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A DEDEKIND CRITERION OVER VALUED FIELDS

L. El Fadil, M. Boulagouaz, and A. Deajim UDC 512.62

Abstract: Let (K, ν) be an arbitrary-rank valued field, let R_ν be the valuation ring of (K, ν), and let $K(\alpha)/K$ be a separable finite field extension generated over K by a root of a monic irreducible polynomial $f \in R_\nu[X]$. We give some necessary and sufficient conditions for $R_\nu[\alpha]$ to be integrally closed. We further characterize the integral closedness of $R_\nu[\alpha]$ which is based on information about the valuations on $K(\alpha)$ extending ν. Our results enhance and generalize some existing results as well as provide applications and examples.

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1. Introduction

Given a valued field (K, ν), we denote by $\overline{K}$ an algebraic closure of K; by R_ν, the valuation ring of ν; by M_ν, the maximal ideal of R_ν; by $k_\nu = R_\nu/M_\nu$, the residue field of ν; and by Γ_ν, the (totally ordered abelian) value group of ν. We denote the set of elements $g \in \Gamma_\nu$ such that $g > 0$ by Γ_ν^+ and a minimum element of Γ_ν^+, if any, by $\min(\Gamma_\nu^+)$. We also denote by ν^G the Gaussian extension of ν to the field $K(X)$ of rational functions; i.e., given $f(X) = \sum_{i=0}^m a_i X^i \in K[X]$, we put $\nu^G(f) = \min\{\nu(a_0), \dots, \nu(a_m)\}$ and extend to $K(X)$ as $\nu^G(f/g) = \nu^G(f) - \nu^G(g)$ for $f, g \in K[X]$ and $g \neq 0$.

Let (K, ν) be a valued field of arbitrary rank, let $f \in R_\nu[X]$ be a monic irreducible separable polynomial, let $\alpha \in \overline{K}$ be a root of f, let $L = K(\alpha)$ be the simple field extension over K generated by α, and let S be the integral closure of R_ν in L. Assume that $\overline{f} = \prod_{i=0}^s \overline{\phi_i}^{\,l_i}$ is the monic irreducible factorization of $\overline{f}$ over k_ν, and $\phi_i \in R_\nu[X]$ is a monic lifting of $\overline{\phi_i}$ for $i = 1, \dots, s$. For the sake of brevity, we will refer to these notations and assumptions as **Assump's**.

Under **Assump's**, if R_ν is a discrete valuation ring and M_ν does not divide the index ideal $[S : R_\nu[\alpha]]$, then the well-known theorem of Dedekind (see [1, Proposition 8.3] for instance) gives the factorization of the ideal $M_\nu S$; namely, $M_\nu S = \prod_{i=1}^s \mathfrak{p}_i^{l_i}$, where $\mathfrak{p}_i = M_\nu S + \phi_i(\alpha)S$ with residue degree $\deg(\phi_i)$. Dedekind in [2] gave a criterion for the divisibility of $[S : R[\alpha]]$ by M_ν that was also extended in [3]. Considering an arbitrary valuation ν in general, Ershov in [4] introduced a nice generalized version of Dedekind's Criterion. Namely, he showed that if we write f in the form

$$f = \prod_{i=1}^s \phi_i^{l_i} + \pi T$$

for some $\pi \in M_\nu$ and $T \in (R_\nu - M_\nu)[X]$; then $R_\nu[\alpha]$ is integrally closed (i.e. $R_\nu[\alpha] = S$) if and only if either $l_i = 1$ for all $i = 1, \dots, s$ or, else, $\nu(\pi) = \min(\Gamma_\nu^+)$ and $\overline{\phi_i}$ does not divide $\overline{T}$ for all those $i = 1, \dots, s$ with $l_i \geq 2$. Khanduja and Kumar gave a different elegant proof of Ershov's result in [5, Theorem 1.1].

Assuming **Assump's**, the following Theorem 2.5 gives a new characterization of the integral closedness of $R_\nu[\alpha]$, where we utilize the Euclidean division of f by ϕ_i for all $i = 1, \dots, s$, $l_i \geq 2$, with a motivation to enhance its application as compared to [5, Theorem 1.1]. Theorem 2.5 further improves [5, Theorem 4.1] as it does not require K to be Henselian. Using our techniques, moreover, we give a simpler proof of some significant result proved in [6, Theorem 1.3] which gives a complete characterization

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of the integral closedness of $R_\nu[\alpha]$ which is based on the valuations of L extending ν and their values at $\phi_i(\alpha)$. We also compute the ramification indices and residue degrees of all valuations of L extending ν (Corollary 2.10). Some further applications and examples are given in Section 3.

2. The Main Results

Keeping the notations of **Assump's**, denote by (K^h, ν^h) a Henselization of (K, ν) and by $\overline{\nu^h}$ the unique extension of ν^h to the algebraic closure $\overline{K^h}$ of K^h.

We begin this section with the following important well-known result which we present without proof (see, for instance, [7, 17.17]). The result asserts a one-to-one correspondence between the valuations on L extending ν and the irreducible factors of f over K^h.

Lemma 2.1. *Keep the notation and assumptions of* **Assump's**. *Let $f = \prod_{j=1}^t f_j$ be the factorization of f into a product of distinct monic irreducible polynomials over K^h. Then there are exactly t extensions $\omega_1, \dots, \omega_t$ of ν to L. Moreover, if α_j is a root of f_j in $\overline{K^h}$ for $j \in \{1, \dots, t\}$, then the valuation ω_j corresponding to f_j is precisely the valuation on L satisfying $\omega_j(h(\alpha)) = \overline{\nu^h}(h(\alpha_j))$ for all $h \in K[X]$.*

The following result is a generalization of [8, Lemma 2.1] to arbitrary-rank valuations.

Lemma 2.2. *Keep the notation and assumptions of* **Assump's** *and Lemma* 2.1.

