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A NEW CHARACTERIZATION OF FINITE σ-SOLUBLE $P\sigma T$-GROUPS

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Abstract: We prove that G is a finite σ-soluble group with transitive σ-permutability if and only if the following hold: (i) G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ and a normal subgroup N with σ-nilpotent quotient G/N such that $H_i \cap N \le Z_{\mathfrak{U}}(H_i)$ for all i; and (ii) every σ_i-subgroup of G is τ_σ-permutable in G for all $\sigma_i \in \sigma(N)$.

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1. Introduction

Throughout this paper, all groups are finite and G stands for a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$; and $\pi(G)$ is the set of all primes dividing $|G|$. Furthermore, $Z_{\mathfrak{U}_\pi}(G)$ is the *π-supersoluble hypercenter of* G, i.e., the product of all normal subgroups N of G such that every chief factor of G below N is either cyclic or a π'-group, and $Z_{\mathfrak{U}}(G) = Z_{\mathfrak{U}_{\mathbb{P}}}(G)$ is the supersoluble hypercenter of G.

In what follows, σ is some partition of $\mathbb{P}$, i.e., $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \ne j$; $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \ne \varnothing\}$ (see [1]).

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G (see [1]) if each nonidentity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathcal{H}$ has exactly one Hall σ_i-subgroup of G for every i.

A subgroup A of G is said to be *σ-permutable* in G (see [2]) if G possesses a complete Hall σ-set and A permutes with every Hall σ_i-subgroup H of G, i.e., $AH = HA$ for all i and A is *σ-semipermutable* in G [3] if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ with $\sigma(A) \cap \sigma(H) = \varnothing$.

The theories of σ-permutable and σ-semipermutable subgroups are closely related to the theories of σ-soluble and σ-nilpotent groups [1–5].

Recall that G is said to be *σ-decomposable* (see [6]) or *σ-nilpotent* (see [2]) if G is σ_i-closed for all i; *σ-soluble* (see [2]) if every chief factor H/K of G is a σ_i-group for some i; and $G^{\mathfrak{N}_\sigma}$ is the *σ-nilpotent residual of* G, i.e., the smallest normal subgroup of G with σ-nilpotent quotient.

Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \setminus \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \ne \varnothing$ for a Hall σ_i-subgroup $H \in \mathcal{H}\}$ (see [7]).

Then we say, following Beidleman and Skiba [7], that a subgroup A of G is as follows:

(i) *τ_σ-permutable in G with respect to $\mathcal{H}$* if $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$;

(ii) *τ_σ-permutable* in G if A is τ_σ-permutable in G with respect to some complete Hall σ-set $\mathcal{H}$ of G.

In the classical case when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \dots\}$ (we use here the notations of [1]), the σ-permutable, σ-semipermutable, and τ_σ-permutable subgroups are called respectively *S-permutable* (see [8]), *S-semipermutable* (see [9]), and *τ-permutable* (see [10]).

Finally, recall that G is said to be a *$P\sigma T$-group* (see [2]) if σ-permutability is a transitive relation in G; i.e., if H is a σ-permutable subgroup of K and K is a σ-permutable subgroup of G, then H is σ-permutable in G. In the case when $\sigma = \sigma^1$, a $P\sigma T$-group is called a *PST-group* [8].

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The theory of $P\sigma T$-groups was developed in [1, 2, 5, 11], and the following theorem is one of the culmination results of the theory.

Theorem A (see Theorem A in [1]). *Let $D = G^{\mathfrak{N}_\sigma}$. If G is a σ-soluble $P\sigma T$-group, then the following hold:*

(i) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ-nilpotent, and every element of G induces a power automorphism in D;*

(ii) *$O_{\sigma_i}(D)$ has a normal complement in a Hall σ_i-subgroup of G for all i.*

Conversely, if (i) *and* (ii) *hold for some subgroups D and M of G, then G is a $P\sigma T$-group.*

In this paper, basing on Theorem A and some results of [7], we obtain the following characterization of σ-soluble $P\sigma T$-groups:

Theorem B. *G is a σ-soluble $P\sigma T$-group if and only if the following hold:*

(i) *G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ and a normal subgroup N with σ-nilpotent quotient G/N such that $H_i \cap N \leq Z_{\mathfrak{U}_\pi}(H_i)$ for all i, where $\pi = \pi(N)$;*

(ii) *Every σ_i-subgroup of G is τ_σ-permutable in G for all $\sigma_i \in \sigma(N)$.*

Since every σ-semipermutable subgroup is τ_σ-permutable, we get from Theorem B the following already-known result:

Corollary 1.1 (see Theorem A in [3]). *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ all members of which are π-supersoluble. If every σ_i-subgroup of G is σ-semipermutable in G for all $\sigma_i \in \sigma(D)$, then G is a σ-soluble $P\sigma T$-group.*

Note that Theorem B remains new for each special partition σ of $\mathbb{P}$. In particular, in the case when $\sigma = \sigma^1$ we get from Theorem B the following new characterization of the soluble PST-groups.

