ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 6, pp. 1002–1038.

THE REGULARITY OF INVERSES TO SOBOLEV MAPPINGS AND THE THEORY OF $\mathscr{Q}_{q,p}$-HOMEOMORPHISMS

S. K. Vodopyanov UDC 517.518:517.54

Abstract: We prove that each homeomorphism $\varphi : D \to D'$ of Euclidean domains in $\mathbb{R}^n$, $n \geq 2$, belonging to the Sobolev class $W^1_{p,\mathrm{loc}}(D)$, where $p \in [1,\infty)$, and having finite distortion induces a bounded composition operator from the weighted Sobolev space $L^1_p(D';\omega)$ into $L^1_p(D)$ for some weight function $\omega : D' \to (0,\infty)$. This implies that in the cases $p > n-1$ and $n \geq 3$ as well as $p \geq 1$ and $n \geq 2$ the inverse $\varphi^{-1} : D' \to D$ belongs to the Sobolev class $W^1_{1,\mathrm{loc}}(D')$, has finite distortion, and is differentiable $\mathscr{H}^n$-almost everywhere in D'. We apply this result to $\mathscr{Q}_{q,p}$-homeomorphisms; the method of proof also works for homeomorphisms of Carnot groups. Moreover, we prove that the class of $\mathscr{Q}_{q,p}$-homeomorphisms is completely determined by the controlled variation of the capacity of cubical condensers whose shells are concentric cubes.

DOI: 10.1134/S0037446620060051

Keywords: quasiconformal analysis, Sobolev space, composition operator, capacity estimate

Introduction

The main content of this article consists in proving the following characteristic feature of Sobolev-class homeomorphisms (see Theorem 25) together with ensuing properties.

Proposition 1. *If a homeomorphism $\varphi = (\varphi_1,\dots,\varphi_n) : D \to D'$, where $D, D' \subset \mathbb{R}^n$ are open domains and $n \geq 2$, belongs to the Sobolev class $W^1_{p,\mathrm{loc}}(D)$ with $p \in [1,\infty)$ and has finite distortion then φ induces the bounded composition operator $\varphi^* : L^1_p(D';\omega) \cap \mathrm{Lip}_l(D') \to L^1_p(D)$ as $\varphi^*(u) = u \circ \varphi$ for $u \in L^1_p(D';\omega) \cap \mathrm{Lip}_l(D')$ with some weight function $\omega : D' \to (0,\infty)$ specified in* (30).

Recall that a function $u : D \to \mathbb{R}$ on some open set $D \subset \mathbb{R}^n$ is of *Sobolev class* $L^1_p(D)$ whenever u is locally summable on D, possesses the generalized derivatives $\frac{\partial u}{dx_j} \in L_{1,\mathrm{loc}}(D)$ for all $j = 1,\dots,n$ (i.e., $\frac{\partial u}{dx_j} \in L_1(U)$ for every compactly embedded domain $U \Subset D$), and has the finite seminorm

$$\|u \mid L^1_p(D)\| = \biggl(\int\limits_D |\nabla u(y)|^p\,dy\biggr)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty.$$

Henceforth we denote the space of locally Lipschitz functions defined on D' by $\mathrm{Lip}_l(D')$. It is obvious that $\mathrm{Lip}_l(D') = W^1_{\infty,\mathrm{loc}}(D') \cap C(D')$.

A mapping $\varphi = (\varphi_1,\dots,\varphi_n)$ is of *Sobolev class* $W^1_{p,\mathrm{loc}}(D)$ whenever $\varphi_j(x)$ and the generalized derivatives $\frac{\partial \varphi_j}{dx_i}$ lie in $L_{p,\mathrm{loc}}(D)$ for all $j,i = 1,\dots,n$.

A mapping $\varphi : D \to \mathbb{R}^n$ of Sobolev class $W^1_{1,\mathrm{loc}}(D)$ is called a mapping with *finite distortion* whenever

$$D\varphi(x) = 0 \quad \mathscr{H}^n\text{-a.e. on the set } Z = \{x \in D : \det D\varphi(x) = 0\}. \tag{1}$$

Henceforth $D\varphi(x) = \bigl(\frac{\partial \varphi_j}{\partial x_i}(x)\bigr)$ stands for the Jacobi matrix of φ at $x \in D$; while $|D\varphi(x)|$, for its Euclidean operator norm and $\det D\varphi(x)$, for its determinant, the Jacobian.

Proposition 1 about a functional characterization of Sobolev-class homeomorphisms underlies our proof of the new properties of the inverse homeomorphism; see Theorem 27.

The author was supported by the Mathematical Center in Akademgorodok and the Ministry of Science and Higher Education of the Russian Federation (Contract 075–15–2019–1613).

Original article submitted July 18, 2020; revised September 26, 2020; accepted October 9, 2020.

Proposition 2. *If a homeomorphism $\varphi = (\varphi_1, \dots, \varphi_n) : D \to D'$ of open domains $D, D' \subset \mathbb{R}^n$, where $n \geq 2$, belongs to $W^1_{p,\mathrm{loc}}(D)$ with $p \in [1,\infty)$ for $n = 2$ or $p \in (n-1,\infty)$ for $n \geq 3$ and has finite distortion then the inverse homeomorphism $\varphi^{-1} : D' \to D$ has the following properties:*

(1) *φ^{-1} belongs to $W^1_{1,\mathrm{loc}}(D')$;*

(2) *φ^{-1} has finite distortion;*

(3) *φ^{-1} is differentiable $\mathscr{H}^n$-almost everywhere in D'.*

Assertions (1) and (2) are proved as Theorems 3.2 and 3.3 in [1] for $n = 2$ and $p = 1$ and as Theorem 1.2 in [2] for $n \geq 3$ and $p > n-1$; these articles are motivated by applications to nonlinear elasticity problems, see [3]; we can obtain assertion (3) from [4; 5, Theorem 2] for $n = 2$ and $p = 1$ and from [6] for $n \geq 3$ and $p = n$; the latter reference also includes a comprehensive bibliography.

The proof of the claims of Proposition 2 (see Theorem 27) is new, and conciser than the proofs cited above. Moreover, the new proof also works on more complicated metric structures; see Section 4 which establishes the analogs of Propositions 1 and 2 for Carnot groups.

This article naturally fits into the cycle of publications [7–12]. Originating in the series [13–20], it resides at the junction of the theory of function spaces and geometric function theory [21–39]. Some results of the articles found applications in nonlinear elasticity; see [40].

The main results of this article were formulated in [11].

1. Preliminaries

Recalling the main result of [10, 12], we can say that it amounts to a "weighted" generalization of the results of [13–16] and [17–20] as regards $1 < q = p < \infty$ and $1 < q < p < \infty$.

Theorem 3 [10, 12]. *Given a homeomorphism $\varphi : D \to D'$ of domains $D, D' \subset \mathbb{R}^n$, where $n \geq 2$, and a locally summable function $\omega : D' \to (0,\infty)$, the following are equivalent:*

(1) *the composition operator $\varphi^* : L^1_p(D';\omega) \cap \mathrm{Lip}_l(D') \to L^1_q(D)$, where $1 \leq q \leq p < \infty$, is bounded;*

(2) *for every annular[1)] condenser $E = (F,U) \subset D'$ such that $\varphi^{-1}(E) = (\varphi^{-1}(F), \varphi^{-1}(U)) \subset D$, we have*

$$\operatorname{cap}^{\frac{1}{q}}\big(\varphi^{-1}(E); L^1_q(D)\big) \leq \begin{cases} K_p \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big), & 1 < q = p < \infty, \\ \Psi_{q,p}(U \setminus F)^{\frac{1}{\sigma}} \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big), & 1 < q < p < \infty, \end{cases} \tag{2}$$

where $\Psi_{q,p}$ is a bounded quasiadditive set function on some open subset of D', while σ henceforth is determined from the relation $\frac{1}{\sigma} = \frac{1}{q} - \frac{1}{p}$ if $1 \leq q < p < \infty$ and $\sigma = \infty$ if $1 \leq q = p < \infty$;

(3) *$\varphi : D \to D'$ belongs to $W^1_{q,\mathrm{loc}}(D)$ and has finite distortion; i.e., $D\varphi(x) = 0$ holds $\mathscr{H}^n$-almost everywhere on the set $Z = \{x \in D \mid J(x,\varphi) = 0\}$, and the operator distortion function*

$$D \ni x \mapsto K^{1,\omega}_{q,p}(x,\varphi) = \begin{cases} \dfrac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{p}}\,\omega^{\frac{1}{p}}(\varphi(x))} & \text{if } \det D\varphi(x) \neq 0, \\ 0 & \text{if } \det D\varphi(x) = 0 \end{cases} \tag{3}$$

is in $L_\sigma(D)$.

Moreover, $\varphi \in W^1_{q,\mathrm{loc}}(D)$ and

$$2^{-\frac{n}{q}}\Big(\frac{3n}{2}\Big)^{-1}\big\|K^{1,\omega}_{q,p}(\cdot) \mid L_\sigma(D)\big\| \leq \|\varphi^*\|$$

$$\leq \big\|K^{1,\omega}_{q,p}(\cdot) \mid L_\sigma(D)\big\| \leq \begin{cases} 3n2^{\frac{n-p}{p}} K_p & \text{for } q = p, \\ 3n2^{\frac{n-q}{q}} \Psi_{q,p}(D')^{\frac{1}{\sigma}} & \text{for } q < p. \end{cases} \tag{4}$$

Remark 4. The equivalence of claims (1)–(3) of Theorem 3 is proved in [10, 12] just for $1 < q \leq p < \infty$, which is due to the range of summability parameters in (2).

[1)] For our description of an annular condenser, see Definition 5.

At the same time, the equivalence of claims (1) and (3) of Theorem 3 is proved in [10, 12, 41, 42] for $1 \le q \le p < \infty$.

In this article we establish (see Corollary 29) that all claims of Theorem 3 are also equivalent for $1 = q \le p < \infty$ and $n = 2$.

Indeed, the implication (1) $\Rightarrow$ (2) in Theorem 3 is proved in [10, 12] for $1 < q$; however, the same argument works for $n = 2$ and $q = 1$.

The implication (2) $\Rightarrow$ (3) for $q = 1$ and $n = 2$ is justified in Corollary 29 of this article.

Let us present the definitions of all concepts used in Theorem 3.

A locally summable function $\omega : D' \to \mathbb{R}$ is called a *weight* whenever $0 < \omega(y) < \infty$ for $\mathscr{H}^n$-almost all $y \in D'$. Recall that $u : D' \to \mathbb{R}$ is said to be of *weighted Sobolev class* $L^1_p(D';\omega)$ with $p \in [1,\infty)$ whenever u is locally summable on D', while the generalized derivatives[2)] $\frac{\partial u}{\partial y_j}$ lie in $L_p(D';\omega)$ for all $j = 1, \dots, n$.

The seminorm of $u \in L^1_p(D';\omega)$ equals

$$\big\|u \mid L^1_p(D';\omega)\big\| = \bigg(\int\limits_{D'} |\nabla u|^p(y)\omega(y)\,dy\bigg)^{\frac{1}{p}}. \tag{5}$$

In the case $\omega \equiv 1$ we simply write $L^1_p(D')$ instead of $L^1_p(D';1)$.

Recall that a homeomorphism $\varphi : D \to D'$ induces the *bounded composition operator*

$$\varphi^* : L^1_p(D';\omega) \cap \operatorname{Lip}_l(D') \to L^1_q(D), \quad 1 \le q \le p < \infty,$$

that acts as follows $D \ni x \mapsto (\varphi^* u)(x) = u(\varphi(x))$ whenever

$$\big\|\varphi^* u \mid L^1_q(D)\big\| \le K_{q,p}\big\|u \mid L^1_p(D';\omega)\big\| \text{ for every function } u \in L^1_p(D') \cap \operatorname{Lip}_l(D')$$

holds with some constant $K_{q,p} < \infty$.

Definition 5. Refer as a *condenser* in a domain $D' \subset \mathbb{R}^n$ to a pair $E = (F_1, F_0)$ of connected compact sets (continua) $F_1, F_0 \subset D'$.

If a continuum F lies in U, where $U \Subset D'$ is a compactly embedded connected open set, then we denote the condenser $E = (F, \partial U)$ by $E = (F, U)$.

A condenser $E = (F, \partial U)$ is called *annular* whenever the complement in $\mathbb{R}^n$ to $U \setminus F$ consists of two closed sets each of which is connected: the bounded connected component is F and the unbounded component is $\mathbb{R}^n \setminus U$.

An annular condenser $E = (F, \partial U)$ in $\mathbb{R}^n$ is called *spherical* or *cubical* whenever U is the ball[3)] $B(x,R) = \{y \in \mathbb{R}^n : |y - x|_2 < R\}$ or the cube $Q(x,R) = \{y \in \mathbb{R}^n : |y - x|_\infty < R\}$, while the continuum $F \subset U$ is the closure of the ball $B(x,r) = \{y \in \mathbb{R}^n : |y - x|_2 \le r\}$ or the cube $Q(x,r) = \{y \in \mathbb{R}^n : |y - x|_\infty < r\}$, where $r < R$.

A continuous function $u : D \to \mathbb{R}$ of class $W^1_{1,\mathrm{loc}}(D)$ is called *admissible* for some condenser $E = (F_1, F_0) \subset D$ if $u \equiv 1$ on F_1 and $u \equiv 0$ on F_0.

Given a condenser $E = (F_1, F_0)$, denote the collection of admissible functions by $\mathscr{A}(E)$.

Define the *capacity* of a condenser $E = (F_1, F_0)$ in the space $L^1_q(D)$ with $q \in [1,\infty)$ as

$$\operatorname{cap}\big(E; L^1_q(D)\big) = \inf_{u \in \mathscr{A}(E)} \big\|u \mid L^1_q(D)\big\|^q, \tag{6}$$

where the infimum is over all admissible functions of class $\mathscr{A}(E)$ for $E = (F_1, F_0) \subset D$.

[2)] The definition of generalized derivatives assumes that $\frac{\partial u}{dy_j} \in L_{1,\mathrm{loc}}(D')$.

[3)] Recall that the norm $|x|_p$ of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is defined as $|x|_p = \big(\sum_{k=1}^n |x_k|^p\big)^{\frac{1}{p}}$ for $p \in [1,\infty)$ and $|x|_\infty = \max_{k=1,\dots,n} |x_k|$. Each ball of the norm $|x|_2$ or $|x|_\infty$ is a Euclidean ball or cube respectively.

Define the *weighted capacity* of a condenser $E = (F_1, F_0) \subset D'$ in $L^1_p(D';\omega)$ as

$$\operatorname{cap}\big(E; L^1_p(D';\omega)\big) = \inf_{u \in \mathscr{A}(E)\cap \operatorname{Lip}_l(D')} \big\|u \mid L^1_p(D';\omega)\big\|^p,$$

where the infimum is over all functions belonging to $\mathscr{A}(E) \cap \operatorname{Lip}_l(D')$ and admissible for $E = (F_1, F_0)$.

Henceforth, in D' we mainly consider the *annular condensers* of the form $E = (F, U)$.

Definition 6. Suppose that D is an open set in $\mathbb{R}^n$. Denote by $\mathscr{O}(D)$ some system of open sets in D with the following properties:

(a) if the closure $\overline{B}$ of an open ball[4)] B lies in D then $B \in \mathscr{O}(D)$;

(b) if $U_1, \dots, U_k \in \mathscr{O}(D)$ is a disjoint system of open sets then $\bigcup_{i=1}^k U_i \in \mathscr{O}(D)$ for arbitrary $k \in \mathbb{N}$.

A mapping $\Phi : \mathscr{O}(D) \to [0,\infty]$ is called a κ-*quasiadditive* set function whenever

(c) for each point $x \in D$ there exists δ with $0 < \delta < \operatorname{dist}(x, \partial D)$ such that $0 < \Phi(B(x,\delta)) < \infty$, and if $D = \mathbb{R}^n$ then the inequality $0 \le \Phi(D(x,\delta)) < \infty$ must hold for all $\delta \in (0, \delta(x))$, where $\delta(x) > 0$ may depend on x;

(d) for every finite disjoint collection of open sets $U_i \in \mathscr{O}(D)$, where $i = 1, \dots, l$, with

$$\bigcup_{i=1}^{l} U_i \subset U, \quad \text{where } U \in \mathscr{O}(D), \text{ we have } \sum_{i=1}^{l} \Phi(U_i) \le \kappa \Phi(U). \tag{7}$$

If (7) holds with $\kappa = 1$ then we refer to Φ as a *quasiadditive* set function instead of 1-*quasiadditive*.

If for every finite collection $\{U_i \in \mathscr{O}(D)\}$ of disjoint open sets we have

$$\sum_{i=1}^{n} \Phi(U_i) = \Phi\Big(\bigcup_{i=1}^{n} U_i\Big) \tag{8}$$

then Φ is called *finitely additive*, while if (8) holds for every countable collection $\{U_i \in \mathscr{O}(D)\}$ of disjoint open sets then Φ is called *countably additive*.

A function Φ is *monotone* whenever $\Phi(U_1) \le \Phi(U_2)$ provided that $U_1 \subset U_2 \subset D$ and $U_1, U_2 \in \mathscr{O}(D)$.

It is obvious that every quasiadditive set function is monotone. A κ-quasiadditive set function $\Phi : \mathscr{O}(D) \to [0,\infty]$ is called *bounded* whenever $\sup_{U \in \mathscr{O}(D)} < \infty$.

Theorem 3 motivates us to select the following scale of mappings as an object of study in its own right. Recall that $f : D' \to \mathbb{R}^n$, where D' is a domain in $\mathbb{R}^n$, is called *continuous, open, and discrete* whenever f is continuous in D', the image of every open set is open, and the preimage $f^{-1}(y)$ of each $y \in f(D)$ is discrete.

Definition 7 [10, 12]. Say that a homeomorphism (or, more generally, a continuous, open, and discrete mapping) $f : D' \to D$ for $D, D' \subset \mathbb{R}^n$, where $n \ge 2$, *is of class* $\mathscr{Q}_{q,p}(D', D; \omega)$, where $1 < q \le p < \infty$ for $n \ge 3$ and $1 \le q \le p < \infty$ for $n = 2$, while $\omega \in L_{1,\mathrm{loc}}(D')$ is a weight function, whenever there exist

(1) a constant K_p for $q = p$,

(2) a bounded quasiadditive function $\Psi_{q,p}$ on an open set in D' for $q < p$ such that for every condenser $E = (F_0, F_1)$ in D' with the image $f(E) = (f(F_0), f(F_1))$ in D we have

$$\begin{cases} \operatorname{cap}^{\frac{1}{p}}\big(f(E); L^1_p(D)\big) \le K_p \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big) & \text{if } q = p, \\ \operatorname{cap}^{\frac{1}{q}}\big(f(E); L^1_q(D)\big) \le \Psi_{q,p}(D' \setminus (F_0 \cup F_1))^{\frac{1}{\sigma}} \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big) & \text{if } q < p. \end{cases} \tag{9}$$

If (9) holds only for *annular condensers* $E = (F, U) \subset D'$,

$$\begin{cases} \operatorname{cap}^{\frac{1}{p}}\big(f(E); L^1_p(D)\big) \le K_p \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big) & \text{if } q = p, \\ \operatorname{cap}^{\frac{1}{q}}\big(f(E); L^1_q(D)\big) \le \Psi_{q,p}(U \setminus F)^{\frac{1}{\sigma}} \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big) & \text{if } q < p, \end{cases} \tag{10}$$

[4)] Instead of balls, we can use cubes as elementary sets.

then we obtain the larger class of homeomorphisms $f : D' \to D$ which we will denote by $\mathscr{R}\mathscr{Q}_{q,p}(D', D; \omega)$.

If (10) holds only for *spherical or cubical annular condensers* then we again obtain the larger class of homeomorphisms $f : D' \to D$ which we will denote by $S\mathscr{R}Q_{q,p}(D', D; \omega)$ or respectively $Q\mathscr{R}Q_{q,p}(D', D; \omega)$.

It is obvious that

$$\mathscr{Q}_{q,p}(D', D; \omega) \subset \mathscr{R}Q_{q,p}(D', D; \omega) \subset S\mathscr{R}Q_{q,p}(D', D; \omega) \quad (Q\mathscr{R}Q_{q,p}(D', D; \omega)).$$

Remark 8. In the case $q = p = n$ the class $\mathscr{Q}_{n,n}(D', D; \omega)$ of homeomorphisms includes [12, Section 4.4] the class of the so-called Q-homeomorphisms [6] defined by a controlled variation of the modulus of a family of curves.

The next theorem gives an analytical description of the mappings whose inverses are in $\mathscr{Q}_{q,p}(D', D; \omega)$.

Theorem 9 [10, 12]. *A homeomorphism $f : D' \to D$ belongs to $\mathscr{R}Q_{q,p}(D', D; \omega)$, where $1 < q \le p < \infty$ for $n \ge 3$ and $1 \le q \le p < \infty$ for $n = 2$, if and only if the inverse homeomorphism $\varphi = f^{-1} : D \to D'$ satisfies either* (1) *or* (3) *of Theorem* 3.

Proof. It is not difficult to observe that the claim that $f \in \mathscr{R}Q_{q,p}(D', D; \omega)$, where $1 < q \le p < \infty$ and $2 \le n$, for a homeomorphism $f : D' \to D$ is equivalent to claim (2) of Theorem 3 for the inverse homeomorphism $\varphi = f^{-1} : D \to D'$.

From this we deduce that claims (1) and (3) of Theorem 3 hold for $\varphi : D \to D'$. Since this argument is reversible, Theorem 9 is established in the case $n > 2$. Its validity in the case $1 = q \le p < \infty$ and $n = 2$ will be proved in Theorem 3, Remark 4, and Corollary 29 below. $\square$

This article presents the new examples of classes of mappings in the family $\mathscr{Q}_{q,p}(D', D; \omega)$.