(i) *For every $i = 1, \dots, s$, there is some $j = 1, \dots, t$ such that $\omega_j(\phi_i(\alpha)) > 0$.*

(ii) *$\omega_j(p(\alpha)) \geq \nu^G(p(X))$ for every $j = 1, \dots, t$ and every nonzero $p \in R_\nu[X]$.*

(iii) *For every $j = 1, \dots, t$, there exists a unique $i = 1, \dots, s$ such that $\omega_j(\phi_i(\alpha)) > 0$. Moreover, $\omega_j(\phi_k(\alpha)) = 0$ for all $k \neq i$, $k = 1, \dots, s$.*

(iv) *Equality holds in* (ii) *if and only if $\overline{\phi_i}$ does not divide $\overline{(p/a)}$ for the unique index i associated to ω_j in* (iii), *where a is any coefficient of p of a minimum ν-valuation.*

Proof. (i): Since $k_{\nu^h} = k_\nu$; therefore, $\prod_{i=1}^s \overline{\phi_i}^{l_i} = \prod_{j=1}^t \overline{f_j}$. So, for a fixed $i = 1, \dots, s$, there is some $j = 1, \dots, t$ such that $\overline{\phi_i}$ divides $\overline{f_j}$. Since f_j is irreducible, it follows from Hensel's Lemma that $\overline{f_j} = \overline{\phi_i}^{u_i}$ for some $1 \leq u_i \leq l_i$. Let $\alpha_j \in \overline{K^h}$ be a root of f_j. As $f_j(\alpha_j) = 0$, we have $\overline{\phi_i(\alpha_j)}^{u_j} = \overline{f_j(\alpha_j)} = \overline{0}$ modulo $M_{\overline{\nu^h}}$. Thus, $\phi_i(\alpha_j)^{u_i} \in M_{\overline{\nu^h}}$ and so $\phi_i(\alpha_j) \in M_{\overline{\nu^h}}$. Now, by Lemma 2.1, $\omega_j(\phi_i(\alpha)) = \overline{\nu^h}(\phi_i(\alpha_j)) > 0$ as desired.

(ii): Set $p_1 = p/a$, where a is a coefficient of p of the least ν-valuation. As $\nu^G(p_1) = 0$, $p_1 \in R_\nu[X]$. Since $S = \bigcap_{j=1}^t R_{\omega_j}$ (see [9, Corollary 3.1.4]), it follows that, for every $j = 1, \dots, t$, we have $p_1(\alpha) \in R_\nu[\alpha] \subseteq S \subseteq R_{\omega_j}$ and

$$\omega_j(p(\alpha)) = \omega_j(a) + \omega_j(p_1(\alpha)) = \nu(a) + \omega_j(p_1(\alpha)) = \nu^G(p(X)) + \omega_j(p_1(\alpha)) \geq \nu^G(p(X))$$

as claimed.

(iii): Fix a $j = 1, \dots, t$. Since $\prod_{i=1}^s \phi_i(\alpha)^{l_i} \equiv f(\alpha) \equiv 0 \pmod{M_{\omega_j}}$; therefore, $\omega_j(\prod_{i=1}^s \phi_i(\alpha)^{l_i}) > 0$. Thus, $\omega_j(\phi_i(\alpha)) > 0$ (and so $\phi_i(\alpha) \in M_{\omega_j}$) for some $i = 1, \dots, s$. For $k = 1, \dots, s$ with $k \neq i$, as $\overline{\phi_i}$ and $\overline{\phi_k}$ are coprime modulo M_ν, we let $s_k, t_k \in R_\nu[X]$ be such that $\overline{s_k}\overline{\phi_i} + \overline{t_k}\overline{\phi_k} \equiv 1 \pmod{M_\nu}$. Then $s_k(\alpha)\phi_i(\alpha) + t_k(\alpha)\phi_k(\alpha) = 1 + h(\alpha)$ for some $h \in M_\nu[X]$. As $\nu^G(h) > 0$, it follows from (ii) that $\omega_j(h(\alpha)) > 0$ and so $h(\alpha) \in M_{\omega_j}$. Since $\phi_i(\alpha) \in M_{\omega_j}$ and $s_k(\alpha) \in R_\nu[\alpha] \subseteq S \subseteq R_{\omega_j}$; therefore, $s_k(\alpha)\phi_i(\alpha)) \in M_{\omega_j}$. Thus, $t_k(\alpha)\phi_k(\alpha) \in R_{\omega_j} - M_{\omega_j}$. Hence, $\omega_j(t_k(\alpha)\phi_k(\alpha)) = 0$ and so $\omega_j(\phi_k(\alpha)) = 0$, yielding the uniqueness of i such that $\omega_j(\phi_i(\alpha)) > 0$.

(iv): Define the map $\psi_j : k_\nu[X] \to R_{\omega_j}/M_{\omega_j}$ by $\overline{p}(X) \mapsto p(\alpha) + M_{\omega_j}$. Since $M_\nu \subseteq M_{\omega_j}$, ψ_j is a well-defined ring homomorphism. As $\omega_j(p(\alpha)) = \nu^G(p(X)) + \omega_j(p_1(\alpha))$ (see (ii)), it follows that $\omega_j(p(\alpha)) = \nu^G(p(X))$ if and only if $\omega_j(p_1(\alpha)) = 0$, if and only if $p_1(\alpha) \in R_{\omega_j} - M_{\omega_j}$, if and only if $\overline{p_1}(X) \notin \ker \psi_j$. By (iii), let ϕ_i be such that $\omega_j(\phi_i(\alpha)) > 0$. Then $\phi_i(\alpha) \in M_{\omega_j}$ and so $\overline{\phi_i} \in \ker \psi_j$. Since

$\ker\psi_j$ is a principal ideal of $k_\nu[X]$ and $\overline{\phi_i}$ is irreducible over k_ν, $\ker\psi_j$ is generated by $\overline{\phi_i}$. It follows that $\omega_j(p(\alpha)) = \nu^G(p)$ if and only if $\overline{\phi_i}$ does not divide $\overline{p_1}$. $\square$

Keeping the notation of **Assump's**, in what follows we let $q_i, r_i \in R_\nu[X]$ be the quotient and the remainder upon the Euclidean division of f by ϕ_i for $i = 1, \ldots, s$.

In [5, Lemma 2.1(b)], it was shown that Γ_ν^+ contains a smallest element in case $R_\nu[\alpha]$ is integrally closed and $l_i \geq 2$ for some $i = 1, \ldots, s$. Below, we prove this fact differently with something more.

