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LOCALLY FINITE PERIODIC GROUPS SATURATED WITH FINITE SIMPLE ORTHOGONAL GROUPS OF ODD DIMENSION

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Abstract: Suppose that n is an odd integer, $n \geq 5$. We prove that a periodic group G, saturated with finite simple orthogonal groups $O_n(q)$ of odd dimension over fields of odd characteristic, is isomorphic to $O_n(F)$ for some locally finite field F of odd characteristic. In particular, G is locally finite and countable.

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Introduction

Let M be some set of finite groups. A group G is said to be *saturated* with groups from M, if each finite subgroup of G lies in a subgroup isomorphic to some element of M.

The main goal of the paper is to prove the following result:

Theorem. *Suppose that m is an integer, $m \geq 2$, and M is a set whose elements are finite simple orthogonal groups of dimension $n = 2m + 1$ over fields of odd characteristic. If G is a periodic group saturated with groups from M, then G is isomorphic to a simple orthogonal group $O_n(F)$ for some locally finite field F.*

A particular case of this theorem was proved in [1].

1. Preliminary Facts

We will be using the notation and results of [2, 3]. Recall some of them.

Suppose that F is a field of odd characteristic, $n = 2m + 1$ is an odd integer, $n \geq 5$, and V is a vector space of dimension n over F, while $e = \{e_1, e_2, \dots, e_n\}$ is a basis for V over F. Let f be a symmetric bilinear form on V such that $f(e_i, e_i) = 1$ and $f(e_i, e_j) = 0$ for all $i, j \in \{1, 2, \dots, n\}$, $i \neq j$. The basis e is called the *standard basis* for f. Suppose that f_1 is a symmetric bilinear form on V such that $f_1(e_i, e_i) = \mu$, where μ is an element of F which is not a square, while $f_1(e_i, e_j) = 0$ for all i and j satisfying $i \neq j$. Then f and f_1 are not isometric, and every nondegenerate form on V is isometric to f or f_1. The group of linear transformations of V that preserve f, preserves f_1 as well and is denoted by $GO(V)$. The subgroup $SO(V) = GO(V) \cap SL(V)$ has index 2 in $GO(V)$ and is called the *special orthogonal group* V. The group $\Omega_n(F) = \Omega(V) = [SO(V), SO(V)]$ is simple and $GO(V)/\Omega(V)$ is an elementary abelian group of order 4. The group $\Omega_n(F) = O_n(F)$ is isomorphic to a simple group $B_m(F)$ of Lie type B.

Suppose that t is an involution (i.e. an element of order 2) from $L = \Omega(V)$. Then V is an orthogonal direct sum of the subspaces $V^+(t) = \{v \in V \mid vt = v\}$ and $V^-(t) = \{v \in V \mid vt = -v\}$. Denote the dimensions of $V^+(t)$ and $V^-(t)$ by $d(t)$ and $r(t)$. It is clear that $d(t)$ and $r(t)$ are the defect and the rank of the transformation $t - 1$ respectively. Two involutions $t, t_1 \in \Omega(V)$ conjugate if and only if $d(t) = d(t_1)$. Therefore, $\Omega_n(F)$ contains exactly $m = (n - 1)/2$ classes of conjugated involutions.

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Lemma 1. *Let A be a maximal elementary abelian subgroup of $L = \Omega(V)$. Then $|A| = 2^{n-1}$ or $|A| = 2^{n-2}$, $C_L(A) = A$, and each involution from L is conjugate to an involution from A. If involutions t and t_1 from L are contained in A and conjugate in L; i.e., $d(t) = d(t_1)$; then t and t_1 are conjugate in $N_L(A)$.*

Let t be an involution from A. If $d(t) = 1$, then $|t^{N_L(A)}|$ is equal to n if $|A| = 2^{n-1}$; and 1, if $|A| = 2^{n-2}$. If $d(t) \neq 1$, then $|t^{N_L(A)}| > n$.