Remark 10. As [13–16] show, in the case $1 < q = p < \infty$ and $\omega \equiv 1$ the composition operator $\varphi^* : L^1_p(D') \cap \mathrm{Lip}_l(D') \to L^1_p(D)$ of Theorem 3 extends by continuity to $L^1_p(D')$ and coincides with the composition operator in the following sense:

$$L^1_p(D') \ni u \mapsto \varphi^* u = \begin{cases} u \circ \varphi, & \text{where } u \text{ is a continuous representative} \\ & \text{of } u \in L^1_p(D') \text{ for } p \in (n, \infty), \\ u \circ \varphi, & \text{where } u \text{ is an arbitrary representative} \\ & \text{of } u \in L^1_p(D') \text{ for } p \in [1, n]. \end{cases}$$

For $p = n$ the mappings of this class are quasiconformal. In [43] the mappings of this class for $p \ne n$ are called p-morphisms.

Recall a few useful concepts: Given $k \ge 0$, $\delta \in (0, \infty]$, and $A \subset \mathbb{R}^n$, put

$$\mathscr{H}^k_\delta(A) = \frac{\omega_k}{2^k} \inf\Big\{\sum_{i\in\mathbb{N}} (\operatorname{diam} T_i)^k : \operatorname{diam} T_i < \delta, A \subset \bigcup_{i\in\mathbb{N}} T_i\Big\},$$

where $\omega_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$, while the infimum is over all countable coverings $\{T_i\}$ of A. If A cannot be countably covered by sets of these sizes then set $\mathscr{H}^k_\delta(A) = \infty$. The limit

$$\mathscr{H}^k(A) = \lim_{\delta\to 0} \mathscr{H}^k_\delta(A)$$

exists and is called the *k-dimensional Hausdorff measure* of A. In the Euclidean space $\mathbb{R}^n$ the n-dimensional Hausdorff measure $\mathscr{H}^n(A)$ of $A \subset \mathbb{R}^n$ coincides with the n-dimensional Lebesgue measure; see [44, Theorem 2.3.4] for instance.

The quasiadditive set function Φ is differentiable in the following sense:

Proposition 11 [45–47]. **I.** *Suppose that a monotone quasiadditive set function* Φ *is defined on some system* $\mathscr{O}(D')$ *of open subsets of a domain* D'*. Then*

(1) *for* $\mathscr{H}^n$*-almost all* $y \in D'$ *the derivative*[5] *exists and is finite:*

$$\lim_{\delta\to 0,\, y\in B_\delta} \frac{\Phi(B_\delta)}{\mathscr{H}^n(B_\delta)} = \Phi'(y);$$

(2) *for every open set* $U \in \mathscr{O}(D')$ *we have*

$$\int\limits_U \Phi'(y)\,dy \le \Phi(U).$$

II. *Suppose that a monotone* κ*-quasiadditive set function* Φ *is defined on some system* $\mathscr{O}(D')$ *of open subsets of a domain* D'*. Then*

(3) *for* $\mathscr{H}^n$*-almost all points* $y \in D'$ *the upper derivative exists and is finite:*

$$\lim_{r\to 0} \sup_{0<\delta<r,\, y\in B_\delta} \frac{\Phi\,(B_\delta)}{\mathscr{H}^n(B_\delta)} = \overline{\Phi}'(y);$$

(4) *for every open set* $U \in \mathscr{O}(D')$ *we have*

$$\int\limits_U \overline{\Phi}'(y)\,dy \le \kappa\Phi(U).$$

In all limits we can replace balls with cubes.

Example 12 (volume derivative). **I.** Consider an open set D' in $\mathbb{R}^n$ and an injective continuous mapping $f : D' \to \mathbb{R}^n$. For each open set $U \subset D'$ the image $f(U)$ is a Borel set, and so the set function $\mathscr{V}_n$ is defined:

$$U \mapsto \mathscr{V}_n(U) = \mathscr{H}^n(f(U)).$$

The function $\mathscr{V}_n$ is defined on the open sets $U \subset D'$ and $\mathscr{V}_n$ is obviously monotone and countably additive. By Proposition 11, the derivative $\mathscr{V}_n'(y)$ exists and coincides for $\mathscr{H}^n$-almost all $y \in D'$ with the density (volume derivative)

$$D' \ni y \mapsto J_f(y) = \lim_{r\to 0} \frac{\mathscr{H}^n(f(B(y,r)))}{\mathscr{H}^n(B(y,r))} \qquad (11)$$

of the set function $\mathscr{B}(D') \ni T \mapsto \mathscr{H}^n(f(T))$ defined on the σ-algebra $\mathscr{B}(D')$ of Borel sets $T \subset D'$.

II. Consider an open set D' in $\mathbb{R}^n$ and a continuous discrete open mapping $f : D' \to \mathbb{R}^n$. Fix some open set $U \Subset D'$. Then $f(U)$ is a bounded open set. Moreover, the multiplicity function (Banach indicatrix)

$$f(U) \ni x \mapsto \mathscr{N}(x, f, U) = \#\{y \in U : f(y) = x\}$$

is bounded; see [48, Proposition 4.1]. Put $\mathscr{N}(f, U) = \sup_{x\in f(U)} \mathscr{N}(x, f, U)$. We have $\mathscr{N}(f, U) < \infty$.

The set function $\mathscr{V}_n$:

$$U \supset V \mapsto \mathscr{V}_n(V) = \mathscr{H}^n(f(V))$$

on the open sets $V \subset U$ is a monotone κ-quasiadditive set function with the constant $\kappa = \mathscr{N}(f, U)$.

Indeed, if $V_i \subset U$ for $i = 1, \dots, l$ is a finite collection of disjoint open sets then

$$\sum_{i=1}^{l} \mathscr{V}_n(V_i) = \sum_{i=1}^{l} \mathscr{H}^n(f(V_i)) = \sum_{i=1}^{l} \int\limits_{f(U)} \chi_{f(V_i)}(x)\,dx \le \int\limits_{f(U)} \mathscr{N}(f,U)\,dx = \mathscr{N}(f,U)\mathscr{V}_n(U).$$

---

[5] Henceforth B_δ is an arbitrary ball $B(z,\delta) \subset D'$ containing y.

By Proposition 11, the upper derivative $\overline{\mathscr{V}}'_n(y)$ exists:

$$U \ni y \mapsto \overline{\mathscr{V}}'_n(y) = \overline{\lim_{r\to 0,\, y\in B_r}} \frac{\mathscr{H}^n(\mathscr{V}_n(B_r))}{\mathscr{H}^n(B_r)} = \overline{\lim_{r\to 0,\, y\in B_r}} \frac{\mathscr{H}^n(f(B_r))}{\mathscr{H}^n(B_r)}, \tag{12}$$

where B_r is a radius r ball containing y that is not necessarily the center of the ball.

Proposition 11 yields the inequality

$$\int\limits_V \overline{\mathscr{V}}'_n(y)\, dy \le \kappa \mathscr{V}_n(V)$$

for every open set $V \subset U$.

Example 13 (Lebesgue's Differentiation Theorem). Consider an open set D in $\mathbb{R}^n$ and a nonnegative function $g \in L_{1,\mathrm{loc}}(D)$. Given an open set $U \subset D$, put $\Phi(U) = \int_U g(x)\, dx$.

The function Φ is defined on the open sets $U \subset D$, is monotone and countably additive. Its derivative $\Phi'(x)$ exists for $\mathscr{H}^n$-almost all $x \in D$ and coincides $\mathscr{H}^n$-almost everywhere with $g(x)$ [49, 50].

Recall that the function $u : D \to \mathbb{R}$ belongs to $\mathrm{ACL}(D)$, i.e., $u \in \mathrm{ACL}(D)$, whenever the restriction $u|_Q$ of $u : D \to \mathbb{R}$ to an arbitrary closed cube $Q \subset D$ whose edges are parallel to the coordinate axes is absolutely continuous on $\mathscr{H}^{n-1}$-almost all segments orthogonal to the facets of this cube. It is known, see [51] for instance, that every $f \in W^1_{1,\mathrm{loc}}(D)$ can be redefined on a zero measure set so that the new function $\tilde{f}$ will belong to $\mathrm{ACL}(D)$, while all its partial derivatives coincide with the generalized derivatives $\mathscr{H}^n$-almost everywhere in D.

Below we apply the following change-of-variables formula for Lebesgue integrals.

Proposition 14 [12, 52]. *Suppose that $\varphi : D \to \mathbb{R}^n$ belongs to $W^1_{1,\mathrm{loc}}(D)$ or $\mathrm{ACL}(D)$. Then*

(1) *there exists a zero measure Borel set $\Sigma \subset D$ such that $\varphi : D\setminus\Sigma \to \mathbb{R}^n$ enjoys Luzin's $\mathscr{N}$-property;*

(2) *$D \setminus \Sigma \ni x \mapsto (u \circ \varphi)(x)|\det D\varphi(x)|$ and $\mathbb{R}^n \ni y \mapsto u(y)\mathscr{N}(y, \varphi, D \setminus \Sigma)$ are measurable functions as soon as so is $u : \mathbb{R}^n \to \mathbb{R}$;*

(3) *if $A \subset D \setminus \Sigma$ is a measurable set then the area formula holds:*

$$\int\limits_A |\det D\varphi(x)|\, dx = \int\limits_{\mathbb{R}^n} \mathscr{N}(y, \varphi, A)\, dy;$$

(4) *if $u \ge 0$ is a nonnegative function then the integrands in* (13) *are measurable and the following change-of-variables formula for Lebesgue integrals holds:*

$$\int\limits_{D\setminus\Sigma} u(\varphi(x))|\det D\varphi(x)|\, dx = \int\limits_{\mathbb{R}^n} \sum_{x\in\varphi^{-1}(y)\setminus\Sigma} u(x)\, dy; \tag{13}$$

(5) *if one of the functions*

$$D \setminus \Sigma \ni x \mapsto (u \circ \varphi)(x)|\det D\varphi(x)| \quad \text{and} \quad \mathbb{R}^n \ni y \mapsto u(y)\mathscr{N}(y, \varphi, D \setminus \Sigma)$$

is integrable then so is the other, and

$$\int\limits_{D\setminus\Sigma} u(\varphi(x))|\det D\varphi(x)|\, dx = \int\limits_{\mathbb{R}^n} u(y)\mathscr{N}(y, \varphi, D \setminus \Sigma)\, dy. \tag{14}$$

Remark 15. Since $\mathscr{H}^n(\Sigma) = 0$ in (13) and (14), on the left-hand side of these formulas we can replace integration over $D \setminus \Sigma$ by integration over D, so that along with (14) we have

$$\int\limits_D u(\varphi(x))|\det D\varphi(x)|\, dx = \int\limits_{\mathbb{R}^n} u(y)\mathscr{N}(y, \varphi, D \setminus \Sigma)\, dy.$$

The last formula is proved in [53] by a different method.

REMARK 16. Observe that every $\varphi \in W^1_{q,\mathrm{loc}}(D)$ for $q > n$ and every homeomorphism $\varphi \in W^1_{n,\mathrm{loc}}(D)$ enjoys Luzin's $\mathscr{N}$-property; see [54–56].

DEFINITION 17. Consider a homeomorphism $\varphi : D \to D'$ of Euclidean domains in $\mathbb{R}^n$ with $n \geq 2$. Example 12 shows that we can express D as the union of three disjoint Borel sets: $D = Z \cup \Sigma \cup (D \setminus (Z \cup \Sigma))$, where

(1) Z includes the set $\{x \in D : J_\varphi(x) = 0\}$ of zeros of the volume derivative and differs from it by a measure zero set: $\mathscr{H}^n(Z \setminus \{x \in D : J_\varphi(x) = 0\}) = 0$,

(2) $\Sigma \subset D$ is a singular set; i.e., $\varphi(\Sigma)$ is of positive measure;

(3) $D \setminus (Z \cup \Sigma)$ is the set on which φ enjoys Luzin's $\mathscr{N}$-property and $0 < J_\varphi(x) < \infty$ at each point of $D \setminus (Z \cup \Sigma)$.

The decomposition of D corresponds to the decomposition of the image

$$D' = (D' \setminus (Z' \cup \Sigma')) \cup Z' \cup \Sigma',$$

where $Z' = \varphi(\Sigma)$ and $\Sigma' = \varphi(Z)$ play the same roles for the inverse homeomorphism $\varphi^{-1} : D' \to D$ as Z and Σ do for φ; for the details, see [12, § 1].

Moreover, if $\varphi : D \to D'$ belongs to $W^1_{1,\mathrm{loc}}(D)$ or $\varphi \in \mathrm{ACL}(D)$ then [12, § 1] shows that

$$\mathscr{H}^n(\{x \in D : J_\varphi(x) = 0\} \Delta \{x \in D : \det D\varphi(x) = 0\}) = 0.$$

Hence, for Sobolev-class mappings we may assume from the outset that $Z \supset \{x \in D : \det D\varphi(x)) = 0\}$ and $\mathscr{H}^n(Z \setminus \{x \in D : \det D\varphi(x) = 0\}) = 0$, while $\det D\varphi(x) \neq 0$ on $D \setminus (Z \cup \Sigma)$.

Below we will use the notation just introduced.

1.1. From a minimal collection of condensers to localization of the distortion function. In this section we show that the localization of distortion function in Theorem 3 can be obtained from a tuple consisting of cubical condensers. In other words, we will verify that if (2) is satisfied only for cubical condensers then claims (1) and (3) of Theorem 3 hold as well. This approach contrasts with the classical tradition in the theory of quasiconformal mappings which usually considers spherical condensers as a minimal collection; see [27].

In the following theorem, as the system of open sets $\mathscr{O}_c(D')$ on which the quasiadditive set function Ψ is defined we take the minimal system of open sets in D' (cp. Definition 6) which contains

(1) D';

(2) every open cube Q with $\overline{Q} \subset D'$;

(3) the complement $Q_2 \setminus \overline{Q}_1$ for two cubes $Q_1, Q_2 \subset D'$ with the common center and $Q_1 \subset Q_2$.

As a bounded quasiadditive set function we consider $\Phi : \mathscr{O}_c(D') \to [0, \infty)$.

Theorem 18. *Given a homeomorphism $\varphi : D \to D'$ of domains $D, D' \subset \mathbb{R}^n$, where $n \geq 2$, and a locally summable weight function $\omega : D' \to (0, \infty)$, if every cubical condenser $E = (\overline{Q(y,r)}, Q(y,R)) \subset D'$ with the preimage $\varphi^{-1}(E) = (\varphi^{-1}(Q(y,r)), \varphi^{-1}(Q(y,R)))$ in D satisfies*

$$\operatorname{cap}^{\frac{1}{q}}\big(\varphi^{-1}(E); L^1_q(D)\big) \leq \begin{cases} K_p \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big), & 1 < q = p < \infty, \\ \Psi^{\frac{1}{\sigma}}_{q,p}(U \setminus F) \operatorname{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big), & 1 < q < p < \infty, \end{cases}$$

where $\Psi_{q,p}$ is some bounded quasiadditive set function on the system $\mathscr{O}_c(D')$, then the following hold:

(1) The homeomorphism $\varphi : D \to D'$ belongs to $W^1_{q,\mathrm{loc}}(D)$ and has finite distortion: $D\varphi(x) = 0$ holds $\mathscr{H}^n$-almost everywhere on $Z = \{x \in D \mid J(x, \varphi) = 0\}$, and the operator distortion function

$$D \ni x \mapsto K^{1,\omega}_{q,p}(x, \varphi) = \begin{cases} \dfrac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{p}} \omega^{\frac{1}{p}}(\varphi(x))} & \textit{if } \det D\varphi(x) \neq 0, \\ 0 & \textit{if } \det D\varphi(x) = 0, \end{cases}$$

belongs to $L_\sigma(D)$*, where* σ *is determined from* $\frac{1}{\sigma}=\frac{1}{q}-\frac{1}{p}$ *if* $1<q<p<\infty$ *and* $\sigma=\infty$ *if* $1<q=p<\infty$*; moreover,*

$$K^{1,\omega}_{q,p}(x,\varphi)\le\begin{cases}7^{\frac{n}{p}}nK_p^p & \text{for } 1<q=p<\infty,\\ 7^{\frac{n}{q}}n\Psi'(\varphi(x))^{\frac{1}{\sigma}} & \text{for } 1<q<p<\infty\end{cases}$$

for $\mathscr{H}^n$*-almost all* $x\in D\setminus(Z\cup\Sigma)$*.*

(2) *The composition operator* $\varphi^*: L^1_p(D';\omega)\cap\operatorname{Lip}_l(D')\to L^1_q(D)$*, where* $1\le q\le p<\infty$*, is bounded; moreover,*

$$\|\varphi^*\|\le\big\|K^{1,\omega}_{q,p}(\cdot)\mid L_\sigma(\varphi^{-1}(A))\big\|\le\begin{cases}7^{\frac{n}{p}}nK_p & \text{in the case } 1<q=p<\infty,\\ 7^{\frac{n}{q}}n\Psi(D')^{\frac{1}{\sigma}} & \text{in the case } 1<q<p<\infty;\end{cases}$$

the quasiadditive function

$$D'\supset A\mapsto\widetilde{\Psi}_{q,p}(A)=\big\|K^{1,\omega}_{q,p}(\cdot)\mid L_\sigma(\varphi^{-1}(A))\big\|^\sigma$$

satisfies the relations:

(a) $\widetilde{\Psi}_{q,p}(U)\le 7^{\frac{n\sigma}{q}}n^\sigma\Psi(U)$ *for every open set* $U\in\mathscr{O}_c(D')$*,*

(b) $\|\varphi^*_A\|\le\Psi^{\frac{1}{\sigma}}_{q,p}(A)$*, where* $\|\varphi^*_A\|$ *is the norm of* $\varphi^*_A: L^1_p(A;\omega)\cap\operatorname{Lip}_l(A)\to L^1_q(\varphi^{-1}(A))$*, while* $A\subset D'$ *is an open set.*

(3) *Every condenser* $E=(F_1,F_0)$ *in* D' *with the preimage* $\varphi^{-1}(E)=(\varphi^{-1}(F_1),\varphi^{-1}(F_0))$ *in* D' *satisfies*

$$\operatorname{cap}^{\frac{1}{q}}\big(\varphi^{-1}(E);L^1_q(D)\big)\le\begin{cases}7^{\frac{n}{p}}nK_p\operatorname{cap}^{\frac{1}{p}}(E;L^1_p(D';\omega)), & 1<q=p<\infty,\\ 7^{\frac{n}{q}}n\Psi^{\frac{1}{\sigma}}_{q,p}(D'\setminus(F_0\cup F_1))\operatorname{cap}^{\frac{1}{p}}\big(E;L^1_p(D';\omega)\big), & 1<q<p<\infty.\end{cases}$$

(4) *The classes of homeomorphisms coincide:*

$$Q\mathscr{R}Q_{q,p}(D',D;\omega)=\mathscr{Q}_{q,p}(D',D;\omega).$$

(5) *The claims of Theorem 18 remain valid in the case* $1=q\le p<\infty$ *and* $n=2$*.*

Proof. Fix a cube $Q(y,r)\Subset D'$ and consider the test function $u(z)=(r-|z-y|_\infty)^+$. It obviously satisfies the hypotheses of Lemma 2.3 of [12], which yields $u\circ\varphi\in L^1_q(\varphi^{-1}(U))$ and

$$\begin{aligned}&\big\|u\circ\varphi\mid L^1_q(\varphi^{-1}(Q(y,r)))\big\|\\ &\le\begin{cases}K_p\big\|u\mid L^1_p(Q(y,r);\omega)\big\|=K_p\omega(Q(y,r))^{\frac{1}{p}} & \text{if } q=p,\\ \Psi(Q(y,r))^{\frac{1}{\sigma}}\big\|u\mid L^1_p(Q(y,r);\omega)\big\|=\Psi(Q(y,r))^{\frac{1}{\sigma}}\omega(Q(y,r))^{\frac{1}{p}} & \text{if } q<p,\end{cases}\end{aligned}\tag{15}$$

where $\omega(Q(y,r))=\int_{Q(y,r)}\omega(z)\,dz$ is the weighted measure of the cube $Q(y,r)$.

Fix an arbitrary positive integer $1\le j\le n$. Define in $Q(y,r)$ the n-dimensional open tetrahedron

$$T_jQ(y,r)=\{z\in Q(y,r): y_j-r<z_j<y_j,\ |z_j-y_j|>\max_{l\ne j}|z_l-y_l|\}.$$

On the preimage $\varphi^{-1}(T_jQ(y,r))$ the composition $u\circ\varphi$ equals $r+\varphi_j(x)-y_j$. From (15) we deduce that $\varphi_j\in L^1_q(\varphi^{-1}(T_jQ(y,r)))$.

Take $z=y-\frac{3}{4}e_j$, where e_j is the jth vector of the standard basis for $\mathbb{R}^n$. For this choice of z we have

$$Q(z,r/4)\subset T_jQ(y,r)\subset Q(y,r)\subset Q(z,7r/4).$$

To each point $z \in W$, where W is an arbitrary open set compactly embedded into D' (the latter is written as $W \Subset D'$) associate the cube $Q(z, r/4)$ such that $Q(z, 2r) \Subset D'$. From (15) we obtain

$$\big\|\nabla\varphi_j \mid L^1_q(\varphi^{-1}(Q(z,r/4)))\big\| \leq \|u\circ\varphi \mid L^1_q(\varphi^{-1}(Q(y,r)))\|$$
$$\leq \begin{cases} K_p\omega(Q(y,r))^{\frac{1}{p}} \\ \Psi(Q(y,r))^{\frac{1}{\sigma}}\omega(Q(y,r))^{\frac{1}{p}} \end{cases} \leq \begin{cases} K_p\omega(Q(z,7r/4))^{\frac{1}{p}} & \text{if } q=p, \\ \Psi(Q(z,7r/4))^{\frac{1}{\sigma}}\omega(Q(z,7r/4))^{\frac{1}{p}} & \text{if } q<p \end{cases} \tag{16}$$

for $j = 1, \dots, n$. Since each compact set in D' can be covered by a finitely many cubes of the form $Q(z, r/4)$, we deduce from (16) that

(1) $\varphi_j \in L^1_{q,\mathrm{loc}}(D)$;

(2) $\varphi \in L^1_{q,\mathrm{loc}}(D)$ because the number $1 \leq j \leq n$ is arbitrary.