Lemma 2.3. *Keep the notation and assumptions of Lemma 2.2. If $R_\nu[\alpha]$ is integrally closed and $I = \{i \mid l_i \geq 2, i = 1, \ldots, s\}$ is not empty, then Γ_ν^+ has a minimum element with $\min(\Gamma_\nu^+) = \nu^G(r_i)$ for every $i \in I$.*

Proof. For $i \in I$, let $q_i^*, r_i^* \in R_\nu[X]$ be the quotient and remainder upon the Euclidean division of q_i by ϕ_i. Since $\overline{\phi_i}$ divides both $\overline{f}$ and $\overline{q_i}\overline{\phi_i}$; therefore, $\overline{\phi_i}$ divides $\overline{r_i}$. But, as ϕ_i is monic, $\deg(\overline{\phi_i}) = \deg(\phi_i) > \deg(r_i) \geq \deg(\overline{r_i})$. This implies that $\overline{r_i}$ is zero and so $\nu^G(r_i) > 0$. Thus, $\nu^G(r_i) \in \Gamma_\nu^+$. Now as $\overline{f} = \overline{q_i}\overline{\phi_i}$ and $\overline{\phi_i}^2$ divides $\overline{f}$, we see that $\overline{\phi_i}$ must divide $\overline{q_i}$. Applying a similar argument to the expression $\overline{q_i} = \overline{q_i^*}\overline{\phi_i} + \overline{r_i^*}$, we get that $\overline{r_i^*}$ is zero. Thus, $\nu^G(r_i^*) > 0$ and so $\nu^G(r_i^*) \in \Gamma_\nu^+$. To the contrary, suppose that $\tau_i \in \Gamma_\nu^+$ is such that $\tau_i < \nu^G(r_i)$, and set $\delta_i = \min\{\tau_i, \nu^G(r_i) - \tau_i, \nu^G(r_i^*)\}$. As $\delta_i \in \Gamma_\nu^+$, let $d_i \in R_\nu$ be such that $\nu(d_i) = \delta_i$ and set $\theta_i = q_i(\alpha)/d_i$. Let ω be a valuation of L extending ν. We show that $\theta_i \in R_\omega$ and, since ω is arbitrary, it would follow that $\theta_i \in S$ [9, Corollary 3.1.4]. As $f(\alpha) = 0$; therefore, $\theta_i = -r_i(\alpha)/(d_i\phi_i(\alpha))$. By Lemma 2.2, let $j \in \{1, \ldots, s\}$ be the unique index such that $\omega(\phi_j(\alpha)) > 0$ and $\omega(\phi_k(\alpha)) = 0$ for all $k \in \{1, \ldots, s\} - \{j\}$. If $i \neq j$, then $\omega(\phi_i(\alpha)) = 0$ and

$$\omega(\theta_i) = \omega(r_i(\alpha)) - \omega(d_i) = \omega(r_i(\alpha)) - \nu(d_i) \geq \nu^G(r_i) - \delta_i > \delta_i - \delta_i = 0,$$

and so $\theta_i \in R_\omega$ in this case. Assume, on the other hand, that $i = j$. If $\omega(\phi_i(\alpha)) > \delta_i$, then as q_i^* is monic and $\omega(q_i^*(\alpha)) \geq \nu^G(q_i^*) = 0$ (Lemma 2.2), we have

$$\omega(q_i(\alpha)) \geq \min\{\omega(q_i^*(\alpha)\phi_i(\alpha)), \omega(r_i^*(\alpha))\} \geq \min\{\omega(\phi_i(\alpha)), \nu^G(r_i^*)\} \geq \delta_i.$$

So, $\omega(\theta_i) = \omega(q_i(\alpha)) - \omega(d_i) \geq \delta_i - \delta_i = 0$, which implies that $\theta_i \in R_\omega$ in this case too. If, on the other hand, $\omega(\phi_i(\alpha)) \leq \delta_i$; then

$$\omega(\theta_i) = \omega(r_i(\alpha)) - \omega(d_i) - \omega(\phi_i(\alpha)) \geq \nu^G(r_i) - \delta_i - \delta_i \geq \nu^G(r_i) - \tau_i - \delta_i \geq \delta_i - \delta_i = 0.$$

So $\theta_i \in R_\omega$ in this case as well. It follows now from the above argument that $\theta_i \in S$. But, as q_i is monic and $1/d_i \notin R_\nu$, it is clear that $\theta_i \notin R_\nu[\alpha]$, contradicting the assumption that $R_\nu[\alpha]$ is integrally closed. Hence, $\nu^G(r_i)$ is the minimum element of Γ_ν^+ as claimed. $\square$

Lemma 2.4. *Keep the notation and assumptions of Lemma 2.2. If $\min(\Gamma_\nu^+) = \sigma$, then $\omega(\phi_i(\alpha)) = \sigma/l_i$ for all $i \in \{1, \ldots, s\}$ with $\nu^G(r_i) = \sigma$ and for every valuation ω of L extending ν such that $\omega(\phi_i(\alpha)) > 0$.*

Proof. Let $i \in \{1, \ldots, s\}$ and let ω be a valuation of L extending ν such that $\omega(\phi_i(\alpha)) > 0$. Write f in the form $f = m_i\phi_i^{l_i} + n_i\phi_i + r_i$, with $m_i, n_i \in R_\nu[X]$ and $\nu^G(m_i) = 0$, while $\overline{\phi_i}$ does not divide $\overline{m_i}$, $\nu^G(n_i) > 0$, and $\deg(r_i) < \deg(\phi_i)$. Notice that if $l_i = 1$ then $m_i = q_i$ and $n_i = 0$. By Lemma 2.2, $\omega(n_i(\alpha)) \geq \nu^G(n_i) \geq \sigma$, $\omega(m_i(\alpha)) = \nu^G(m_i) = 0$, and $\omega(r_i(\alpha)) = \nu^G(r_i) = \sigma$ as $\overline{\phi_i}$ divides neither $\overline{m_i}$ nor $\overline{r_i}$. We then have

$$l_i\omega(\phi_i(\alpha)) = \omega(m_i(\alpha)\phi_i^{l_i}(\alpha)) = \omega(n_i(\alpha)\phi_i(\alpha) + r_i(\alpha)) = \omega(r_i(\alpha)) = \nu^G(r_i) = \sigma$$

as claimed. $\square$

Now we get to our first main result which computationally enhances [5, Theorem 1.1] as well as improves [5, Theorem 4.1] in the sense that K is not assumed to be Henselian.