Corollary 1.2. *Let $D = G^{\mathfrak{N}}$ be the nilpotent residual of G and $\pi = \pi(D)$. Then G is a soluble PST-group if and only if every p-subgroup of G is τ-permutable in G for all $p \in \pi$.*

The proof of Theorem B consists of many steps and the following theorem is one of them.

Theorem C. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $H_i \cap D \leq Z_{\mathfrak{U}_\pi}(H_i)$ for all i. If all maximal subgroups of every noncyclic Sylow p-subgroup of G are τ_σ-permutable in G for all $p \in \pi$, then*

(i) *D is a nilpotent Hall subgroup of G, $D \leq Z_{\mathfrak{U}}(G)$;*

(ii) *$(p - 1, |G|) \neq 1$ for every prime p dividing $|D|$. Hence, $p \in \pi(G/D)$ for the smallest prime p dividing $|G|$.*

Corollary 1.3 (see Theorem 10.3 in [12, VI]). *If every Sylow subgroup of G is cyclic, then G is supersoluble.*

Corollary 1.4 (see Theorem B in [3]). *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G possesses a complete Hall σ-set $\mathcal{H}$ such that every member H of $\mathcal{H}$ with $H \cap D \neq 1$ is π-supersoluble. If all maximal subgroups of every noncyclic Sylow p-subgroup of G are σ-semipermutable in G for all $p \in \pi$, then D is a nilpotent Hall subgroup of G of odd order and every chief factor of G below D is cyclic.*

The unexplained terminology and notation are standard. The reader is referred to [9, 12, 13] if need be.

2. Proof of Theorem C

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

Lemma 2.1 [2, Corollary 2.4 and Lemma 2.5]. *The class $\mathfrak{N}_\sigma$ is closed under direct products, homomorphic images and subgroups. Moreover, if E is a normal subgroup of G and $E/(E \cap \Phi(G))$ is σ-nilpotent, then E is σ-nilpotent.*

In view of Proposition 2.2.8 in [14], we get from Lemma 2.1 the following

Lemma 2.2. *If N is a normal subgroup of G, then $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}N/N$.*

Lemma 2.3 [15]. *Let H, K, and N be pairwise permutable subgroups of G and let H be a Hall subgroup of G. Then $N \cap HK = (N \cap H)(N \cap K)$.*

Recall that G is a *D_π-group* if G possesses a Hall π-subgroup E and every π-subgroup of G lies in some conjugate of E; a *σ-full group of Sylow type* (see [16]), if every subgroup E of G is a D_{σ_i}-group for every $\sigma_i \in \sigma(E)$, and *σ-full* (see [16]), provided that G possesses a complete Hall σ-set.

In view of Theorems A and B in [16], the following is true:

Lemma 2.4. *If G is σ-soluble, then G is a σ-full group of Sylow type.*

Lemma 2.5 [2, Lemma 3.1]. *Let H be a σ_i-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_i}(G) \le N_G(H)$.*

Lemma 2.6 [7, Lemma 2.6]. *Suppose that G has a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ such that the subgroups H and K of G are τ_σ-permutable in G with respect to $\mathcal{H}$. Let R be a normal subgroup of G and $H \le L \le G$. Then*

(1) *$\mathcal{H}_0 = \{H_1R/R, \dots, H_tR/R\}$ is a complete Hall σ-set of G/R. Moreover, if $\sigma(H) = \sigma(HR/R)$, then HR/R is τ_σ-permutable in G/N with respect to $\mathcal{H}_0$.*

(2) *If $HK = KH$ and $\sigma(H \cap K) = \sigma(H) = \sigma(K)$, then $H \cap K$ is τ_σ-permutable in G with respect to $\mathcal{H}$.*

(3) *If $H \le O_{\sigma_i}(G)$ for some i, then H is σ-permutable in G.*

(4) *If G is a σ-full group of Sylow type, then H is τ_σ-permutable in L.*

Lemma 2.7. *Let $Z = Z_{\mathfrak{U}_\pi}(G)$. Then*

(1) *each chief factor of G below Z is either cyclic or a π'-group;*

(2) *$Z \cap E \le Z_{\mathfrak{U}_\pi}(E)$ for every subgroup E of G;*

(3) *$NZ/N \le Z_{\mathfrak{U}_\pi}(G/N)$ for every normal subgroup N of G.*

Proof. (1): In fact, it suffices to prove that if A and B are normal subgroups of G such that each chief factor of G below A is either cyclic or a π'-group and each chief factor of G below B is either cyclic or a π'-group, then each chief factor H/K of G below AB is either cyclic or a π'-group. Moreover, in view of the Jordan–Hölder Theorem for chief series, it suffices to show that if $A \le K < H \le AB$, then H/K is either cyclic or a π'-group. But this follows from $H = A(H \cap B) = K(H \cap B)$ and the G-isomorphism $K(H \cap B)/K \simeq (H \cap B)/(K \cap B)$. Therefore, each chief factor of G below Z is either cyclic or a π'-group.