Verify also that

(3) $\nabla\varphi_j(x) = 0$ at $\mathscr{H}^n$-almost all points of $\varphi^{-1}(E)$, where E is a measure zero set in D' for $1 \leq j \leq n$.

It suffices to see (3) for a measure zero set $E \subset D'$ with $\operatorname{diam} E < \infty$ and $\operatorname{dist}(E, \mathbb{R}^n \setminus D') > 0$. There exists a bounded open set $V \Subset D'$ such that $E \subset V$ and $\mathscr{H}^n(V) < \varepsilon$ for some $\varepsilon > 0$ prescribed in advance. Applying the Besicovitch Covering Theorem [50] to the open set V, by analogy with [57] we find some countable collection $\mathscr{W} = \{Q_k\}$ of cubes $Q_k = Q_k(z_k, r_k)$ such that

(a) $\bigcup_{k=1}^{\infty} Q_k = V$;

(b) for $Q_k = Q_k(z_k, r_k) \in \mathscr{W}$ the condition $|z_k - \mathbb{R}^n \setminus V|_\infty = 32r_k$ holds, where $|x - F|_\infty = \inf_{y\in F}|x-y|_\infty$ is the distance from x to $F \neq \varnothing$;

(c) we can subdivide the family $\mathscr{W}$ into a finite number N_n, depending only on the dimension n, of subfamilies such that in each of them the cubes are disjoint, and a similar property holds for the family $\mathscr{W}^* = \{8Q_k = Q_k(z_k, 8r_k)\}$ of cubes.

In accordance with the last property, we can subdivide the sequence $\{8Q_k\}$ into N_n subfamilies $\{8Q_{1m}\}_{m=1}^{\infty}, \dots, \{8Q_{N_n m}\}_{m=1}^{\infty}$ so that in each subfamily the cubes are disjoint: $8Q_{km} \cap 8Q_{kl} = \varnothing$ whenever $m \neq l$ for $k = 1, \dots, N_n$.

Apply the last property to estimate the left-hand side in (16) for $q = p$:

$$\int\limits_{\varphi^{-1}(V)} |\nabla\varphi_j(x)|^p\,dx \leq \sum_{k\in\mathbb{N}}\int\limits_{\varphi^{-1}(Q_k(z_k,r_k))} |\nabla\varphi_j(x)|^p\,dx \leq K_p^p\sum_{k\in N_n}\sum_{m\in\mathbb{N}}\omega(8Q_{km}) \leq N_nK_p^p\omega(V).$$

Since ε is an arbitrary positive real, while the Lebesgue integral of a summable function is absolutely continuous, (3) in the case $q = p$ is established.

In the case $q < p$ we have to apply Hölder's inequality: From (16) we infer that

$$\int\limits_{\varphi^{-1}(V)} |\nabla\varphi_j(x)|^q\,dx \leq \sum_{k\in\mathbb{N}}\int\limits_{\varphi^{-1}(Q_k(z_k,r_k))} |\nabla\varphi_j(x)|^q\,dx$$
$$\leq \sum_{k\in N_n}\sum_{m\in\mathbb{N}}\Psi(8Q_{km})^{\frac{q}{\sigma}}\omega(8Q_{km})^{\frac{q}{p}} \leq \sum_{k\in N_n}\Big(\sum_{m\in\mathbb{N}}\Psi(8Q_{km})\Big)^{\frac{q}{\sigma}}\Big(\sum_{m\in\mathbb{N}}\omega(8Q_{km})\Big)^{\frac{q}{p}}$$
$$\leq \sum_{k\in N_n}\Psi(D')^{\frac{q}{\sigma}}\omega(V)^{\frac{q}{p}} \leq N_n\Psi(D')^{\frac{q}{\sigma}}\omega(V)^{\frac{q}{p}}.$$

For the reason described above, (3) is also established in the case $q < p$.

(4) The mapping φ has finite distortion; i.e., $D\varphi(x) = 0$ at $\mathscr{H}^n$-almost all points of Z because (14) yields $\mathscr{H}^n(\varphi(Z \setminus \Sigma)) = 0$, where $\Sigma \subset D$ is the singularity set of φ of measure zero.

(5) From the inclusions $Q(z,r) \subset T_jQ(y,4r) \subset Q(y,4r) \subset Q(z,7r)$ and (16), using the change-of-variables formula (14) and property (3), we deduce that

$$\int\limits_{Q(z,r)} \frac{|\nabla\varphi_j(\varphi^{-1}(y))|^q}{J(\varphi^{-1}(y),\varphi)}\chi_{D'\setminus\varphi(\Sigma)}(y)\,dy \leq \begin{cases} K_p^p\omega(Q(z,7r)), & 1 \leq q = p < \infty, \\ \Psi(Q(z,7r))^{\frac{q}{\sigma}}\omega(Q(z,7r))^{\frac{q}{p}}, & 1 \leq q < p < \infty, \end{cases}$$

where $J(x,\varphi)$ is the Jacobian of φ. It remains to divide both sides of the last inequality by $\mathscr{H}^n(Q(z,7r))$, recall that $\frac{q}{\sigma}+\frac{q}{p}=1$, and pass to the limit as $r\to 0$ by Proposition 11 and the Lebesgue Differentiation Theorem; see Example 13. In the limit for $\mathscr{H}^n$-almost all $z\in D'\setminus\varphi(\Sigma)$ we obtain the pointwise relation

$$\frac{|\nabla\varphi_j(\varphi^{-1}(z))|}{J(\varphi^{-1}(z),\varphi)^{\frac{1}{q}}\omega(z)^{\frac{1}{p}}}\le\begin{cases}7^{\frac{n}{p}}K_p^p, & 1\le q=p<\infty,\\ 7^{\frac{n}{q}}\Psi'(z)^{\frac{1}{\sigma}}, & 1\le q<p<\infty.\end{cases}$$

Taking the inequality $|D\varphi(x)|\le\sum_{j=1}^n|\nabla\varphi_j(x)|$ into account, we conclude that for $q=p$ the distortion function $D\ni x\mapsto K^{1,\omega}_{p,p}(x,\varphi)$ (cp. (3)) is bounded by the constant $7^{\frac{n}{p}}nK_p^p$, while for $q<p$ we find that

$$\left(\frac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{q}}\omega^{\frac{1}{p}}(\varphi(x))}\right)^{\sigma}|\det D\varphi(x)|\le 7^{\frac{n\sigma}{q}}n^{\sigma}\Psi'(\varphi(x))|\det D\varphi(x)|$$

for almost all points $x\in D\setminus(Z\cup\Sigma)$. Integrating both sides over $D\setminus(Z\cup\Sigma)$ and changing the variable on the right-hand side, by (14) we arrive at

$$\left\|K^{1,\omega}_{q,p}(\cdot)\mid L_\sigma(D)\right\|\le 7^{\frac{n}{q}}n\Psi^{\frac{1}{\sigma}}(D'). \tag{17}$$

Claim (1) of Theorem 18 is proved.

The estimate in (17) means that the hypotheses of claim (3) of Theorem 3 hold. Consequently, Theorem 3 implies claim (2) of Theorem 18. The implication (2) ⇒ (3) in Theorem 18 is the implication (1) ⇒ (2) in Theorem 3; for the details see [12].

From claim (3) of Theorem 18 we deduce claim (4).

Claim (5) of Theorem 18 can be justified in the same fashion as in Theorem 9. □

2. Continuous Discrete Open Mappings of Class $\mathscr{Q}_{q,p}$: Their Properties and Analytical Description

Our next goal is to obtain the properties of $f\in S\mathscr{R}Q_{q,p}(D',D;\omega)$ and $f\in\mathscr{R}Q_{q,p}(D',D;\omega)$ which follow directly from Definition 5.

In order to prove the theorems of this section, we need some auxiliary statements established in the following subsection.

2.1. Auxiliary relations. Every locally summable function $\omega: D'\to(0,\infty)$ with $D'\subset\mathbb{R}^n$ determines the weighted measure of measurable sets $A\subset D'$ as

$$\omega(A)=\int\limits_A\omega(y)\,dy.$$

Lemma 19. *For $1\le p<\infty$ each condenser $E=(F,U)$ in D' satisfies the upper bound*

$$\operatorname{cap}\bigl(E;L^1_p(U;\omega)\bigr)\le\frac{\omega(U\setminus F)}{\operatorname{dist}(F,\partial U)^p},$$

where $\operatorname{dist}(F,\partial U)$ is the Euclidean distance between F and the boundary of U.

Proof. Put $r=\operatorname{dist}(F,\partial U)$. As an admissible function for the capacity $\operatorname{cap}\bigl(E;L^1_p(U;\omega)\bigr)$ take

$$u(y)=\max\left(0,1-\frac{\operatorname{dist}(y,F)}{r}\right).$$

Indeed, $u(y)=1$ at $y\in F$ and $u(y)=0$ at $y\notin U$, while $|\nabla u(y)|\le r^{-1}$ for almost all $y\in U$. Consequently,

$$\operatorname{cap}\bigl(E;L^1_p(U;\omega)\bigr)\le\int\limits_{U\setminus F}|\nabla u(y)|^p\omega(y)\,dy\le r^{-p}\omega(U\setminus F).\quad\square$$

Lemma 20. *Assume that $n-1<q<\infty$ if $n\geq 3$ and $1\leq q<\infty$ if $n=2$. Every condenser $E=(F,U)$ in D with a connected set F satisfies*

$$\operatorname{cap}^{n-1}\left(E;L^1_q(U)\right)\geq c_1^{n-1}\frac{(\operatorname{diam}F)^q}{\mathscr{H}^n(U)^{q-(n-1)}},\tag{18}$$

where c_1 is the constant in Morrey's inequality (see (19)) which depends only on n and q.

Proof. If $n=2$ and $q=1$ then (18) is a corollary of the following property: *The 1-capacity of an arbitrary condenser $E=(F,U)\subset\mathbb{R}^2$ equals $\operatorname{cap}\left(E;L^1_1(U)\right)=\inf_\gamma\mathscr{H}^1(\gamma)$, where $\mathscr{H}^1(\gamma)$ stands for the $\mathscr{H}^1$-measure, or the length, of a smooth closed curve $\gamma\subset U$ the bounded connected component $\mathbb{R}^2\setminus\gamma$ of whose complement includes F, and the lower bound is taken over all such curves γ* [51]. It is obvious that $\inf_\gamma\mathscr{H}^1(\gamma)\geq\operatorname{diam}F$.

Now we apply a modified form of the method of [18, Lemma 5]. Since both sides of the sought inequality are invariant under motions and have the same homogeneous degrees with respect to homothety, it suffices to prove the lemma in the case that $\operatorname{diam}F$ equals the distance between the two points $0,T\in F$, where $T=(0,0,\dots,0,1)$ lies on the axis x_n.

Thus, $\operatorname{diam}F=|T|=1$. Consequently, each plane P_A of dimension $n-1$ orthogonal to the axis x_n and passing through some point $A=(0,0,\dots,0,a_n)$ with $0<a_n<1$ crosses F at some point x_A.

Denote by

$$B_A=B(x_A,\operatorname{dist}(x_A,(\mathbb{R}^n\setminus U)\cap P_A))$$

the maximal $(n-1)$-dimensional ball centered at x_A lying in $U\cap P_A$.

Every function $u\in L^1_p(U)\cap\operatorname{Lip}_l(U)$ with $u=1$ on F whose support lies in U takes the value 0 on the sphere

$$S(x_A,\operatorname{dist}(x_A,(\mathbb{R}^n\setminus U)\cap P_A))\cap P_A.$$

Therefore, expressing $x\in P_A$ as $x=(\xi,a_n)$, use Morrey's inequality [50, Subsection 4.5.3] for $\mathscr{H}^1$-almost all $a_n\in(0,1)$ in the form

$$\int\limits_{P_A\cap U}|\nabla u(\xi,a_n)|^q\,d\xi\geq c_1\mathscr{H}^{n-1}(B_A)^{1-\frac{q}{n-1}},\tag{19}$$

where the Hausdorff measure $\mathscr{H}^{n-1}(B_A)$ coincides with the $(n-1)$-dimensional Lebesgue measure of B_A, while c_1 is a constant depending only on n and q. Applying Hölder's inequality with exponents $\frac{q}{q-(n-1)}$ and $\frac{q}{n-1}$ to the second integral in the first line, we infer that

$$\begin{aligned}(\operatorname{diam}F)^q&=\Bigg(\int\limits_0^1 da_n\Bigg)^q=\Bigg(\int\limits_0^1\mathscr{H}^{n-1}(B_A)^{\frac{q-(n-1)}{q}}\cdot\mathscr{H}^{n-1}(B_A)^{\frac{n-1-q}{q}}\,da_n\Bigg)^q\\&\leq\Bigg(\int\limits_0^1\mathscr{H}^{n-1}(B_A)\,da_n\Bigg)^{q-(n-1)}\Bigg(\int\limits_0^1\mathscr{H}^{n-1}(B_A)^{1-\frac{q}{n-1}}\,da_n\Bigg)^{n-1}&(20)\\&\leq\frac{1}{c_1^{n-1}}\mathscr{H}^n(U)^{q-(n-1)}\Bigg(\int\limits_U|\nabla u(x)|^q\,dx\Bigg)^{n-1}.&(21)\end{aligned}$$

While passing from (20) to (21), we apply the Cavalieri–Lebesgue formula to find an upper bound on the first integral in (20). This yields (18). ☐

2.2. Properties of continuous discrete open mappings of class $S\mathscr{R}Q_{q,p}$. Our main goal in this subsection is to obtain the properties of $f\in S\mathscr{R}Q_{q,p}(D',D;\omega)$ which are straightforward from Definition 5.

Theorem 21. *Assume that $n-1<q<\infty$ if $n\geq 3$ and $1\leq q<\infty$ if $n=2$. Every continuous discrete open mapping $f: D'\to D$ of class $S\mathscr{R}Q_{q,p}(D',D;\omega)$ with $q\leq p<\infty$ enjoys the following properties:*

(1) *f is differentiable $\mathscr{H}^n$-almost everywhere in the domain D';*

(2) *f has finite distortion;*

(3) *we have the pointwise estimate*

$$|Df(y)|\leq c_2\begin{cases} K_p^{n-1}\cdot|\det Df(y)|^{\frac{p-(n-1)}{p}}\omega(y)^{\frac{n-1}{p}} & \text{for } q=p,\\ \left(\Psi'_{q,p}(y)\right)^{\frac{n-1}{\sigma}}|\det Df(y)|^{\frac{q-(n-1)}{q}}\omega(y)^{\frac{n-1}{p}} & \text{for } q<p\end{cases}\tag{22}$$

for $\mathscr{H}^n$-almost all $y\in D'$ with the constant $c_2=2^n\alpha(n)c_1^{\frac{1-n}{q}}$, where

$$\alpha(n)=\mathscr{H}^n(B(0,1))=\Gamma\left(\frac{1}{2}\right)^n/\Gamma\left(\frac{n}{2}+1\right)$$

and $\frac{1}{\sigma}=\frac{1}{q}-\frac{1}{p}$, and for every open set $U\subset D'$ the ensuing relations

$$\int\limits_U|Df(y)|\,dy\leq c_3\cdot\begin{cases} K_p^{n-1}\cdot\mathscr{H}^n(f(U))^{\frac{p-(n-1)}{p}}\cdot\omega(U)^{\frac{n-1}{p}} & \text{for } q=p,\\ \Psi_{q,p}(U)^{\frac{n-1}{\sigma}}\cdot\mathscr{H}^n(f(U))^{\frac{q-(n-1)}{q}}\cdot\omega(U)^{\frac{n-1}{p}} & \text{for } q<p\end{cases}\tag{23}$$

with the constant $c_3=c_2\cdot\mathscr{N}(f,U)^{\frac{q-(n-1)}{q}}$.

Proof. **I.** At step 1 we establish that f is differentiable.

Use the scheme of proof in [58] for the case $q=p=n$, see [59, Lemma 1] for $n-1<q<p=n$ and $\omega\equiv 1$; a different method is obtained in [9, Theorem 2]. Associate to each point $y\in D'$ some spherical condenser $E_r=(\overline{B(y,r)},B(y,2r))$ with $B(y,2r)\subset D'$. Considering the definition of $\mathscr{Q}_{q,p}(D',D;\omega)$ for $q<p$ and Lemma 19, we obtain

$$\begin{aligned}\operatorname{cap}^{\frac{1}{q}}\left(f(E_r);L^1_q(f(B(y,2r)))\right)&\leq\Psi_{q,p}(B(y,2r))^{\frac{1}{\sigma}}\operatorname{cap}^{\frac{1}{p}}\left(E_r;L^1_p(U;\omega)\right)\\&\leq\Psi_{q,p}(B(y,2r))^{\frac{1}{\sigma}}\frac{\omega(B(y,2r))^{\frac{1}{p}}}{r}\end{aligned}$$

because the image of a condenser is also a condenser for the class of mappings under consideration; and for $q=p$ we should write K_p instead of $\Psi_{q,p}(B(y,2r))^{\frac{1}{\sigma}}$.

Using Lemma 20 to estimate capacity on the left,

$$\operatorname{cap}^{\frac{1}{q}}\left(f(E_r);L^1_q(f(B(y,2r)))\right)\geq c_1^{\frac{1}{q}}\frac{(\operatorname{diam}f(\overline{B(y,r)}))^{\frac{1}{n-1}}}{\mathscr{H}^n(f(B(y,2r)))^{\frac{q-(n-1)}{q(n-1)}}},$$

we infer that

$$\frac{\operatorname{diam}f(\overline{B(y,r)})}{r}\leq\frac{2^nc_1^{\frac{1-n}{q}}}{(2r)^n}\Psi_{q,p}(B(y,2r))^{\frac{n-1}{\sigma}}\mathscr{H}^n(f(B(y,2r)))^{\frac{q-(n-1)}{q}}\omega(B(y,2r))^{\frac{n-1}{p}}.$$

Letting $r\to 0$, for $\mathscr{H}^n$-almost all $y\in D'$ we obtain

$$\varlimsup_{z\to y}\frac{|f(z)-f(y)|}{|z-y|}\leq 2^n\alpha(n)c_1^{\frac{1-n}{q}}\left(\Psi'_{q,p}(y)\right)^{\frac{n-1}{\sigma}}\overline{\mathscr{V}}'_n(y)^{\frac{q-(n-1)}{q}}\omega(y)^{\frac{n-1}{p}},\tag{24}$$

where the values of $\overline{\mathscr{V}}'_n(y)$ and $\Psi'_{q,p}(y)$, appearing in (12) and Proposition 11 part **I**, are finite $\mathscr{H}^n$-almost everywhere in D'. Since the right-hand side of (24) is finite $\mathscr{H}^n$-almost everywhere in D', by

Stepanov's Theorem (see [49, 50] for instance), the mapping f is differentiable $\mathscr{H}^n$-almost everywhere in D'. It is known that at the differentiability points of f the left-hand side of (24) equals $|Df(y)|$, while $\overline{\mathscr{V}}'_n(y) = |\det Df(y)|$; see [49] for instance.

In (24), as well as (26) and the inequality in part **II** of the proof below, in the case $q = p$ we should write K_p^{n-1} instead of $(\Psi'_{q,p}(\cdot))^{\frac{n-1}{\sigma}}$.

II. Appreciating the above, rearrange (24) as

$$|Df(y)| \leq c_2(\Psi'_{q,p}(y))^{\frac{n-1}{\sigma}} |\det Df(y)|^{\frac{q-(n-1)}{q}} \omega(y)^{\frac{n-1}{p}},$$

where $c_2 = 2^n \alpha(n) c_1^{\frac{1-n}{q}}$. This yields (22). Furthermore, $Df(y) = 0$ obviously holds $\mathscr{H}^n$-almost everywhere on the zero set $Z' = \{y \in D' : \det Df(y) = 0\}$ of the Jacobian $\det Df(y)$. Consequently, f has finite distortion.

III. To prove (23), we have to integrate (22) and apply Hölder's inequality while remembering that $\frac{p-(n-1)}{p} + \frac{n-1}{p} = 1$ for $q = p$ and $\frac{n-1}{\sigma} + \frac{q-(n-1)}{q} + \frac{n-1}{p} = 1$ for $q < p$. This yields

$$\int\limits_U |Df(y)|\, dy \leq c_2 \biggl(\int\limits_U |\det Df(y)|\, dy \biggr)^{\frac{q-(n-1)}{q}} \biggl(\int\limits_U \Psi'_{q,p}(y)\, dy \biggr)^{\frac{n-1}{\sigma}} \biggl(\int\limits_U \omega(y)\, dy \biggr)^{\frac{n-1}{p}}.$$

Since $\int_U |\det Df(y)|\, dy \leq \mathscr{N}(f, U)\mathscr{H}^n(f(U))$ and $\int_U \Psi'_{q,p}(y)\, dy \leq \Psi_{q,p}(U)$ (see Example 12 and Proposition 11), we obtain (23).