The normalizer $N_L(A)$ in L acts transitively by conjugation on each of the sets $D_s = \{t \in A \mid d(t) = s\}$, $s = 1, 3, \dots$. In the case when $|A| = 2^{n-1}$, the subgroup $N_L(A)$ has a subgroup $N_0 \geq A$ such that $|N_L(A) : N_0| \leq 2$, $N_0 \simeq A : \mathrm{Alt}(n)$ and N_0 acts transitively by conjugation on each of the sets D_s.

Proof. The subgroup A is diagonal in some orthogonal basis $e = \{e_1, \dots, e_n\}$ for the space V; therefore, $A = A^* \cap \Omega(V)$, where $A^* = \{a^* \in GO(V) \mid e_i a^* = \pm e_i\}$. Since A^* contains $-1 \notin \Omega(V)$, the order of A is at most 2^{n-1}, and we may assume that either the basis e is standard or $(e_i, e_i) = 1$ for $i > 1$ and $(e_1, e_1) = \mu$, where μ is not a square in F. We start with the second case. Thus, the transformation c, satisfying

$$e_1 c = -e_1, \quad e_2 c = -e_2, \quad e_i c = e_i \text{ for } i > 2,$$

does not belong to $\Omega(V)$; i.e., $\langle c, -1 \rangle \cap A = 1$. On the other hand, every transformation $a \in A^*$ such that $e_1 a = e_1$ and $d(a)$ is odd, is contained in A. Hence, $|A| = 2^{n-2}$ and

$$A = \{a = [1, \alpha_2, \dots, \alpha_n] \mid \alpha_i = \pm 1, i = 2, \dots, n;\ d(a) \text{ is odd}\},$$

where $[\alpha_1, \alpha_2, \dots, \alpha_n]$ denotes the diagonal matrix with the element α_i in the entry (i, i), $i = 1, 2, \dots, n$. It is clear that the number of elements $a \in A$ such that $d(a) = d$ is equal to 1, if $d = 1$; and

$$C_{n-1}^{d-1} = \frac{(n-1)(n-2)\dots(n-d+1)}{(d-1)!},$$

if $d > 1$. If $2 \leq i < j < k \leq n$ and $c = c(i, j, k)$ is a transformation such that $e_i c = e_j$, $e_j c = e_k$, $e_k c = e_i$, and $e_l c = e_l$, given $l \notin \{i, j, k\}$; then $[1, \alpha_2, \dots, \alpha_n]^c = [1, \beta_2, \dots, \beta_n]$, where $\beta_i = \alpha_k$, $\beta_j = \alpha_i$, $\beta_k = \alpha_j$, and $\beta_l = \alpha_l$, given $l \notin \{i, j, k\}$. Obviously, $c \in N_{GO(V)}(A)$ and the order of c is equal to 3; therefore, $c \in \Omega(V)$. Conjugating $a = [1, \alpha_2, \dots, \alpha_n]$ by an element g, equal to the product $c(i_1, j_1, k_1) \dots c(i_l, j_l, k_l)$ for a suitable i_s, j_s, k_s, there is no difficulty in obtaining the element $a^g = [1, 1, \dots, 1, -1, \dots, -1]$, where the first $d(a)$ of the diagonal elements equal 1, and the rest of them equal -1. This shows that every two involutions $t, t_1 \in A$, that are conjugate in L, i.e. involutions with condition $d(t) = d(t_1)$, are conjugate in $N_L(A)$. Moreover, there exists only one involution t in A such that $d(t) = 1$, and the number of involutions t with condition $d(t) = d > 1$ is equal to

$$C_{n-1}^{d-1} = \frac{(n-1)(n-2)\dots(n-d+1)}{(d-1)!} = \frac{(n-1)(n-2)}{2} \prod_{i=3}^{d-1} \frac{n-i}{i}.$$