Theorem 2.5. *Keep the notation and assumptions of Lemma 2.2.*

(i) *If $l_i = 1$ for all $i = 1, \ldots, s$, then $R_\nu[\alpha]$ is integrally closed.*

(ii) *If $I = \{i \mid l_i \geq 2, i = 1, \ldots, s\}$ is not empty, then $R_\nu[\alpha]$ is integrally closed if and only if $\nu^G(r_i) = \min(\Gamma_\nu^+)$ for every $i \in I$.*

PROOF. (i): Assume that $l_i = 1$ for all $i = 1, \dots, s$. An arbitrary element of S is of the form $\theta = h(\alpha)/b$ for some $b \in R_\nu$ and $h \in R_\nu[X]$, with $\nu^G(h) = 0$ and $\deg(h) < \deg(f)$. Since f is monic, $\deg(\overline{h}) \leq \deg(h) < \deg(f) = \deg(\overline{f})$. As $l_i = 1$ for all $i = 1, \dots, s$, there is some $i = 1, \dots, s$ such that $\overline{\phi_i}$ does not divide $\overline{h}$. For such a fixed i, let ω be a valuation of L extending ν such that $\omega(\phi_i(\alpha)) > 0$, which exists by Lemma 2.2. Hence, $\omega(h(\alpha)) = \nu^G(h) = 0$. If $\nu(b) > 0$, then $\omega(\theta) = \omega(h(\alpha)) - \omega(b) = 0 - \nu(b) < 0$. Thus $\theta \notin S$, which is a contradiction. Hence, $\nu(b) = 0$, which implies that $\theta \in R_\nu[\alpha]$. This shows that $S = R_\nu[\alpha]$ and so $R_\nu[\alpha]$ is integrally closed.

(ii): Assume that $I \neq \varnothing$. If $R_\nu[\alpha]$ is integrally closed, then it follows from Lemma 2.3 that $\nu^G(r_i)$ is the minimum element of Γ_ν^+ for every $i \in I$, as claimed.

Conversely, put $\min(\Gamma_\nu^+) = \sigma$ and let $\pi \in R_\nu$ be such that $\nu(\pi) = \sigma$. Assume that $\nu^G(r_i) = \sigma$ for every $i \in I$. We aim at proving that $R_\nu[\alpha]$ is integrally closed. By an appropriate choice of a lifting of $\overline{\phi_i}$, we begin by showing that we can also assume that $\nu^G(r_i) = \sigma$ for $i \notin I$. Let $i \notin I$, and assume that $\nu^G(r_i) > \sigma$. If $\delta \in \Gamma_\nu^+$ with $\sigma < \delta < 2\sigma$, then $\delta - \sigma \in \Gamma_\nu^+$ with $\delta - \sigma < 2\sigma - \sigma = \sigma$ contradicting the minimality of σ. So there is no element of Γ_ν^+ lying strictly between σ and 2σ. So, $\nu^G(r_i) \geq 2\sigma$. Let $q_i^*, r_i^* \in R_\nu[X]$ be the quotient and remainder upon the Euclidean division of q_i by ϕ_i. Put $\phi_i^{**} = \phi_i + \pi$, $q_i^{**} = q_i - \pi q_i^*$, and $r_i^{**} = r_i - \pi r_i^* + \pi^2 q_i^*$. Then

$$q_i^{**}\phi_i^{**} + r_i^{**} = (q_i - \pi q_i^*)(\phi_i + \pi) + r_i - \pi r_i^* + \pi^2 q_i^* = q_i\phi_i + r_i = f.$$

It can be easily checked that q_i^{**} and r_i^{**} are the quotient and remainder upon the Euclidean division of f by ϕ_i^{**} (if $\deg(r_i^{**}) \geq \deg(\phi_i^{**})$; then we replace r_i^{**} by the remainder upon the Euclidean division of r_i^{**} with ϕ_i^{**} and replace q_i^{**} with $q_i^{**} + Q_i$, where Q_i is the quotient upon the Euclidean division of r_i^{**} by ϕ_i^{**}). Since $l_i = 1$, $\overline{r_i^*}$ is nonzero, and so $\nu^G(\pi r_i^*) = \nu(\pi) = \sigma$. As $\nu^G(r_i) \geq 2\sigma$ and $\nu^G(\pi^2 q_i^*) \geq \nu(\pi^2) = 2\sigma$, it follows that $\nu^G(r_i^{**}) = \nu^G(\pi r_i^*) = \sigma$. So, replacing ϕ_i by $\phi_i + \pi$, we can assume that $\nu^G(r_i) = \sigma$. We thus assume in the remainder of the proof that $\nu^G(r_i) = \sigma$ for all $i = 1, \dots, s$. We finally get to proving that $R_\nu[\alpha]$ is integrally closed. Assume to the contrary that there exists some $\theta \in S - R_\nu[\alpha]$. Then θ can be written as $\theta = g(\alpha)/b$ for some $b \in R_\nu$ and $g \in R_\nu[X]$ with $\nu(b) \geq \sigma$, $\nu^G(g) = 0$, and $\deg(g) < \deg(f)$. Given $i = 1, \dots, s$, let $m_i \geq 0$ be the highest power of $\overline{\phi_i}$ dividing $\overline{g}$. Since $\deg(g) < \deg(f)$, there must exist some $i = 1, \dots, s$ such that $m_i \leq l_i - 1$. For such an i, apply the Euclidean division of g by $\phi_i^{m_i}$ to get $g = S_i\phi_i^{m_i} + T_i$, where $S_i, T_i \in R_\nu[X]$, while $\overline{\phi_i}$ does not divide $\overline{S_i}$, and $\nu^G(T_i) \geq \sigma$. By Lemma 2.2, let ω be a valuation of L extending ν such that $\omega(\phi_i(\alpha)) > 0$. Since $\overline{\phi_i}$ does not divide $\overline{S_i}$ and S_i is monic, it follows from Lemma 2.2 that $\omega(S_i(\alpha)) = \nu^G(S_i) = 0$. Using Lemma 2.4, we then have $\omega(S_i(\alpha)\phi_i(\alpha)^{m_i}) = m_i\omega(\phi_i(\alpha)) = m_i\sigma/l_i$. Since $\omega(T_i(\alpha)) \geq \nu^G(T_i) \geq \sigma$ (by Lemma 2.2), it follows that

$$\omega(g(\alpha)) = \min\{\omega(S_i(\alpha)\phi_i(\alpha)^{m_i}), \omega(T_i(\alpha))\} = \min\{m_i\sigma/l_i, \sigma\} = m_i\sigma/l_i < \sigma.$$