(2): Let $1 = Z_0 < Z_1 < \dots < Z_{t-1} < Z_t = Z$ be a chief series of G below Z. Then each factor Z_i/Z_{i-1} of the series is either cyclic or a π'-group by (1).

Consider the normal series

$$1 = Z_0 \cap E \le Z_1 \cap E \le \dots \le Z_{t-1} \cap E \le Z_t \cap E = Z \cap E$$

in E. Assume that $(Z_i \cap E)/(Z_{i-1} \cap E)$ is not a π'-group. Then, in view of the isomorphism,

$$(Z_i \cap E)/(Z_{i-1} \cap E) \simeq (Z_i \cap E)Z_{i-1}/Z_{i-1} \le Z_i/Z_{i-1}$$

we get that Z_i/Z_{i-1} is cyclic, and so $(Z_i \cap E)/(Z_{i-1} \cap E)$ is cyclic. Therefore, in view of the Jordan–Hölder Theorem, each chief factor of E below $Z \cap E$ is either cyclic or a π'-group. Hence $Z \cap E \le Z_{\mathfrak{U}_\pi}(E)$.

(3): Let $(H/N)/(K/N)$ be a chief factor of G/N such that $H/N \le NZ/N$. Then, in view of the isomorphism $(H \cap Z)K/K \simeq (H \cap Z)/(K \cap Z)$, we have that $H/K = (H \cap Z)K/K$ is a chief factor of G such that H/K is either cyclic or a π'-group by (1). Hence $NZ/N \le Z_{\mathfrak{U}_\pi}(G/N)$. The lemma is proved.

The following lemma is a corollary of Theorem 6.7 in [13, IV].

Lemma 2.8. *Let $N \le E$ be normal subgroups of G such that $N \le \Phi(E)$ and every chief factor of G between E and N is cyclic. Then each chief factor of G below E is cyclic.*

A group G is said to be *σ-primary* (see [2]) if G is a σ_i-group for some i.

Lemma 2.9. *Let $D = G^{\mathfrak{N}_\sigma}$ and $p \in \pi = \pi(D)$, where p is the smallest prime dividing $|D|$. If all maximal subgroups of every Sylow p-subgroup of G are τ_σ-permutable in G, then D is p-soluble.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Then $D \neq 1$. Assume that $p \in \sigma_k$.

We show first that $DR/R \simeq D/(D \cap R)$ is p-soluble for every minimal normal subgroup R of G. Indeed, in case p does not divide $|DR/R|$, it is clear. Suppose that $p \in \pi(DR/R)$. Then p is the smallest prime dividing $|DR/R|$, where $DR/R = (G/R)^{\mathfrak{N}_\sigma}$ by Lemma 2.2.

Let V/R be a maximal subgroup of a Sylow p-subgroup P/R of G/R. Then $P/R = G_pR/R$ and $V = R(V \cap G_p)$ for some Sylow p-subgroup G_p of G. Hence

$$p = |(P/R) : (V/R)| = |G_pR : R(V \cap G_p)| = |G_p| : |V \cap G_p| = |G_p : (V \cap G_p)|,$$

and so $V \cap G_p$ is a maximal subgroup of G_p. Therefore, $V \cap G_p$ is τ_σ-permutable in G by hypothesis, and so $V/R = R(V \cap G_p)/R$ is τ_σ-permutable in G/R by Lemma 2.6(1). The choice of G implies that $(G/R)^{\mathfrak{N}_\sigma} = DR/R \simeq D/(D \cap R)$ is p-soluble.

Hence $R \leq D$ and R is nonabelian. It is easy to see that R is the unique minimal normal subgroup of G and $C_G(R) = 1$. By [12, IV, Theorem 2.8], a Sylow p-subgroup Q of R is not cyclic. Hence $|Q| > p$.

Let P be a Sylow p-subgroup of G such that $Q = P \cap R$. Then by the Tate Theorem [12, IV, Theorem 4.7] there exists some maximal subgroup V of P such that $Q \nleq V$, which implies that $P = QV$ and so $V \cap R < P \cap R = Q$. If $V \cap R = 1$, then $V \cap R = P \cap V \cap R = Q \cap V = 1$ and so $|Q| = p$; a contradiction. Hence $V \cap R \neq 1$. Since $R = R_1 \times \cdots \times R_n$, where $R_1 \simeq \cdots \simeq R_n$ are nonabelian simple groups, $Q = (P \cap R_1) \times \cdots \times (P \cap R_n)$ and so $V \cap R_i < P \cap R_i$ for some i. Note also that $V \cap R_i \neq 1$. Otherwise from the isomorphism

$$V(P \cap R_i)/V \simeq (P \cap R_i)/(V \cap (P \cap R_i)) = (P \cap R_i)/1$$

we get that the order of a Sylow p-subgroup of $P \cap R_i$ divides p and so $P \cap R_i$ is p-nilpotent by [12, IV, Theorem 2.8], which implies that R is p-nilpotent.