Thus, $|Df(y)|$ is locally summable on D'. $\square$

Remark 22. The above proof uses instead of (9) the weaker relations

$$\begin{cases} \operatorname{cap}^{\frac{1}{p}}\bigl(f(E_r); L^1_p(D)\bigr) \leq K_p A(r,\omega) & \text{if } q = p, \\ \operatorname{cap}^{\frac{1}{q}}\bigl(f(E_r); L^1_q(D)\bigr) \leq \Psi_{q,p}(B(y,2r) \setminus \overline{B(y,r)})^{\frac{1}{\sigma}} A(r,\omega) & \text{if } q < p, \end{cases}$$

where

$$A(r,\omega) = \biggl(\int\limits_{B(y,2r)} |\nabla u(y)|^p \omega(y)\, dy \biggr)^{\frac{1}{p}}, \quad E_r = (\overline{B(y,r)}, B(y,2r)),$$

while $u(y) = \max\bigl(0, 1 - \frac{\operatorname{dist}(y, \overline{B(y,r)})}{r}\bigr)$ is a test function for an upper bound on the capacity $\operatorname{cap}^{\frac{1}{p}}(E_r)$; see Lemma 19.

2.3. Regularity properties of continuous discrete open mappings of class $\mathscr{R}Q_{q,p}$. In this subsection we continue studying the regularity properties of the mappings of class $\mathscr{R}Q_{q,p}$ and point out conditions under which $f \in \mathscr{R}Q_{q,p}(D', D; \omega)$ belongs to a Sobolev class. Observe that the method of proof generalizes the classical approach and was used several times by many authors in particular cases; e.g., see the weightless case in [58] for $q = p = n$, and in [59] for $n - 1 < q < p = n$, as well as the weighted case in [60] for $q = p = n = 2$, in [61] for $q = p = n$, in [62] for $n - 1 < q = p < \infty$, and so forth.

Theorem 23. *Assume that $n - 1 < q < \infty$ if $n \geq 3$ and $1 \leq q < \infty$ if $n = 2$. Every continuous discrete open mapping $f : D' \to D$ in the family $\mathscr{R}Q_{q,p}(D', D; \omega)$ with $q \leq p < \infty$ enjoys the properties:*

(1) *f belongs to $W^1_{1,\mathrm{loc}}(D')$;*

(2) *f has finite distortion;*

(3) *f is differentiable $\mathscr{H}^n$-almost everywhere on D';*

(4) *f satisfies the estimates in* (22) *and* (23).

Proof. Claims (2)–(4) in case every mapping $f \in \mathscr{Q}_{q,p}(D', D; \omega)$ for $q \leq p < \infty$, where $n - 1 < q < \infty$ if $n \geq 3$ and $1 \leq q < \infty$ if $n = 2$, are justified in Theorem 21.

It remains to justify claim (1). Verify that $f \in \mathrm{ACL}(D')$. Using the local summability of the partial derivatives (see (24)), we obtain the required containment (cp. the equivalent description of $f \in W^1_{1,\mathrm{loc}}(D')$ in [51, § 1.1.3, Theorems 1 and 2]).

To prove that $f \in \mathrm{ACL}(D')$, take an arbitrary n-dimensional open cube $P \Subset D'$ with edges parallel to coordinate axes and verify, for instance, that f is absolutely continuous on $\mathscr{H}^{n-1}$-almost all sections of P by the straight lines parallel to the axis x_j, for $j = 1, \dots, n$. Since there exists at most countably many cubes of this form, let us prove the absolute continuity of f on the intersections of $\mathscr{H}^{n-1}$-almost all curves parallel to the axis x_j with D'. Since $j = 1, \dots, n$ is arbitrary, the theorem will be established.

Take the projection P_j of P to the subspace $y_j = 0$ and the projection I of P to the coordinate axis y_j. Then $P = P_j \times I = \{(z, y_j) : z \in P_j,\ y_j \in I\}$.

The quasiadditive function $\Psi_{q,p}$ in Definition 5 of $f \in \mathscr{R}Q_{q,p}(D', D; \omega)$ induces the bounded quasiadditive function $\Psi_{q,p}(A, P)$ of open sets $A \subset \mathscr{O}(P_j)$ with $A \times I \in \mathscr{O}(P)$ as

$$\mathscr{O}(P_j) \ni A \mapsto \Psi_{q,p}(A, P) = \Psi_{q,p}(A \times I);$$

see Definition 6. By Proposition 11, for $\mathscr{H}^{n-1}$-almost all points $z \in P_j$; i.e., for all points $z \in P_j \setminus \Sigma_0$, where $\Sigma_0 \subset P_j$ is some set of $\mathscr{H}^{n-1}$-measure zero, the derivative

$$\Psi'_{q,p}(z, P) = \lim_{r \to 0} \frac{\Psi_{q,p}\left(B_j(z, r), P\right)}{\mathscr{H}^{n-1}(B_j(z, r))}$$

exists and is finite, where $B_j(z, r)) \subset P_j$ stands for the $(n-1)$-dimensional ball of radius r centered at z.

The set function $\mathscr{V}_n$: $\mathscr{O}(P) \ni G \mapsto \mathscr{H}^n(f(G))$ is a bounded monotone κ-quasiadditive function defined on the open sets $G \in \mathscr{O}(P)$, and $\mathscr{V}_n$ induces the monotone κ-quasiadditive function

$$\mathscr{O}(P_j) \ni A \mapsto \mathscr{V}_n(A, P) = \mathscr{V}_n(A \times I) = \mathscr{H}^n(f(A \times I))$$

of the open sets $A \subset \mathscr{O}(P_j)$ (while $A \times I \in \mathscr{O}(P)$) with the constant $\kappa = \mathscr{N}(f, P)$. Proposition 11 yields $\overline{\mathscr{V}}'_n(z, P) < \infty$ for $\mathscr{H}^{n-1}$-almost all $z \in P_j$ such that $\overline{\mathscr{V}}'_n(z, P) < \infty$ at all $z \in P_j \setminus \Sigma'$, where $\Sigma' \subset P_j$ is some set of $\mathscr{H}^{n-1}$-measure zero.

On the cross-section $I_z = \{z\} \times I$ of the cube, P takes arbitrary disjoint segments $\Delta_1, \Delta_2, \dots, \Delta_k$ of length $b_1, b_2, \dots, b_k$ respectively with rational endpoints. It is obvious that the collection of all these segments is *countable*.

Denote the open set $\bigcup_{y \in \Delta_i} B(y, r)$ by U_i. Choose $r > 0$ so that the open sets $U_1, U_2, \dots, U_k$ are disjoint and $U_i \subset P$ for $i = 1, \dots, k$.

Consider the condensers $E_i = (\Delta_i, U_i)$. Then Lemma 19 yields

$$\operatorname{cap}\left(E_i; L^1_p(U_i; \omega)\right) \le \frac{\omega(U_i)}{r^p} = \frac{\int\limits_{U_i} \omega(y)\, dy}{r^p}, \quad i = 1, \dots, k.$$

On the other hand, for $n - 1 < q < \infty$ Lemma 20 implies that

$$\operatorname{cap}^{\frac{n-1}{q}}\left(f(E_i); L^1_q(U_i)\right) \ge c_1^{\frac{n-1}{q}} \frac{\operatorname{diam} f(\Delta_i)}{\mathscr{H}^n(f(U_i))^{\frac{q-(n-1)}{q}}}.$$

From these two inequalities and the condition $f \in \mathscr{R}Q_{q,p}(D', \omega)$ we infer that

$$\operatorname{diam} f(\Delta_i) \le \frac{c_1^{\frac{1-n}{q}}}{r^{n-1}} \mathscr{H}^n(f(U_i))^{\frac{q-(n-1)}{q}} \left(\Psi_{q,p}\left(U_i\right)\right)^{\frac{n-1}{\sigma}} \omega(U_i)^{\frac{n-1}{p}}.$$

Summing over $i = 1, \dots, k$, applying Hölder's inequality, and using the properties of quasiadditive functions, we arrive at

$$\sum_{i=1}^{k} \operatorname{diam} f(\Delta_i) \le \frac{c_1^{\frac{1-n}{q}}}{r^{n-1}} \Bigl(\sum_{i=1}^{k} \mathscr{H}^n(f(U_i)) \Bigr)^{\frac{q-(n-1)}{q}} \Bigl(\sum_{i=1}^{k} \Psi_{q,p}(U_i) \Bigr)^{\frac{n-1}{\sigma}} \Bigl(\sum_{i=1}^{k} \omega(U_i) \Bigr)^{\frac{n-1}{p}}$$

$$\le c_4 \left(\frac{\mathscr{V}_n(B_j(z,r),P)}{\mathscr{H}^{n-1}(B_j(z,r))} \right)^{\frac{q-(n-1)}{q}} \left(\frac{\Psi_{q,p}(B_j(z,r),P)}{\mathscr{H}^{n-1}(B_j(z,r))} \right)^{\frac{n-1}{\sigma}} \left(\frac{\sum_{i=1}^{k} \omega(U_i)}{\mathscr{H}^{n-1}(B_j(z,r))} \right)^{\frac{n-1}{p}}, \tag{25}$$

where $c_4 = 2^{n-1}\alpha(n-1)c_1^{\frac{1-n}{q}} \cdot \mathscr{N}(f,P)^{\frac{q-(n-1)}{q}}$, while $\alpha(n-1) = \mathscr{H}^{n-1}(B_j(0,1))$.

Letting r tend to 0, we obtain the following inequality whose validity for $\mathscr{H}^{n-1}$-almost all $z \in P_j$ is guaranteed by the existence of limits in the three expressions in parentheses in (25) for $\mathscr{H}^{n-1}$-almost all $z \in P_j$:

$$\sum_{i=1}^{k} \operatorname{diam} f(\Delta_i) \le c_4 (\overline{\mathscr{V}}'_n(z,P))^{\frac{q-(n-1)}{q}} (\Psi'_{q,p}(z,P))^{\frac{n-1}{\sigma}} \Biggl(\int\limits_{\bigcup_{i=1}^{k} \Delta_i} \omega(z,y_n)\, dy_j \Biggr)^{\frac{n-1}{p}}. \tag{26}$$

The first quotient in parentheses in (25) has finite upper limit at all $z \in P_j \setminus \Sigma'$. The second quotient has finite limit at all $z \in P_j \setminus \Sigma_0$ (see Proposition 11); here $\mathscr{H}^{n-1}(\Sigma_0) = \mathscr{H}^{n-1}(\Sigma') = 0$. The third expression in parentheses in (25) also has finite limit for $\mathscr{H}^{n-1}$-almost all $z \in P_j$. In order to verify this, consider any term in the third expression in parentheses in (26), for instance, with index i; the existence of the limit for each term implies the same for the sum of finitely many terms. Recall that $U_i = \bigcup_{y \in \Delta_i} B(y,r)$. For this reason,

$$U_i \subset (B_j(z,r) \times \Delta_i) \cup (B_j(z,r) \times \alpha) \cup (B_j(z,r) \times \beta), \tag{27}$$

where α and β in (27) are the length r subintervals of the interval $(\{z\} \times I) \cap U_i$ complementary to Δ_i. Applying Fubini's Theorem, express the term chosen above as

$$\frac{\omega(U_i)}{\mathscr{H}^{n-1}(B_j(z,r))} = \frac{1}{\mathscr{H}^{n-1}(B_j(z,r))} \int\limits_{U_i} \omega(y)\, dy = \int\limits_{\Delta_i} \omega(z,y_j)\, dy_j \tag{28}$$

$$+ \frac{1}{\mathscr{H}^{n-1}(B_j(z,r))} \Biggl(\int\limits_{B_j(z,r)} \Bigl(\int\limits_{\Delta_i} \omega(w,y_j)\, dy_j - \int\limits_{\Delta_i} \omega(z,y_j)\, dy_j \Bigr) dw \Biggr) + \mathfrak{R}(r) \tag{29}$$

as $r \to 0$; by Fubini's Theorem, for all $z \in P_j \setminus D_j$, with $\mathscr{H}^{n-1}(D_j) = 0$, the integral $\int_{z \times I} \omega(z,y_j)\, dy_j$ exists, which at the same points $z \in P_j$ ensures that the integral on the right-hand side of (28) is finite. By the Lebesgue Differentiation Theorem the first expression in (29) vanishes[6)] for $\mathscr{H}^{n-1}$-almost all $z \in P_j$; i.e., for all $z \in P_j \setminus \Sigma_i$ outside some set $\Sigma_i \subset P_j$ of $\mathscr{H}^{n-1}$-measure zero. The remainder $\mathfrak{R}(r)$ in (29) is nonnegative and contains two terms that are dominated by the sum

$$\frac{1}{\mathscr{H}^{n-1}(B_j(z,r))} \Biggl(\int\limits_{B_j(z,r)} \Bigl(\int\limits_{\alpha} \omega(w,y_j)\, dy_j + \int\limits_{\beta} \omega(w,y_j)\, dy_j \Bigr) dw \Biggr),$$

[6)] Here the Lebesgue Differentiation Theorem is applied in the form

$$\lim_{r \to 0} \frac{1}{\mathscr{H}^{n-1}(B_j(z,r))} \int\limits_{B_j(z,r)} \Bigl| \int\limits_{\Delta_i} \omega(w,y_j)\, dy_j - \int\limits_{\Delta_i} \omega(z,y_j)\, dy_j \Bigr| dw = 0$$

for $\mathscr{H}^{n-1}$-almost all $z \in P_j$.

where α and β are the length r subintervals of the interval $(\{z\} \times I) \cap U_i$ complementary to Δ_i. Since the integral in the inner parentheses vanishes as $r \to 0$, it follows that $\mathfrak{R}(r) = o(1)$ as $r \to 0$.

Since the collection $\{\Delta_i\}$ of segments is at most countable, we see that (26) is justified for all $z \in P_j \setminus \bigl(D_j \cup \Sigma' \cup \bigcup_{i=0}^{\infty} \Sigma_i\bigr)$, where $D_j \cup \Sigma' \cup \bigcup_{i=0}^{\infty} \Sigma_i \subset P_j$ is of $\mathscr{H}^{n-1}$-measure zero.

Moreover, (26) shows that the absolute continuity of $f : \{z\} \times I \to D$ for z fixed is guaranteed by that of the integral $\int_{\{z\}\times I} \omega(z, y_j)\, dy_j$ on the interval I. Consequently, we can extend (26) to every finite, and consequently every countable, collection of segments $\{\Delta_i\}$, not necessarily with rational endpoints.

Since j can be any positive integer from 1 to n, the absolute continuity of $f : D' \to D$ is established. With (23), this also implies that $f \in W^1_{1,\mathrm{loc}}(D')$; see the details in [51].[7] $\square$

3. New Examples of Homeomorphisms of Class $\mathscr{Q}_{q,p}$

In this section we add to Theorem 9 the new examples of $\mathscr{Q}_{q,p}(D', D; \omega)$-homeomorphisms and establish some new properties of the latter.

Example 24. Consider a homeomorphism $\varphi : D \to D'$ from $W^1_{p,\mathrm{loc}}(D)$, where $1 < p < \infty$ for $n \geq 3$ and $1 \leq p < \infty$ for $n = 2$, with finite distortion. The inverse homeomorphism $f = \varphi^{-1} : D' \to D$ belongs to $\mathscr{Q}_{p,p}(D', D; \omega)$ with the constant $K_p = 1$ and weight function (30); see below.

In order to verify the validity of Example 24, we establish a few properties of interest in their own rights.

Theorem 25. *Consider a homeomorphism $\varphi : D \to D'$ from $W^1_{p,\mathrm{loc}}(D)$, where $1 \leq p < \infty$, with finite distortion. Then the weight function defined by the relation*

$$D' \ni y \mapsto \omega(y) = \begin{cases} \frac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|} & \text{if } y \in D' \setminus (Z' \cup \Sigma'), \\ 1 & \text{otherwise} \end{cases} \tag{30}$$

is locally summable, $\omega \in L_{1,\mathrm{loc}}(D')$, and the composition operator

$$\varphi^* : L^1_p(D'; \omega) \cap \mathrm{Lip}_l(D') \to L^1_p(D), \quad 1 \leq p < \infty, \tag{31}$$

is bounded; furthermore $\|\varphi^*\| \leq \bigl\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\bigr\| = 1$.

Proof. Take a homeomorphism $\varphi : D \to D'$ from $W^1_{p,\mathrm{loc}}(D)$, where $1 \leq p < \infty$, having finite distortion. Let us study the conditions on the weight $\omega : D' \to (0, \infty)$ which ensure that (31) is a bounded operator.

Suppose that there exists a locally summable weight $\omega : D' \to (0, \infty)$ such that φ induces the bounded composition operator $\varphi^* : L^1_p(D'; \omega) \cap \mathrm{Lip}_l(D') \to L^1_p(D)$, where $1 \leq p < \infty$. Then $\varphi : D \to D'$ satisfies claims (1) and (3) of Theorem 3, and so the operator distortion function

$$D \ni x \mapsto K^{1,\omega}_{p,p}(x, \varphi) = \begin{cases} \frac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{p}} \omega^{\frac{1}{p}}(\varphi(x))} & \text{if } \det D\varphi(x) \neq 0, \\ 0 & \text{if } \det D\varphi(x) = 0 \end{cases}$$

(see (3)) belongs to $L_\infty(D)$. The estimate $\|\varphi^*\| \leq \bigl\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\bigr\|$ follows from (4). The condition $\bigl\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\bigr\| = 1$ guarantees obviously that $\|\varphi^*\| \leq 1$. In other words, these relations imply the

[7] As [51, § 1.1.3, Theorems 1 and 2] shows, a locally summable function $u : \Omega \to \mathbb{R}$ belongs to $L^1_p(\Omega)$ with $p \geq 1$ if and only if u can be changed on a set of $\mathscr{H}^n$-measure zero so that the modified function is absolutely continuous on $\mathscr{H}^{n-1}$-almost all straight lines parallel to each coordinate axis, and has ordinary partial derivatives belonging to $L_p(\Omega)$. Furthermore, the weak gradient ∇u of u in the sense of generalized functions coincides $\mathscr{H}^n$-almost everywhere with the ordinary gradient.

equalities $K^{1,\omega}_{p,p}(x,\varphi) = 0$ on Z and $K^{1,\omega}_{p,p}(x,\varphi) = 1$ for $\mathscr{H}^n$-almost all $x \in D \setminus Z$. Consequently, on $D' \setminus (Z' \cup \Sigma')$ we can take as the weight the measurable function

$$\omega(y) = \frac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|}, \tag{32}$$

defined for $\mathscr{H}^n$-almost all $y \in D' \setminus (Z' \cup \Sigma')$.

By (14),

$$\int\limits_{\varphi(W)\setminus(Z'\cup\Sigma')} \omega(y)\,dy = \int\limits_{\varphi(W)\setminus(Z'\cup\Sigma')} \frac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|}\,dy = \int\limits_{W\setminus(Z\cup\Sigma)} |D\varphi(x)|^p\,dx < \infty$$

for a compactly embedded domain $W \Subset D$. Observe that these relations impose no constraints on the behavior of (31) on $Z' = \varphi(\Sigma)$.

Using the above, define the weight in the case $|Z'| > 0$ according to (30). For this choice, the weight $\omega : D' \to (0,\infty)$ is locally summable, $\omega \in L_{1,\mathrm{loc}}(D')$, while the outer operator distortion function equals

$$D \ni x \mapsto K^{1,\omega}_{p,p}(x,\varphi) = \begin{cases} 1 & \text{if } \det D\varphi(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, (3) is met, and therefore so is the equivalent condition (1) of Theorem 3. Moreover, we have the estimate $\|\varphi^*\| \leq \left\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\right\| = 1$ for the composition operator. $\square$

Theorems 3 and 9 imply the following statement:

Corollary 26. *Consider a homeomorphism $\varphi : D \to D'$ from $W^1_{p,\mathrm{loc}}(D)$, where $1 < p < \infty$ for $n \geq 3$ and $1 \leq p < \infty$ for $n = 2$, having finite distortion. Then the inverse mapping $f = \varphi^{-1}$ belongs to $\mathscr{Q}_{p,p}(D', D; \omega)$.*

Proof. By Theorem 9, $f = \varphi^{-1}$ belongs $\mathscr{Q}_{p,p}(D', D; \omega)$ for $1 < q$. Observe that the proof of the implication $1 \Rightarrow 2$ in Theorem 3 obtained in [10, 12] for $1 < q$ works for $n = 2$ and $q = 1$ as well. $\square$

Example 24, Theorems 21, 23, and 25, together with Corollary 26 lead to the following statement of the regularity properties of a homeomorphism whose inverse $\varphi : D \to D'$ belongs to $W^1_{p,\mathrm{loc}}(D)$ and has finite distortion.

Theorem 27. *Consider a homeomorphism $\varphi : D \to D'$ from $W^1_{p,\mathrm{loc}}(D)$, where $n - 1 < p < \infty$ for $n \geq 3$ and $1 \leq p < \infty$ for $n = 2$, having finite distortion. Then the inverse homeomorphism $f = \varphi^{-1} : D' \to D$ enjoys the properties:*

(1) f belongs to $W^1_{1,\mathrm{loc}}(D')$;
(2) f has finite distortion;
(3) f is differentiable $\mathscr{H}^n$-almost everywhere in D';
(4) for $\mathscr{H}^n$-almost everywhere $y \in D'$ we have the estimate

$$|Df(y)| \leq 2^n \alpha(n) c_1 |\det Df(y)|^{\frac{p-(n-1)}{p}} \omega(y)^{\frac{n-1}{p}}; \tag{33}$$

and also for every open set $U \subset D'$ we have the inequality

$$\int\limits_U |Df(y)|\,dy \leq c_2 \mathscr{H}^n(f(U))^{\frac{p-(n-1)}{p}} \omega(U)^{\frac{n-1}{p}} \tag{34}$$

with the weight function (30) *and the constant $c_2 = 2^n \alpha(n) c_1$, where*

$$\alpha(n) = \mathscr{H}^n(B(0,1)) = \Gamma\left(\frac{1}{2}\right)^n \Big/ \Gamma\left(\frac{n}{2} + 1\right),$$

while c_1 is the constant in (20).

