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SOME NOTES ON THE SECOND MAXIMAL SUBGROUPS OF FINITE GROUPS

J. Zhang, Zh. Gao, and L. Miao UDC 512.542

Abstract: Under study are the arithmetic properties of second maximal subgroups of finite groups. Generally speaking, we investigated the problem by Monakhov [1, Problem 19.54] and developed the research of Meng and Guo [2, Theorem B] by weakening the condition of solvability.

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1. Introduction

All groups is finite. We will adhere to the notation of [3, 4]. In particular, $|G|$ denotes the order of a group G (or a set G), while $\pi(G)$ denotes the set of all prime divisors of $|G|$. Let H_G be the core of H in G when $H \leq G$ and let $M < \cdot G$ signify that M is a maximal subgroup of G. Put

$$\mathrm{Max}(G,H) = \{M < \cdot\, G \mid H \leq M\}, \quad \mathrm{Max}_1(G,H) = \{M < \cdot\, G \mid H < \cdot\, M\}.$$

Let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of G. An element x of G is called *nonvanishing* if $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.

Next, we recall some known research that is tightly related to our study.

Firstly, we need to list the following problem by Monakhov [1, Problem 19.54] which motivated our research.

Problem. *What are the chief factors of a finite group whose no* 2*-maximal subgroup is* m*-maximal for any* $m \geq 3$*?*

Addressing the problem, Meng and Guo [2] considered the properties of the second maximal subgroup of a group and the structure of a WSM-group under the universe of solvable groups, where the WSM-group is equal to the group satisfying the condition of the above problem.

On the other hand, Isaacs, Navarro, and Wolf conjectured in [5] that every nonvanishing element of a solvable group G is contained in the Fitting subgroup $F(G)$. In [6], Guo, Skiba, and Tang introduced the concept of boundary factors and traces of subgroups in finite groups and investigated the solvability of a group by considering the traces of maximal subgroups.

Continuing to study the Problem and developing the research of Meng and Guo [2], we will investigate the numerical structure of a second maximal subgroup of a group by weakening the condition of solvability. Meanwhile, viewing from the conjecture in [5] and the result in [6], we also consider the relationship between the conjecture and the traces of second maximal subgroups of a group. Here we obtained the following results:

Theorem 1.1. *Let* G *be a* WSM*-group and let* x *be a nonvanishing element of* G*. If each second maximal subgroup of* G *has a nilpotent trace, then* G *is solvable and* $x \in F(G)$*.*

Theorem 1.2. *Let* $H < \cdot\, M < \cdot\, G$ *and* $|M : H| = p^{\alpha}$*, where* $p \in \pi(G)$*. If* H *is a* CAP*-subgroup of* G*, then* $|\mathrm{Max}(G,H) \setminus \mathrm{Max}_1(G,H)| \leq 1$*.*

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2. Preliminaries

For the sake of convenience, we start with listing some known results that will be useful in this paper.

Lemma 2.1 [2, Lemma 1]. *Let G be a group and let H be a subgroup of G. If there exists M, $X \in \mathrm{Max}(G,H)$ such that H is maximal in M and H is not maximal in X, then $H_G = M_G$.*

Lemma 2.2 [2, Theorem B]. *Let G be a solvable group and let H be a weak second maximal subgroup of G. Then there exists at most one $X \in \mathrm{Max}(G,H)$ such that H is not maximal in X.*

Lemma 2.3. *Let G be a group. If every second maximal subgroup of G is nilpotent, then G is either solvable or isomorphic to $PSL(2,5)$ or $SL(2,5)$.*

Proof. See [7] and [8].

3. The Main Results

In [6], Guo, Skiba, and Tang introduced the concept of boundary factors and traces of subgroups in finite groups. Next, we will study the construction of a nonvanishing element of a group by the nilpotency of the traces of second maximal subgroups of G.

Theorem 3.1. *Let G be a WSM-group and let x be a nonvanishing element of G. If each second maximal subgroup of G has a nilpotent trace, then G is solvable and $x \in F(G)$.*

Proof. By [2, Theorem A], we only need to prove that G is solvable.

