists a set $F_{g_i} \subseteq E$ such that $g_i\rho_{\mathcal{R}(E\setminus F_{g_i})} \notin H_{-\varepsilon_i e_i}\rho_{\mathcal{R}(E\setminus F_{g_i})}$. Put $F_{g_i} = \varnothing$ if $g_i \in H_{-\varepsilon_i e_i}$ and $F_g = \bigcup_{i=1}^n F_{g_i}$. Then for every $i \in \{1, \dots, n\}$ from $g_i \notin H_{-\varepsilon_i e_i}$ it follows that $g_i\rho_{\mathcal{R}(E\setminus F_g)} \notin H_{-\varepsilon_i e_i}\rho_{\mathcal{R}(E\setminus F_g)}$. Consequently, $g\rho_{\mathcal{R}(E\setminus F_g)}$ in the HNN-extension $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus F_g)}(\Gamma))$ has a reduced form of length $n > 0$ and so it cannot lie in the base group $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus F_g)}(T))$ of this HNN-extension which contains all edge subgroups. Thus, F_g is the required set.

Proposition 2. *Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type* (1). *Then for each $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus \{1\}$ there is a finite subgraph $\Gamma' = (V', E')$ of the graph Γ such that $\Gamma' \cap T$ is a tree, $g\rho_{\mathcal{R}(E\setminus E')} \in \pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma')) \setminus \{1\}$, the graph of groups $\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma')$ is of type* (1), *and the subgroup $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma'))$ is a retract of the group $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma))$.*

PROOF. Take $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus \{1\}$ and some word w in the generators of $\pi_1(\mathcal{G}(\Gamma))$ representing g.

Define $V_g \subseteq V$ and $E_g \subseteq E \setminus E_T$ as follows: $v \in V_g$ if and only if w involves some generator of G_v or its inverse; $e \in E_g$ if and only if w involves the symbol t_e or t_e^{-1}.

According to Proposition 1, there is a finite set of edges F_g such that $g\rho_{\mathcal{R}(E\setminus F_g)} \neq 1$. Take a finite subtree T' of T containing all vertices in the set $V_g \cup \{e(\varepsilon) \mid e \in E_g \cup F_g,\ \varepsilon = \pm 1\}$ and the subgraph Γ' of Γ obtained by adding to T' all edges in $E_g \cup F_g$, with V' and E' standing for the vertex and edge sets of Γ' respectively. Then $g \in \pi_1(\mathcal{G}(\Gamma'))$ and $g\rho_{\mathcal{R}(E\setminus E')} \in \pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma'))$. Since $F_g \subseteq E'$, it follows that $\ker \rho_{\mathcal{R}(E\setminus E')} \leq \ker \rho_{\mathcal{R}(E\setminus F_g)}$, and so $g\rho_{\mathcal{R}(E\setminus E')} \neq 1$.

The definition of the system of subgroups $\mathcal{R}(E \setminus E') = \{R_v \mid v \in V\}$ shows that $R_v = H_v$ for each $v \in V \setminus V'$. Therefore, $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma))$ is isomorphic to the free product of $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma'))$, the groups G_v/R_v for $v \in V \setminus V'$, and the free group with generators $\{t_e \mid e \in E \setminus E'\}$.

Consequently, $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma'))$ is a retract of $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma))$. As indicated above, the property that the graph of groups $\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma')$ is of type (1) follows from the definition of $\mathcal{R}(E \setminus E')$.

4. Proofs of Theorems 1 and 2

For every graph of groups $\mathcal{G}(\Gamma)$ we can formally consider the group

$$\mathrm{GDP}(\mathcal{G}(\Gamma)) = \langle G_v\ (v \in V);\ H_{+e} = H_{-e}\ (e \in E),\ [G_v, G_w] = 1\ (v, w \in V,\ v \neq w)\rangle,$$

whose generators are the generators of G_v for $v \in V$ and the defining relations are those of G_v for $v \in V$ as well as all possible relations of the form $h_e\varphi_{+e} = h_e\varphi_{-e}$ for $e \in E$ and $h_e \in H_e$ and $[g_v, g_w] = 1$ for $v, w \in V$ with $v \neq w$, where g_v and g_w are arbitrary words in the generators of G_v and G_w respectively, $h_e\varphi_{\varepsilon e}$ for $\varepsilon = \pm 1$ is a word in the generators of $G_{e(\varepsilon)}$ defining the image of h_e under the homomorphism $\varphi_{\varepsilon e}$. It is obvious that $\mathrm{GDP}(\mathcal{G}(\Gamma))$ is a homomorphic image of $\pi_1(\mathcal{G}(\Gamma))$; its properties are considered in detail in [1]. Here we need only the following:

Proposition 3. *Suppose that Γ has no multiple edges and loops and the graph of groups $\mathcal{G}(\Gamma)$ is of type* (1) *or* (2). *Then the following hold:*

(1) *For each $v \in V$ the identity mapping of the generators of G_v into GDP($\mathcal{G}(\Gamma)$) determines an injective homomorphism, and so the natural homomorphism $\pi_1(\mathcal{G}(\Gamma)) \to$ GDP($\mathcal{G}(\Gamma)$) is injective on all groups G_v for $v \in V$.*

(2) *GDP($\mathcal{G}(\Gamma)$) is a torsion-free group provided that G_v is torsion-free for all $v \in V$ and at least one of the following conditions is met:*

(a) *$\mathcal{G}(\Gamma)$ is a graph of groups of type* (1) *and the subgroup H_v is isolated in G_v for each $v \in V$;*

(b) *$\mathcal{G}(\Gamma)$ is a graph of groups of type* (2) *and the subgroup $H_{\varepsilon e}$ is isolated in $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

Proof. Everything follows from Theorems 1 and 2 of [1]. We should only note that in Theorem 2 of [1], dealing with a graph of groups of type (1), we need the product of however many subgroups in the family $\{H_{\varepsilon e} \mid (e, \varepsilon) \in \Theta_v\}$ to be isolated in G_v. By Proposition 6 below, this is equivalent to H_v being isolated.