We show first that R is σ-primary. Suppose the contrary. We can assume without loss of generality that V is τ_σ-permutable in G with respect to $\mathcal{H}$. Then there exists some $j \neq k$, and for $H = H_j$ we have $H \cap R_i \neq 1$ because R is not σ-primary. Note also that $\sigma_k \in \sigma(H^G)$. If not, then $R \cap H^G = 1$, which implies that $1 < H^G \leq C_G(R) = 1$. Therefore $\sigma_k \in \tau_{\mathcal{H}}(V)$, and so $VH^x = H^xV$ for all $x \in G$. By [13, Chapter A, Lemma 14.1(a)], $L = VH^x \cap R_i$ is a subnormal subgroup of VH^x, where V is a Hall σ_k-subgroup of VH^x and H^x is a Hall σ_j-subgroup of VH^x. Therefore, $L = (L \cap V)(L \cap H^x)$ by [13, I, Lemma 3.2]. Hence,

$$\begin{aligned} L &= (L \cap V)(L \cap H^x) = (VH^x \cap R_i \cap V)(VH^x \cap R_i \cap H^x) \\ &= (R_i \cap V)(R_i \cap H^x) = (V \cap R_i)(H \cap R_i)^x = (H \cap R_i)^x(V \cap R_i) \end{aligned}$$

for all $x \in R_i$, where $(H \cap R_i)(V \cap R_i) \neq R_i$ because $V \cap R_i < P \cap R_i$. Therefore, R_i is not simple by [8, Lemma 1.1.9(1)] because $H \cap R_i \neq 1$ and $V \cap R_i \neq 1$. This contradiction shows that R is σ-primary.

Then $H \cap R_i \neq 1$ for some $j \neq k$ and $H = H_j$. Therefore, $V \cap R$ is τ_σ-permutable in G by Lemma 2.6(2). But $V \cap R \leq R \leq O_{\sigma_k}(G)$ and so $V \cap R$ is σ-permutable in G by Lemma 2.6(3). Because $R \leq D \leq O^{\sigma_i}(G)$ and so $R \leq N_G(V \cap R)$ by Lemma 2.5, it follows that $V \cap R \leq O_p(R) = 1$; a contradiction. Thus R is abelian, and so D is p-soluble. The lemma is proved.

Lemma 2.10. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. If $\mathcal{H} = \{H_1, \dots, H_t\}$ is a complete Hall σ-set of G such that $H_i \cap D \leq Z_{\mathfrak{U}_\pi}(H_i)$ for all i, then $\mathcal{H}_0 = \{H_1N/N, \dots, H_tN/N\}$ is a complete Hall σ-set of G/N such that $(H_iN/N) \cap (G/N)^{\mathfrak{N}_\sigma} \leq Z_{\mathfrak{U}_{\pi_0}}(H_iN/N)$ for all i, where $\pi_0 = \pi((G/N)^{\mathfrak{N}_\sigma})$.*

Proof. It is clear that $\mathcal{H}_0$ is a complete Hall σ-set of G/N. Put $D_0 = (G/N)^{\mathfrak{N}_\sigma}$. Then $D_0 = DN/N$ by Lemma 2.2, and so

$$\pi_0 = \pi(D_0) = \pi(DN/N) = \pi(D/(D \cap N)) \subseteq \pi(D) = \pi.$$

Hence, $Z_{\mathfrak{U}_\pi}(H_iN/N) \leq Z_{\mathfrak{U}_{\pi_0}}(H_iN/N)$. On the other hand, $D \cap H_iN = (D \cap H_i)(D \cap N)$ by Lemma 2.3. Thus,

$$D_0 \cap (H_iN/N) = (D \cap H_i)N/N.$$

Note that, in view of Lemma 2.7(3),

$$(D \cap H_i)(N \cap H_i)/(N \cap H_i) \leq Z_{\mathfrak{U}_\pi}(H_i/(N \cap H_i))$$

since $D \cap H_i \leq Z_{\mathfrak{U}_\pi}(H_i)$. Hence

$$f((D \cap H_i)(N \cap H_i)/(N \cap H_i)) = (D \cap H_i)N/N \leq Z_{\mathfrak{U}_\pi}(H_iN/N),$$

where $f : H_i/(N \cap H_i) \to H_iN/N$ is the canonical isomorphism, since

$$f(Z_{\mathfrak{U}_\pi}(H_i/(N \cap H_i))) = Z_{\mathfrak{U}_\pi}(H_iN/N).$$

Therefore, $(D \cap H_i)N/N \leq Z_{\mathfrak{U}_{\pi_0}}(H_iN/N)$ for all i. The lemma is proved.