PROOF. Indeed, Example 24 and Corollary 26 show that under the hypotheses of Theorem 27 the inverse homeomorphism $f = \varphi^{-1} : D' \to D$ belongs to $\mathscr{Q}_{p,p}(D', D; \omega)$ with the weight function (30) and the constant $K_p = 1$. Thus, Theorem 23 implies claims (1)–(4) of Theorem 27. $\square$

REMARK 28. The presented proof of Theorem 27 is new, although some particular situations had already been considered. Namely, claims (1) and (2) of Theorem 27 were established in [1, Theorems 3.2 and 3.3] for $n = 2$ and $p = 1$, and in [2, Theorem 1.2] for $n \ge 3$ and $p > n - 1$; we can extract claim (3) of Theorem 27 for $n \ge 3$ and $p = n$ from the book [6] which includes a comprehensive bibliography. The method for proving the absolute continuity and differentiability in Theorems 21 and 23 stems essentially from Menshov [4]; see the exposition of his results in [63].

Using Theorem 27, we verify Theorem 3 in the case $1 = q \le p < \infty$ and $n = 2$.

Corollary 29. *Suppose that a homeomorphism $f : D' \to D$ of domains D' and D in $\mathbb{R}^2$ belongs to $\mathscr{Q}_{q,p}(D', D; \omega)$ with $1 = q \le p < \infty$. Then the inverse mapping $\varphi = f^{-1} : D \to D'$ enjoys properties* (1) *and* (3) *of Theorem* 3.

PROOF. By Theorem 23, the homeomorphism $f : D' \to D$ in $\mathscr{Q}_{q,p}(D', D; \omega)$, where $1 \le q \le p < \infty$ and $n = 2$, belongs to $W^1_{1,\text{loc}}(D')$ and has finite distortion. By Theorem 27 applied to the homeomorphism $f : D' \to D$ from $W^1_{1,\text{loc}}(D')$ which has finite distortion (see the definition in (1)), the inverse mapping $\varphi = f^{-1} : D \to D'$ belongs to $W^1_{1,\text{loc}}(D)$ and has finite distortion. Now we have to apply to $\varphi \in W^1_{1,\text{loc}}(D)$ the arguments of [12, Lemma 2.5 and Section 2.5] which imply[8)] that claims (1) and (3) of Theorem 3 are valid for $\varphi : D \to D'$. $\square$

EXAMPLE 30. Suppose that $n - 1 < s < \infty$ and consider a homeomorphism $f : D' \to D$ of open domains $D', D \subset \mathbb{R}^n$, where $n \ge 2$, such that

(1) $f \in W^1_{n-1,\text{loc}}(D')$;

(2) f has finite distortion; i.e., $Df(y) = 0$ $\mathscr{H}^n$-almost everywhere on $Z = \{y \in D' \mid \det Df(y) = 0\}$;

(3) the outer operator distortion function

$$D' \ni y \mapsto K^{1,1}_{n-1,s}(y, f) = \begin{cases} \frac{|Df(y)|}{|\det Df(y)|^{\frac{1}{s}}} & \text{if } \det Df(y) \ne 0, \\ 0 & \text{if } \det Df(y) = 0 \end{cases} \tag{35}$$

belongs to $L_\sigma(D)$, where $\sigma = (n-1)p$ and $p = \frac{s}{s-(n-1)}$.

Then the inverse homeomorphism $\varphi = f^{-1} : D \to D'$ enjoys the following properties:

(4) $\varphi \in W^1_{p,\text{loc}}(D)$, $p = \frac{s}{s-(n-1)}$;

(5) φ has finite distortion,

while $f : D' \to D$

(6) belongs to $\mathscr{Q}_{p,p}(D', D; \omega)$ with the constant $K_p = 1$ and the weight function $\omega \in L_{1,\text{loc}}(D')$ defined as

$$\omega(y) = \begin{cases} \frac{|\operatorname{adj} Df(y)|^p}{|\det Df(y)|^{p-1}} & \text{if } y \in D' \setminus Z', \\ 1 & \text{otherwise}, \end{cases} \tag{36}$$

where $Z' = \{y \in D' : Df(y) = 0\}$.

PROOF. It is known (see [20, Theorem 4]) that if a homeomorphism $f : D' \to D$ meets the above-stated requirements, then the inverse homeomorphism $\varphi = f^{-1} : D \to D'$ enjoys the following properties:

(4) $\varphi \in W^1_{p,\text{loc}}(D)$, $p = \frac{s}{s-(n-1)}$;

(5) φ has finite distortion.

[8)] The arguments in the indicated fragments of [12] are applicable to every homeomorphism $\varphi : D \to D'$ from $W^1_{1,\text{loc}}(D)$ having finite distortion and satisfying (2).

By (35), the function $K^{1,1}_{n-1,s}(y,f)$, where $n-1<s<\infty$, belongs to $L_\sigma(D')$ with $\sigma=(n-1)p$ and $p=\frac{s}{s-(n-1)}$. Consequently, $K^{1,1}_{n-1,s}(y,f)^\sigma$ is summable on $D'\setminus Z'$. Furthermore, it is straightforward that $\frac{(n-1)p}{s}=p-1$, and so the available inequality $|Df(y)|^{n-1}\geq|\operatorname{adj}Df(y)|$ yields

$$K^{1,1}_{n-1,s}(y,f)^\sigma=\frac{(|Df(y)|^{n-1})^p}{|\det Df(y)|^{\frac{(n-1)p}{s}}}\geq\frac{|\operatorname{adj}Df(y)|^p}{|\det Df(y)|^{p-1}}\quad\text{for } y\in D'\setminus Z'. \tag{37}$$

The summability of the left-hand side of (37) implies that of the right-hand side on $D'\setminus Z'$; consequently, weight function (36) is locally summable.

Observe that for $\mathscr{H}^n$-almost all $y\in D'\setminus Z'$ we have

$$\frac{|D\varphi(f(y))|^p}{|\det D\varphi(f(y))|}=\frac{|\operatorname{adj}Df(y)|^p}{|\det Df(y)|^{p-1}}. \tag{38}$$

Thus, the distortion function

$$D\ni x\mapsto K^{1,\omega}_{p,p}(x,\varphi)=\begin{cases}\frac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{p}}\omega^{\frac{1}{p}}(\varphi(x))} & \text{if } \det D\varphi(x)\neq 0,\\ 0 & \text{if } \det D\varphi(x)=0\end{cases} \tag{39}$$

equals 1 $\mathscr{H}^n$-almost everywhere on $\{x\in D:\det D\varphi(x)\neq 0\}$.

Theorems 3 and 9 show that the homeomorphism $\varphi:D\to D'$ has the inverse $f=\varphi^{-1}:D'\to D$ belonging to $\mathscr{Q}_{p,p}(D',D;\omega)$ with the weight function (36). $\square$

Remark 31. Example 30, as well as Corollaries 33 and 34 below, yields the new properties of f and its inverse. Some result close to Example 30 for $s=p=n$ and the weight function $\omega(y)=K^{1,1}_{n-1,n}(y,f)^{(n-1)n}=\big(\frac{|Df(y)|^n}{|\det Df(y)|}\big)^{n-1}$ at $y\in D'\setminus Z'$ instead of the right-hand side of (38) is stated in the language of moduli of families of curves in [64].

Example 32. Suppose that $n-1<s<\infty$ and consider some homeomorphism $f:D'\to D$ of open domains $D',D\subset\mathbb{R}^n$, where $n\geq 2$, such that

(1) $f\in W^1_{n-1,\mathrm{loc}}(D')$;

(2) f has finite codistortion: $\operatorname{adj}Df(y)=0$ holds $\mathscr{H}^n$-almost everywhere on $Z=\{y\in D'\mid \det Df(y)=0\}$;

(3) the inner operator distortion function

$$D'\ni y\mapsto \mathscr{K}^{1,1}_{n-1,s}(y,f)=\begin{cases}\frac{|\operatorname{adj}Df(y)|}{|\det Df(y)|^{\frac{n-1}{s}}} & \text{if } \det Df(y)\neq 0,\\ 0 & \text{if } \det Df(y)=0\end{cases} \tag{40}$$

belongs to $L_p(D')$, where $p=\frac{s}{s-(n-1)}$ and $n-1<s<\infty$.

Then the inverse homeomorphism $\varphi=f^{-1}:D\to D'$ enjoys the following properties:

(4) $\varphi\in W^1_{p,\mathrm{loc}}(D)$, $p=\frac{s}{s-(n-1)}$;

(5) φ has finite distortion;

while the homeomorphism $f:D'\to D$ is such that

(6) f belongs to $\mathscr{Q}_{p,p}(D',D;\omega)$ with the constant $K_p=1$ and weight function (36);

(7) f has finite distortion for $n-1<s<n+\frac{1}{n-2}$.

Proof. It is known (see [20, Theorem 3]) that if a homeomorphism $f:D'\to D$ meets the above-stated requirements then the inverse homeomorphism $\varphi=f^{-1}:D\to D'$ enjoys the properties

(4) $\varphi\in W^1_{p,\mathrm{loc}}(D)$,

(5) φ has finite distortion.

Observe that by (40) the function $\mathscr{K}^{1,1}_{n-1,s}(y,f)$ belongs to $L_p(D')$, where $n-1<s<\infty$. Hence, $\mathscr{K}^{1,1}_{n-1,s}(y,f)^p$ is summable on $D'\setminus Z'$. Furthermore, we verify directly that

$$\mathscr{K}^{1,1}_{n-1,s}(y,f)^p=\left(\frac{|\operatorname{adj} Df(y)|}{|\det Df(y)|^{\frac{n-1}{s}}}\right)^p=\frac{|\operatorname{adj} Df(y)|^p}{|\det Df(y)|^{p-1}}\quad\text{for } y\in D'\setminus Z'. \tag{41}$$

Consequently, although the premises in Example 32 differ from those in Example 30, we arrive at the same weight function (36) which is locally summable by (41) and condition (3) of Example 32.

Observe that (38) holds for $\mathscr{H}^n$-almost all $y\in D'\setminus Z'$. Therefore, the distortion function $D\ni x\mapsto K^{1,\omega}_{p,p}(x,\varphi)$ defined in (39) equals 1 $\mathscr{H}^n$-almost everywhere on $\{x\in D:\det D\varphi(x)\neq 0\}$. Thus, the distortion function $D\ni x\mapsto K^{1,\omega}_{p,p}(x,\varphi)$ is well defined and $\|K^{1,\omega}_{p,p}(\cdot,\varphi)\|=1$.

By claim (1) of Theorem 3, the mapping $\varphi:D\to D'$ induces the bounded composition operator $\varphi^*:L^1_p(D';\omega)\cap \operatorname{Lip}_l(D')\to L^1_q(D)$. From this we infer that $f=\varphi^{-1}:D'\to D$ belongs to $\mathscr{Q}_{p,p}(D',D;\omega)$ with the weight function (36).

The relation $n-1<s<n+\frac{1}{n-2}$ yields $n-1<p$. Consequently, by Theorem 27 the homeomorphism $f:D'\to D$ has finite distortion.

Claims (6) and (7) are also justified. $\square$

Corollary 33. *Every homeomorphism $f:D'\to D$ in Examples* 30 *and* 32 *enjoys the following additional properties:*

(1) (9) *holds for each condenser $E=(F_1,F_0)$ in D';*

(2) *for $n-1<s<n+\frac{1}{n-2}$ the homeomorphism $f:D'\to D$ is differentiable $\mathscr{H}^n$-almost everywhere in D'.*

Proof. Claim (1) follows from Remark 10 and (4). Claim (2) follows from Theorem 27 because $n-1<s<n+\frac{1}{n-2}$ guarantees that $n-1<p$. $\square$

Corollary 34. *Suppose that a homeomorphism $f:D\to D'$ enjoys properties* (1)–(3) *of Examples* 30 *and* 32. *Then*

(1) *under the condition $n-1<s<n+\frac{1}{n-2}$ the restriction of the homeomorphism $f:U'\to U$, where $U'\Subset D'$ is a compactly embedded domain, while $U=f(U')$, induces the bounded composition operator*

$$f^*:L^1_{\frac{p}{p-(n-1)}}(U)\to L^1_1(U'); \tag{42}$$

(2) *under the condition $n\le s<n+\frac{1}{n-2}$ the homeomorphism f enjoys Luzin's $\mathscr{N}^{-1}$-property;*

(3) *under the condition $n\le s<n+\frac{1}{n-2}$ the homeomorphism f has nonzero Jacobian $\mathscr{H}^n$-almost everywhere in D'.*

Proof. (1): Consider a compactly embedded domain $U'\Subset D'$ and some function $u\in\operatorname{Lip}(U)$, where $U=f(U')$.

The condition $n-1<s<n+\frac{1}{n-2}$ yields the inequality $n-1<p<\infty$ for the parameter p. Hence, by the properties stated in Examples 30 and 32, the inverse homeomorphism $\varphi=f^{-1}:D\to D'$ satisfies the hypotheses of Theorem 27. Applying (33) in the subsequent estimates, we obtain

$$\int\limits_{U'} |\nabla(u\circ f)(y)|\,dy \le \int\limits_{U'} |\nabla u|(f(y))|Df(y)|\,dy \tag{43}$$

$$\le c_2 \int\limits_{U'} |\nabla u|(f(y))|\det Df(y)|^{\frac{p-(n-1)}{p}}\omega(y)^{\frac{n-1}{p}}\,dy$$

$$\le c_2\Bigl(\int\limits_{U'} (|\nabla u|(f(y)))^{\frac{p}{p-(n-1)}}|\det Df(y)|\,dy\Bigr)^{\frac{p-(n-1)}{p}}\Bigl(\int\limits_{U'}\omega(y)\,dy\Bigr)^{\frac{n-1}{p}}$$

$$\le c_2\Bigl(\int\limits_{U'}\omega(y)\,dy\Bigr)^{\frac{n-1}{p}}\Bigl(\int\limits_{U}|\nabla u(x)|^{\frac{p}{p-(n-1)}}\,dx\Bigr)^{\frac{p-(n-1)}{p}}, \tag{44}$$

where as in (34) we put $c_2 = 2^n\alpha(n)c_1$. The inequality between (43) and (44) means that $f : U' \to U$ induces the bounded composition operator (42).

(2): The condition $n \le s < n + \frac{1}{n-2}$ yields the inequality $n-1 < p \le n$ for the parameter p. By [19, Theorem 4], the homeomorphism f enjoys Luzin's $\mathscr{N}^{-1}$-property. For the reader's convenience, let us state this here:

Proposition 35 [19, Theorem 4]. *If a measurable mapping $\varphi : D \to D'$ induces the bounded composition operator*

$$\varphi^* : L^1_p\left(D'\right) \cap \mathrm{Lip}_l(D') \to L^1_q(D), \quad 1 \le q \le p \le n,$$

then f enjoys Luzin's $\mathscr{N}^{-1}$-property.

(3) From [19, Corollary 4] we deduce that the Jacobian of f is nonzero $\mathscr{H}^n$-almost everywhere in U'; we deduce this property from property (2) of the corollary using the change-of-variables formula (13). Since $U' \subset D'$ is an arbitrary domain, Corollary 34 is justified. □

Remark 36. The properties of the homeomorphism f and its inverse which are stated in Example 32 and Corollaries 33 and 34 are new with the exception of the differentiability of f in Example 32 for $s = n$ and $n \ge 3$; the case $s = n$ is considered in [65].

4. Regularity of Inverse Homeomorphisms to Sobolev Mappings on a Carnot Group

In this section we generalize Propositions 1 and 2 to homeomorphisms of Carnot groups. The method for proving the ACL-property of quasiconformal mappings on Heisenberg groups was developed in [66, 67] and differs substantially in the details from the classical one available in Euclidean space. This method was later applied to prove the ACL-property of more complicated analytical objects; see [18] for instance among others.

Below we adapt our new proof of Propositions 1 and 2 to demonstrate the validity of their generalizations to Carnot groups. Essentially, we show that on Carnot groups we can successfully apply the arguments stemming from the classical article by Menshov [68].

4.1. Definitions of the main structures on Carnot groups.

4.1.1. A *Carnot group* [69–72] is a connected simply-connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathscr{G}$ decomposes as the direct sum $V_1 \oplus \dots \oplus V_m$ of vector spaces so that $[V_1, V_k] = V_{k+1}$ for $1 \le k \le m-1$ and $[V_1, V_m] = \{0\}$, while $\dim V_1 \ge 2$. Below we use the notation $x \cdot y$ for the product of two elements x and y of a group $\mathbb{G}$ and e for the neutral element of the group. The subspace $V_1 \subset \mathscr{G}$ is called *horizontal*.

4.1.2. The Lie algebra of a Carnot group and the exponential mapping. Take *left-invariant vector fields* $X_{11}, \dots, X_{1n_1}$ constituting a basis for V_1. Since they generate $\mathscr{G}$, for each i with $1 < i \le m$ we can choose a basis X_{ij} for $1 \le j \le n_i = \dim V_i$ in V_i consisting of order $i-1$ commutators of the basis fields $X_{1k} \in V_1$. Since $\mathscr{G}$ is nilpotent, we can identify each element $x \in \mathbb{G}$ with a point of the space $\mathbb{R}^{n_1+\dots+n_m}$ via the exponential mapping:

$$x = \exp\Big(\sum_{\substack{1\le i\le m\\ 1\le j\le n_i}} x_{ij} X_{ij} \Big).$$

The diffeomorphism $\exp : \mathscr{G} \to \mathbb{G}$ provides a global coordinate system: each element $x \in \mathbb{G}$ corresponds to a unique tuple of numbers $\{x_{ij}\} \in \mathbb{R}^N$, where $N = n_1 + \dots + n_m$. Moreover, it is convenient to identify the elements of $\mathscr{G}$ with the points in $\mathbb{R}^N$ so that the exponential mapping $\exp : \mathscr{G} \to \mathbb{G}$ is the identity [71]. The latter means that the elements of the algebra $\mathscr{G}$ and the group $\mathbb{G}$ are the same points in $\mathbb{R}^N$ subjected to the operations depending on the choice of structure. With this choice of a coordinate system the neutral element e of the group is 0, while the inverse x^{-1} to $x \in \mathbb{G}$ is $-x$.

The *dilations* δ_t defined as $x \mapsto \delta_t x = (t^i x_{ij})_{1\le i\le m,\, 1\le j\le n_i}$ are automorphisms of both the algebra $\mathscr{G}$ and the group $\mathbb{G}$ for each $t > 0$.

4.1.3. Example. The Euclidean space $\mathbb{R}^n$ with its standard structure is an example of an abelian group: The vector fields $\frac{\partial}{\partial x_i}, i = 1, \dots, n$, lack nontrivial commutation relations and constitute a basis for the corresponding Lie algebra.

The Heisenberg group $\mathbb{H}^n$ is an example of a nonabelian Carnot group. The Lie algebra of $\mathbb{H}^n$ has dimension $2n+1$. The vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad i = 1, \dots, n,$$

constitute a basis for the Heisenberg algebra; here we identify the Heisenberg group $\mathbb{H}^n$ with the space $\mathbb{R}^{2n+1} = \{(x, y, t) : x, y, \in \mathbb{R}^n, t \in \mathbb{R}\}$. The only nontrivial commutation relations are $[X_i, Y_i] = -4T$ for $i = 1, \dots, n$.

Thus, $V = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ and $V_2 = \operatorname{span}\{T\}$ are one-dimensional vector subspaces. The image $\exp(V_2)$ is the center of $\mathbb{H}^n$. The group operation is defined as

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2\langle y, x'\rangle - 2\langle x, y'\rangle).$$

4.1.4. The metric structures on Carnot groups. A *homogeneous norm* [70, 71] on a group $\mathbb{G}$ is a continuous function $\rho : \mathbb{G} \to [0, \infty)$ with the following properties:

(a) $\rho(x) = 0$ if and only if $x = e$;

(b) $\rho(x) = \rho\left(x^{-1}\right)$ and $\rho\left(\delta_t(x)\right) = t\rho(x)$;

(c) there exists a constant $C > 0$ such that $\rho\left(x \cdot y\right) \le c\left(\rho\left(x\right) + \rho\left(y\right)\right)$ for all $x, y \in \mathbb{G}$.

Naturally, the homogeneous norm is not uniquely determined; however, two arbitrary homogeneous norms ρ_1 and ρ_2 are *equivalent* [70] to each other: there exist two reals $\alpha, \beta \in (0, \infty)$ such that $\alpha \le \rho_1(x)/\rho_2(x) \le \beta$ independently of $x \in \mathbb{G}\backslash\{e\}$.

A homogeneous norm determines a *homogeneous quasimetric*: for two points $x, y \in \mathbb{G}$ put $\rho(x, y) = \rho(x^{-1}y)$. The quasimetric enjoys the following properties implied by the properties (a)–(c) of homogeneous norms:

(a_1) $\rho(x, y) \ge 0$ and $\rho(x, y) = 0$ if and only if $x = y$;

(b_1) $\rho(x, y) = \rho(y, x)$ and $\rho(\delta_t x, \delta_t y) = t\rho(x, y)$;

(c_1) the generalized triangle inequality $\rho\left(x, y\right) \le C\left(\rho\left(x, z\right) + \rho\left(z, y\right)\right)$ holds for all $x, y, z \in \mathbb{G}$ with the constant $C > 0$ from property (c) above.

The equivalence of the homogeneous norms ρ_1 and ρ_2 yields the equivalence of metrics: $\alpha\rho_2(x, y) \le \rho_1(x, y) \le \beta\rho_2(x, y)$ for all $x, y \in \mathbb{G}$.