We will show that this number is more than n by induction on $d \geq 3$ (recall that d is odd and $n \geq 5$). For $d = 3$,

$$C_{n-1}^{d-1} = \frac{(n-1)(n-2)}{2} = \frac{n^2 - 3n + 2}{2} \geq n + 1 > n.$$

Moreover, $C_{n-1}^{d-1} = C_{n-1}^{n-d}$; therefore, we may assume that $d - 1 \leq n - d$, i.e. $d \leq \frac{n+1}{2}$. Now, given $i - 1 \leq d - 1$, we have $\frac{n-i}{i} > 1$; i.e., $\prod_{i=3}^{n-1} \frac{n-i}{i} > 1$, which implies that $C_{n-1}^{d-1} > n$.

Direct checking shows that $C_{GO(V)}(A) = A^*$, and hence $C_L(A) = A$.

Consider the standard basis e. Then

$$A = \{a = [\alpha_1, \alpha_2, \dots, \alpha_n] \mid \alpha_1, \alpha_2, \dots, \alpha_n \in \{1, -1\};\ d(a) \text{ is odd}\}.$$

In particular, $C_L(A) = A$. Clearly, $N^* = N_{GO(V)}(A^*) = A^* : \mathrm{Sym}(n)$ and $N = N^* \cap L = A : H$, where $H \simeq \mathrm{Sym}(n)$ or $\mathrm{Alt}(n)$, acts transitively on the set of involutions $t \in A$ with the common parameter $d(t)$. Thus, if $t \in A$ and $d(t) = d$, then $|t^N| = C_n^d$. As before, it is an easy check that $|t^N| > n$ if $d(t) > 1$, and $|t^N| = n$ if $d(t) = 1$.

Lemma 2. *Suppose that $|F| = q$, while A is a maximal elementary abelian subgroup $L = \Omega(V)$ with order 2^{n-1}, and a and b are two distinct involutions from A such that $d(a) = d(b) = 1$ and $K = \langle a, b\rangle$. Then $C_L(a) = L_1 : \langle b\rangle$, where $L_1 \simeq \Omega^\varepsilon_{n-1}(V)$, $\varepsilon 1 \equiv q (\operatorname{mod} 4)$, and $C_L(K) = L_0 \times K$, where $L_0 \simeq \Omega_{n-2}(q)$. Moreover, $C_L(a)$ is maximal in L, and $C_L(K)$ is maximal in $C_L(a)$.*

Proof. By Lemma 1, A contains n involutions t for which $d(t) = 1$ and all of them are conjugate in $N_L(A)$. Moreover, $N_L(A)$ acts double transitively on the set of such involutions; hence, all subgroups K of the statement of Lemma 2 are conjugate in $N_L(A)$.

Since $C_{GO(V)}(a) = GO(V^+) \times GO(V^-)$, where

$$V^+ = \{v \in V \mid va = v\}, \quad V^- = \{v \in V \mid va = -a\};$$

therefore, $C_L(a) = C_{GO(V)}(a) \cap \Omega(V)$ includes $\Omega(V^-)$ as a subgroup of index 2. The proof of Lemma 11.53 in [2] implies that $\Omega(V^-) = \Omega^\varepsilon(V^-)$, where ε is defined by a congruence $q \equiv \varepsilon 1 (\operatorname{mod} 4)$ and a is the only involution in $\Omega(V^-)$ such that $d(a) = 1$; hence, $b \notin \Omega(V^-)$ and $C_L(a) = \Omega(V^-)\langle b\rangle$. Further,

$$C_L(K) = C_{C_L(a)}(b) = \langle b\rangle \times C_{\Omega(V^-)}(b) = L_0 \times K,$$

where $L_0 \simeq \Omega_{n-2}(q)$. The maximality of $C_L(K)$ in $C_{C_L(a)}(b)$ and of $C_L(a)$ in L follows from the maximality of geometric subgroups of $\Omega_n(q)$ and $\Omega^\varepsilon_{n-1}(q)$ (see Tables 3.5.D, E, and F in [4] and Tables 8.31, 33, 39, 50, 52, 58, 66, 68, 74, 82, 84, and 85 in [3].