Lemma 2.11. *Let $D = G^{\mathfrak{N}_\sigma}$ and $\pi = \pi(D)$. Suppose that G is σ-soluble and all maximal subgroups of every noncyclic Sylow p-subgroup of G are τ_σ-permutable in G for all $p \in \pi$. Then*

(1) *the hypothesis holds for G/L for every minimal normal subgroup L of G;*

(2) *if D is nilpotent, then D is a Hall subgroup of G.*

Proof. (1): See the proof of Lemma 2.9.

(2): Suppose that this assertion is false. Let P be a Sylow p-subgroup of D and let G_p be a Sylow p-subgroup of G such that $1 < P < G_p$. We can assume without loss of generality that $G_p \leq H_1$.

(a) *$D = P$ is a minimal normal subgroup of G. Hence $D \leq G_p = H_1 \trianglelefteq G$.*

Let R be a minimal normal subgroup of G lying in D. Since D is nilpotent by hypothesis, R is a q-group for some prime q. Moreover, by (1) and the choice of G we have that $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G/R. Suppose now that $PR/R \neq 1$. Then PR/R is a Sylow p-subgroup of G/R. If $q \neq p$, then P is a Sylow p-subgroup of G. This contradicts the fact that $P < G_p$. Hence $q = p$ and so $R \leq P$. It implies that P/R is a Sylow p-subgroup of G/R, and so P is a Sylow p-subgroup of G. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is the unique minimal normal subgroup of G lying in D. Since D is nilpotent, a p'-complement E of D is characteristic in D and so E is normal in G. Hence $E = 1$. This implies that $R = D = P$. Finally, G/D is σ-nilpotent by Lemma 2.1 and so H_1/D is normal in G/D. Hence (a) holds.

(b) *$D \nleq \Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = D \rtimes M$.* (This follows from (2) and Lemma 2.1 because G is not σ-nilpotent.)

(c) *If G has a minimal normal subgroup $L \neq D$, then $G_p = D \times (L \cap G_p)$. Hence $O_{p'}(G) = 1$.*

By Lemma 2.2, $(G/L)^{\mathfrak{N}_\sigma} = LD/L$. Therefore, by (1), (a), and the choice of G we have that $LD/L \simeq D$ is a Hall subgroup of G/L. Hence $G_pL/L = DL/L$, and so $G_p = D \times (L \cap G_p)$. Since $D < G_p$ by (a), $O_{p'}(G) = 1$.

(d) *$V = C_G(D) \cap M$ is a normal subgroup of G and $C_G(D) = D \times V \leq H_1$.*

In view of (a) and (b), $C_G(D) = D \times V$, where $V = C_G(D) \cap M$ is a normal subgroup of G. By (a), $V \cap D = 1$ and so $V \simeq DV/D$ is σ-nilpotent by Lemma 2.1. Let W be a σ_1-complement of V. Then W is characteristic in V and so it is normal in G. Therefore, (d) holds in view of (c).

(e) *$G_p \neq H_1$.*

Assume that $G_p = H_1$. Then $D < G_p \leq C_G(D)$ by (a) and [13, Chapter A, Theorem 10.6(b)]. It follows from (d) that $L \leq C_G(D) \cap M \leq G_p$ for some minimal normal subgroup L of G. Hence $G_p = D \times L$ is a normal elementary abelian p-subgroup of G by (c). This ensues from Lemmas 2.6(3) and 2.5 that every maximal subgroup of G_p is normal in G. It follows that every subgroup of G_p is normal in G.

Hence $|D| = |L| = p$. Let $D = \langle a \rangle$, $L = \langle b \rangle$, and $N = \langle ab \rangle$. Then $N \nleq D$ and so, in view of the G-isomorphisms

$$DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D,$$

we get that $G/C_G(D) = G/C_G(N)$ is a p-group since $G_p = H_1$ and G/D is σ-nilpotent by Lemma 2.1. It follows from (d) that G is a p-group. This contradiction shows that we have (e).

Final contradiction for (2). By Theorem A in [16], G has a σ_1-complement E such that $W = EG_p = G_pE$. Then $D \leq G_p \leq W$ by (a). Moreover, since $W/D \leq G/D \in \mathfrak{N}_\sigma$ and $\mathfrak{N}_\sigma$ is a hereditary class by Lemma 2.1, $W/D \in \mathfrak{N}_\sigma$, and thereby $V = W^{\mathfrak{N}_\sigma} \leq D$. Therefore, in view of Lemmas 2.4 and 2.6(4), the hypothesis holds for W. From (e) we derive that $W \neq G$. Hence the conclusion of the lemma holds for W by the choice of G, which implies that V is a Hall subgroup of W. Moreover, $V \leq D$ and so $|V_p| \leq |P| < |G_p|$ for a Sylow p-subgroup V_p of V. Hence V is a p'-group. It implies from (d) that $V \leq C_G(D) \leq H_1 \cap W$. Therefore $V = 1$, which shows that $W = EG_p = E \times G_p$ is σ-nilpotent and so $E \leq C_G(D) \leq H_1$. Hence $E = 1$. It follows that $D = 1$, which is a contradiction. Thus D is a Hall subgroup of G. The lemma is proved.

