

## ON THE DECOMPOSITION OF PARTIALLY COMMUTATIVE GROUPS OF VARIETIES AND THEIR DEFINING GRAPHS

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UDC 512.5

**Abstract:** Considering the partially commutative groups of varieties which contain the variety of all nilpotent groups of class 2 as well as the variety of all groups, we study the problems of their decomposability into a direct product and a free product (in the variety).

**DOI:** 10.1134/S0037446621040169

**Keywords:** partially commutative group, graph join, graph union, soluble group

### 1. Introduction

The notion of direct product is one of the most important in group theory. The question of the possibility of decomposing a group into a direct product has arisen constantly starting with the development of the theory of finite and abelian groups. Alongside the classical studies, some comparatively recent papers are also devoted to the study of this question for nilpotent groups (see, for example, [1, 2]). Much attention has been paid to the question of a direct product decomposition of free groups of various varieties. The following Problem 22 is formulated in [3]:

*Is every nonabelian relatively free group of exponent equal to zero or a prime power not decomposable into a direct product?*

Neumann [4] and independently Olshanskii [5] gave a negative solution to the problem. However, the free group  $F(\mathfrak{M})$  of a variety  $\mathfrak{M}$  does often admit no nontrivial direct decomposition. Some varieties with this property are indicated in [3]. For example, if  $F(\mathfrak{M})$  is nonabelian, residually nilpotent and has exponent zero or a prime power then it is not decomposable into a direct product. In [6] there is observed in particular that the free groups of varieties containing the product of the variety of abelian groups of finite exponent by the variety of all abelian groups do not admit nontrivial direct products either.

Let  $\Delta = (X; E)$  be a graph, where  $X$  is a nonempty set of vertices and  $E$  is the edge set. Henceforth, all graphs are finite,  $X = \{x_1, \dots, x_n\}$ , undirected, and have no loops. A graph with a sole vertex and empty edge set is called *trivial*.

Some graph  $\Delta$  and a variety of groups  $\mathfrak{M}$  uniquely defined the partially commutative group  $F(\mathfrak{M}, \Delta)$  of  $\mathfrak{M}$ ; i.e.,

$$F(\mathfrak{M}, \Delta) = \langle X \mid x_i x_j = x_j x_i \text{ if } (x_i, x_j) \in E; \mathfrak{M} \rangle.$$

The vertex set of the graph is simultaneously the generator set of the group. The graph  $\Delta$  is called the *defining graph* of  $F(\mathfrak{M}, \Delta)$ .

The *union of graphs*  $\Gamma_1 = (V_1; E_1)$  and  $\Gamma_2 = (V_2; E_2)$  is the graph  $\Gamma_1 \sqcup \Gamma_2 = (V_1 \cup V_2; E_1 \cup E_2)$ . The *join of*  $\Gamma_1$  and  $\Gamma_2$  with disjoint vertex sets is the graph denoted by  $\Gamma_1 \bowtie \Gamma_2$ , with vertex set  $V_1 \cup V_2$  and edge set  $E_1 \cup E_2 \cup \{(v; u); v \in V_1; u \in V_2\}$ . It is easy to check that  $\Gamma \bowtie \Delta = (\Gamma^c \cup \Delta^c)^c$ , where  $\Sigma^c$  is the complement of a graph  $\Sigma$ . A graph  $\Delta$  is a graph join if and only if the graph  $\Delta^c$  is disconnected.

There is a relationship between the direct decomposition of a partially commutative group and the representation of the defining graph as a join. Obviously,  $F(\mathfrak{M}, \Gamma \bowtie \Delta) \cong F(\mathfrak{M}, \Gamma) \times F(\mathfrak{M}, \Delta)$ . The main question we are interested in is as follows:

*For what group varieties  $\mathfrak{M}$  is the possibility of decomposing the group  $F(\mathfrak{M}, \Delta)$  into a direct product equivalent to the possibility of representing the defining graph as a graph join?*

Let  $\mathfrak{N}_2$  be the variety of all nilpotent groups of nilpotency class  $\leq 2$ , let  $\mathfrak{G}$  be the variety of all groups, and let  $\mathfrak{S}$  be a variety of soluble groups containing  $\mathfrak{N}_2$ .