Proposition 4. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients, that $\mathcal{G}(\Gamma)$ is a graph of groups of type* (1), (2), *or* $(3)_{+\mathcal{C}}$ *and that the direct product D of groups G_v for $v \in V$ lies in $\mathcal{C}$. Then the following hold:*

(1) *There exists a homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a group of class $\mathcal{C}$ injective on G_v for all $v \in V$.*

(2) *We can choose the homomorphism σ so that its image is a torsion-free group if G_v for all $v \in V$ are torsion-free and at least one of the following holds:*

(a) *$\mathcal{G}(\Gamma)$ is a graph of groups of type* (1) *and the subgroup H_v is isolated in G_v for each $v \in V$;*

(b) *$\mathcal{G}(\Gamma)$ is a graph of groups of type* (2) *or* $(3)_{+\mathcal{C}}$ *and the subgroup $H_{\varepsilon e}$ is isolated in $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

Proof. Put $E' = \{e \in E \backslash E_T \mid H_{+e} = 1 = H_{-e}\}$ and take the graph Γ' obtained from Γ by removing all edges in E'. Then $\pi_1(\mathcal{G}(\Gamma))$ amounts to the free product of $\pi_1(\mathcal{G}(\Gamma'))$ and the free group with basis $\{t_e \mid e \in E'\}$. Clearly, it suffices to justify the claim for $\pi_1(\mathcal{G}(\Gamma'))$; hence, we assume henceforth that $E' = \varnothing$. Observe also that if G_v is trivial for all $v \in V$ then the claim obviously holds, and therefore we can exclude this case from further consideration.

The group $\mathrm{GDP}(\mathcal{G}(T))$ belongs to the class $\mathcal{C}$ because it is a homomorphic image of D. Hence, if $\Gamma = T$, then by Proposition 3 the natural homomorphism $\eta\colon \pi_1(\mathcal{G}(T)) \to \mathrm{GDP}(\mathcal{G}(T))$ is the required one. Assume henceforth that $\Gamma \neq T$ and so $E \setminus E_T \neq \varnothing$.

Since the class $\mathcal{C}$ contains at least one nontrivial group and is closed under subgroups and extensions, then $\mathcal{C}$ contains some cyclic group $\mathcal{Z}$ whose order we may assume to exceed 2 if $\mathcal{C}$ consists of periodic groups and equal to infinity otherwise. Fix some generator z of $\mathcal{Z}$ and denote by $\mathcal{I}$ the set of all functions from $E \setminus E_T$ to $\mathcal{Z}$ with pointwise multiplication; i.e., a Cartesian power of the group $\mathcal{Z}$ whose exponent equals the cardinality of $E \setminus E_T$. For $i \in \mathcal{I}$ take an isomorphic copy $\mathcal{G}(T)_i$ of the graph of groups $\mathcal{G}(T)$ and the corresponding isomorphism $\tau_i\colon \mathcal{G}(T) \to \mathcal{G}(T)_i$ assuming that its restrictions to G_v for $v \in V$ are group isomorphisms. Denote the disjoint union of the graphs $\mathcal{G}(T)_i$ for $i \in \mathcal{I}$ by Σ. Given $i \in \mathcal{I}$ and $e \in E \setminus E_T$,

define the function $j \in \mathcal{I}$ as

$$j(e) = i(e)z^{-1}; \quad j(e') = i(e'), \quad e' \in E \setminus (E_T \cup \{e\});$$

connect the vertices $e(1)\tau_i$ and $e(-1)\tau_j$ of Σ with a new edge f;

associate to the edge f the group H_e and the homomorphisms $\psi_{+f} = \varphi_{+e}\tau_i$ and $\psi_{-f} = \varphi_{-e}\tau_j$.

Denote the resulting graph of groups by Δ. Observe that if we treat its subgraphs $\mathcal{G}(T)_i$ for all $i \in \mathcal{I}$ as macrovertices then for $|\mathcal{Z}| = \infty$ the corresponding macrograph amounts to the integer lattice in the space of dimension $\mathfrak{c} = \operatorname{card} E \setminus E_T$, while for $|\mathcal{Z}| < \infty$ it is the $\mathfrak{c}$-dimensional "integer torus" obtained by identifying the lattice with its image translated along some coordinate axis over any distance that is a multiple of $|\mathcal{Z}|$.

The definition of the graph of groups Δ implies immediately that if $\mathcal{G}(\Gamma)$ is of type (1) then so is Δ, and by the choice of the order of $\mathcal{Z}$ it has neither multiple edges nor loops. Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type $(3)_{+\mathcal{C}}$. Then $E \setminus E_T$ consists precisely of one edge, the class $\mathcal{C}$ contains nonperiodic groups, and $\mathcal{Z}$ is an infinite cyclic group. This implies that the above macrograph amounts to an infinite chain, and so Δ is of type (2). Thus, in both cases according to Proposition 3 all vertex groups $G_v\tau_i$ of Δ for $v \in V$ and $i \in \mathcal{I}$ embed into the group $\mathrm{GDP}(\Delta)$ via the identity mapping of generators, and therefore, may be treated as subgroups of the latter. Observe also that if $\mathcal{G}(\Gamma)$ satisfies the hypotheses of claim (2) then so does Δ, and by the same Proposition 3 $\mathrm{GDP}(\Delta)$ is a torsion-free group.