Given a metric $\rho(x,y)$, we define the spheres $S_\rho(x,t)=\{y\in\mathbb{G}:\rho(x,y)=t\}$ and the balls $B_\rho(x,t)=\{y\in\mathbb{G}:\rho(x,y)<t\}$; furthermore, the spheres are closed and the balls are open in the topology of $\mathbb{G}$.

Now we fix the homogeneous norm of $x=(x_1;\dots;x_i;\dots;x_m)\in\mathbb{R}^N$ with $x_i=(x_{i1},\dots,x_{in_i})\in V_i$, defined as

$$\rho(x)=\max(|x_1|,|x_2|^{\frac{1}{2}},\dots,|x_m|^{\frac{1}{m}}), \tag{45}$$

where $|x_i|=\left(x_{i1}^2+\dots+x_{in_i}^2\right)^{\frac{1}{2}}$ for $x_i\in V_i$, and $i=1,\dots,m$.

Assume that the Lie algebra $\mathscr{G}$ is equipped with an inner product with respect to which the basis left-invariant vector fields $\{X_{ij}\}$ for $1\le i\le m$ and $1\le j\le n_i$ are orthonormal.

The *Carnot–Carathéodory distance* $d(x,y)$ between two points $x,y\in\mathbb{G}$ is the greatest lower bound of the lengths of all *horizontal curves* with endpoints x and y, where the length of a tangent vector is measured by the chosen Riemannian metric on $\mathbb{G}$, while a horizontal curve is a piecewise smooth path whose tangent vector belongs to V_1. We can show that $d(x,y)$ is always a finite left-invariant metric with respect to which the automorphism group δ_t is the group of dilations with the coefficient t, namely, $d(\delta_t x,\delta_t y)=td(x,y)$ [69, 70]. By definition, we put $d(x)=d(0,x)$.

We can show that $d(x)$ is a homogeneous norm; therefore, the distances $d(x,y)$ and $\rho(x,y)$ are equivalent. Denote the sphere and ball of radius $t\ge 0$ in the Carnot–Carathéodory metric by $S_c(0,t)$ and $B_c(0,t)$ respectively.

The equivalence of the metric functions $d(x,y)$ and $\rho(x,y)$ leads to the property that the identity mapping between the metric spaces $(\mathbb{G},d(\cdot,\cdot))$ and $(\mathbb{G},\rho(\cdot,\cdot))$ is a quasi-isometry.

4.1.5. Measures on Carnot groups. Fix the bi-invariant Haar measure on $\mathbb{G}$ which is obtained by transferring the Lebesgue measure from the Lie algebra $\mathscr{G}$ onto the group $\mathbb{G}$ via the exponential mapping; i.e., the Haar measure of a measurable set $A\subset\mathbb{G}$ equals the Lebesgue measure of $\exp^{-1}(A)$ in $\mathscr{G}$; see [70, Proposition 1.2]. Normalize the Haar measure so that the Lebesgue measure of the ball $B_c(0,1)$ equals 1. Observe that we can choose the normalizing factor in the definition of the Hausdorff measure $\mathscr{H}^N(A)$ of a measurable set A in $\mathbb{R}^N$ with the Euclidean metric so that the Hausdorff measure $\mathscr{H}^N(A)$ equals the Lebesgue measure of A. Using that, when we speak below about the Lebesgue measure of some set $A\subset\mathbb{G}$, we mean the Hausdorff measure $\mathscr{H}^N(A)$.

This yields the relation $\mathscr{H}^N(\delta_t A)=t^\nu\mathscr{H}^N(A)$ for each measurable set $A\subset\mathbb{G}$, where the number $\nu=\sum_{i=1}^m in_i$ is called the *homogeneous dimension* of $\mathbb{G}$.

By analogy with the Hausdorff measure on $\mathbb{R}^n$ (see Section 1), consider the Hausdorff measure on the metric space $(\mathbb{G},d)$.

Given $k\ge 0$ and $\delta\in(0,\infty]$, as well as $A\subset\mathbb{G}$, define

$$\mathscr{H}^k_\delta(A)=\frac{\omega_k}{2^k}\inf\Big\{\sum_{i\in\mathbb{N}}(\operatorname{diam}A_i)^k:\operatorname{diam}A_i<\delta, A\subset\bigcup_{i\in\mathbb{N}}A_i\Big\},$$

where ω_k is a normalizing factor and $\operatorname{diam}A_i=\sup\{d(x,y):x,y\in A_i\}$, while the infimum is taken over all countable coverings $\{A_i\}$ of A. If A cannot be covered by a countable collection of sets of these sizes then we put $\mathscr{H}^k_\delta(A)=\infty$. The limit $\mathscr{H}^k(A)=\lim_{\delta\to 0}\mathscr{H}^k_\delta(A)$ exists and is called the *k-dimensional Hausdorff measure* of A on $(\mathbb{G},d)$.

Choose the normalizing factor ω_ν in the definition of Hausdorff measure so that $\mathscr{H}^\nu(B_c(0,1))=1$, where $B_c(0,1)$ is a ball in the Carnot–Carathéodory metric. Then $\mathscr{H}^\nu(B_c(0,r))=r^\nu$. Moreover, if $E\subset\mathbb{G}$ is a measurable set then $\mathscr{H}^\nu(\delta_t E)=t^\nu\mathscr{H}^\nu(E)$. Observe that the homogeneous dimension ν of $\mathbb{G}$ equals the Hausdorff dimension of $(\mathbb{G},d)$.

Owing to the appropriate normalizing factors, the ν-dimensional Hausdorff measure $\mathscr{H}^\nu(A)$ of each measurable set $A\subset\mathbb{G}$ equals the Lebesgue measure $\mathscr{H}^N(A)$ of A.

4.1.6. Horizontal foliations on Carnot groups. Fix $1\le j\le n_1$ and consider a family Γ_j of curves amounting to a smooth foliation of an open set $A\subset\mathbb{G}$. The leaves $\gamma\in\Gamma_j$ are the integral curves of the horizontal vector field $X_{1j}\in V_1$. If we denote the flow corresponding to this field by g_s then the leaf assumes the form $\gamma(s) = g_s(p) = p \exp sX_{1j}$, where p lies on a surface P transversal to the vector field X_{1j}, while the parameter s lies in some interval $I \subset \mathbb{R}$.

Assume that the foliation Γ_j of A is equipped with a measure $d\gamma_j$ satisfying

$$c_7 r^{\nu-1} \le \int\limits_{\gamma\in\Gamma_j,\, \gamma_j\cap B_c(x,r)\neq\varnothing} d\gamma \le c_8 r^{\nu-1} \tag{46}$$

for sufficiently small balls $B_c(x,r) \subset \mathbb{G}$ with constants c_7 and c_8 independent of $B_c(x,r)$. For the foliation determined by the vector field $X_{1j} \in V_1$ we can obtain the measure $d\gamma_j$ as the contraction $i(X_{1j})$ of X_{1j} with the bi-invariant volume form dx; see [67].[9)]

Horizontal foliations on Carnot groups are convenient for defining absolutely continuous functions (or mappings) on Carnot groups: for these, the analog of the expression "a mapping is absolutely continuous on almost all lines parallel to coordinate axes," available in the Euclidean spaces, is the expression "a mapping is absolutely continuous on $d\gamma_j$-almost all lines of the horizontal foliation Γ_j for $1 \le j \le n_1$."

4.1.7. Differentiability on Carnot groups. Consider two Carnot groups $\mathbb{G}$ and $\widetilde{\mathbb{G}}$, as well as a domain D in $\mathbb{G}$. A mapping $\varphi : D \to \widetilde{\mathbb{G}}$ is called *$\mathscr{P}$-differentiable* [69] at $x \in D$ whenever there exists a homomorphism $L : \mathbb{G} \to \widetilde{\mathbb{G}}$ of Carnot groups such that $L(\exp V_1) \subset \exp \widetilde{V}_1$ and the "divided difference"

$$\tilde{\delta}_t^{-1}(\varphi(x)^{-1}\varphi(x\delta_t u)) \quad \text{converges to } L(u) \tag{47}$$

as $t \to 0+$ uniformly in $u \in B_c(0,1)$. Here $\tilde{\delta}_t$ is a one-parameter group of dilations on $\widetilde{\mathbb{G}}$, while $\widetilde{V}_1$ is a horizontal subspace of the Lie algebra $\widetilde{\mathscr{G}}$ of $\widetilde{\mathbb{G}}$.

Pansu proved in [69] that every Lipschitz mapping $\varphi : D \to \widetilde{\mathbb{G}}$ defined on an open set D is $\mathscr{P}$-differentiable $\mathscr{H}^\nu$-almost everywhere in D.

Specifying the convergence in (47) to other topologies, we arrive at the distinct concepts of differentiability. For instance, the convergence in measure on the ball $B_c(0,1)$ leads to the concept of *approximative* differentiability (see [73]), while convergence in the Sobolev space topology leads to differentiability in the Sobolev space topology (see [56]).

4.2. Sobolev classes on Carnot groups. Suppose that D is a domain in $\mathbb{G}$. A locally summable function $f : D \to \mathbb{R}$ belongs to the Sobolev class $L^1_p(D)$ with $p \in [1,\infty]$ whenever the generalized derivatives $X_{1j}f$ for $j = 1,\dots,n$ along the vector fields X_{1j} lie in $L_p(D)$. We endow $L^1_p(D)$, for $p \in [1,\infty]$, with the seminorm

$$\|f \mid L^1_p(D)\| = \biggl(\int\limits_D |\nabla_{\mathscr{L}} f|^p(x)\, d\mathscr{H}^\nu(x)\biggr)^{\frac{1}{p}};$$

the vector $\nabla_{\mathscr{L}} f(x) = (X_{11}f(x),\dots,X_{1n_1}f(x)) \in V_1$ is called the *subgradient* of f.

Given two Carnot groups $\mathbb{G}$ and $\mathbb{G}'$, as well as a domain D in $\mathbb{G}$, we call a mapping $\varphi : D \to \mathbb{G}'$ *absolutely continuous on lines* and write $\varphi \in \mathrm{ACL}(D)$ whenever for every domain U with $\overline{U} \subset D$ and the foliation Γ_j determined by the left-invariant vector field X_{1j} for $j = 1,\dots,n_1$, the mapping φ is absolutely continuous on $\gamma \cap U$ with respect to the Hausdorff $\mathscr{H}^1$-measure for $d\gamma_j$-almost all curves $\gamma \in \Gamma_j$. For a mapping of the class, the derivatives $X_{1j}\varphi$ along the vector fields X_{1j}, for $j = 1,\dots,n_1$, such that $X_{1j}\varphi(x) \in V_1(x)$ exist $\mathscr{H}^\nu$-almost everywhere in D [69, Proposition 4.1].

The matrix $D_h\varphi(x) = (X_{1i}\varphi_{1j}(x))$ with $i = 1,\dots,n_1$ and $j = 1,\dots,n'_1$ defined almost everywhere in D determines the linear operator $D_h\varphi(x) : V_1 \to V'_1$, where $\dim V'_1 = n'_1$, from the horizontal space V_1 into the horizontal space V'_1 of $\mathbb{G}'$, called the *horizontal differential* of φ at x, and $|D_h\varphi(x)|$ stands for the norm of $D_h\varphi(x)$.

[9)] If dx is a volume form on $\mathbb{G}$ of degree N then $i(X_{1j})$ is a form of degree $N-1$, which at the smooth vector fields $Y_1, Y_2, \dots, Y_{N-1}$ on $\mathbb{G}$ takes the value $i(X_{1j})(Y_1, Y_2, \dots, Y_{N-1}) = dx(X_{1j}, Y_1, Y_2, \dots, Y_{N-1})$.

The algebraical-analytical specificity of a Carnot group is reflected in the fact that the horizontal differential $D_h\varphi(x) : V_1 \to V_1'$ induces [73, Theorem 1.2] the homomorphism $D\varphi(x) : \mathbb{G} \to \mathbb{G}'$ of Carnot groups called the $\mathscr{P}$-differential (see Subsection 4.1.7); moreover, their norms can be estimated in terms of each other: $|D_h\varphi(x)| \leq |D\varphi(x)| \leq C|D_h\varphi(x)|$, where C depends only on the algebraical structure of $\mathbb{G}$. It is known [73, Theorem 1.2] that for almost all $x \in D$ the homomorphism $D\varphi(x)$ is an approximative differential of $\varphi : D \to \mathbb{G}'$ provided that $\varphi \in W^1_{1,\mathrm{loc}}(D)$.

The mapping $\varphi : D \to \mathbb{G}'$ belong to the Sobolev class $W^1_{p,\mathrm{loc}}(D)$ with $p \in [1,\infty)$ whenever $\varphi \in \mathrm{ACL}(D)$ and the quantity

$$\|\varphi \mid W^1_p(U)\| = \|\rho(\varphi(\cdot)) \mid L_p(U)\| + \Bigg(\int\limits_U |D_h\varphi(x)|^p \, d\mathscr{H}^\nu(x)\Bigg)^{\frac{1}{p}}$$

is finite on each domain $U \subset D$ with $\overline{U} \subset D$. As regards the descriptions of Sobolev-class mappings equivalent to the above, see [73].

4.3. Change of variables for mappings of ACL-classes on Carnot groups. Consider the coinciding groups $\mathbb{G}$ and $\mathbb{G}'$ and some mapping $\varphi : D \to \mathbb{G}$, where $D \subset \mathbb{G}$, that belongs to $W^1_{p,\mathrm{loc}}(D)$ with $p \in [1,\infty)$.

As in $\mathbb{R}^n$, the determinant $\det D\varphi(x)$ of the matrix of the homomorphism $D\varphi(x)$ is called the *Jacobian* of φ at x. The geometric meaning of the Jacobian is analogous to (11): If $\varphi : D \to D'$, where $D, D' \subset \mathbb{G}$, is a homeomorphism of a Sobolev class then

$$D \ni x \mapsto \lim_{r\to 0} \frac{\mathscr{H}^\nu(\varphi(B_c(x,r)))}{\mathscr{H}^\nu(B_c(x,r))} = \lim_{r\to 0} \frac{\mathscr{H}^N(\varphi(B_c(x,r)))}{\mathscr{H}^N(B_c(x,r))} = |\det D\varphi(x)| \tag{48}$$

for $\mathscr{H}^\nu$-almost all $x \in D$. We can obtain (48) using the change-of-variables formula (50) presented below and the Lebesgue Differentiation Theorem on Carnot groups; see [47, Corollary 3] for instance.

Definition 37. Given a mapping $\varphi : D \to \mathbb{G}$ on a Carnot group belonging to $W^1_{p,\mathrm{loc}}(D)$, define the Borel zero set $Z = \{x \in D : \det D\varphi(x) = 0\}$ of the Jacobian and the singularity set

$$\Sigma = D \setminus \{x \in D : \text{the approximative differential } D\varphi(x) \text{ is defined}\} \tag{49}$$

of measure zero, which we may assume to be a Borel set. It is obvious that $Z \cap \Sigma = \varnothing$.

By analogy with Definition 17, introduce the sets $Z' = \varphi(\Sigma)$ and $\Sigma' = \varphi(Z)$.

We can prove the change-of-variables formula for Lebesgue integrals on Carnot groups by repeating verbatim the arguments in the proof of Proposition 15 of [52] with the only difference that instead of the results of [49] which are used in this proof we should apply the corresponding theorems of [73].

Let us recall here just one simple formula [73, Corollary 5.1] useful below.

Proposition 38. *If a homeomorphism $\varphi : D \to D'$, where $D, D' \subset \mathbb{G}$, belongs to $W^1_{1,\mathrm{loc}}(D)$, or* ACL(D), *then outside the measure zero Borel set $\Sigma \subset D$ (see (49)) the mapping $\varphi : D \setminus \Sigma \to \mathbb{G}$ enjoys Luzin's $\mathscr{N}$-property, and every integrable function $u : D' \to \mathbb{R}$ satisfies*

$$\int\limits_D u(\varphi(x))|\det D\varphi(x)| \, d\mathscr{H}^\nu(x) = \int\limits_{D'\setminus Z'} u(y) \, d\mathscr{H}^\nu(y), \quad \text{where } Z' = \varphi(\Sigma). \tag{50}$$

4.4. Regularity properties of Sobolev-class homeomorphisms on Carnot groups. Say that a mapping $\varphi : D \to \mathbb{G}$ on a Carnot group belonging to $W^1_{1,\mathrm{loc}}(D)$, or ACL(D), has *finite distortion* whenever $D\varphi(x) = 0$ almost everywhere on the zero set $Z = \{x \in D : \det D\varphi(x) = 0\}$ of the Jacobian.

4.4.1. A Sobolev-class homeomorphism on a Carnot group as a composition operator. The next statement generalizes Proposition 1 to Carnot groups.

Proposition 39. *Consider two domains D and D' on a Carnot group $\mathbb{G}$ and a homeomorphism $\varphi : D \to D' \in W^1_{p,\mathrm{loc}}(D)$, where $1 \le p < \infty$, having finite distortion. Then the weight function defined as*

$$D' \ni y \mapsto \omega(y) = \begin{cases} \frac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|} & \text{if } y \in D' \setminus (Z' \cup \Sigma'), \\ 1 & \text{otherwise} \end{cases} \tag{51}$$

is locally summable, $\omega \in L_{1,\mathrm{loc}}(D')$, and $\Sigma' = \varphi(z)$. The composition operator

$$\varphi^* : L^1_p(D'; \omega) \cap \mathrm{Lip}_l(D') \to L^1_p(D), \quad 1 \le p < \infty, \tag{52}$$

is bounded and, furthermore, $\|\varphi^*\| \le \left\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\right\| = 1$, where the operator distortion function $K^{1,\omega}_{p,p}(\cdot)$ is defined in (53).

Proof. Verify that the weight function (51) is locally summable, the operator distortion function $K^{1,\omega}_{p,p}(\cdot)$ (see (53)) lies in $L_\infty(D)$, and operator (52) is bounded.

Formula (50) yields

$$\int\limits_{\varphi(W)\setminus(Z'\cup\Sigma')} \omega(y)\, d\mathscr{H}^\nu(y) = \int\limits_{\varphi(W)\setminus(Z'\cup\Sigma')} \frac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|}\, d\mathscr{H}^\nu(y)$$
$$= \int\limits_{W\setminus(Z\cup\Sigma)} |D\varphi(x)|^p\, d\mathscr{H}^\nu(x) < \infty$$

for a compactly embedded domain $W \Subset D$. Thus, $\omega \in L_{1,\mathrm{loc}}(D')$.

The operator distortion function

$$D \ni x \mapsto K^{1,\omega}_{p,p}(x, \varphi) = \begin{cases} \frac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{p}}\, \omega^{\frac{1}{p}}(\varphi(x))} & \text{if } \det D\varphi(x) \ne 0, \\ 0 & \text{if } \det D\varphi(x) = 0 \end{cases} \tag{53}$$

lies in $L_\infty(D)$. Moreover, $\left\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\right\| = 1$.

To estimate the norm of (52), apply the change-of-variables formula (50) (see the integral in (54) below):

$$\begin{aligned} \left\|u \circ \varphi \mid L^1_p(D)\right\|^p &\le \int\limits_D |\nabla_{\mathscr{L}}(u \circ \varphi)(x)|^p\, d\mathscr{H}^\nu(x) \\ &\le \int\limits_D |\nabla_{\mathscr{L}} u(\varphi(x))|^p \frac{|D\varphi(x)|^p}{|\det D\varphi(x)|} |\det D\varphi(x)|\, d\mathscr{H}^\nu(x) \qquad (54) \\ &\le \int\limits_{D'} |\nabla_{\mathscr{L}} u(y)|^p \frac{|D\varphi(\varphi^{-1}(y))|^p}{|\det D\varphi(\varphi^{-1}(y))|\omega(y)} \omega(y)\, d\mathscr{H}^\nu(y) \\ &\le \left\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\right\|^p \cdot \left\|u \mid L^1_p(D'; \omega)\right\|^p, \qquad (55) \end{aligned}$$

valid for every $u \in L^1_p(D'; \omega) \cap \mathrm{Lip}_l(D')$. Hence, $\|\varphi^*\| \le \left\|K^{1,\omega}_{p,p}(\cdot) \mid L_\infty(D)\right\| = 1$. $\square$

4.4.2. Sobolev-class homeomorphisms on Carnot groups and the regularity of their inverses. The next statement generalizes Proposition 2 to Carnot groups.

Proposition 40. *Consider two domains D and D' on a Carnot group $\mathbb{G}$, and a homeomorphism $\varphi : D \to D' \in W^1_{p,\mathrm{loc}}(D)$, where $\nu - 1 < p < \infty$, with finite distortion.*

Then the inverse homeomorphism $f = \varphi^{-1} : D' \to D$ enjoys the following properties:

(1) *f is differentiable $\mathscr{H}^\nu$-almost everywhere in the domain D';*
(2) *f has finite distortion;*
(3) *for $\mathscr{H}^\nu$-almost all $y \in D'$ we have*

$$|Df(y)| \le c_9 |\det Df(y)|^{\frac{p-(\nu-1)}{p}} \omega(y)^{\frac{\nu-1}{p}}; \tag{56}$$

(4) *for every open set $U \subset D'$ we have*

$$\int_U |Df(y)|\, d\mathscr{H}^\nu(y) \le c_9 \mathscr{H}^\nu(f(U))^{\frac{p-(n-1)}{p}} \cdot \omega(U)^{\frac{\nu-1}{p}} \tag{57}$$

with the weight function (51) *and constant $c_9 = 2^\nu\big(\frac{1}{c_{10}}\big)^{\frac{\nu-1}{p}}$, where the last equality is guaranteed by the normalization $\mathscr{H}^\nu(B(0,r)) = r^\nu$ and c_{10} is defined in* (59).
(5) *f is in $W^1_{1,\mathrm{loc}}(D')$.*

We can prove Proposition 40 following the scheme of the proof of Theorems 21 and 23. Below we present the main arguments of the proof of Proposition 40, emphasizing the features of the geometry of Carnot groups.