2. Proof of the Theorem

Let n be an odd integer, $n \geq 5$, while G is a periodic group such that its every subgroup lies in a subgroup isomorphic to $\Omega_n(q)$ for some odd q which is a power of a prime. Our goal is to show that G is isomorphic to $\Omega_n(F)$ for a suitable locally finite field F.

Because $\Omega_5(q) \simeq S_4(q)$, according to [5] it is true for $n = 5$. By induction we may assume that $n \geq 7$, and the claim is true when n is replaced with $n - 1$.

Let $M(G)$ be the set of all subgroups of G isomorphic to elements of M, while $L = \{\Omega_n(q) \mid q \text{ is odd}\}$. If $M(G)$ has a subgroup isomorphic to $\Omega_n(q)$, where $q \equiv 1 (\operatorname{mod} 4)$, then we fix and denote by $L = L(q)$ one of such subgroups. If there are no such subgroups, then we fix and denote as $L = L(q)$ some (arbitrary) element of $M(G)$. In both cases we will identify L with $\Omega(V)$, where V is an orthogonal space of dimension n over a field of order q.

We will also fix an elementary abelian 2-subgroup A of order 2^{n-1} from L, elements a and b from A, and a subgroup K as they are defined in Lemma 2. If L_1 is a subgroup of G isomorphic to $\Omega(V_1)$ for some space V_1 of dimension n over a field of order q_1, and t is an involution from L_1; then denote by $d_{L_1}(t)$ the dimension of the space of fixed points of t in V_1.

Lemma 3. *$C_G(A) = A$, $N_G(A)$ contains a subgroup of index 1 or 2 coinciding with $N_0 \simeq A : \mathrm{Alt}(n)$ from Lemma 1.*

Proof. Suppose that $c \in C_G(A)$. Then $C = \langle c, A\rangle$ is a finite subgroup lying in some element $L_1 \in M(G)$. Applying Lemma 1 with L_1 instead of L, we get that $c \in C_{L_1}(A) = A$. So, $C_G(A) = A$, and therefore $N_G(A)$ is a finite subgroup lying in some $L_2 \in M(G)$. Now we use Lemma 1 with L_2 in place of L and derive that $N_G(A)$ includes N_0 as a subgroup of index 1 or 2.

Lemma 4. *If $K \leq L_1 \in M(G)$, then $d_{L_1}(a) = d_{L_1}(b) = d(a) = 1$.*

Proof. Let A_1 be a maximal elementary abelian 2-subgroup of L_1 including K.

If $A_1 = A$, Lemma 1 with L_1 in place of L implies that the subgroup N_0 lies in L_1. Because $|a^{N_0}| = |b^{N_0}| = n$, we have $d_{L_1}(a) = d_{L_1}(b) = 1 = d_L(a)$. In this case the claim of the lemma is true.

Suppose that $A_1 \leq A$ and $A_1 \neq A$. Then $|A : A_1| = 2$ by Lemma 1. Set $C = C_G(A_1)$. It is clear that $A \leq C$; and, if $t \in A \setminus A_1$, then

$$C_C(t) = C_G(t) \cap C = C_G(\langle t, A_1\rangle) = C_G(A) = A.$$

By Shunkov's Theorem [6], C is locally finite. Let $N_1 = N_{L_1}(A_1) \leq N_G(A_1)$. Since A_1 is finite, $N_G(A_1)/C$ is finite, and so $N_G(A_1)$ is a locally finite group. It follows that $N_G(A_1) = CH$, where H is a finite subgroup including A. Suppose that $H \leq L_2 \in M(G)$.