It is easy to see that for each $i \in \mathcal{I}$ the mapping of the generators of $\mathrm{GDP}(\Delta)$ extending the isomorphisms $\tau_j^{-1}\tau_{j\cdot i}$ for $j \in \mathcal{I}$ determines the automorphism α_i of this group, where $j \cdot i$ stands for the pointwise product of functions j and i. Hence, the semidirect product $X = \mathrm{GDP}(\Delta) \leftthreetimes \mathcal{I}$ is defined, in which the inner automorphism corresponding to $i \in \mathcal{I}$ acts on $\mathrm{GDP}(\Delta)$ as α_i. Observe that the group X is quite similar to the direct wreath product of $\mathrm{GDP}(\mathcal{G}(T))$ with $\mathcal{I}$; the difference is that as the base we use not the usual but a generalized direct product of isomorphic copies of $\mathrm{GDP}(\mathcal{G}(T))$. The group G_v lies in $\mathcal{C}$ as a subgroup of D for all $v \in V$, and according to our assumption at least one of G_v's is nontrivial. Consequently, if $\mathcal{G}(\Gamma)$ satisfies the hypotheses of claim (2) then $\mathcal{C}$ contains a nonperiodic group, and so $\mathcal{Z}$, $\mathcal{I}$, and X are torsion-free groups.

Given $e \in E \setminus E_T$, define the function $\dot{e} \in \mathcal{I}$ with $\dot{e}(e) = z$ and $\dot{e}(e') = 1$ for all $e' \in E \setminus (E_T \cup \{e\})$. Take also the function $u \in \mathcal{I}$ identically equal to 1. We can verify directly that the mapping of words to the generators of $\pi_1(\mathcal{G}(\Gamma))$ acting on the generators of G_v for $v \in V$ as the isomorphism τ_u and transforming each symbol t_e for $e \in E \setminus E_T$ into $\dot{e}$ carries all defining relations of $\pi_1(\mathcal{G}(\Gamma))$ into equalities valid in X, and so it determines a homomorphism $\sigma: \pi_1(\mathcal{G}(\Gamma)) \to X$. By the above argument, this homomorphism is injective on G_v for all $v \in V$. Thus, to complete the proof of the proposition it remains to show that $X \in \mathcal{C}$. To this end, use the closedness of $\mathcal{C}$ under subgroups, quotients, direct products, extensions, and Cartesian powers.

If $\mathcal{G}(\Gamma)$ is a graph of groups of type (3) then $E \setminus E_T$ amounts to precisely one edge, and so $\mathcal{I} \cong \mathcal{Z} \in \mathcal{C}$. Suppose that $\mathcal{G}(\Gamma)$ is a graph of groups of type (1). According to the above assumption, $H_{+e} \neq 1 \neq H_{-e}$ for all $e \in E \setminus E_T$ and consequently each edge $e \in E \setminus E_T$ corresponds uniquely to the pair of subgroups (H_{+e}, H_{-e}) of D, i.e., a subgroup of the $\mathcal{C}$-group $P = D \times D$. Hence, $E \setminus E_T$ can be embedded into the set of all subsets of P, which in turn embeds into the Cartesian power $Q = P^{|P|}$ lying in $\mathcal{C}$. Thus,

$$\mathcal{I} \leq \mathcal{Z}^{|Q|} \in \mathcal{C},$$

and so again $\mathcal{I} \in \mathcal{C}$.

The group $\mathrm{GDP}(\Delta)$ amounts to a homomorphic image of the direct product

$$R = \prod_{i \in \mathcal{I}} \mathrm{GDP}(\mathcal{G}(T)_i)$$

whose every factor is isomorphic to the group $\mathrm{GDP}(\mathcal{G}(T))$. As indicated above, $\mathrm{GDP}(\mathcal{G}(T)) \in \mathcal{C}$, and, since $\mathcal{I} \in \mathcal{C}$, the group R embeds into the Cartesian power $\mathrm{GDP}(\mathcal{G}(T))^{|\mathcal{I}|}$ lying in $\mathcal{C}$. Hence, $R \in \mathcal{C}$ and $\mathrm{GDP}(\Delta) \in \mathcal{C}$, while $X \in \mathcal{C}$ as an extension of the $\mathcal{C}$-group $\mathrm{GDP}(\Delta)$ by using the $\mathcal{C}$-group $\mathcal{I}$.

Proposition 5 [23, Proposition 2]. *Suppose that $\mathcal{C}$ is a class of groups closed under subgroups and direct products of finitely many factors. Given a group X, the intersection of finitely many subgroups of the family $\mathcal{C}^*(X)$ is again a subgroup in this family, and if X is $\mathcal{C}$-residual then for each finite subgroup Y there is a subgroup $Z \in \mathcal{C}^*(X)$ such that $Y \cap Z = 1$.*

The following can be verified directly:

Proposition 6. *Consider some torsion-free group X with two trivially intersecting central subgroups Y and Z. If the subgroup YZ is isolated in X then so is Z.*

Proposition 7. *Suppose that $\mathcal{C}$ is a class of groups closed under subgroups, quotients, and direct products of finitely many factors. Denote the class of all torsion-free $\mathcal{C}$-groups by $\mathcal{C}_{\mathrm{tf}}$. Consider some group X, two central subgroups Y and Z with $Y \cap Z = 1$, and a homomorphism σ of X onto a group of class $\mathcal{C}$ injective on YZ. Then the following hold:*