4.4.3. Capacity estimates on Carnot groups. The proof of Proposition 40 rests on estimates for the capacity of two condensers on a Carnot group similar to those in Subsection 2.1.

A continuous function $u : D \to \mathbb{R}$ from $W^1_{1,\mathrm{loc}}(D)$ is called *admissible* for a condenser $E = (F, U)$ whenever $u \equiv 1$ on F and $u \equiv 0$ outside U. Denote the collection of admissible functions for $E = (F, U)$ by $\mathscr{A}(E)$.

Let us generalize the concept of capacity which was defined in (6) for Euclidean space to Carnot groups.

Define the *capacity* of a condenser $E = (F, U)$ in $L^1_q(D)$, where $q \in [1, \infty)$, on a Carnot group as

$$\operatorname{cap}\big(E; L^1_q(D)\big) = \inf_{u \in \mathscr{A}(E) \cap L^1_q(D)} \big\|u \mid L^1_q(D)\big\|^q,$$

where the infimum is taken over all functions in $\mathscr{A}(E) \cap L^1_q(D)$.

The weight functions, the weighted Sobolev space $L^1_p(D'; \omega)$, and the space of locally Lipschitz functions $\mathrm{Lip}_l(D')$ on a domain D' of the metric space $(\mathbb{G}, d)$ are defined by analogy with the Euclidean space $\mathbb{R}^n$; see (5).

The *weighted capacity* of a condenser $E = (F, U) \subset D'$ in $L^1_p(D'; \omega)$ on a Carnot group is

$$\operatorname{cap}\big(E; L^1_p(D'; \omega)\big) = \inf_{u \in \mathscr{A}(E) \cap \mathrm{Lip}_l(D')} \big\|u \mid L^1_p(D'; \omega)\big\|^p,$$

where the infimum is taken over all $u \in \mathscr{A}(E) \cap \mathrm{Lip}_l(D')$.

The distance between two nonempty sets $A_1, A_2 \subset \mathbb{G}$ in a Carnot group equals $\operatorname{dist}(A_1, A_2) = \inf\{d_c(x, y) : x \in A_1,\, y \in A_2\}$.

Below we will use the notation $\omega(E) = \int_E \omega(y)\, d\mathscr{H}^\nu(y)$ for a measurable set $E \subset \mathbb{G}$.

Lemma 41. *For $1 \le p < \infty$ the weighted capacity of each condenser $E = (F, U)$ in D' satisfies the upper bound*

$$\operatorname{cap}\big(E; L^1_p(U; \omega)\big) \le \frac{\omega(U \setminus F)}{\operatorname{dist}(F, \partial U)^p}. \tag{58}$$

Proof. Demonstration of this property is similar to the proof of Lemma 19 with the only difference that instead of the Euclidean distance we have to use the Carnot–Carathéodory distance.

Lemma 42 [18, Lemma 5]. *Assume that $\nu-1<q<\infty$. Each condenser $E=(F,U)$ in D with a connected set F satisfies*

$$\operatorname{cap}^{\nu-1}\big(E;L^1_q(U)\big)\geq c_{10}^{\nu-1}\frac{(\operatorname{diam}F)^q}{\mathscr{H}^\nu(U)^{q-(\nu-1)}},\tag{59}$$

where c_{10} is a constant depending only on q and the geometry of the group.

4.4.4. Proof of Proposition 40. **I.** At this step we will establish the differentiability of f. To this end, use the scheme of proof of Theorem 21. Associate to each point $y\in D'$ some spherical condenser $E_r=(\overline{B_c(y,r)},B_c(y,2r))$ with $B_c(y,2r)\subset D'$; i.e., a condenser whose boundary consists of two connected components which are concentric spheres. With the norm $\|\varphi^*\|\leq 1$, we have

$$\operatorname{cap}\big(f(E_r);L^1_p(f(B_c(y,2r)))\big)\leq\operatorname{cap}\big(E_r;L^1_p(U;\omega)\big)\leq\frac{\omega(B_c(y,2r))}{r^p}.$$

To estimate capacity on the left-hand side, use (59) with $q=p$,

$$c_{10}\frac{(\operatorname{diam}f(\overline{B_c(y,r)}))^{\frac{p}{\nu-1}}}{\mathscr{H}^\nu(f(B_c(y,2r)))^{\frac{p-(\nu-1)}{\nu-1}}}\leq\operatorname{cap}\big(f(E_r);L^1_p(f(B_c(y,2r)))\big),$$

and infer that

$$\frac{\operatorname{diam}f(\overline{B(y,r)})}{r}\leq\frac{2^\nu c_{10}^{\frac{1-\nu}{p}}}{(2r)^\nu}\mathscr{H}^\nu(f(B(y,2r)))^{\frac{p-(\nu-1)}{p}}\omega(B(y,2r))^{\frac{\nu-1}{p}}$$
$$=2^\nu c_{10}^{\frac{1-\nu}{p}}\bigg(\frac{\mathscr{H}^\nu(f(B(y,2r)))}{\mathscr{H}^\nu(B(y,2r))}\bigg)^{\frac{p-(\nu-1)}{p}}\bigg(\frac{\omega(B(y,2r)}{\mathscr{H}^\nu(B(y,2r))}\bigg)^{\frac{\nu-1}{p}}.$$

Letting r tend to 0, for $\mathscr{H}^\nu$-almost all $y\in D'$ we obtain

$$\varlimsup_{z\to y}\frac{d(f(z),f(y))}{d(z,y)}\leq 2^\nu\left(\frac{1}{c_{10}}\right)^{\frac{\nu-1}{p}}\mathscr{V}'_\nu(y)^{\frac{p-(\nu-1)}{p}}\omega(y)^{\frac{\nu-1}{p}},\tag{60}$$

where the value $\mathscr{V}'_\nu(y)=\lim_{r\to0}\frac{\mathscr{H}^\nu(f(B(y,r)))}{\mathscr{H}^\nu(B(y,r))}$, equal to the volume derivative, is finite $\mathscr{H}^\nu$-almost everywhere in D'; see (48). The Lebesgue Differentiation Theorem yields $\lim_{r\to0}\frac{\omega(B(y,2r)}{\mathscr{H}^\nu(B(y,2r))}=\omega(y)$ for $\mathscr{H}^\nu$-almost all $y\in D'$; for instance, see [72, Chapter 1, § 3.1, Corollary; 46; 47, Corollary 3].

Since the right-hand side of (60) is finite $\mathscr{H}^\nu$-almost everywhere in D' by a Stepanov-type theorem on Carnot groups (see [73, Theorem 3.1]), f is differentiable $\mathscr{H}^\nu$-almost everywhere in D'. It is known that at the differentiability points of f the left-hand side of (60) equals $|Df(y)|$ (see [73, Corollary 2.1] for instance), while $\mathscr{V}'_\nu(y)=|\det Df(y)|$ (cp. (48)).

II, **III.** Appreciating the above, rearrange (60) as

$$|Df(y)|\leq c_9|\det Df(y)|^{\frac{p-(\nu-1)}{p}}\omega(y)^{\frac{\nu-1}{p}},$$

where $c_9=2^\nu\big(\frac{1}{c_{10}}\big)^{\frac{\nu-1}{p}}$. This yields the pointwise estimate (56). Then the following obviously holds: $Df(y)=0$ on the zero set $Z'=\{y\in D':\det Df(y)=0\}$ of the Jacobian $\det Df(y)$ everywhere outside a set of $\mathscr{H}^\nu$-measure zero. Thus, f has finite distortion.

IV. To justify (57), we have to integrate (56) and apply Hölder's inequality, remembering that $\frac{p-(\nu-1)}{p}+\frac{\nu-1}{p}=1$:

$$\int\limits_U|Df(y)|\,d\mathscr{H}^\nu(y)\leq c_9\bigg(\int\limits_U|\det Df(y)|\,d\mathscr{H}^\nu(y)\bigg)^{\frac{p-(\nu-1)}{p}}\bigg(\int\limits_U\omega(y)\,d\mathscr{H}^\nu(y)\bigg)^{\frac{\nu-1}{p}}.$$

Since $\int_U |\det Df(y)|\, d\mathscr{H}^\nu(y) \le \mathscr{H}^\nu(f(U))$ (see (50)), we arrive at (57). Thus, $|Df(y)|$ is locally summable on D'.

V. It remains to prove property (5). To this end, it suffices to verify that $f \in \mathrm{ACL}(D')$. Since the horizontal derivatives are locally summable (cf. (57)), the claim will follow. See [73, Proposition 4.2] for an equivalent description of $f \in W^1_{1,\mathrm{loc}}(D')$ on Carnot groups.

STEP 1. Recall that $B_\rho(y,t)$ stands for the radius t ball in the metric space $(\mathbb{G},\rho)$ which is centered at y; see (45).

To show that $f \in \mathrm{ACL}(D')$, fix $1 \le j \le n$. For some number[10] $M \in (0,\infty)$, define the set

$$\begin{gathered} P_{j0}(0,t) = \{y \in B_\rho(0,t) : \ y_j = 0\}, \quad t \in (0,M), \\ P_j(0,t) = \{z = y \exp y_j X_{1j} : y \in P_{j0}(0,t),\, |y_j| < t,\}, \quad t \in (0,M), \\ P_j(w,t) = wP_j(0,t) \subset D' \quad \text{for } w \in D' \text{ and } t \in (0,M). \end{gathered} \tag{61}$$

Observe that $P_j(0,t)$ is a ball in the metric $\rho_j(x,y)$ defined as follows: $y \in P_j(0,t)$ if and only if $\rho_j(y) < t$, where the homogeneous norm ρ_j is defined by the condition

$$\mathscr{G} \ni y \mapsto \rho_j(y) = \max(\{\rho(y) : y_j = 0\}, |y_j|) \tag{62}$$

on the Lie algebra $\mathscr{G}$ and carried over to the group by the global coordinate system $\exp(y) = \exp(\sum y_{ik} X_{ij}) \cdot \exp(y_j X_{1j})$, where the ranges of summation are $1 \le i \le m$ and $1 \le k \le n_1$ with $k \ne j$ for $i = 1$ and $1 \le k \le n_i$ for $i \ge 2$. Consequently, the quasimetric $\rho_j(x,y)$ is equivalent in the sense of Subsection 4.1.4 to both the quasimetric $\rho(x,y)$ and the metric $d_c(x,y)$.

Applying the Covering Lemma of [70, Lemma 1.66] in the case $D' = \mathbb{G}$ or of [72, § 3.1, Lemma 2] in the case $D' \ne \mathbb{G}$, we infer the existence of an at most countable family

$$\{P_j(w_k,t_k) : P_j(0,t_k) \Subset P_j(0,M)\}, \quad k \in \mathbb{N}, \tag{63}$$

of sets covering D' of the form indicated in (61): $D' = \bigcup_{k\in\mathbb{N}} P_j(w_k,t_k)$.

Therefore, it suffices to prove the absolute continuity of f on the intersection of $\mathscr{H}^{\nu-1}$-almost every integral line of the horizontal vector field X_{1j} with the ball $P_j(w_k,t_k)$; here $\mathscr{H}^{\nu-1}$ is the Hausdorff measure on the surface $w_k P_{j0}(0,t_k)$ transversal to the foliation Γ_j; see (46). Since $k \in \mathbb{N}$ and $j = 1,\dots,n$ is arbitrary, the ACL-property of f on D' will be established.

STEP 2. Fix some set $P_j(w_k,t_k)$ in the family (63). The question of absolute continuity of the mapping $f : P_j(w_k,t_k) \to D'$ reduces to that of the composition $f \circ l_{w_k} : P_j(0,t_k) \to D'$, where l_{w_k} is the left translation: $\mathbb{G} \ni x \mapsto w_k \cdot x$.

To avoid bulky formulas, put

$$Q_0 = P_{j0}(0,t_k), \quad Q = P_j(0,t_k). \tag{64}$$

It is obvious that the part of the plane $Q_0 \subset \{y \in \mathbb{G} : y_{1j} = 0\}$ is transversal to the foliation Γ_j.

Consider the restriction $\tilde\rho = \rho|_{Q_0}$ of the metric ρ to Q_0. Note some useful properties of the metric structure $(Q_0,\tilde\rho)$ and the measures $\mathscr{H}^{\nu-1}$ and $\mathscr{H}^{N-1}$ on Q_0.

(1) *The metric function $\tilde\rho$ is a quasimetric in the sense of Subsection* 4.1.4*; i.e., $\tilde\rho : Q_0 \times Q_0 \to [0,\infty)$ enjoys the properties* (a_1)–(c_1) *of Subsection* 4.1.4 *for all points $x, y \in Q_0$.*

(2) *The Hausdorff measure $\mathscr{H}^{\nu-1}$ (see Subsection* 4.1.5*) satisfies the doubling condition: for the ball $B_{\tilde\rho}(y,t)$ and the Hausdorff measure $\mathscr{H}^{\nu-1}$ on Q_0 we have*

$$\mathscr{H}^{\nu-1}(B_{\tilde\rho}(y,2t)) \le \mu \mathscr{H}^{\nu-1}(B_{\tilde\rho}(y,t)) \tag{65}$$

[10] The value of M is governed by the necessity to ensure (65)–(67), while the exact value of M is not used in the proof.

for all $y \in Q_0$ *and* $B_{\tilde\rho}(y,2t) \subset Q_0$ *with some constant* $\mu \in (0,\infty)$ *independent of* $y \in Q_0$ *and* $B_{\tilde\rho}(y,2t) \subset Q_0$.

(3) *There exist constants* $c_{11}, c_{12} \in (0,\infty)$ *such that*

$$c_{11}\mathscr{H}^{\nu-1}(A) \le \mathscr{H}^{N-1}(A) \le c_{12}\mathscr{H}^{\nu-1}(A) \tag{66}$$

for every measurable set $A \subset Q_0$.

(4) *The quasimetric space* $(Q_0,\tilde\rho)$ *with the Hausdorff measure* $\mathscr{H}^{\nu-1}$ *satisfying* (65) *is a space of homogeneous type; for the definition, see* [72, 74] *for instance.*

(5) *The metric space* $(Q_0,\tilde\rho)$ *endowed with the measure* $\mathscr{H}^{N-1}$ *is also a space of homogeneous type.*

PROOF. Since the restriction of a quasimetric to an arbitrary set inherits the properties of the quasimetric, claim (1) is justified.

The measure doubling property follows from [75, Theorem 3.17]: it shows that $\mathscr{H}^{\nu-1}(B_{\tilde\rho}(z,r)) \sim r^{\nu-1}$ uniformly[11)] in all $z \in Q_0$ and $B_{\tilde\rho}(z,r) \subset Q$.

Moreover, [75, Theorems 3.7 and 3.17] imply that the Hausdorff measure $\mathscr{H}^{\nu-1}(A)$ of $A \subset Q_0$ is comparable to the Hausdorff measure $\mathscr{H}^{N-1}(A)$, as the inequalities in (66) follow from the equivalences

$$\mathscr{H}^{\nu-1}(B_{\tilde\rho}(z,r)) \sim r^{\nu-1} \quad\text{and}\quad \mathscr{H}^{N-1}(B_{\tilde\rho}(z,r)) \sim r^{\nu-1}, \tag{67}$$

uniform in $z \in Q_0$ and $B_{\tilde\rho}(z,r) \subset Q_0$; see [75, Theorems 3.7 and 3.17].

Property 5 follows from (66) and property 4. □

STEP 3. The mapping $h : Q_0 \times I_k \mapsto Q$, where $I_k = (-t_k, t_k)$, see (64), defined as

$$Q_0 \times I_k \ni (z, y_j) \mapsto h(z,y_j) = (z, z\exp y_j X_{1j}) \tag{68}$$

is a diffeomorphism. We can choose $M > 0$ in (63) so that

$$0 < \varkappa_1 \le |\det h(z,y_j)| \le \varkappa_2 < \infty \tag{69}$$

for all points $(z,y_j) \in Q_0 \times I_k$ with constants $\varkappa_1$, $\varkappa_2 \in (0,\infty)$, independent of $k \in \mathbb{N}$, where k is from (63).

Using (68), we can transport integration over Q to the open set $Q_0 \times I_k$. Consider on $Q_0 \times I_k$ the tensor product Λ of the measure $\mathscr{H}^{\nu-1}$ on Q_0 and the measure $\mathscr{H}^1$ on I_k. By (66),

the measure Λ is comparable with the measure $\mathscr{H}^N$:

$$\varkappa_3\Lambda(E) \le \mathscr{H}^N(E) \le \varkappa_4\Lambda(E)$$

for every measurable set $E \subset Q_0 \times I_k$. The constants $\varkappa_3$, $\varkappa_4 \in (0,\infty)$ are independent of the choice of $E \subset Q_0 \times I_k$. Consequently, the measures Λ and $\mathscr{H}^N$ are absolutely continuous with respect to each other. Therefore, the Radon–Nikodým derivative $\mathfrak{D}(x)$ of $\mathscr{H}^N$ with respect to Λ exists and satisfies

$$\varkappa_3 \le \mathfrak{D}(x) = \frac{d\mathscr{H}^N}{d\Lambda}(x) \le \varkappa_4 \quad \text{for } \mathscr{H}^N\text{-almost all } x \in Q_0 \times I_k. \tag{70}$$

For the function $v(x) = |\det Dh(x)| \cdot \mathfrak{D}(x)$ of (69) and (70) we have

$$\varkappa_1 \cdot \varkappa_3 \le v(x) \le \varkappa_2 \cdot \varkappa_4 \quad \text{for } \mathscr{H}^N\text{-almost all } x \in Q_0 \times I_k. \tag{71}$$

If $u \in L_1(Q)$ is a nonnegative function then (71) implies the same for the product $u \cdot v$. Using the above and (13), we infer that

$$\begin{aligned}\int\limits_Q u(y)\,d\mathscr{H}^\nu(y) &= \int\limits_{Q_0\times I_k} u(h(x))|\det Dh(x)|\,d\mathscr{H}^\nu(x)\\
&= \int\limits_{Q_0\times I_k} u(h(x))|\det Dh(x)|\cdot\mathfrak{D}(x)\,d\Lambda(x) = \int\limits_{Q_0} dz \int\limits_{I_k} u(h(x))v(z,t)\,dt\\
&= \int\limits_{Q_0} dz \int\limits_{I_k} u(z,\gamma_z(\tau))v(h^{-1}(z,\gamma_z(\tau)))\,d\mathscr{H}^1(\tau),\end{aligned} \tag{72}$$

[11)]In the context of this article $\mathscr{H}^{\nu-1}(B_{\tilde\rho}(z,r))$ is equivalent to $r^{\nu-1}$ on Q_0 if and only if there exist positive reals ζ_1 and ζ_2 such that $\zeta_1 r^{\nu-1} \le \mathscr{H}^{\nu-1}(B_{\tilde\rho}(z,r)) \le \zeta_2 r^{\nu-1}$ for all $z \in Q_0$ and all r with $B_{\tilde\rho}(z,r) \subset Q_0$.

where the curve $\gamma_z : I_k \to \mathbb{G}$, defined by the condition $\gamma_z(\tau) = z \exp \tau X_{1j}$, has tangent vector $|\dot{\gamma}_z(\tau)| = 1$ for $\tau \in I_k$. Here we write the points of $Q_0 \times I_k$ as the pairs (z, t) with $z_0 \in Q_0$ and $t \in I_k$, while the points of Q as the pairs $(z, \gamma_z(\tau))$ with $z_0 \in Q_0$ and $\tau \in I_k$.

STEP 4. Given $z \in Q_0$ (see (64)), and $r \in (0, t_k)$ with $B_{\tilde{\rho}}(z, r) \subset Q_0$, consider the tubular neighborhood

$$E(z,r) = \bigcup_{\tau \in (-t_k, t_k)} B_c(z \exp \tau X_{1j}, r).$$

We are interested in the behavior of the ratio

$$\frac{\mathscr{H}^\nu(f(E(z,r)))}{r^{\nu-1}}$$

as $r \to 0$. As [67, the Main Lemma] and [76, Lemma 3] show for Heisenberg and Carnot groups, the upper limit of this ratio is finite for almost all $z \in Q_0$:

$$\overline{\mathscr{V}}'(z) = \varlimsup_{r \to 0} \frac{\mathscr{H}^\nu(f(E(z,r)))}{r^{\nu-1}} < \infty \tag{73}$$

for $\mathscr{H}^{\nu-1}$-almost all points $z \in Q_0$; henceforth $\mathscr{V}(z,r) = \mathscr{H}^\nu(f(E(z,r)))$. Thus, $\overline{\mathscr{V}}'(z) < \infty$ at all points $z \in Q_0 \setminus T'$, where $T' \subset Q_0$ is some set of $\mathscr{H}^{\nu-1}$-measure zero.

STEP 5. On the interval $I_k = (-t_k, t_k)$ (see (64)), take arbitrary disjoint segments $\Delta_1 = [a_1, b_1]$, $\Delta_2 = [a_2, b_2], \dots, \Delta_l = [a_l, b_l]$, of lengths $b_1, b_2, \dots, b_l$ such that $b_1 + b_2 + \dots + b_l < 2t_k$.

Fix a sufficiently small positive real $r \in (0, t_k)$ with $B_{\tilde{\rho}}(z, r) \subset Q_0$. To a point z and the interval Δ_i associate the open set

$$U_i(z,r) = \bigcup_{\tau \in \Delta_i} B_c(z \exp \tau X_{1j}, r). \tag{74}$$

Assume that $r > 0$ is chosen so small that the open sets $U_1(z,r), \dots, U_l(z,r)$ are disjoint and $U_i(z,r) \subset Q$ for $i = 1, \dots, l$; see (63) and (64).