Because $A \leq L_2$, we have $d_{L_2}(a) = d_{L_2}(b) = 1$. Lemma 1 implies that $1 = |a^{N_G(A_1)}| = |a^H|$, which yields that $d_{L_1}(a) = d_{L_1}(b) = 1$. On the other hand, by Lemma 1 A_1 has the only involution i with condition $d(i) = 1$; therefore, the case under consideration when $A \neq A_1 \leq A$ is impossible.

By induction on $s = |A : (A_1 \cap A)| + |A_1 : (A_1 \cap A)|$, we will show that $|A_1| = |A| = 2^{n-1}$ and $d_{L_1}(a) = d_{L_1}(b) = 1$.

If $s \leq 3$, then either $A = A_1 \cap A$, or $A_1 = A_1 \cap A$, and these cases are already done. Hence, we can assume that $A_1 \neq A_1 \cap A \neq A$.

Let $t \in A \setminus (A_1 \cap A)$, $t_1 \in A_1 \setminus (A_1 \cap A)$. Then $R = \langle t, t_1, A_1 \cap A\rangle$ is finite, and so $R \leq L_2 \in M(G)$. Suppose that $\langle t, A_1 \cap A\rangle \leq A_2$, where A_2 is a maximal elementary abelian subgroup in L_2. Then

$$|A_2| \geq |A_2 \cap A| > |A_1 \cap A|.$$

By the inductive hypothesis, $|A_2| = |A|$ and $d_{L_2}(a) = d_{L_2}(b) = d(a) = 1$. Further, $\langle t_1, A_1 \cap A\rangle \leq L_1 \cap L_2$ and $\langle t_1, A_1 \cap A\rangle \leq A_3$, where A_3 is a maximal elementary abelian 2-subgroup of L_2. Because $a, b \in A_3$, we have $|A_3| = |A|$. Moreover, $|A_3 \cap A_1| > |A \cap A_1|$. By the inductive hypothesis, $d_{L_1}(a) = d_{L_1}(b) = d_{L_2}(a) = 1$. The lemma is proved.

Lemma 5. *$C_G(K) = K \times R$, where $R \simeq \Omega_{n-2}(F)$ for some locally finite field F of odd characteristic.*

Proof. Let $\overline{C} = C_G(K)/K$. We want to show that $\overline{C}$ is saturated with groups from the set $M_1 = \{\Omega_{n-2}(q) \mid q \text{ is odd}\}$.

Suppose that $\overline{X}$ is a finite group from $\overline{C}$, while X is the full preimage of $\overline{X}$ in G. By condition, $K \leq X \leq L_1 \in M(G)$; and by Lemma 2 $C_{L_1}(K) = K \times R_1$, where $R_1 \simeq \Omega_{n-2}(q_1)$ for odd q_1. Thus, $\overline{C}$ is saturated with groups from the set $M_1 = \{\Omega_{n-2}(q) \mid q \text{ is odd}\}$. By the inductive hypothesis, $\overline{C} \simeq \Omega_{n-2}(F)$ for some locally finite field F of odd characteristic. In particular, C is a locally finite group. We will show that $[C, C] \cap K = 1$. Let $c \in [C, C]$. Then $c = [c_1, c_2][c_3, c_4] \dots [c_{p-1} c_p]$ for some p and suitable elements $c_1, \dots, c_p \in C$. The subgroup $\langle K, c_1, \dots, c_p\rangle$ is finite and lies in $K \times Y$, where $Y \simeq \Omega_{n-2}(q_2)$ for some q_2. It is clear that $c \in Y$ and $c \notin K$. Since $C = K[C, C]$; therefore, $C = K \times [C, C]$, and the lemma is proved.