(1) *There exists a homomorphism $\overline{\sigma}$ of the group X/Z onto a $\mathcal{C}$-group injective on YZ/Z.*

(2) *If X is $\mathcal{C}$-residual then so is X/Z.*

(3) *Suppose that the group $X\sigma$ is torsion-free and the subgroup $(YZ)\sigma$ is isolated in it. Then*

(a) *we can choose the homomorphism $\overline{\sigma}$ so that the group $(X/Z)\overline{\sigma}$ is torsion-free and the subgroup $(YZ/Z)\overline{\sigma}$ is isolated in it;*

(b) *if X is a $\mathcal{C}_{\mathrm{tf}}$-residual group then so is X/Z.*

Proof. Put $T = \ker \sigma$. Then $X/T \in \mathcal{C}$ and $T \cap YZ = 1$.

(1): Observe that if $N \in \mathcal{C}^*(X)$ then the group

$$(X/Z)/(NZ/Z) \cong X/NZ \cong (X/N)/(NZ/N)$$

belongs to the class $\mathcal{C}$ as a quotient of the $\mathcal{C}$-group X/N.

In particular, we have the inclusion $(X/Z)/(TZ/Z) \in \mathcal{C}$. The relation $T \cap YZ = 1$ implies that $TZ/Z \cap YZ/Z = 1$. Hence, the natural homomorphism of X/Z onto $(X/Z)/(TZ/Z)$ is the required one.

(2): Take $x \in X$ with $xZ \neq 1$. If a subgroup $N \in \mathcal{C}^*(X)$ satisfies $x \notin NZ$, then $xZ \notin NZ/Z$ and $(X/Z)/(NZ/Z) \in \mathcal{C}$. Thus, it remains to find a subgroup N with these properties. If $x \notin TZ$ then T is the required subgroup. Suppose that $x = tz$ for some $t \in T$ and $z \in Z$. Since $xZ \neq 1$, it follows that $t \neq 1$ and, using the $\mathcal{C}$-residuality of X, we can find a subgroup $M \in \mathcal{C}^*(X)$ avoiding t. Put $N = M \cap T$. Then $N \in \mathcal{C}^*(X)$ by Proposition 5, and if $x = t'z'$ for some $t' \in N$ and $z' \in Z$ then the equality $T \cap Z = 1$ yields $t = t' \in N \leq M$ in contradiction with the choice of M. Consequently, N is the required subgroup.

(3a): Since

$$(X/Z)\overline{\sigma} = (X/Z)/(TZ/Z), \quad (YZ/Z)\overline{\sigma} = (YZT/Z)/(TZ/Z),$$

it suffices to show that YZT and TZ are isolated in X.

The equalities $T \cap YZ = 1$ and $Y \cap Z = 1$ imply that YZT amounts to the direct product of Y, Z, and T. If $(YZ)\sigma = YZT/T$ is isolated in $X\sigma = X/T$, then YZT is isolated in X; thus, Proposition 6 shows that TZ is isolated in this group.

(3b): The argument follows the same scheme as the proof of claim (2). We should only choose N to satisfy $(X/Z)/(NZ/Z) \in \mathcal{C}_{\mathrm{tf}}$. Since $X\sigma$ is torsion-free and X is $\mathcal{C}_{\mathrm{tf}}$-residual, it follows that $T \in \mathcal{C}_{\mathrm{tf}}^*(X)$, and we may also assume that M belongs to $\mathcal{C}_{\mathrm{tf}}^*(X)$.

Since the class $\mathcal{C}_{\mathrm{tf}}$ is closed under subgroups and direct products, Proposition 5 implies that $N = M \cap T \in \mathcal{C}_{\mathrm{tf}}^*(X)$. Since $TZ/NZ \cong T/N(T \cap Z) = T/N$, the subgroup NZ is isolated in TZ; while the latter, as we showed above, is isolated in X. Thus,

$$(X/Z)/(NZ/Z) \cong X/NZ \in \mathcal{C}_{\mathrm{tf}}.$$

Proposition 8 [3, Proposition 7]. *Suppose that $\mathcal{C}$ is a root class of groups. If each group G_v for $v \in V$ is $\mathcal{C}$-residual and there exists a homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a $\mathcal{C}$-group injective on all subgroups H_v for $v \in V$ then $\pi_1(\mathcal{G}(\Gamma))$ is also $\mathcal{C}$-residual.*

Proof of Theorems 1 and 2. Assume firstly that the direct product D of $G_v\sigma_v$ for $v \in V$ lies in $\mathcal{C}$.

Put $S_v = \ker \sigma_v$ for $v \in V$. The family $\mathcal{S} = \{S_v \mid v \in V\}$ is obviously a system of compatible normal subgroups and $\pi_1(\mathcal{G}(\Gamma))\rho_{\mathcal{S}}$ satisfies the hypotheses of Proposition 4. Therefore, the composition of $\rho_{\mathcal{S}}$ and the homomorphism of Proposition 4 is the required mapping σ. Since $\mathcal{C}_{\mathrm{tf}}$ is also a root class, Proposition 8 guarantees the residuality of $\pi_1(\mathcal{G}(\Gamma))$.

Therefore, Theorem 2 and claims (1) and (3a) of Theorem 1 are justified completely, while claims (2) and (3b) only on assuming that $D \in \mathcal{C}$. Let us verify that the latter two claims also hold in the absence of this assumption.