Thus, $\omega(\theta) = \omega(g(\alpha)) - \omega(b) = \omega(g(\alpha)) - \nu(b) < \sigma - \sigma = 0$. Hence, $\theta \notin R_\omega$ and so $\theta \notin S$. This contradiction leads to the conclusion that $S = R_\nu[\alpha]$, as desired. □

The following corollary is immediate.

Corollary 2.6. *Keep the assumptions of Theorem* 2.5. *If Γ_ν^+ does not have a minimum element, then $R_\nu[\alpha]$ is integrally closed if and only if $l_i = 1$ for all $i = 1, \dots, s$.*

The following corollary shows, in particular, that Theorem 2.5 is a new version of the generalized Dedekind criterion which computationally improves [4, Theorem 1] and [5, Theorem 1.1] in the case of separable extensions.

Corollary 2.7. *Keep the assumptions of Theorem* 2.5. *If Γ_ν^+ has a minimum element σ and $I = \{i \mid l_i \geq 2,\ i = 1, \dots, s\}$ is not empty, then $R_\nu[\alpha]$ is integrally closed if and only if $\overline{\phi_i}$ does not divide $\overline{M}$ for every $i \in I$, where $M = \frac{f - \prod_{i=1}^{s}\phi_i^{l_i}}{\pi}$ for any $\pi \in R_\nu$ with $\nu(\pi) = \sigma$.*

PROOF. Let $i \in I$. Since $\overline{r_i} = \overline{f} - \overline{q_i}\ \overline{\phi_i}$ and $\overline{\phi_i}$ divides $\overline{f}$; therefore, $\overline{r_i}$ is divisible by $\overline{\phi_i}$. But as $\deg(\overline{r_i}) \le \deg(r_i) < \deg(\phi_i) = \deg(\overline{\phi_i})$, $\overline{r_i}$ must be zero. Thus,

$$\overline{q_i} = \overline{\phi_i^{l_i-1}} \prod_{j=1, j\neq i}^{s} \overline{\phi_j^{l_j}}.$$

Let $H_i \in R_\nu[X]$ be such that $q_i = \phi_i^{l_i-1}\prod_{j=1,j\neq i}^{s}\phi_j^{l_j} + \pi H_i$ with $\pi \in R_\nu$ such that $\nu(\pi) = \sigma$. Then

$$f = \left(\phi_i^{l_i-1}\prod_{j=1,j\neq i}^{s}\phi_j^{l_j} + \pi H_i\right)\phi_i + r_i.$$

Put

$$M = \frac{f - \prod_{j=1}^{s}\phi_j^{l_j}}{\pi} \in R_\nu[X].$$

Then

$$M = \frac{\left(\phi_i^{l_i-1}\prod_{j=1,j\neq i}^{s}\phi_j^{l_j} + \pi H_i\right)\phi_i + r_i - \prod_{j=1}^{s}\phi_j^{l_j}}{\pi} = H_i\phi_i + \frac{r_i}{\pi}.$$

Since $M, H_i\phi_i \in R_\nu[X]$, we must have $\frac{r_i}{\pi} \in R_\nu[X]$ and so $\nu^G\left(\frac{r_i}{\pi}\right) \ge 0$. Clearly, $\overline{\phi_i}$ divides $\overline{M}$ if and only if $\overline{\phi}$ divides $\overline{(\frac{r_i}{\pi})}$. As $\deg(\overline{(\frac{r_i}{\pi})}) \le \deg(\overline{r_i}) < \deg(\overline{\phi_i})$ (see above), we conclude that $\overline{\phi_i}$ divides $\overline{M}$ if and only if $\overline{(\frac{r_i}{\pi})}$ is zero; i.e., $\nu^G(r_i) > \sigma$. Contrapositively, $\overline{\phi_i}$ does not divide $\overline{M}$ if and only if $\nu^G(r_i) = \sigma$. □

Our second main result, Theorem 2.9 below, gives a characterization of the integral closedness of $R_\nu[\alpha]$ which is based on characterization of the extensions of ν to L (see also [6, Theorem 1.3], where the proof of our result is simpler and selfcontained).

In 1850, Eisenstein introduced his infamous criterion for testing irreducibility of polynomials over valued fields in [10]. In 2008, Brown gave a simple proof of the most general version of Eisenstein–Schönemann irreducibility criterion in [11]. Namely, if $p \in \mathbb{Z}$ is prime and $f \in \mathbb{Z}[x]$ is such that $f = \phi^n + a_{n-1}\phi^{n-1} + \cdots + a_0$ for some monic polynomial $\phi \in \mathbb{Z}[x]$ whose reduction modulo p is irreducible and $a_i \in \mathbb{Z}[x]$ with $\deg(a_i) < \deg(\phi)$ for $i = 0, \ldots, n-1$, then f is irreducible over $\mathbb{Q}$ if $\gcd(\nu_p^G(a_0), n) = 1$ and $n\nu_p^G(a_i) \ge (n-i)\nu_p^G(a_0) > 0$ for every i where ν_p is the p-adic valuation. In preparation for Theorem 2.9, we introduce the following definition and prove some lemma that partially generalizes the Eisenstein–Schönemann irreducibility criterion.

DEFINITION. We say that a monic polynomial $g \in R_\nu[X]$ is *ν-Eisenstein–Schönemann* if there exists a monic polynomial $\psi \in R_\nu[X]$ such that $\overline{\psi}$ is irreducible, $\overline{g}$ is a positive power of $\overline{\psi}$, and $\nu^G(r) = \min(\Gamma_\nu^+)$, where $r \in R_\nu[X]$ is the remainder upon the Euclidean division of g by ψ. In particular, if $\psi(x) = x$, then g is said to be *ν-Eisenstein*.