If there exists a second maximal subgroup H of G such that $H = 1$, then there is a maximal subgroup M of G such that $|M|$ is a prime. By [9, Chapter IV, Theorem 7.4], G is solvable. Hence every second maximal subgroup of G is nontrivial. Now we assert that G is not nonabelian simple. Otherwise, G is nonabelian simple. Since each second maximal subgroup of G has a nilpotent trace, every second maximal subgroup of G is nilpotent. By Lemma 2.3 $G \cong A_5$, where A_5 is the alternating group of degree 5. However, A_5 is not a WSM-group by [10].

Further, we may choose a minimal normal subgroup L of G and consider the quotient group G/L. If L is maximal in G, then G/L is of order q, with q a prime. If L is not maximal in G, then G/L satisfies the hypothesis and G/L is solvable by induction on $|G|$. Hence G/L is solvable for every minimal normal subgroup L of G. Further, $L \nleq \Phi(G)$ and L is the unique minimal normal subgroup of G.

Let L_p be a Sylow p-subgroup of L where p is a largest prime divisor of $|L|$. Clearly, $p > 3$. By the Frattini argument, $G = LN_G(L_p) = LM$, where M is maximal in G and $N_G(L_p) \leq M$. Further, $L \cap M \neq 1$ and $M_G = 1$. Hence there is a maximal subgroup H of M such that $L \cap M \leq H$ and $L \cap H = L \cap M$ is nilpotent by hypothesis. Then $N_L(L_p) = L \cap N_G(L_p) \leq L \cap M = L \cap H$ is nilpotent and $N_L(L_p)/C_L(L_p)$ is a p-subgroup. Further, $O^p(G) < L$ by [11, Chapter X, Theorem 8.13] and L is a p-subgroup of G. Hence, G is solvable since G/L is solvable.

In view of [12, Theorem 3.7], we can weaken the condition of solvability in [2, Theorem B] by the following condition and arithmetic description of a second maximal subgroup of a group.

Theorem 3.2. *Let $H < \cdot M < \cdot G$ and $|M : H| = p^{\alpha}$, where $p \in \pi(G)$. If H is a CAP-subgroup of G, then $|\mathrm{Max}(G,H) \backslash \mathrm{Max}_1(G,H)| \leq 1$.*

Proof. Clearly, we may assume that $H_G = 1$ and $X_i \in \mathrm{Max}(G,H) \backslash \mathrm{Max}_1(G,H)$, where i=1,2. Since $H < \cdot M$, $H = M \cap X_1 = M \cap X_2$. Also, $H_G = M_G = 1$ by Lemma 2.1.

Since G is primitive, G has one of the following structures by [3, Chapter A, Theorem 15.2]:

(1) $G = LM$, $C_G(L) = L$ and L is abelian, where L is the unique minimal normal subgroup of G;

(2) $G = LM$ and L is nonabelian, where L is the unique minimal normal subgroup of G;

(3) G has exactly two minimal normal subgroups L and L^* of G, while $G = LM = L^*M$ and $L \cap M = L^* \cap M = 1$. Also, $C_G(L) = L^*$, $C_G(L^*) = L$ and $L \cong L^* \cong LL^* \cap M$. Moreover, if $V < G$ and $LV = L^*V = G$, then $L \cap V = L^* \cap V = 1$.

By Lemma 2.2, we may assume that G is not solvable. Since H is a CAP-subgroup of G, we assert that G is not simple. Otherwise, $H = 1$. Then M is a maximal subgroup of G of prime order and G is solvable by [9, Chapter IV, Theorem 7.4]; a contradiction. To prove, we will coincide the following cases:

Case I: G has the structure (1) above.

If $L \nleq X_i$ for some $i \in \{1,2\}$, then $G = LX_i$ and $L \cap X_i = 1$. Further, $M \cong ML/L$ and $X_i \cong LX_i/L$. Since $HL/L < \cdot ML/L = LX_i/L$, $H < \cdot X_i$; a contradiction. So $L \leq X_1$ and $L \leq X_2$. Then $LM \cap X_1 = LM \cap X_2 = LH$ and $|\operatorname{Max}(G,H) \setminus \operatorname{Max}_1(G,H)| \leq 1$.