Take some $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus \{1\}$. According to Proposition 2, there exists a finite subgraph $\Gamma' = (V', E')$ of Γ such that $\Gamma' \cap T$ is a tree, $g\rho_{\mathcal{R}(E\setminus E')} \in \pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma')) \setminus \{1\}$, the graph of groups $\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma')$ is of type (1), and the subgroup $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma'))$ is a retract of $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma))$.

The definition of the system of subgroups $\mathcal{R}(E \setminus E') = \{R_v \mid v \in V\}$ implies that the subgroup R_v is a direct factor of H_v for each $v \in V$. Hence, Proposition 7 applied to $X = G_v$, the subgroups $YZ = H_v$ and $Z = R_v$, and the homomorphism $\sigma = \sigma_v$ shows that

(a) there exists a homomorphism $\overline{\sigma}_v$ of G_v/R_v onto a $\mathcal{C}$-group injective on H_v/R_v, and if G_v is $\mathcal{C}$-residual then so is G_v/R_v;

(b) if $G_v\sigma_v$ is a torsion-free group and the subgroup $H_v\sigma_v$ is isolated in it then $(G_v/R_v)\overline{\sigma}_v$ and $(H_v/R_v)\overline{\sigma}_v$ enjoy the same properties, and the $\mathcal{C}_{\mathrm{tf}}$-residuality of G_v implies the same property of G_v/R_v.

Since the graph Γ' is finite, the direct product of $(G_v/R_v)\overline{\sigma}_v$ for $v \in V'$ lies in $\mathcal{C}$. Consequently, $\pi_1(\mathcal{G}_{\mathcal{R}(E\setminus E')}(\Gamma'))$ is approximated by $\mathcal{C}$ or $\mathcal{C}_{\mathrm{tf}}$ by the above. Thus, the composition of $\rho_{\mathcal{R}(E\setminus E')}$ and the retracting homomorphism can be extended to a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a $\mathcal{C}$-group carrying g into a nontrivial element.

5. Proof of Theorems 3 and 4

Call a system $\mathcal{R} = \{R_v \mid v \in V\}$ of compatible normal subgroups $\mathcal{C}$-*admissible* whenever there exists a homomorphism of $\pi_1(\mathcal{G}_{\mathcal{R}}(\Gamma))$ onto a $\mathcal{C}$-group injective on all vertex groups G_v/R_v for $v \in V$. The proof of Theorems 3 and 4 is based on Proposition 4 and the next statement which is straightforward from Theorems 1–3 of [3].

Proposition 9. *Suppose that $\mathcal{C}$ is a root class of groups and that for all $u \in V$ and $L \in \mathcal{C}^*(G_u)$ there exists a $\mathcal{C}$-admissible system of compatible normal subgroups $\mathcal{R} = \{R_v \mid v \in V\}$ with $R_u \leq L$.*

(1) *If G_v is $\mathcal{C}$-residual for all $v \in V$ and the subgroup $H_{\varepsilon e}$ is $\mathcal{C}$-separable in $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$ then $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}$-residual.*

(2) *If $H_{\varepsilon e}$ is a proper central subgroup of $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$ then the converse holds: the $\mathcal{C}$-residuality of $\pi_1(\mathcal{G}(\Gamma))$ implies that each group G_v is $\mathcal{C}$-residual for $v \in V$ and the subgroup $H_{\varepsilon e}$ is $\mathcal{C}$-separable in $G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

Proposition 10. *Suppose that $\mathcal{C}$ is a root class of groups closed under quotients and at least one of the following holds:*

(1) *$\mathcal{G}(\Gamma)$ is a finite graph of groups of type (1) and G_v is $\mathcal{C}$-regular with respect to the subgroup H_v for each $v \in V$;*

(2) *$\mathcal{G}(\Gamma)$ is a finite graph of groups of type (2) and $G_{e(\varepsilon)}$ is $\mathcal{C}$-regular with respect to the subgroup $H_{\varepsilon e}$ for all $e \in E$ and $\varepsilon = \pm 1$.*

Then for all $u \in V$ and $L \in \mathcal{C}^*(G_u)$ there exists a $\mathcal{C}$-admissible system of compatible normal subgroups $\mathcal{R} = \{R_v \mid v \in V\}$ with $R_u \leq L$.

Proof. Take $u \in V$ and a subgroup $L \in \mathcal{C}^*(G_u)$. Let us point out a system of compatible normal subgroups $\mathcal{R} = \{R_v \mid v \in V\}$ with $R_u \leq L$ and $R_v \in \mathcal{C}^*(G_v)$ for all $v \in V$ such that the graph of groups $\mathcal{G}_{\mathcal{R}}(\Gamma)$ has the same type as $\mathcal{G}(\Gamma)$. Since the graph Γ is finite, in this case the direct product of $\mathcal{C}$-groups G_v/R_v for $v \in V$ lies in $\mathcal{C}$, and according to Proposition 4 the system $\mathcal{R}$ is $\mathcal{C}$-admissible.

Assume firstly that condition (1) holds. For all $e \in E$ and $\varepsilon = \pm 1$ define the subgroup $L_{\varepsilon e} \leq H_{\varepsilon e}$ as follows: If the edge e is not a loop and $u = e(\varepsilon)$ for some $\varepsilon = \pm 1$, put $L_{\varepsilon e} = L \cap H_{\varepsilon e}$ and $L_{-\varepsilon e} = L_{\varepsilon e}\varphi_{\varepsilon e}^{-1}\varphi_{-\varepsilon e}$. If e is a loop and $e(-1) = u = e(1)$ then

$$L_{+e} = (L \cap H_{+e}) \cap (L \cap H_{-e})\varphi_{-e}^{-1}\varphi_{+e},$$
$$L_{-e} = (L \cap H_{+e})\varphi_{+e}^{-1}\varphi_{-e} \cap (L \cap H_{-e}) = L_{+e}\varphi_{+e}^{-1}\varphi_{-e}.$$

In the remaining cases $L_{\varepsilon e} = H_{\varepsilon e}$.