Lemma 2.8. *Keep the assumptions of Theorem* 2.5*. If $g \in R_\nu[X]$ is monic and ν-Eisenstein–Schönemann, then g is irreducible over K.*

PROOF. Let $\psi \in R_\nu[X]$ be monic such that $\overline{\psi}$ is irreducible, $\overline{g} = \overline{\psi}^l$, and $\nu^G(r) = \min(\Gamma_\nu^+) = \sigma$, where $r \in R_\nu[X]$ is the remainder upon the Euclidean division of g by ψ. Suppose to the contrary that $g = h_1h_2$ for some nonconstant and monic $h_1, h_2 \in R_\nu[X]$. Then $\overline{h_1} = \overline{\psi}^{l_1}$ and $\overline{h_2} = \overline{\psi}^{l_2}$ for some positive l_1 and l_2 with $l_1 + l_2 = l$. Assume that the Euclidean division of each of g, h_1, and h_2 by ψ yields

$$g = q\psi + r, \quad h_1 = q_1\psi + r_1, \quad h_2 = Q_2\psi + r_2.$$

It is clear that r is the remainder upon the Euclidean division of the product r_1r_2 by ψ. Since both $\overline{h_1}$ and $\overline{h_2}$ are positive powers of $\overline{\psi}$, both of $\overline{r_1}$ and $\overline{r_2}$ must be zero. So, $\nu^G(r_1) \ge \sigma$ and $\nu^G(r_2) \ge \sigma$. Thus, $\nu^G(r) \ge 2\sigma > \sigma$ (as $\sigma > 0$), which is a contradiction. Hence, g is irreducible over R_ν and, consequently, irreducible over K (by Gauss's Lemma as R_ν is integrally closed). □

Theorem 2.9. *Keep the assumptions of Theorem 2.5. The following are equivalent:*

(i) $R_\nu[\alpha]$ *is integrally closed.*

(ii) ν *has exactly* s *distinct extensions* $\omega_1, \dots, \omega_s$ *to* L, *and if* $I = \{i \mid l_i \geq 2, i = 1, \dots, s\}$ *is not empty; then* $l_i \omega_i(\phi_i(\alpha))$ *is the minimum element of* Γ_ν^+ *for every* $i \in I$, *where* ω_i *is a valuation satisfying* $\omega_i(\phi_i(\alpha)) > 0$ *which exists by Lemma 2.2.*

Proof. Assume that $R_\nu[\alpha]$ is integrally closed. Since $k_\nu = k_{\nu^h}$ and $\overline{f} = \prod_{i=1}^s \overline{\phi_i^{l_i}}$, Hensel's Lemma yields a factorization $f = \prod_{i=1}^s f_i$ over K^h such that $\overline{f_i} = \overline{\phi_i^{l_i}}$ for $i = 1, \dots, s$. In order for us to invoke Lemma 2.1, we need to show that the factors $f_1, \dots, f_s$ are all irreducible over K^h. If $i \in \{1, \dots, s\} - I$, then f_i is immediately irreducible over K^h since $\overline{f_i} = \overline{\phi_i}$ is irreducible. If $i \in I$, then we set to show that f_i is ν^h-Eisenstein–Schönemann and thus irreducible by Lemma 2.8. Since $R_\nu[\alpha]$ is integrally closed and $l_i \geq 2$, it follows from Lemma 2.3 that Γ^+ has a minimum element σ and $\nu^G(r_i) = \sigma$. Notice that as $\Gamma_\nu = \Gamma_{\nu^h}$; therefore, σ is the minimum element of $\Gamma_{\nu^h}^+$ as well. Let $q_i^*, r_i^* \in R_{\nu^h}[X]$ be, respectively, the quotient and remainder upon the Euclidean division of f_i by ϕ_i. Letting $G_i = \prod_{j=1, j\neq i}^s f_j$, we write $f = f_i G_i = q_i^* \phi_i G_i + r_i^* G_i$. Using the Euclidean division again to divide $r_i^* G_i$ by ϕ_i, let $r_i^* G_i = q_i^{**} \phi_i + r_i^{**}$, with $q_i^{**}, r_i^{**} \in R_{\nu^h}[X]$. Then

$$f = q_i^* \phi_i G_i + q_i^{**} \phi_i + r_i^{**} = (q_i^* G_i + q_i^{**})\phi_i + r_i^{**}.$$

Owing to the uniqueness of the remainder, $r_i = r_i^{**}$. Thus, ${\nu^h}^G(r_i^{**}) = {\nu^h}^G(r_i) = \nu^G(r_i) = \sigma$. If ${\nu^h}^G(r_i^*) > \sigma$, then ${\nu^h}^G(r_i^* G_i) > \sigma$ and so ${\nu^h}^G(r_i^{**}) > \sigma$; a contradiction. Thus, ${\nu^h}^G(r_i^*) = \sigma$ and we conclude that f_i is ν^h-Eisenstein–Schönemann as desired. It follows now by Lemma 2.1 that there are exactly s valuations $\omega_1, \dots, \omega_s$ of L extending ν; and by Lemma 2.4 $l_i \omega_i(\phi_i(\alpha)) = \sigma$ for the valuation ω_i of L extending ν with $\omega_i(\phi_i(\alpha)) > 0$.