The above question is answered by Theorem 2 which strengthens Proposition 1 in [7].

**Proposition 1.** *If a graph  $\Gamma$  is disconnected then the group  $G = F(\mathfrak{S}, \Gamma)$  is not decomposable into a direct product.*

**Theorem 2.** *Suppose that the variety  $\mathfrak{M}$  coincides with  $\mathfrak{G}$  or  $\mathfrak{S}$ . The group  $F(\mathfrak{M}, \Delta)$  has a nontrivial direct product if and only if  $\Delta$  is the join of graphs with disjoint vertex sets.*

Assume for example that  $\Delta$  is a linear graph on four vertices or a cycle of length 5. Then the graphs  $\Delta \cong \Delta^c$  are connected. If  $\mathfrak{M} = \mathfrak{G}$  or  $\mathfrak{M} = \mathfrak{S}$  then the group  $F(\mathfrak{M}, \Delta)$  admits only the trivial direct decomposition.

The results of [4, 5] imply the existence of the nonabelian relatively free groups of exponent zero which admit a nontrivial direct product decomposition. The free group of a variety may be regarded as a partially commutative group whose defining graph is totally disconnected. Therefore, Theorem 2 fails if it contains no constraints on the variety.

Theorem 2 yields

**Corollary 3.** *Suppose that the variety  $\mathfrak{M}$  coincides with  $\mathfrak{G}$  or  $\mathfrak{S}$ . Then the question of the possibility of a nontrivial direct decomposition of the group  $F(\mathfrak{M}, \Delta)$  is algorithmically solvable.*

The group  $F(\mathfrak{M}, \Delta)$  is called *directly decomposable in the class of partially commutative groups of  $\mathfrak{M}$*  if  $F(\mathfrak{M}, \Delta)$  is not representable as the direct product of two nontrivial partially commutative groups in  $\mathfrak{M}$ .

Theorem 2 implies that if  $\mathfrak{M}$  coincides with  $\mathfrak{G}$  or  $\mathfrak{S}$  then so do the properties of a group to admit a nontrivial direct decomposition and to be directly decomposable in the class of the partially commutative groups of  $\mathfrak{M}$ .

Let  $G = \langle X; R, \mathfrak{M} \rangle$  and  $H = \langle Y; Q, \mathfrak{M} \rangle$  be presentations in  $\mathfrak{M}$  of some groups  $G$  and  $H$  and assume that the generator sets  $X$  and  $Y$  are disjoint. The group  $G \star H = \langle X \sqcup Y; R \sqcup Q, \mathfrak{M} \rangle$  is called the  $\mathfrak{M}$ -product of  $G$  and  $H$ .

Call the group  $F(\mathfrak{M}, \Delta)$   $\mathfrak{M}$ -decomposable if it can be obtained as the  $\mathfrak{M}$ -product of nontrivial partially commutative groups in  $\mathfrak{M}$ .

A graph  $\Delta$  is called *elementary* if  $\Delta$  and  $\Delta^c$  are connected graphs. A graph is elementary if and only if it is not representable as a nontrivial union or join of graphs. Note that the trivial graph is elementary.

Theorem 2 gives

**Corollary 4.** *Suppose that the variety  $\mathfrak{M}$  coincides with  $\mathfrak{G}$  or  $\mathfrak{S}$ . Then a graph  $\Delta$  is elementary if and only if the group  $F(\mathfrak{M}, \Delta)$  has no nontrivial direct decomposition and is not  $\mathfrak{M}$ -decomposable either.*

Alongside the decomposition of partially commutative groups of varieties into a direct product, we will be interested in the question of decomposing these groups into an  $\mathfrak{M}$ -decomposition of partially commutative groups and the related question about the representation of the defining graph as a union and a join of elementary graphs. The answer is given by

**Theorem 5.** *Suppose that the variety  $\mathfrak{M}$  coincides with  $\mathfrak{G}$  or  $\mathfrak{S}$ . Then the group  $F(\mathfrak{M}, \Delta)$  can be obtained from the partially commutative groups  $F(\mathfrak{M}, \Gamma_i), i = 1, \dots, m$ , by taking direct products and  $\mathfrak{M}$ -products; moreover, all graphs  $\Gamma_i$  are elementary.*

**Corollary 6.** *Suppose that the variety  $\mathfrak{M}$  coincides with  $\mathfrak{G}$  or  $\mathfrak{S}$ . Suppose that the graph  $\Delta$  does not include the linear graph  $L_4$  on four vertices as a subgraph. Then the group  $F(\mathfrak{M}, \Delta)$  can be obtained from infinite cyclic groups by taking direct products and  $\mathfrak{M}$ -products.*

## 2. Auxiliary Results

Each graph uniquely defines a partially commutative group of the variety. The following proposition

states that for a wide class of varieties only one defining graph corresponds to a partially commutative group.