Put
$$M_v = \prod_{(e,\varepsilon)\in\Theta_v} L_{\varepsilon e}$$
for every $v \in V$.

Since $L \in \mathcal{C}^*(G_u)$, for every pair $(e,\varepsilon) \in \Theta_u$ the quotient $H_{\varepsilon e}/L \cap H_{\varepsilon e}$ is isomorphic to a subgroup of the $\mathcal{C}$-group G_u/L, and so $L \cap H_{\varepsilon e} \in \mathcal{C}^*(H_{\varepsilon e})$. Combined with Proposition 5, this implies that $L_{\varepsilon e} \in \mathcal{C}^*(H_{\varepsilon e})$ for all $(e,\varepsilon) \in \Theta_u$. Since $\mathcal{C}$ is a nonempty class of groups closed under subgroups; $\mathcal{C}$ contains the trivial group, and consequently, $L_{\varepsilon e} \in \mathcal{C}^*(H_{\varepsilon e})$ for all remaining pairs (e,ε). Therefore, since Γ is a finite graph, H_v/M_v amounts to the direct product of finitely many $\mathcal{C}$-groups $H_{\varepsilon e}/L_{\varepsilon e}$, and so $M_v \in \mathcal{C}^*(H_v)$ for all $v \in V$.

Given $v \in V$ and using the $\mathcal{C}$-regularity of G_v with respect to the subgroup H_v, find a subgroup $R_v \in \mathcal{C}^*(G_v)$ with $M_v = R_v \cap H_v$. Then $R_u \cap H_u = M_u \leq L$, and since the natural homomorphism $G_v \to G_v/R_v$ extends the natural homomorphism $H_v \to H_v/M_v$, the subgroup H_vR_v/R_v amounts to the direct product of $H_{\varepsilon e}R_v/R_v$ for $(e,\varepsilon) \in \Theta_v$. Thus, $\{R_v \mid v \in V\}$ is the required system.

Assume that condition (2) holds. Given $v \in V$, define the subgroup $R_v \in \mathcal{C}^*(G_v)$ using induction on the length of the (unique) path connecting v to u. Put $R_u = L$. Assume that $v \neq u$, while e is the edge of the path from v to u incident to v, and $v = e(\varepsilon)$ for $\varepsilon = \pm 1$, provided that the subgroup $R_{e(-\varepsilon)} \in \mathcal{C}^*(G_{e(-\varepsilon)})$ is already defined. Then

$$R_{e(-\varepsilon)} \cap H_{-\varepsilon e} \in \mathcal{C}^*(H_{-\varepsilon e}), \quad (R_{e(-\varepsilon)} \cap H_{-\varepsilon e})\varphi_{-\varepsilon e}^{-1}\varphi_{\varepsilon e} \in \mathcal{C}^*(H_{\varepsilon e})$$

and by the $\mathcal{C}$-regularity of $G_{e(\varepsilon)} = G_v$ with respect to the subgroup $H_{\varepsilon e}$ there is $R_v \in \mathcal{C}^*(G_v)$ such that

$$R_v \cap H_{\varepsilon e} = (R_{e(-\varepsilon)} \cap H_{-\varepsilon e})\varphi_{-\varepsilon e}^{-1}\varphi_{\varepsilon e}.$$

It is clear that $\{R_v \mid v \in V\}$ is the required system.

Proof of Theorem 4. All is straightforward from Propositions 9 and 10.

Proposition 11 [23, Proposition 3]. *Suppose that $\mathcal{C}$ is a class of groups closed under quotients. Consider a group X with a normal subgroup Y. The subgroup Y is $\mathcal{C}$-separable in X if and only if X/Y is $\mathcal{C}$-residual.*

Proposition 12. *Suppose that $\mathcal{C}$ is a class of groups closed under subgroups and quotients. Consider a group X and two central subgroups Y and Z with $Y \cap Z = 1$. Suppose also that X is $\mathcal{C}$-residual and $\mathcal{C}$-regular with respect to the subgroup YZ, while YZ is $\mathcal{C}$-separable in X. Then*

(1) Z is $\mathcal{C}$-separable in X and X/Z is $\mathcal{C}$-residual;

(2) X/Z is $\mathcal{C}$-regular with respect to YZ/Z and YZ/Z is $\mathcal{C}$-separable in X/Z.

Proof. (1): Take some $x \in X \setminus Z$ and verify that there exists a subgroup $N \in \mathcal{C}^*(X)$ with $x \notin ZN$. If $x \notin YZ$; then, since YZ is $\mathcal{C}$-separable, there is a subgroup $N \in \mathcal{C}^*(X)$ with $x \notin YZN$. It is obvious now that $x \notin ZN$. Suppose that $x = yz$ for some $y \in Y$ and $z \in Z$. Since $x \notin Z$, we see that y is distinct from 1 and by the $\mathcal{C}$-residuality of X it lies outside some subgroup $L \in \mathcal{C}^*(X)$. The condition $Y \cap Z = 1$ implies that $yz \notin (L \cap Y)Z$. Since

$$YZ/(L \cap Y)Z \cong Y/L \cap Y \cong YL/L \leq X/L$$

and the class $\mathcal{C}$ is closed under subgroups, it follows that

$$(L \cap Y)Z \in \mathcal{C}^*(YZ).$$

Thus, since X is $\mathcal{C}$-regular with respect to YZ, there is a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap YZ = (L \cap Y)Z$. Then $yz \notin N$ and $Z \leq N$, whence $x = yz \notin ZN$.