Lemma 6. *$C_G(K)$ lies in a subgroup P of $C_G(a)$ which is the union of an ascending sequence of subgroups P_i, $i = 1, 2, \dots$, isomorphic to $\Omega^{\lambda}_{n-1}(q_i).2$, with $q_i = \lambda 1 \pmod 4$, $\lambda \in \{+, -\}$, and λ depends on the choice of L and is common for all i.*

Proof. By Lemma 5, $C_G(K)$ is locally finite and countable. If $C_G(K)$ is finite, then we may assume that $C_G(K) = C_L(K)$, and the lemma is true by Lemma 2. Suppose that $C_G(K)$ is infinite and $C_G(K) = \{g_i \mid i \in \mathbb{N}\}$. Put $P_0 = C_L(a)$. Let g_{i_1} be an element of $C_G(K)$ not belonging to P_1, and the number i_1 is the smallest of those subject to that condition. The subgroup $\langle C_{P_0}(K), N_0\rangle$ coincides with P_0 by Lemma 2. Let L_1 be an element of $M(G)$ containing $C_{P_0}(K)$ and let g_i be the first element in order not belonging to $C_{P_0}(K)$. By condition, $L_1 \simeq \Omega_n(q_1)$ for some q_1. The subgroup $C_{L_1}(K)$ is maximal in $C_{L_1}(a)$; and, because $N_0 \not\leq C(K)$, the subgroup $\langle C_{L_1}(K), N_0\rangle$ coincides with $C_{L_1}(a) \simeq \Omega^{\lambda}_{n-1}(q_1).2$. Since $C_{L_0}(K) < C_{L_1}(K)$; therefore, $P_0 = C_{L_0}(a) < C_{L_1}(a) = P_1$.

Similarly, let L_2 be an element of $M(G)$ including $C_{P_1}(K)$ and let g_{i_2} be the first element in order not belonging to $C_{P_1}(K)$. As before, $P_2 = C_{L_2}(a)$, and P_2 includes $C_{L_1}(a)$. Proceeding this construction of the subgroups P_i's in a similar way, we will get an ascending sequence of subgroups $P_i \simeq \Omega^{\lambda}_{n-1}(q_i)$ whose union P includes $C_G(a)$. The lemma is proved.

Lemma 7. *$P = C_G(a)$.*

Proof. Suppose the contrary. Let $t \in C_G(a) \setminus P$. The subgroup $\langle K, K^t\rangle$ is generated by the elements a, b, and b^t. Since $\langle b, b^t\rangle$ is a finite group, so is $\langle K, K^t\rangle$. Because $\langle K, K^t\rangle$ lies in the subgroup L^*

isomorphic to $\Omega_n(q^*)$ for some q^*; therefore,

$$\langle K, K^t\rangle \leq D = C_{L^*}(a) \simeq \Omega^{\lambda}_{n-1}(q^*)\cdot 2.$$

Now, K and K^t are conjugate in D, because A and A^t are conjugate in D and $N_D(A)$ acts double transitively on the set of involutions $a^* \in A$ with condition $\alpha_D(a^*) = 1$. Let Γ be a graph with vertex set $\Sigma = \{K^d \mid d \in D\}$ such that two vertices K^r and K^s are adjacent if and only if $[K^r, K^s] = 1$. Suppose that Δ is a connected component in Γ that includes K. Since $K \leq A$ and $C_N(K) \neq C_N(a)$, we have $K^{C_N(a)} \neq \{K\}$, and so $|\Delta| \geq 2$. Thus, if $\Delta \neq \Sigma$, then D acts on Σ by conjugation transitively and imprimitively. Hence, the stabilizer of the vertex K in D equal to $N_D(K)$ is not maximal in D, which is untrue. Therefore, $\Delta = \Sigma$ and there is a sequence $t_1, t_2, \dots, t_r = t$ of elements from $C_{L^*}(a)$ such that $1 = [K, K^{t_1}] = [K^{t_i}, K^{t_{i+1}}]$, $i = 1, 2, \dots, r-1$. By induction on r, we will show that $K^{t_r} \leq P$. If $r = 1$, then $K^t \leq C_G(K)$; and by Lemma 6 $K^t \leq P$. Suppose that $r > 1$ and $K^{t_{r-1}} \leq P$. There exists $u \in P$ such that $K^{t_1 u} = K$ and $1 = [K, K^{t_2 u}] = \dots = [K^{t_{r-1}u}, K^{t_r u}]$, $i = 2, \dots, r$. By the inductive hypothesis, $K^{t_r u} \leq P$ and $K^{t_r} \leq P$.