**Proposition 7** [8]. *Suppose that the variety of groups  $\mathfrak{M}$  contains the variety  $\mathfrak{N}_2$ . The groups  $F(\Delta, \mathfrak{M})$  and  $F(\Gamma, \mathfrak{M})$  are isomorphic if and only if so are the graphs  $\Delta$  and  $\Gamma$ .*

Suppose that the vertex sets of graphs  $\Delta_1$  and  $\Delta_2$  are disjoint. If  $\mathfrak{N}_2 \subseteq \mathfrak{M}$  then Proposition 7 implies that the group  $F(\mathfrak{M}, \Delta)$  is the  $\mathfrak{M}$ -product of  $F(\mathfrak{M}, \Delta_1)$  and  $F(\mathfrak{M}, \Delta_2)$  if and only if  $\Delta = \Delta_1 \cup \Delta_2$ .

Let

$$X^\perp = \{x \in X \mid (x, y) \in E \text{ for all } x \neq y \in X\}.$$

The following proposition will be used in the proof of Theorem 2:

**Proposition 8** [7]. *Let  $\mathfrak{M}$  be a variety of groups containing  $\mathfrak{N}_2$ . Suppose that  $G = F(\mathfrak{M}, \Delta)$  decomposes into a direct product  $H \times A$ , where  $A$  is an abelian group. Then  $H \cong F(\mathfrak{M}, \Gamma)$ , where  $\Gamma$  is the subgraph in  $\Delta$  induced by some set of vertices including  $X \setminus X^\perp$ .*

### 3. The Proofs of the Lemmas and Main Results

**Lemma 9.** *Let  $n, n \geq 2$ , be the number of vertices in a graph  $\Delta$  and let  $\Delta_i$  be obtained from  $\Delta$  by removing the  $i$ th vertex and the incident edges. Suppose that each graph  $\Delta_i$  is a graph join. Then  $\Delta$  is also a graph join.*

PROOF. The hypothesis implies that all graphs  $\Delta_i^c$  are disconnected. Assume that  $\Delta^c$  is a connected graph. By [9, Theorem 3.4], each nontrivial connected graph contains at least two vertices that are not articulation points. Let the  $i$ th vertex of  $\Delta^c$  be its articulation point. Hence,  $(\Delta^c)_i$  is a connected graph but  $(\Delta^c)_i = (\Delta_i)^c$ . Since the last graph is disconnected, we get a contradiction.

**Lemma 10.** *Let  $G = F(\mathfrak{N}_2, \Delta)$  and let  $A$  and  $B$  be normal subgroups in  $G$  such that  $G = AB$ ,  $G' < A$ ,  $G' < B$ , and  $[A, B] = 1$ . Then  $\Delta$  is the join of some graphs  $\Delta_1$  and  $\Delta_2$  with disjoint vertex sets.*

PROOF. If  $G$  is an abelian group then  $\Delta$  is a complete graph, and the lemma holds. Therefore, we will assume that  $G$  is nonabelian. Then at least one of the groups  $A$  and  $B$  is nonabelian either. Let  $A$  be a nonabelian group.

The proof will be carried out by induction on the number of vertices  $n$  of  $\Delta$ . Let  $n = 2$ . Then  $G$  is a free abelian group or a free group of rank  $\mathfrak{N}_2$ . In the first case, the claim holds. Consider the second case. Denote a basis for  $G$  by  $X = \{x_1, x_2\}$ . Since  $A$  is nonabelian,  $A$  contains noncommuting elements  $a_1$  and  $a_2$ . Then

$$a_1 \equiv x_1^{l_1} x_2^{l_2} \pmod{G'}, \quad a_2 \equiv x_1^{r_1} x_2^{r_2} \pmod{G'}, \quad [a_1, a_2] \neq 1.$$

Since  $[a_1, B] = [a_2, B] = 1$ ; therefore,  $B \leq G'$ , which contradicts the hypothesis.