Thus, the subgroup Z is $\mathcal{C}$-separable in X and by Proposition 11 the quotient X/Z is $\mathcal{C}$-residual.

(2): Take some subgroup $M/Z \in \mathcal{C}^*(YZ/Z)$. Then $M \in \mathcal{C}^*(YZ)$ and, since X is $\mathcal{C}$-regular with respect to YZ, there is a subgroup $N \in \mathcal{C}^*(X)$ with $N \cap YZ = M$. This easily implies that $N/Z \in \mathcal{C}^*(X/Z)$ and $N/Z \cap YZ/Z = M/Z$. Thus, X/Z is $\mathcal{C}$-regular with respect to YZ/Z. It remains to observe that according to Proposition 11 the $\mathcal{C}$-separability of YZ in X is equivalent to the $\mathcal{C}$-residuality of $X/YZ \cong (X/Z)/(YZ/Z)$, which in turn is equivalent to the $\mathcal{C}$-separability of YZ/Z in X/Z.

Proof of Theorem 3. According to Proposition 2, given $g \in \pi_1(\mathcal{G}(\Gamma) \setminus \{1\}$, there is a finite subgraph $\Gamma' = (V', E')$ of Γ such that $\Gamma' \cap T$ is a tree, $g\rho_{\mathcal{R}(E \setminus E')} \in \pi_1(\mathcal{G}_{\mathcal{R}(E \setminus E')}(\Gamma')) \setminus \{1\}$, the graph of groups $\mathcal{G}_{\mathcal{R}(E \setminus E')}(\Gamma')$ is of type (1), and the subgroup $\pi_1(\mathcal{G}_{\mathcal{R}(E \setminus E')}(\Gamma'))$ is a retract of $\pi_1(\mathcal{G}_{\mathcal{R}(E \setminus E')}(\Gamma))$. The definition of $\mathcal{R}(E \setminus E') = \{R_v \mid v \in V\}$ implies that for each $v \in V$ the subgroup R_v is a direct factor of H_v, while the group H_v/R_v amounts to the direct product of the subgroups $H_{\varepsilon e}R_v/R_v$ for $(e, \varepsilon) \in \Theta_v$. Thus, by Proposition 12 G_v/R_v is $\mathcal{C}$-residual and $\mathcal{C}$-regular with respect to H_v/R_v, while H_v/R_v is $\mathcal{C}$-separable in G_v/R_v.

By the same Proposition 12, the listed properties imply the $\mathcal{C}$-separability in G_v/R_v of all direct factors $H_{\varepsilon e}R_v/R_v$ for $(e, \varepsilon) \in \Theta_v$ of the subgroup H_v/R_v.

Thus, $\pi_1(\mathcal{G}_{\mathcal{R}(E \setminus E')}(\Gamma'))$ satisfies the hypotheses of Theorem 4, and so it is $\mathcal{C}$-residual. This implies that the composition of the mapping $\rho_{\mathcal{R}(E \setminus E')}$ and the retracting homomorphism can be extended to a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a $\mathcal{C}$-group carrying g to a nontrivial element.

6. Proofs of Corollaries

Proof of Corollary 1. It is easy to see that the classes $\mathcal{S}$ of all solvable groups and $\mathcal{PS}_\rho$ of periodic solvable ρ-groups of finite exponent are root classes closed under quotients. Since the solvability lengths and exponents of G_v for $v \in V$ are jointly bounded, their direct product lies in $\mathcal{S}$ or $\mathcal{PS}_\rho$ depending on the claim in question. Thus, the residuality of $\pi_1(\mathcal{G}(\Gamma))$ follows from Theorems 1 and 2.

Proof of Corollary 2. The group G_v, for each $v \in V$, admits a homomorphism σ_v onto a group of class $\mathcal{C}$ injective on the subgroup H_v: If $G_v \in \mathcal{C}$ then σ_v is the identity mapping of G_v; in the remaining two cases its existence is guaranteed by Proposition 5 above and Proposition 11 of [24] respectively. If Γ is a finite graph then the direct product of $G_v\sigma_v$ for $v \in V$ lies in $\mathcal{C}$. Hence, again we can apply Theorems 1 and 2.

Proposition 13. *Suppose that $\mathcal{C}$ is a root class of groups consisting of periodic groups. Then the following hold:*

(1) *$\mathcal{C}$ contains all finite solvable $\rho(\mathcal{C})$-groups* [25, Proposition 10].

(2) *Each group in $\mathcal{C}$ has finite exponent* [23, Proposition 17].

(3) *If $\mathcal{C}$ is closed under quotients then every $\rho(\mathcal{C})$-bounded solvable $\mathcal{C}$-group is finite* [23, Proposition 18].

Proposition 14. *Suppose that $\mathcal{C}$ is a root class of groups consisting of periodic groups and $\rho(\mathcal{C})$ contains all primes. Then in every $\rho(\mathcal{C})$-bounded solvable group all subgroups are $\mathcal{C}$-separable.*

PROOF. Take a $\rho(\mathcal{C})$-bounded solvable group X and a subgroup Y of X. Theorem 6 of [17] shows that Y is finitely separable in X. Since every homomorphic image of X is a solvable group, Y turns out separable in it by the class $\mathcal{FS}$ of all finite solvable groups. Proposition 13 yields $\mathcal{FS} \subseteq \mathcal{C}$.