Conversely, assume that there are exactly s valuations $\omega_1, \dots, \omega_s$ of L extending ν, and if $I = \{i \mid l_i \geq 2, i = 1, \dots, s\}$ is not empty, then $l_i \omega_i(\phi_i(\alpha))$ is the minimum element of Γ_ν^+ for every $i \in I$ and every ω_i satisfying $\omega_i(\phi_i(\alpha)) > 0$. If $I = \varnothing$, then $R_\nu[\alpha]$ is integrally closed by Theorem 2.5. Assume that $I \neq \varnothing$. Following Theorem 2.5, in order to show that $R_\nu[\alpha]$ is integrally closed, it suffices to prove that $\nu^G(r_i) = \sigma$ for every $i \in I$, where $\sigma = \min(\Gamma_\nu^+)$. Let ω_i be the valuation of L extending ν such that $\omega_i(\phi_i(\alpha)) > 0$ (by Lemma 2.2). Then, by assumption, $l_i \omega_i(\phi_i(\alpha)) = \sigma$. Write f in the form $f = m_i \phi_i^{l_i} + n_i \phi_i + r_i$ for $m_i, n_i \in R_\nu[X]$ with $\nu^G(m_i) = 0$. Thus $\overline{\phi_i}$ does not divide $\overline{m_i}$, $\nu^G(n_i) > 0$, and $\deg(r_i) < \deg(\phi_i)$. Since $f(\alpha) = 0$, we have $r_i = -m_i \phi_i^{l_i} - n_i \phi_i$. We can see (using Lemma 2.2(ii)) that

$$\omega_i(n_i(\alpha)\phi_i(\alpha)) = \omega_i(n_i(\alpha)) + \omega_i(\phi_i(\alpha)) > \omega_i(n_i(\alpha)) \geq \nu^G(n_i) \geq \sigma,$$

and (where $\omega_i(m_i(\alpha)) = \nu^G(m_i) = 0$ by Lemma 2.2(iv))

$$\omega_i(m_i(\alpha)\phi_i(\alpha)^{l_i}) = \omega_i(\phi_i(\alpha)^{l_i}) = l_i \omega_i(\phi_i(\alpha)) = \sigma.$$

So,

$$\omega_i(r_i(\alpha)) = \omega_i\left(-m_i(\alpha)\phi_i(\alpha)^{l_i} - n_i(\alpha)\phi_i(\alpha)\right) = \sigma.$$

Since $\deg(r_i) < \deg(\phi_i)$, $\overline{\phi_i}$ does not divide $\overline{r_i}$. So, by Lemma 2.2(iv), $\nu^G(r_i) = \omega_i(r_i(\alpha)) = \sigma$ and the proof is complete. □

With the notation of Theorem 2.9, given a valuation ω_i of L extending ν, we denote the ramification index $[\Gamma_{\omega_i} : \Gamma_\nu]$ by $e(\omega_i/\nu)$ and the residue degree $[k_{\omega_i} : k_\nu]$ by $f(\omega_i/\nu)$. The following *fundamental inequality* is well known (see [9, Theorem 3.3.4] for instance):

$$\sum_{i=1}^s e(\omega_i/\nu) f(\omega_i/\nu) \leq [L : K].$$

When $R_\nu[\alpha]$ is integrally closed, we calculate in the next corollary the ramification indices $e(\omega_i/\nu)$ and residue degrees $f(\omega_i/\nu)$ and show consequently that the above inequality is indeed an equality.

Corollary 2.10. *Keep the notation and assumptions of Theorem 2.9. If $R_\nu[\alpha]$ is integrally closed, then $e(\omega_i/\nu) = l_i$ and $f(\omega_i/\nu) = \deg(\phi_i)$ for all $i = 1, \ldots, s$ and, furthermore, $\sum_{i=1}^{s} e(\omega_i/\nu)f(\omega_i/\nu) = [L : K]$.*

Proof. We show first that $e(\omega_i/\nu) \geq l_i$ and $f(\omega_i/\nu) \geq \deg(\phi_i)$ for every $i = 1, \ldots, s$. If $l_i = 1$ for some $i = 1, \ldots, s$, then clearly $e(\omega_i/\nu) \geq l_i$. Since $\overline{f_i} = \overline{\phi_i}$, it follows that, for any root α_i of f_i, $\overline{\phi_i}$ is the minimal polynomial of $\overline{\alpha_i}$ over k_ν and so

$$\deg(\phi_i) = \deg(\overline{\phi_i}) = [k_\nu(\overline{\alpha_i}) : k_\nu] \leq [k_{\omega_i} : k_\nu] = f(\omega_i/\nu).$$

If $l_i \geq 2$ for some $i = 1, \ldots, s$, then it follows from Theorem 2.9(ii) that $\omega_i(\phi_i(\alpha)) = \sigma/l_i$, where $\sigma = \min(\Gamma_\nu^+)$. So, $\Gamma_\nu \subseteq \Gamma[\sigma/l_i] \subseteq \Gamma_{\omega_i}$ and

$$l_i = [\Gamma_\nu[\sigma/l_i] : \Gamma_\nu] \leq [\Gamma_{\omega_i} : \Gamma_\nu] = e(\omega_i/\nu).$$

Also, for a root α_i of f_i, we have $\overline{\phi_i}(\overline{\alpha_i})^{l_i} = \overline{f_i}(\overline{\alpha_i}) = 0$ implying that $\overline{\phi_i}(\overline{\alpha_i}) = 0$ in k_{ω_i}. Since $\overline{\phi_i}$ is monic and irreducible over k_ν; therefore,

$$\deg(\phi_i) = \deg(\overline{\phi_i}) = [k_\nu(\overline{\alpha_i}) : k_\nu] \leq [k_{\omega_i} : k_\nu] = f(\omega_i/\nu).$$

Now, the above argument yields

$$\sum_{i=1}^{s} e(\omega_i/\nu)f(\omega_i/\nu) \geq \sum_{i=1}^{s} l_i \deg(\phi_i) = \sum_{i=1}^{s} l_i \deg(\overline{\phi_i}) = \deg(\overline{f}) = \deg(f) = [L : K].$$

Thus, by this and the fundamental inequality, we get the claimed equality

$$\sum_{i=1}^{s} e(\omega_i/\nu)f(\omega_i/\nu) = [L : K].$$

Furthermore, since $l_i \leq e(\omega_i/\nu)$ and $\deg(\phi_i) \leq f(\omega_i/\nu)$ for all $i = 1, \ldots, s$ with

$$\sum_{i=1}^{s} l_i \deg(\phi_i) = \sum_{i=1}^{s} e(\omega_i/\nu)f(\omega_i/\nu),$$

we conclude that $l_i = e(\omega_i/\nu)$ and $\deg(\phi_i) = f(\omega_i/\nu)$ for every $i = 1, \ldots, s$. □

3. Applications and Examples

Corollary 3.1. *Keep the assumptions of Theorem 2.5 with $f(X) = X^n - a \in R_\nu[X]$ irreducible of degree $n \geq 2$ and $a \in M_\nu$.*