Case II: G has the structure (3) above.

Since $G = LM = L^*M$, $L \cap M = L^* \cap M = 1$, If there exists some X_i such that $(X_i)_G = 1$ for some $i \in \{1,2\}$, then $G = LX_i = L^*X_i$ and $L \cap X_i = L^* \cap X_i = 1$. Further, $HL/L < \cdot ML/L = LX_i/L$, $H < \cdot X_i$; a contradiction. So, $(X_1)_G \neq 1$ and $(X_2)_G \neq 1$. Then we assert that $L \leq (X_1)_G \cap (X_2)_G$ and $L^* \leq (X_1)_G \cap (X_2)_G$. Otherwise, there exists $R \in \{L, L^*\}$ and X_i for some $i \in \{1,2\}$ such that $RX_i = G$ and $R \cap X_i = 1$. With the similar discussion of the above, $H < \cdot X_i$; a contradiction. Hence $LM \cap X_1 = LM \cap X_2 = LH$ and $|\operatorname{Max}(G,H) \setminus \operatorname{Max}_1(G,H)| \leq 1$.

Case III: G has the structure (2) above.

Since H is a CAP-subgroup of G, $H \cap L = 1$. Further, we consider the subgroup HL.

If $HL = G$, then $M = M \cap G = M \cap HL = H(M \cap L)$ and $M \cap L$ is a p-subgroup since $|M : H| = p^\alpha$. Also, $M \cap L$ is a minimal normal subgroup of M since $H \cap L = 1$ and $H < \cdot M$. Now we assert that $N_G(L \cap M) = M$. Otherwise, $N_G(L \cap M) = G$ and $L = L \cap M$ by the minimal normality of L. Further, $HL = G = M$; a contradiction. Hence $N_L(L \cap M) = L \cap N_G(L \cap M) = L \cap M$. Clearly, $L \cap M$ is a Sylow p-subgroup of L. Thus, $N_L(L \cap M) = L \cap M = C_L(L \cap M)$. By the Burnside Theorem, L is p-nilpotent. Since $L \cap M \neq 1$, L is a p-subgroup of G and $L = L \cap M \leq M$. Then $HL = G = M$; a contradiction.

If $HL < G$, then $M = M \cap HL = H(M \cap L)$ since $L \cap M \neq 1$ and $H < \cdot M$. Hence $L \leq M$ and $LM = G = M$; a contradiction.

The authors proved in [2, Lemma 1] that $H_G = M_G$, where $M \in \operatorname{Max}_1(G,H)$ and $\operatorname{Max}_1(G,H)$ is properly included in $\operatorname{Max}(G,H)$. By dual consideration, we will show the following relationship between H and X, where $X \in \operatorname{Max}(G,H) \setminus \operatorname{Max}_1(G,H)$:

Theorem 3.3. *Let G be a group and let H be a subgroup of G. If there exist $M \in \operatorname{Max}_1(G,H)$ and $X \in \operatorname{Max}(G,H) \setminus \operatorname{Max}_1(G,H)$, then either $H_G = X_G$ or $HX_G = X$.*

Proof. Assume that $H_G \neq X_G$. Prove that $HX_G = X$.

If $H_G = 1$, then $H_G = M_G = 1$ by Lemma 2.1. Further, $X_G \neq 1$, G is primitive and $G = LM$, where L is a minimal normal subgroup L of G which lies in X_G. Since H is maximal in M, $H = M \cap X$ and $LH = X$. Hence $HX_G = X$.

If $H_G \neq 1$, then $1 < H_G < H$ or $H_G = H$. To proceed the proof, we will consider the following cases:

Case I: $1 < H_G < H$.