Suppose that the lemma holds for graphs whose number of vertices is  $\leq n - 1$ .

Let  $\Delta$  be a graph on vertices  $X = \{x_1, \dots, x_n\}$ . Given  $i = 1, \dots, n$ , denote by  $\Delta_i$  the subgraph in  $\Delta$  induced by  $X \setminus \{x_i\}$ . Let  $\varphi_i$  be the homomorphism (retraction)

$$\varphi_i : F(\mathfrak{M}, \Delta) \longrightarrow F(\mathfrak{M}, \Delta_i)$$

such that  $x_i \mapsto 1$  and  $x_j \mapsto x_j$  if  $i \neq j$ . Put  $\varphi_i(G) = G_i$ ,  $\varphi_i(A) = A_i$ , and  $\varphi_i(B) = B_i$ .

We have  $G_i = A_i B_i$ . Since  $G' < A$ , it follows that  $G'_i \leq A_i$ . Suppose that  $G'_i = A_i$  for some  $i$ . Denote by  $\langle x, y, \dots \rangle$  the subgroup in  $G$  generated by elements  $x, y, \dots$  and designate as  $\langle x, y, \dots \rangle^G$  the normal subgroup generated by these elements. We get

$$A \leq \varphi_i^{-1}(A_i) = \varphi_i^{-1}(G'_i) = \langle G', \ker \varphi_i \rangle = \langle G', \langle x_i \rangle^G \rangle = \langle G', x_i \rangle.$$

It follows that  $A$  is an abelian group; a contradiction. Hence,  $G'_i < A_i$ . If  $B$  is nonabelian then  $G'_i < B$  for all  $i$ . Assume that  $G'_i = B_i$  for some  $i$  and the group  $B$  is abelian. Then

$$B \leq \varphi_i^{-1}(B_i) = \varphi_i^{-1}(G'_i) = \langle G', \ker \varphi_i \rangle = \langle G', \langle x_i \rangle^G \rangle = \langle G', x_i \rangle.$$

Choose  $b \in B \setminus G'$ . There are  $c \in G'$  and  $0 \neq l \in \mathbb{Z}$  such that  $b = cx_i^l$ . From  $[A, B] = 1$  it follows that  $x_i^l$  lies in the center  $\mathcal{Z}(G)$  of  $G$ . Proposition 4 in [10] gives the description of  $\mathcal{Z}(G)$ :

$$\mathcal{Z}(G) = G' \times \langle x_{i_1} \rangle \times \cdots \times \langle x_{i_m} \rangle,$$

where for  $j = 1, \dots, m$  the vertex  $x_{i_j}$  is adjacent to each vertex of the graph different from  $x_{i_j}$ . Consequently,  $\mathcal{Z}(G)$  is isolated in  $G$ . Hence,  $x_i \in \mathcal{Z}(G)$ . Therefore,  $\Delta$  is the join of  $\Gamma_i = (\{x_i\}; \emptyset)$  and  $\Delta_i$ .

Thus, we may assume that  $G'_i < A_i$  and  $G'_i < B_i$  for all  $i$ . By the induction assumption, each graph  $\Delta_i$  is a join. Hence, by Lemma 9, so is  $\Delta$ . The lemma is proved.

**PROOF OF THEOREM 2.** Suppose that the number  $n$  of vertices in the graph is 2. If the vertices are adjacent then  $F(\mathfrak{M}, \Delta)$  is a free abelian group and the graph is the join of trivial graphs. If the vertices are nonadjacent then  $F(\mathfrak{M}, \Delta)$  is a free group of rank 2 that has no nontrivial direct decompositions or a free group of rank 2 in  $\mathfrak{S}$ . Free nilpotent groups have no nontrivial direct decompositions. Therefore,  $F(\mathfrak{S}, \Delta)$  is indecomposable into a nontrivial direct product either. We may assume that  $n > 2$ .

If  $\Delta$  is the join of two graphs then the group  $F(\mathfrak{M}, \Delta)$  is directly decomposable for every variety  $\mathfrak{M}$ .