1. *If Γ_ν^+ has no minimum element, then $R_\nu[\alpha]$ is not integrally closed.*
2. *If $\min(\Gamma_\nu^+) = \sigma$, then $R_\nu[\alpha]$ is integrally closed if and only if $\nu(a) = \sigma$.*

Proof. This is a direct application of Theorem 2.5. □

Corollary 3.2. *Keep the assumptions of Theorem 2.5. Let $\min(\Gamma_\nu^+) = \sigma$ and let $g \in R_\nu[X]$ be monic. If g is ν-Eisenstein and $L = K(\theta)$ for some root θ of g, then $R_\nu[\theta]$ is integrally closed.*

Proof. By Lemma 2.8, g is irreducible over K. Now, the remaining part is straightforward from Theorem 2.5. □

Corollary 3.3. *Let $f(X) = X^n - a \in R_\nu[X]$, $\min(\Gamma^+) = \sigma$, $\nu(a) = m\sigma$ for some $m \in \mathbb{N}$. Let $L = K(\theta)$ for a root θ of $f(X)$. If m and n are coprime; then f is irreducible over R and $R[\theta^v/\pi^u]$ is the integral closure of R in L, where $\pi \in R_\nu$ is such that $\nu(\pi) = \sigma$, and $u, v \in \mathbb{Z}$ are the unique integers such that $mv - nu = 1$ and $0 \leq v < n$.*

Proof. Let $A = a^v/\pi^{nu}$. Then $\nu(A) = (mv - nu)\sigma = \sigma$. By Lemma 2.8, $g(X) = X^n - A$ is irreducible over R_ν. Furthermore, θ^v/π^u is a root of g. So $[K(\theta^v/\pi^u) : K] = n$. Therefore, $K(\theta) = K(\theta^v/\pi^u)$ and f is irreducible over K. By Corollary 3.2, $R_\nu[\theta^v/\pi^u]$ is integrally closed. □

EXAMPLE 1. Let $\geq$ be the lexicographic order on $\mathbb{Z}^2$; i.e., $(a, b) \geq (c, d)$ if and only if $(a < c)$ or $(a = c$ and $b \leq d)$. Then $(\mathbb{Z}^2, \geq)$ is a totally ordered abelian group. Let F be a field and $K = F(X, Y)$, the field of rational functions over F in indeterminates X and Y. Define the valuation $\nu : K \to \mathbb{Z}^2 \cup \{\infty\}$ by $0 \neq \sum_{i,j} a_{i,j} X^i Y^j \mapsto \min\{(i, j) \mid a_{i,j} \neq 0\}$ for $\sum_{i,j} a_{i,j} X^i Y^j \in F[X, Y]$, $0 \mapsto \infty$, and $\nu^G(f/g) = \nu^G(f) - \nu^G(g)$ for $f, g \in F[X, Y]$ with $g \neq 0$. Then, obviously, ν is a discrete valuation on K of rank 2 whose value group is $\Gamma_\nu = (\mathbb{Z}^2, \geq)$. Let $f(Z) = Z^3 + aZ + b \in R_\nu[Z]$ be irreducible and $L = K(\alpha)$ for some root α of f. Assume that $\nu(a) > (0, 0)$ and $\nu(b) > (0, 0)$. Then $\overline{f}(Z) = Z^3$. Let r be the remainder upon the Euclidean division of f by Z. Noting that $\min(\Gamma_\nu^+) = (0, 1)$, it follows from Theorem 2.5 that $R_\nu[\alpha]$ is integrally closed if and only if $\nu^G(r) = (0, 1)$. In particular, if $f(Z) = Z^3 + Y$, then $R_\nu[\alpha]$ is integrally closed; while if $f(Z) = Z^3 + YZ + X$, then $R_\nu[\alpha]$ is not integrally closed.

EXAMPLE 2. Let (F, ν) be a valued field and let $K = F(X)$ be the field of rational functions over F in an indeterminate X. Given some positive irrational real λ, define the valuation $\omega : K \to \mathbb{R} \cup \{\infty\}$ as follows: $\omega(0) = \infty$, for $0 \neq f(X) = \sum_{i=0}^n a_i X^i \in F[X]$, set $\omega(f) = \min\{\nu(a_i) + i\lambda, i\}$, and for $f, g \in F[X]$ with $g \neq 0$, $\omega(f/g) = \omega(f) - \omega(g)$ (see [9, Theorem 2.2.1]). Let $f(Z) = Z^3 + aZ + b \in R_\omega[Z]$ be irreducible and $L = K(\alpha)$ for some root α of f. If (F, ν) is the trivial valued field, then $\Gamma_\omega = \lambda\mathbb{Z}$. So, in this case, if $\nu(a) > 0$ and $\nu(b) > 0$, then $\overline{f}(Z) = Z^3$. Hence, by Theorem 2.5, $R_\omega[\alpha]$ is integrally closed if and only if $\nu(b) = \lambda$. In particular, if $f(Z) = Z^3 + X$, then $R_\omega[\alpha]$ is integrally closed. If $F = \mathbb{Q}$ and ν is the p-adic valuation on $\mathbb{Q}$ for some prime integer p, then $\Gamma_\omega = \mathbb{Z} + \lambda\mathbb{Z}$, which is dense in $\mathbb{R}$ and, thus, $\inf(\Gamma_\omega^+) = 0$. So, according to Theorem 2.5, $R_\omega[\alpha]$ is integrally closed if and only if $\overline{f}$ is square-free. In particular, if $\nu(a) > 0$ and $\nu(b) > 0$; then $\overline{f}(Z) = Z^3$ and so $R_\omega[\alpha]$ is not integrally closed.

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L. EL FADIL
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES DHAR-MAHRAZ
UNIVERSITY OF SIDI MOHAMED BEN ABDELLAH, FES, MOROCCO
E-mail address: lhouelfadil2@gmail.com

M. BOULAGOUAZ
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND TECHNOLOGIES
UNIVERSITY OF SIDI MOHAMED BEN ABDELLAH, FES, MOROCCO
E-mail address: boulag@rocketmail.com

A. DEAJIM
DEPARTMENT OF MATHEMATICS, KING KHALID UNIVERSITY, ABHA, SAUDI ARABIA
E-mail address: deajim@kku.edu.sa; deajim@gmail.com