Proof of Theorem C. Suppose that this theorem is false and let G be a counterexample of minimal order. Then $D \neq 1$. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$. Let R be a minimal normal subgroup of G.

(1) *The hypothesis holds for* G/R (see the proof of Lemma 2.9 and use Lemma 2.10).

(2) *D is soluble, and so G is σ-soluble. Hence G is a σ-full group of Sylow type* (in view of Theorem 2.8 in [12, IV], this follows from Lemmas 2.4, 2.9, and the Feit–Thompson Theorem).

(3) *D is nilpotent.*

Assume that this is false. Note that $(G/R)^{\mathfrak{N}_\sigma} = RD/R$ is nilpotent by (1) and the choice of G. Therefore $R \leq D$, while R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by Lemma 2.1. It implies from (2) that R is a p-group for some prime p. Therefore, by [13, Chapter A, Theorem 15.2] $R = C_G(R)$, $G = R \rtimes M$ for some maximal subgroup M of G and $|R| > p$, if not, then $G/C_G(R) = G/R$ is a cyclic group and so D is nilpotent, contrary to our assumption on D.

Clearly, $R \leq H_i \cap D$ for some i. Then $H_i = R \rtimes (H_i \cap M)$ and $R \leq Z_{\mathfrak{U}_\pi}(H_i)$ by hypothesis. It shows that there exists a maximal subgroup V of R such that V is normal in H_i because $p \in \pi$. Let P be a Sylow p-subgroup of $H_i \cap M$. Then RP is a Sylow p-subgroup of G, and VP is a maximal subgroup of RP. Hence, by the hypothesis of the theorem VP is τ_σ-permutable in G. It follows from Lemma 2.6(2)(3) that $V = V(R \cap P) = R \cap VP$ is σ-permutable in G. Therefore $O^{\sigma_i}(G) \leq N_G(V)$ by Lemma 2.5, and thereby $G = H_iO^{\sigma_i}(G) \leq N_G(V)$. The minimality of R implies that $V = 1$ and so $|R| = p$; a contradiction. Hence, we have (3).

(4) *D is a Hall subgroup of G.* (This is straightforward from (2), (3), and Lemma 2.11.)

(5) *If p is a prime such that $(p-1, |G|) = 1$, then p does not divide $|D|$. In particular, the smallest prime divisor of $|G|$ divides $|G : D|$.*

Assume the contrary and let P be the Sylow p-subgroup of D. Then, arguing as in the proof of (3), we can show that some maximal subgroup E of P is normal in G. Hence $C_G(D/E) = G$ because $(p-1, |G|) = 1$ by hypothesis. Since D is a Hall subgroup of G by (4), D has a complement M in G. Therefore $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is σ-nilpotent. Thus, G/E is σ-nilpotent. It follows that $D \leq E$; a contradiction. Hence p does not divide $|D|$. In particular, the smallest prime divisor of $|G|$ divides $|G : D|$.

(6) *Every chief factor of G below D is cyclic.*

Suppose the contrary. Assume that $\Phi(D) \neq 1$ and let $R \leq \Phi(D)$. Then the choice of G and (1) imply that every chief factor of G/R below $(G/R)^{\mathfrak{N}_\sigma} = D/R$ is cyclic, and so every chief factor of G below D is cyclic by Lemma 2.8. Hence $\Phi(D) = 1$, and so every Sylow subgroup of D is elementary. Moreover, there is $p \in \pi(D)$ such that the Sylow p-subgroup P of D has a minimal normal subgroup N of G such that $|N| > p$. Let V be a maximal subgroup of P such that $P = NV$. Then $N \cap V \neq 1$. Since D is a Hall subgroup of G, P is the Sylow p-subgroup of G. Therefore V is τ_σ-permutable in G, and so $N \cap V$ is σ-permutable in G by Lemma 2.6(2)(3). Arguing as in the proof of (3), we can show that $N \cap V$ is normal in G. The minimality of N implies that $N \cap V = 1$, and so $|N| = p$. This contradiction completes the proof of (6).

Claims (3)–(6) show that the conclusion of the theorem holds for G, which contradicts the choice of G. The theorem is proved.