Hence, the subgroup Y is $\mathcal{C}$-separable.

The following combines Propositions 5 and 8 of [24]:

Proposition 15. *Suppose that $\mathcal{C}$ is a root class of groups consisting of periodic groups and consider a $\rho(\mathcal{C})$-bounded nilpotent group X. A subgroup of X is $\mathcal{C}$-separable in X if and only if it is $\rho(\mathcal{C})'$-isolated in X.*

Proposition 16. *Suppose that $\mathcal{C}$ is a root class of groups consisting of periodic groups and closed under quotients. Consider a $\rho(\mathcal{C})$-bounded solvable group X and suppose that at least one of the following holds:*

(1) *$\rho(\mathcal{C})$ contains all primes;*

(2) *X is $\rho(\mathcal{C})$-bounded nilpotent.*

Then X is $\mathcal{C}$-regular with respect to each central subgroup Y.

PROOF. Taking an arbitrary subgroup $M \in \mathcal{C}^*(Y)$, put $\overline{X} = X/M$ and $\overline{Y} = Y/M$. Let us find a subgroup $\overline{N} = N/M$ satisfying $\overline{N} \in \mathcal{C}^*(\overline{X})$ and $\overline{N} \cap \overline{Y} = 1$. Then $N \in \mathcal{C}^*(X)$ and $N \cap Y = M$.

Since the classes $\mathcal{BS}_{\rho(\mathcal{C})}$ and $\mathcal{BN}_{\rho(\mathcal{C})}$ of $\rho(\mathcal{C})$-bounded solvable groups and $\rho(\mathcal{C})$-bounded nilpotent groups are closed under subgroups and quotients [21, Proposition 2], it follows that $\overline{X}, \overline{Y} \in \mathcal{BS}_{\rho(\mathcal{C})}$, and by Proposition 13 the $\mathcal{C}$-group $\overline{Y}$ is finite. If $\rho(\mathcal{C})$ contains all primes then by Proposition 14 $\overline{X}$ is $\mathcal{C}$-residual and Proposition 5 guarantees that the required subgroup $\overline{N}$ exists.

Assume that $X \in \mathcal{BN}_{\rho(\mathcal{C})}$ and take the set $\overline{T}$ of elements of $\overline{X}$ such that $x \in \overline{T}$ if and only if the order of x is finite and not divisible by any number in $\rho(\mathcal{C})$. Then $\overline{T}$ is a normal subgroup of $\overline{X}$ [26, § 4], while $\overline{X}/\overline{T} \in \mathcal{BN}_{\rho(\mathcal{C})}$ in view of the properties of the class $\mathcal{BN}_{\rho(\mathcal{C})}$ mentioned above, and $\overline{T} \cap \overline{Y} = 1$ because $\overline{Y}$ lies in $\mathcal{C}$, is finite, and consequently $\overline{Y}$ is a $\rho(\mathcal{C})$-group. Since the trivial subgroup of $\overline{X}/\overline{T}$ is $\rho(\mathcal{C})'$-isolated, by Proposition 15 $\overline{Y}$ is $\mathcal{C}$-residual. Hence, by Proposition 5 there exists a subgroup $\overline{N}/\overline{T} \in \mathcal{C}^*(\overline{X}/\overline{T})$ satisfying $\overline{N}/\overline{T} \cap \overline{Y}\,\overline{T}/\overline{T} = 1$. It is easy to see that then $\overline{N}$ is the required subgroup.

PROOF OF COROLLARY 3. If Γ is a finite graph then making the transformation described in Section 2 we can fulfil the relation $H_{\varepsilon e} \neq G_{e(\varepsilon)}$ for all $e \in E$ and $\varepsilon = \pm 1$. It is clear that all vertex groups remain solvable and $\rho(\mathcal{C})$-bounded. The class $\mathcal{FS}$ of all finite solvable groups is a root class closed under quotients, and by Proposition 13 $\mathcal{FS}$ is included in $\mathcal{C}$. According to Propositions 14 and 16, for each $v \in V$ the group G_v is $\mathcal{FS}$-regular with respect to each central subgroup, and all subgroups of G_v are $\mathcal{FS}$-separable. Thus, Theorems 3 and 4 imply that $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{FS}$-residual and therefore $\mathcal{C}$-residual.

PROOF OF COROLLARY 4. Denote by $\mathcal{C}_1$ the class of finite solvable $\rho(\mathcal{C})$-groups; and by $\mathcal{C}_2$, the class of periodic $\rho(\mathcal{C})$-groups of finite exponent. It is easy to see that $\mathcal{C}_1$ and $\mathcal{C}_2$ are root classes closed under quotients. Thus, the claim holds for them by Theorems 3 and 4 as well as Propositions 15 and 16.

Proposition 13 implies that $\mathcal{C}_1 \subseteq \mathcal{C} \subseteq \mathcal{C}_2$. It is also obvious that

$$\rho(\mathcal{C}_1) = \rho(\mathcal{C}) = \rho(\mathcal{C}_2).$$

Thus, if the conditions of $\rho(\mathcal{C})'$-isolation of subgroups in the statement hold then $\pi_1(\mathcal{G}(\Gamma))$ is $\mathcal{C}_1$-residual, and so $\mathcal{C}$-residual.

Conversely, if $\pi_1(\mathcal{G}(\Gamma))$ is a $\mathcal{C}$-residual group then it is $\mathcal{C}_2$-residual and satisfies the conditions of $\rho(\mathcal{C})'$-isolation of subgroups.

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E. V. Sokolov
Ivanovo State University, Ivanovo, Russia
E-mail address: ev-sokolov@yandex.ru