We consider the quotient subgroup G/H_G. So $(H/H_G)_{G/H_G} = (X/H_G)_{G/H_G}$ or $(H/H_G)(X/H_G)_{G/H_G} = X/H_G$ by the induction on $|G|$. Further, $H_G = X_G$ or $HX_G = X$. Hence $HX_G = X$ by assumption.

Case II: $H_G = H$.

Since H is maximal in M; therefore, $|M : H| = p$ and $H = M \cap X = M \cap X_G$. Then $G = MX_G$ and $|G : X_G| = |M : M \cap X_G| = |M : H| = p$. Hence $X_G = X$ and $HX_G = X$.

Corollary 3.4. *Under the hypothesis of Theorem 3.3, if H is subnormal in G, then either $H_G = X_G$ or $X \trianglelefteq G$.*

Proof. If G is simple then $H_G = X_G = 1$. Assume now that G is not simple. Since H is subnormal in G; therefore, $\operatorname{Soc}(G) \leq N_G(H)$ by [3, Chapter A, Theorem 14.3] and $N_G(H) = M$ or $N_G(H) = G$. If $N_G(H) = G$, then H is normal in G and $H = X_G$ or $X \trianglelefteq G$ by Theorem 3.3. If $N_G(H) = M$, then $\operatorname{Soc}(G) \leq N_G(H) = M$ and $M_G = H_G \neq 1$ by Lemma 2.1. Hence $1 < H_G < H$. Next, we consider the quotient group G/H_G. By induction on $|G|$, we see that $(H/H_G)_{G/H_G} = (X/H_G)_{G/H_G}$ or $X/H_G \trianglelefteq G/H_G$. Then $H_G = X_G$ or $X \trianglelefteq G$.

References

1. *The Kourovka Notebook: Unsolved Problems in Group Theory*. 19th ed., Khukhro E. I. and Mazurov V. D. (eds.), Sobolev Inst. Math., Novosibirsk (2018).
2. Meng H. and Guo X., "Weak second maximal subgroups in solvable groups," J. Algebra, vol. 517, 112–118 (2019).
3. Doerk K. and Hawkes T., *Finite Soluble Groups*, De Gruyter, Berlin and New York (1992).
4. Guo W., *The Theory of Classes of Groups*, Science Press and Kluwer Acad. Publ., Beijing, New York, Dordrecht, Boston, and London (2000).
5. Isaacs I. M., Navarro G., and Wolf T. R., "Finite group elements where no irreducible character vanishes," J. Algebra, vol. 222, no. 2, 413–423 (1999).
6. Guo W., Skiba A. N., and Tang X., "On boundary factors and traces of subgroups of finite groups," Commun. Math. Stat., vol. 2, no. 3–4, 349–361 (2014).
7. Janko Z., "Endliche Gruppen mit lauter nilpotenten zweitmaximalen Untergruppen," Math. Z., vol. 79, no. 1, 422–424 (1962).
8. Berkovič Ya. G., "The existence of subgroups of a finite non-solvable group," Dokl. Akad. Nauk, vol. 156, no. 6, 1255–1257 (1964).
9. Huppert B., *Endliche Gruppen. I*, Springer, Berlin and New York (1967).
10. Conway J. H., Curtis R. T., Norton S. P., Parker R. A., and Wilson R. A., *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups*, Clarendon, Oxford (1985).
11. Huppert B. and Blackburn N., *Finite Groups. III*, Springer, Berlin and New York (1982).
12. Guo X. and Shum K. P., "Cover-avoidance properties and the structure of finite groups," J. Pure Appl. Algebra, vol. 181, no. 2–3, 297–308 (2003).

J. Zhang
School of Mathematics and Information
China West Normal University
Nanchong, People's Republic of China

Zh. Gao
School of Mathematical Sciences, Yangzhou University
Yangzhou, People's Republic of China

L. Miao (the corresponding author)
Basic Course Department, Jiangsu Vocational College of Information Technology
Wuxi, People's Republic of China
School of Mathematical Sciences, Yangzhou University
Yangzhou, People's Republic of China
E-mail address: lmiao@yzu.edu.cn