3. Proof of Theorem B

Lemma 3.1. *Suppose that $D = G^{\mathfrak{N}_\sigma}$ is a nilpotent Hall subgroup of G. If every σ_i-subgroup of G is τ_σ-permutable in G for all $\sigma_i \in \sigma(D)$, then D is an abelian group of odd order and each element of G induces a power automorphism in D.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \dots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \dots, t$.

Note first that

$$(G/N)^{\mathfrak{N}_\sigma} = DN/N \simeq D/(D \cap N)$$

is a nilpotent Hall subgroup of G/N for every minimal normal subgroup N of G by Lemma 2.2. Let V/N be a nonidentity σ_i-subgroup of G/N for some

$$\sigma_i \in \sigma((G/N)^{\mathfrak{N}_\sigma}) = \sigma(DN/N) = \sigma(D/(D \cap N)) \subseteq \sigma(D).$$

Let U be a minimal supplement to N in V. Then $U \cap N \leq \Phi(U)$, and so U is a σ_i-subgroup of G since $V/N = UN/N \simeq U/(U \cap N)$. Thus, U is τ_σ-permutable in G by hypothesis and $\sigma(U) = \sigma(UN/N) = \{\sigma_i\}$, which implies that $V/N = UN/N$ is τ_σ-permutable in G/N by Lemma 2.6(1). Hence the hypothesis holds for G/N.

Let H be a subgroup of the Sylow p-subgroup P of D for some prime $p \in \pi$. We show that H is normal in G. For some i we have $P \leq O_{\sigma_i}(D) = H_i \cap D$. On the other hand, $D = O_{\sigma_i}(D) \times O^{\sigma_i}(D)$ since D is nilpotent. Assume that $O^{\sigma_i}(D) \neq 1$ and let N be a minimal normal subgroup of G lying in $O^{\sigma_i}(D)$. Then $HN/N \leq DN/N = (G/N)^{\mathfrak{N}_\sigma}$, and so the choice of G implies that HN/N is normal in G/N. Hence, $H = H(N \cap O_{\sigma_i}(D)) = HN \cap O_{\sigma_i}(D)$ is normal in G.

Assume now that $O^{\sigma_i}(D) = 1$. Then D is a σ_i-group. Since G/D is σ-nilpotent by Lemma 2.1, H_i/D is normal in G/D and so H_i is normal in G. It follows from Lemma 2.6(3) and the hypothesis of the theorem that all subgroups of H_i are σ-permutable in G. Since D is a normal Hall subgroup of H_i; therefore, D has a complement S in H_i by the Schur–Zassenhaus Theorem. It implies from Lemma 2.5 that $D \leq O^{\sigma_i}(G) \leq N_G(S)$. Hence $H_i = D \times S$, and so

$$G = H_i O^{\sigma_i}(G) = S O^{\sigma_i}(G) \leq N_G(H).$$

This implies that H is normal in G. Hence D is a Dedekind group, and so $|D|$ is odd by Theorem C. Hence, D is abelian and each element of G induces a power automorphism in D. The lemma is proved.

The following lemma is a corollary of Theorem A of this paper and Theorem B in [2].

A subgroup A of G is said to be *σ-subnormal* in G [2] if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \dots, n$.

Lemma 3.2. *The following hold:*

(i) *G is a $P\sigma T$-group if and only if every σ-subnormal subgroup of G is σ-quasinormal in G.*

(ii) *If G is a $P\sigma T$-group, then every quotient G/N of G is also a $P\sigma T$-group.*

Proof of Theorem B. *Sufficiency:* Assume the contrary and let G be a counterexample with $|G| + |N|$ minimal. By Lemma 2.1, $D := G^{\mathfrak{N}_\sigma}$ is the smallest normal subgroup of G with σ-nilpotent quotient. Therefore $D \leq N$ and so the hypothesis holds for (G, D). Hence the choice of G shows that $D = N$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \dots, t$.

(1) *$G = D \rtimes M$, where D is an abelian Hall subgroup of G of odd order, M is σ-nilpotent, and every element of G induces a power automorphism in D.* (This is straightforward from Lemma 3.1 and Theorem C.)

(2) *If R is a nonidentity normal subgroup of G, then the hypothesis holds for G/R, and so G/R is a σ-soluble $P\sigma T$-group* (see the proof of Lemma 3.1 and use Lemma 2.10).