Suppose the contrary; i.e.,  $F(\mathfrak{M}, \Delta)$  admits a nontrivial direct decomposition. Consider the two cases separately:

1. Let  $\mathfrak{M} = \mathfrak{G}$ . Then  $F(\mathfrak{G}, \Delta)$  is a free partially commutative group. For this group, the following proposition was proved in [11]: If the number of vertices in  $\Delta$  is greater than 2 and the complement  $\Delta^c$  of  $\Delta$  is a connected graph then for all nontrivial elements  $x, y \in F(\mathfrak{G}, \Delta)$  there is an element  $z$  such that  $[x, y^z] \neq 1$ .

It follows immediately that  $F(\mathfrak{G}, \Delta)$  is not directly decomposable if  $\Delta^c$  is a connected graph. Therefore,  $\Delta^c$  is a disconnected graph, which is equivalent to the fact that  $\Delta$  is a graph join.

2. Let  $\mathfrak{M} = \mathfrak{S}$ . Then the group  $F(\mathfrak{N}_2, \Delta)$  admits a nontrivial direct decomposition  $A \times B$ . Let at least one of the factors  $A$  and  $B$  be abelian, for example,  $A$ . By Proposition 8,  $B \cong F(\mathfrak{N}_2, \Gamma)$ . The group  $A$  is defined by a complete graph  $\Sigma$  whose vertex set is disjoint from the vertex set of  $\Gamma$ . Consequently,

$$F(\mathfrak{N}_2, \Delta) \cong F(\mathfrak{N}_2, \Sigma) \times F(\mathfrak{N}_2, \Gamma) \cong F(\mathfrak{N}_2, \Gamma \bowtie \Sigma).$$

By Proposition 7, we conclude that  $\Delta \cong \Gamma \bowtie \Sigma$ .

If the groups  $A$  and  $B$  are nonabelian then  $G = AG' \cdot BG'$ , and the theorem follows from Lemma 10. The theorem is proved.

**PROOF OF THEOREM 5.** If  $\Delta$  is nonelementary then one of the graphs  $\Delta$  or  $\Delta^c$  is disconnected.

Suppose that  $\Delta$  is disconnected. Then

$$\Delta = \bigsqcup_{i=1}^l \Delta_i, \quad l > 1.$$

If some connected component  $\Delta_i$  possesses the property that its complement  $\Delta_i^c$  is not a connected graph, then  $\Delta_i$  is representable as the join of some graphs  $\Delta_{ij}$  where each of the graphs  $\Delta_{ij}$  cannot be representable as a join of graphs. Continuing this process, we arrive at a representation of  $\Delta$  through elementary graphs  $\Gamma_m$ , by using the operations of union and join. Therefore,  $F(\mathfrak{M}, \Delta)$  can be obtained from the groups  $F(\mathfrak{M}, \Gamma_m)$ ,  $m = 1, \dots, r$ , by direct products and  $\mathfrak{M}$ -products. The theorem is proved.

**PROOF OF COROLLARY 6.** We may assume that  $\Delta = (X; E)$  be a connected nontrivial graph. Since  $\Delta$  does not include  $L_4$  as a subgraph, the diameter  $d(\Delta)$  is at most 2. The case of  $d(\Delta) = 1$  is trivial. Let  $d(\Delta) = 2$ . Denote the distance between vertices  $x$  and  $y$  in  $\Delta$  by  $r(x, y)$ . Choose a vertex  $x_0$  in  $\Delta$  that is not an articulation point. Such vertex exists by [9, Theorem 3.4]. Let

$$X_1 = \{x \in X \mid r(x_0, x) = 1\}, \quad X_2 = \{x \in X \mid r(x_0, x) = 2\}.$$

If  $X_2 = \emptyset$  then  $\Delta$  is a star. If  $X_2 \neq \emptyset$  then  $x_0$  is an articulation point. This contradicts the hypothesis. The corollary is proved.

**Acknowledgment.** While writing the article, the author asked Viktor Petrovich Il'ev about the validity of the assertion formulated as Lemma 9 here and got an affirmative answer from him as well as a proof, for which the author is grateful to V. P. Il'ev. The proof of Lemma 9 given in the article is somewhat different from V. P. Il'ev's proof.

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