So, $K^t \leq P$ for every $t \in C_G(a)$. The subgroup P is locally finite; therefore, $\langle K, t\rangle$ is finite and lies in $L^* \in M(G)$. Suppose that $H = C_{L^*}(a)$. Then

$$H = \langle C_{L^*}(K), N_G(A) \cap C_{L^*}(a)\rangle \leq P.$$

Because $t \in H$, we have $t \in P$, and the lemma is proved.

Lemma 8. *$C_G(a)$ lies in a subgroup $Z \simeq \Omega_n(F)$ of G for some locally finite field F.*

Proof. By Lemma 7, $C_G(a)$ is countable and locally finite. Let $L_0 = L$ and

$$C_G(a) = C_{L_0}(a) \cup \{g_i \in C_G(a) \mid i \in \mathbb{N}\}.$$

The subgroup $C_1 = \langle C_L(a), g_{i_1}\rangle$, with g_{i_1} the first element in order not belonging to L_0, is finite and hence lies in $L_1 \in M(G)$. Because L_0 includes $N_G(A)$ and $C_{L_1}(a)$ is maximal in L_1, the subgroup $\langle C_1, N_G(A)\rangle$ coincides with L_1.

Let $C_2 = \langle C_1, g_{i_2}\rangle$, where g_{i_2} is the first element in order which is not contained in L_1. By condition, $C_2 \leq L_2 \in M(G)$. It is clear that $C_2 \leq C_{L_2}(a)$ and L_2 includes $N_L(A) = N_{L_2}(A)$. Since $C_{L_2}(a)$ is maximal in L_2 and $N_{L_2}(A) \not\leq C_{L_2}(a)$, the subgroup L_2 coincides with $\langle C_{L_2}(a), N_{L_2}(A)\rangle$ and includes L_1.

Reasoning similarly, we construct subgroups $L_3, L_4, \dots \in M(G)$ with condition $L_i \leq L_{i+1}$, $i = 3, 4, \dots$. The union Z of the so-obtained sequence includes $C_G(a)$. By the main result of each of the papers [7–11], $Z \simeq \Omega_n(F)$ for some locally finite field F, and the lemma is proved.

Lemma 9. *$Z = G$.*

Proof. By Lemma 8, Z is countable and locally finite. Suppose that $g \in G$ and $a^g \neq a$. The group $\langle a, a^g\rangle$ is finite. Therefore, $\langle a, a^g\rangle$ lies in such a subgroup R of the group G that is isomorphic to $\Omega_n(r)$ for some r. Let Δ be a set of involutions belonging to R and conjugate to a in G. Let Γ be a graph with vertex set Δ such that two vertices a^{g_1} and a^{g_2} are adjacent if and only if $[a^{g_1}, a^{g_2}] = 1$. Since $C_R(a)$ is a maximal subgroup of R and $|C_R(a) \cap \Delta| \geq 2$; we have by analogy to Lemma 8 that the graph Γ is connected. This implies as in Lemma 8 that $a^G \subseteq Z$, $\langle a^G\rangle = Z$, and $Z \trianglelefteq G$. Because Z is locally finite, $\langle a, g\rangle$ is finite, and we may assume that $\langle a, g\rangle \leq R$. Since $\langle a^R\rangle = R$; therefore, $g \in \langle c^R\rangle \leq Z$. The lemma is proved, which completes the proof of the theorem.

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