(3) *$H_i = O_{\sigma_i}(D) \times S$ for some subgroup S of H_i for each $\sigma_i \in \sigma(D)$.*

Since D is an abelian Hall subgroup of G by (1), $D = L \times N$, where $L = O_{\sigma_i}(D)$ and $N = O^{\sigma_i}(D) = O_{\sigma_i'}(D)$ are Hall subgroups of G. Assume first that $N \neq 1$. Then

$$O_{\sigma_i}((G/N)^{\mathfrak{N}_\sigma}) = O_{\sigma_i}(D/N) = LN/N$$

has a normal complement V/N in $H_iN/N \simeq H_i$ by (2) and Theorem A. On the other hand, N has a complement S in V by the Schur–Zassenhaus Theorem. Hence $H_i = H_i \cap LSN = LS$ and $L \cap S = 1$ since

$$(L \cap S)N/N \leq (LN/N) \cap (V/N) = (LN/N) \cap (SN/N) = 1.$$

It is clear that V/N is a Hall subgroup of H_iN/N, and so V/N is characteristic in H_iN/N. On the other hand, H_iN/N is normal in G/N by Lemma 2.2 since $D/N \leq H_iN/N$. Hence V/N is normal in G/N. Thus $H_i \cap V = H_i \cap NS = S(H_i \cap N) = S$ is normal in H_i, and so $H_i = O_{\sigma_i}(D) \times S$.

Assume that $D = O_{\sigma_i}(D)$. Then H_i is normal in G, and so all subgroups of H_i are σ-permutable in G by Lemma 2.6(3). Since D is a normal Hall subgroup of H_i, D has a complement S in H_i. Using Lemma 2.5, we imply that $D \leq O^{\sigma_i}(G) \leq N_G(S)$. Hence, $H_i = D \times S = O_{\sigma_i}(D) \times S$.

It follows from Theorem A, (2), and (3) that G is a σ-soluble $P\sigma T$-group, contrary to our assumption on G. This completes the proof of sufficiency.

Assume now that G is a σ-soluble $P\sigma T$-group and let $D = G^{\mathfrak{N}_\sigma}$. Then G possesses a complete σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$ by Lemma 2.4. Moreover, G/D is σ-nilpotent by Lemma 2.1 and every subgroup of D is normal in G by Theorem A. Then $H_i \cap N \leq Z_{\mathfrak{U}}(H_i) \leq Z_{\mathfrak{U}_\pi}(H_i)$, where $\pi = \pi(N)$ for all i. Therefore, (i) holds for G.

We show now that every σ_i-subgroup of G is τ_σ-permutable in G for each $\sigma_i \in \sigma(D)$. It suffices to show that if H is a σ_i-subgroup of G, and so H permutes with every Hall σ_j-subgroup of G for all $j \neq i$. Assume the contrary and let G be a counterexample of minimal order. Then $D \neq 1$ and there are σ_i and σ_j $(i \neq j)$ such that $\sigma_i \in \sigma(D)$ and $HE \neq EH$ for some σ_i-subgroup H and some Hall σ_j-subgroup E of G. Then H is not σ-subnormal in G by Lemma 3.2. Hence a Hall σ_i-subgroup H_i of G is not normal in G since otherwise $H \leq H_i$ and so H is σ-subnormal in G by Lemma 2.6. Note that $|\sigma(D)| > 1$. Indeed, if $|\sigma(D)| = 1$, then $\sigma(D) = \{\sigma_i\}$ and so $D \leq H_i$, which implies that H_i/D is normal in G/D because G/D is σ-nilpotent. Hence H_i is normal in G; a contradiction.

We show now that EHN is a subgroup of G for every minimal normal subgroup N of G. Note first that the hypothesis holds for G/N by Lemma 3.2. Moreover, $HN/N \simeq H/(H \cap N)$ is a σ_i-subgroup of G/N. Therefore, if $\sigma_i \in \sigma(DN/N) = \sigma((G/N)^{\mathfrak{N}_\sigma})$, then the choice of G implies that

$$(HN/N)(EN/N) = (EN/N)(HN/N) = EHN/N$$

is a subgroup of G/N. Hence EHN is a subgroup of G. Assume now that $\sigma_i \notin \sigma(DN/N)$. Then a Hall σ_i-subgroup H_i of G lies in N. Clearly, $H_i = N$ because N is σ-primary. It follows that $H \leq N$ and so H is σ-subnormal in G; a contradiction. Hence EHN is a subgroup of G. Since $|\sigma(D)| > 1$ and D is abelian by Theorem A, G has at least two σ-primary minimal normal subgroups R and N such that $R, N \leq D$ and $\sigma(R) \neq \sigma(N)$. Then at least one of the subgroups R or N, say R, is a σ_k-group for some $k \neq j$. Moreover,

$$R \cap E(HN) = (R \cap E)(R \cap HN) = R \cap HN$$

by Lemma 2.3 and $R \cap HN \leq O_{\sigma_k}(HN) \leq V$, where V is a Hall σ_k-subgroup of H, because N is a σ_k'-group and G is a σ-full group of Sylow type by Lemma 2.4. Hence

$$EHR \cap EHN = EH(R \cap E(HN)) = EH(R \cap HN) = EH(R \cap H) = EH$$

is a subgroup of G, i.e., $HE = EH$. This contradicts $HE \neq EH$. Therefore, (ii) holds for G. Hence the necessity of the condition of the theorem holds for G. The theorem is proved.

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