Consider the continuum $F_i = \{z \exp \tau X_{1j} : \tau \in \Delta_i\}$ and the condenser $E_i = (F_i, U_i(z,r))$. Lemma 41 yields

$$\operatorname{cap}\bigl(E_i; L^1_p(U_i(z,r);\omega)\bigr) \le \frac{\omega(U_i(z,r))}{r^p} = \frac{\int\limits_{U_i(z,r)} \omega(y)\,dy}{r^p}, \quad i = 1, \dots, l.$$

On the other hand, for $\nu - 1 < p < \infty$ Lemma 42 implies that

$$c_{10}^{\frac{\nu-1}{p}} \frac{\operatorname{diam} f(F_i)}{\mathscr{H}^\nu(f(U_i(z,r)))^{\frac{p-(\nu-1)}{p}}} \le \operatorname{cap}^{\frac{\nu-1}{p}}\bigl(f(E_i); L^1_p(U_i)\bigr).$$

Using (55), we deduce from the last two relations that

$$\operatorname{diam} f(F_i)) \le \Bigl(\frac{1}{c_{10}}\Bigr)^{\frac{\nu-1}{p}} \frac{1}{r^{\nu-1}} \mathscr{H}^\nu(f(U_i(z,r)))^{\frac{p-(\nu-1)}{p}} \omega(U_i(z,r))^{\frac{\nu-1}{p}}. \tag{75}$$

Summing inequalities (75) for $i = 1, \dots, l$, applying Hölder's inequality, and using the properties of quasiadditive functions, we arrive at

$$\begin{aligned} \sum_{i=1}^l \operatorname{diam} f(F_i) &\le \Bigl(\frac{1}{c_{10}}\Bigr)^{\frac{\nu-1}{p}} \frac{1}{r^{\nu-1}} \Bigl(\sum_{i=1}^l \mathscr{H}^\nu(f(U_i(z,r)))\Bigr)^{\frac{p-(\nu-1)}{p}} \Bigl(\sum_{i=1}^l \omega(U_i(z,r))\Bigr)^{\frac{\nu-1}{p}} \\ &\le c_{13} \cdot c_{14} \Bigl(\frac{\mathscr{V}_\nu(z,r)}{r^{\nu-1}}\Bigr)^{\frac{q-(\nu-1)}{q}} \Bigl(\frac{\sum_{i=1}^l \omega(U_i(z,r))}{r^{\nu-1}}\Bigr)^{\frac{\nu-1}{p}}, \end{aligned} \tag{76}$$

where $c_{13} = \left(\frac{1}{c_{10}}\right)^{\frac{\nu-1}{p}}$ and $c_{14} = (\beta_\nu(2\alpha)^\nu)^{\frac{\nu-1}{p}}$, while the constant $\beta_\nu(2\alpha)^\nu$ is defined in (81).

Since the left-hand side of (76) is independent of r, we can pass to the limit on the right-hand side using an arbitrary sequence $r_p \to 0$ as $p \to \infty$. Letting $r \to 0$, we will prove the following inequality whose validity for $\mathscr{H}^{\nu-1}$-almost all $z \in Q_0$ is ensured by the existence of limits in the two parentheses in (76) for $\mathscr{H}^{\nu-1}$-almost all $z \in Q_0$, as we explain below:

$$\sum_{i=1}^{l} \operatorname{diam} f(F_i) \le l c_{13} \cdot c_{14} (\overline{\mathscr{V}}'(z))^{\frac{p-(\nu-1)}{p}} \Bigg(\int\limits_{\bigcup_{i=1}^{l} \Delta_i} \omega(z,\tau)\, d\gamma_z(\tau) \Bigg)^{\frac{\nu-1}{p}}, \tag{77}$$

where $\Delta_i \ni \tau \mapsto \gamma_z(\tau) = z \exp(\tau X_{1j})$. The first quotient in parentheses in (76) has finite limit (73) at all $z \in Q_0 \setminus T'$, where T' is of $\mathscr{H}^{\nu-1}$-measure zero, as we mentioned at step 4.

Let us show that the second factor in (76) has finite limit (see details after (77)) for $\mathscr{H}^{\nu-1}$-almost all $z \in Q_0$ and some sequence $r_p \to 0$ as $p \to \infty$.

To this end, consider a sole term in the second factor of (76), for instance, with index i; the existence of a limit for each term would yield the same for finitely many terms. Recalling the definition of $U_i(z,r)$ in (74), we see that

$$\omega(U_i(z,r)) = \int\limits_{U_i(z,r)} \omega(y)\, dy.$$

Given a sufficiently small $r > 0$, put $\tau_m = a_i + mr$ for $m = 0, 1, \dots, m_r$, where m_r is the smallest positive integer with $\tau_{m_r} \ge b_i$. For all $\zeta \in (\tau_{m-1}, \tau_m)$ and some $\zeta_m \in (\tau_{m-1}, \tau_m)$ chosen in (79), we obtain

$$B_c(z \exp \zeta X_{1j}, r) \subset B_c(z \exp \zeta_m X_{1j}, 2r) \subset B_\rho(z \exp \zeta_m X_{1j}, 2\alpha r), \tag{78}$$

where α is a positive constant such that $B_c(y,r) \subset B_\rho(y, \alpha r)$ for all $y \in \mathbb{G}$ and $\rho > 0$.

For r fixed and all $m = 1, 2, 3, \dots, m_r$ we have

$$\frac{1}{r} \int\limits_{\tau_{m-1}}^{\tau_m} \omega(B_\rho(z \exp \tau X_{1j}, 2\alpha r))\, d\tau = \omega(B_\rho(z \exp \zeta_m X_{1j}, 2\alpha r)) \tag{79}$$

because the function $\tau \mapsto \omega(B_\rho(z \exp \tau X_{1j}, 2\alpha r))$ is continuous; the number $\zeta_m \in (\tau_{m-1}, \tau_m)$ exists by the Mean Value Theorem. For this choice of $\{\zeta_m\}$, for $m = 1, 2, \dots, m_r$ we infer from (74) and (78) that

$$U_i(z,r) \subset \bigcup_m B_\rho(z \exp \zeta_m X_{1j}, 2\alpha r),$$

where the union is over the numbers m's mentioned above. Hence, taking (79) into account, we arrive at

$$\omega(U_i(z,r)) \le \sum_m \omega(B_\rho(z \exp \zeta_m X_{1j}, 2\alpha r)) = \frac{1}{r} \int\limits_{a_i}^{\tau_{m_r}} \omega(B_\rho(z \exp \tau X_{1j}, 2\alpha r))\, d\tau. \tag{80}$$

This yields

$$\frac{\omega(U_i(z,r))}{r^{\nu-1}} \le \beta_\nu (2\alpha)^\nu \int\limits_{a_i}^{\tau_{m_r}} \frac{\omega(B_\rho(z \exp \tau X_{1j}, 2\alpha r))}{\mathscr{H}^N(B_\rho(e, 2\alpha r))}\, d\tau, \tag{81}$$

where $\beta_\nu = \mathscr{H}^N(B_\rho(e,1))$ is the volume of $B_\rho(e,1)$. The integrand in (81) amounts to the value of

$$\omega_r(y) = \frac{1}{\mathscr{H}^N(B_\rho(e, 2\alpha r))} \int\limits_{B_\rho(y, 2\alpha r)} \omega(w)\, dw$$

at $y = z\exp\tau X_{1j}$, while the function $\omega_r(y)$ itself is an analog of the Steklov average of $\omega(y)$ on the Carnot group. It is known (see [70, Proposition 1.20] for instance) that

$$\|\omega_r(\cdot) - \omega(\cdot) \mid L_1(Q)\| \to 0 \quad \text{as } r \to 0.$$

Moreover, applying (72) we arrive at the convergence

$$\int\limits_{I_k} |\omega_{r_p}(z\exp\tau X_{1j}) - \omega(z\exp\tau X_{1j})|\, d\tau \to 0 \tag{82}$$

for some sequence $r_p \to 0$ as $p \to \infty$ for all $z \in Q_0 \setminus \Sigma_j$, where $\mathscr{H}^{\nu-1}(\Sigma_j) = 0$. In particular, this implies that the function $I_k \ni \tau \mapsto \omega(z\exp\tau X_{1j})$ is summable on I_k for all $z \in Q_0 \setminus \Sigma_j$.

It is obvious that (82) also holds on every interval $(a_i, b_i + \delta_i) \subset I_k$, where δ_i is an arbitrary real in $(0, t_k - b_i)$, while the function $(a_i, b_i + \delta_i) \ni \tau \mapsto \omega(z\exp\tau X_{1j})$ is summable on $(a_i, b_i + \delta_i)$ for all $z \in Q_0 \setminus \Sigma_j$.

Basing on that, from (81) and (82) for all $z \in Q_0 \setminus \Sigma_j$ we obtain

$$\overline{\lim_{p\to\infty}} \frac{\omega(U_i(z, r_p))}{r_p^{\nu-1}} \le \beta_\nu (2\alpha)^\nu \lim_{p\to\infty} \int\limits_{a_i}^{b_i+\delta} \omega_{r_p}(z\exp\tau X_{1j})\, d\tau = \beta_\nu (2\alpha)^\nu \int\limits_{a_i}^{b_i+\delta} \omega(z\exp\tau X_{1j})\, d\tau \tag{83}$$

for every number $\delta \in (0, \delta_i)$; here we appreciate that $\tau_{m_r} \ge b_i$ and $\tau_{m_r} \to b_i$ as $r \to 0$. Since $\delta > 0$ is arbitrary and the Lebesgue integral is absolutely continuous, (83) yields

$$\overline{\lim_{p\to\infty}} \frac{\omega(U_i(z, r_p))}{r_p^{\nu-1}} \le \beta_\nu (2\alpha)^\nu \int\limits_{a_i}^{b_i} \omega(z\exp\tau X_{1j})\, d\tau.$$

Since the second factor in (76) involves finitely many terms, (77) is justified for all points $z \in Q_0 \setminus (T' \cup \Sigma_j)$, where $T' \cup \Sigma_j \subset Q_0$ has $\mathscr{H}^{\nu-1}$-measure zero.

Step 6. We also see from (77) that the absolute continuity of $f : \gamma_z \to D$ for each fixed z is guaranteed by that of the integral

$$\int\limits_{I_k} \omega(z\exp\tau X_{1j})\, d\tau$$

on the interval $\{z\exp tX_{1j} : t \in I_k\}$ of the integral line of the vector field X_{1j}. Consequently, we can extend (77) to arbitrary countable disjoint collections of segments $\Delta_i \subset I_k$.

Since j can be an arbitrary positive integer from 1 to n, the absolute continuity of $f : D' \to D$ is established. Thus, in view of (57), we proved that $f \in W^1_{1,\mathrm{loc}}(D')$; for the details, see [73, Proposition 4.2].

4.5. Applications. The following theorem is stated in [18]:

Theorem 43 [18, Theorem 9]. *Consider two domains D and D' on a Carnot group. Suppose that a homeomorphism $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_p(D') \cap \mathrm{Lip}_l(D') \to L^1_q(D)$, where $\nu - 1 < q \le p < \infty$. Then the inverse mapping $\varphi^{-1} : D' \to D$ induces the bounded composition operator $\varphi^{-1*} : L^1_r(D) \cap \mathrm{Lip}_l(D) \to L^1_s(D')$, where $r = \frac{q}{q-(\nu-1)}$ and $s = \frac{p}{p-(\nu-1)}$.*

However, the proof in [18] contains some gaps that were imported from [17] but can be repaired by using the results of Section 4. The gap in [17] is also filled by [20, Theorem 6] and, by a different method, Theorem 23 of this article.

Indeed, if $\varphi : D \to D'$ induces the bounded composition operator $\varphi^* : L^1_p(D') \cap \mathrm{Lip}_l(D') \to L^1_q(D)$ then $\varphi \in L^1_{q,\mathrm{loc}}(D)$ and φ has finite distortion; see [18, Proposition 1 and Theorem 2]). By Proposition 40, the inverse mapping φ^{-1} belongs to $W^1_{1,\mathrm{loc}}(D')$ and has finite distortion. The last property is necessary in order for $\varphi^{-1} : D' \to D$ to induce the bounded composition operator $\varphi^{-1*} : L^1_r(D) \cap \mathrm{Lip}_l(D) \to L^1_s(D')$, where $r = \frac{q}{q-(\nu-1)}$ and $s = \frac{p}{p-(\nu-1)}$. No proof that the distortion of φ^{-1} is finite is given in [18].

4.6. Generalizations. The method of Section 4 also applies to the generalizations of Theorems 21 and 23 to continuous discrete open mappings of Carnot groups.

Acknowledgment. The author is grateful to the referee for critical consideration of the manuscript and comments, which helped improve the first version of this article.

References

1. Hencl S. and Koskela P., "Regularity of the inverse of a planar Sobolev homeomorphism," Arch. Rat. Mech. Anal., vol. 180, 75–95 (2006).
2. Onninen J., "Regularity of the inverse of spatial mappings with finite distortion," Calc. Var. Partial Differential Equations, vol. 26, no. 3, 331–341 (2006).
3. Ball J. M., "Global invertibility of Sobolev functions and the interpenetration of matter," Proc. Roy. Soc. Edinburgh Sect. A, vol. 88, no. 3–4, 315–328 (1981).
4. Menchoff D., "Sur les différentielles totales des fonctions univalentes," Math. Ann., vol. 105, no. 1, 75–85 (1931).
5. Gehring F. W. and Lehto O., "On the total differentiability of functions of a complex variable," Ann. Acad. Sci. Fenn. Ser. AI, vol. 272, 1–9 (1959).
6. Martio O., Ryazanov V., Srebro U., and Yakubov E., *Moduli in Modern Mapping Theory*, Springer, New York (2008).
7. Vodopyanov S. K., "Basics of the quasiconformal analysis of a two-index scale of spatial mappings," Dokl. Akad. Nauk, vol. 484, no. 2, 142–146 (2019).
8. Vodopyanov S. K., "Basics of the quasiconformal analysis of a two-index scale of spatial mappings," Sib. Math. J., vol. 59, no. 5, 805–834 (2018).
9. Vodopyanov S. K., "Differentiability of mappings of the Sobolev space W^1_{n-1} with conditions on the distortion function," Sib. Math. J., vol. 59, no. 6, 983–1005 (2018).
10. Vodopyanov S. K., "Composition operators in weighted Sobolev spaces and the theory of ${}_p$-homeomorphisms," Dokl. Akad. Nauk, vol. 494, no. 5, 21–25 (2020).
11. Vodopyanov S. K., "On analytical and geometrical properties of mappings in the theory of ${}_{q,p}$-homeomorphisms," Mat. Zametki, vol. 108, no. 6, 924–928 (2020).
12. Vodopyanov S. K. and Tomilov A. O., "Functional and analytical properties of a class of mappings of the quasiconformal analysis," Izv. Ross. Akad. Nauk Ser. Mat., vol. 85 (2021). doi 10.4213/im9082.
13. Vodopyanov S. K., *The Taylor Formula and Function Spaces* [Russian], Novosibirsk Univ., Novosibirsk (1988).
14. Vodopyanov S. K., "Mappings of homogeneous groups and imbeddings of functional spaces," Sib. Math. J., vol. 30, no. 5, 685–698 (1989).
15. Vodopyanov S. K., "Weighted Sobolev spaces and mapping theory," in: *Abstracts: All-Russia Mathematical School "Potential Theory" (Katsiveli, 26 June–3 July 1991)*, Inst. Mat. Akad. Nauk USSR, Kiev (1991), 7.
16. Vodopyanov S. K., *Geometric Aspects of Spaces of General Differentiable Functions* [Russian]. Extended Abstract of Doct. Sci. Dissertation, Sobolev Institute of Mathematics, Novosibirsk (1992).
17. Ukhlov A. D., "On mappings generating the embeddings of Sobolev spaces," Sib. Math. J., vol. 34, no. 1, 165–171 (1993).
18. Vodopyanov S. K. and Ukhlov A. D., "Sobolev spaces and (P,Q)-quasiconformal mappings of Carnot groups," Sib. Math. J., vol. 39, no. 4, 665–682 (1998).
19. Vodopyanov S. K. and Ukhlov A. D., "Superposition operators in Sobolev spaces," Russian Math. (Iz. VUZ), vol. 46, no. 10, 9–31 (2002).
20. Vodopyanov S. K., "Regularity of mappings inverse to Sobolev mappings," Sb. Math., vol. 203, no. 10, 1383–1410 (2012).
21. Sobolev S. L., "On some transformation groups of an n-dimensional space," Dokl. Akad. Nauk SSSR, vol. 32, no. 6, 380–382 (1941).
22. Mazya V. G., *Classes of Sets and Embedding Theorems of Function Classes. Some Problems of the Theory of Elliptic Operators* [Russian]. Extended Abstract of Cand. Sci. Dissertation, Leningrad Univ., Leningrad (1961).
23. Reshetnyak Yu. G., *Space Mappings with Bounded Distortion*, Amer. Math. Soc., Providence (1989).
24. Mostow G. D., "Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms," Inst. Hautes Études Sci. Publ. Math., vol. 34, no. 1, 53–104 (1968).
25. Reiman M., "Über harmonishe Kapazität und quasikonforme Abbildungen in Raum," Comm. Math. Helv., vol. 44, 284–307 (1969).

26. Väisälä J., *Lectures on n-Dimensional Quasiconformal Mappings*, Springer, Berlin (1971) (Lecture Notes in Math.; V. 229).
27. Gehring F. W., "Lipschitz mappings and the p-capacity of rings in n-space," in: *Advances in the Theory of Riemann Surfaces*, Princeton Univ., Princeton (1971), 175–193.
28. Lelong-Ferrand J., "Etude d'une classe d'applications liées à des homomorphismes d'algèbres de fonctions et généralisant les quasi-conformes," Duke Math., vol. 40, 163–186 (1973).
29. Vodopyanov S. K. and Goldshtein V. M., "Lattice isomorphisms of the spaces W_n^1 and quasiconformal mappings," Sib. Math. J., vol. 16, no. 2, 174–189 (1975).
30. Vodopyanov S. K. and Goldshtein V. M., "Functional characteristics of quasi-isometric mappings," Sib. Math. J., vol. 17, no. 4, 580–584 (1976).
31. Romanov A. S., "A change of variable in the Bessel and Riesz potential spaces," in: *Functional Analysis and Mathematical Physics* [Russian], Inst. Mat., Novosibirsk (1985), 117–133.
32. Vodopyanov S. K., "L_p-Potential theory and quasiconformal mappings on homogeneous groups," in: *Modern Problems of Geometry and Analysis* [Russian], Nauka, Novosibirsk (1989), 45–89.
33. Vodopyanov S. K., "Composition operators on Sobolev spaces," in: *Complex Analysis and Dynamical Systems. II:* A Conf. in Honor of Professor Lawrence Zalcman's Sixtieth Birthday, June 9–12, 2003, Nahariya, Israel. Eds.: M. Agranovsky, L. Karp, D. Shoikhet, Amer. Math. Soc., Ann Arbor, 327–342, 2005 (AMS Contemp. Math.; V. 382).
34. Vodopyanov S. K. and Evseev N. A., "Isomorphisms of Sobolev spaces on Carnot groups and quasi-isometric mappings," Sib. Math. J., vol. 55, no. 5, 817–848 (2014).
35. Vodopyanov S. K. and Evseev N. A., "Isomorphisms of Sobolev spaces on Carnot groups and metric mapping properties," Dokl. Math., vol. 82, no. 2, 232–236 (2015).
36. Vodopyanov S. K. and Evseev N. A., "Isomorphisms of Sobolev spaces on Carnot groups and quasiconformal mappings," Sib. Math. J., vol. 56, no. 5, 789–821 (2015).
37. Vodopyanov S. K., "On admissible changes of variables for Sobolev functions on (sub)Riemannian manifolds," Dokl. Math., vol. 93, no. 3, 318–321 (2016).
38. Vodopyanov S. K., "Admissible changes of variables for Sobolev functions on (sub-)Riemannian manifolds," Sb. Math., vol. 210, no. 1, 59–104 (2019).
39. Vodopyanov S. K., "Isomorphisms of Sobolev spaces on Riemannian manifolds and quasiconformal mappings," Sib. Math. J., vol. 60, no. 5, 774–804 (2019).
40. Molchanova A. and Vodopyanov S., "Injectivity almost everywhere and mappings with finite distortion in nonlinear elasticity," Calculus of Variations and PDE, vol. 59, no. 1 (2020). doi 10.1007/s0052601916714
41. Baykin A. N. and Vodopyanov S. K., "Capacity estimates, Liouville's Theorem, and singularity removal for mappings with bounded (p,q)-distortion," Sib. Math. J., vol. 56, no. 2, 237–261 (2015).
42. Ukhlov A. D. and Vodop'yanov S. K., "Mappings associated with weighted Sobolev spaces," in: *Complex Analysis and Dynamical Systems. III*, Amer. Math. Soc., Ann Arbor and Providence, 2008, 369–382 (AMS Contemp. Math; V. 455).
43. Koskela P., Panka P., and Zhang Yi Ru-Ya., "Ahlfors reflection theorem for p-morphisms," arXiv: 1912.09200v2.
44. Ambrosio L. and Tilli P., *Topics on Analysis in Metric Spaces*, Oxford University, New York (2004).
45. Rado I. and Reichelderfer P. V., *Continuous Transformations in Analysis with an Introduction to Algebraic Topology*, Springer, Berlin (1955).
46. Vodopyanov S. K. and Ukhlov A. D., "Superposition operators in the Lebesgue spaces and differentiability of quasiadditive set functions," Vladikavkaz. Mat. Zh., vol. 4, no. 1, 11–33 (2002).
47. Vodop'yanov S. K. and Ukhlov A. D., "Set functions and their applications in the theory of Lebesgue and Sobolev spaces. I," Siberian Adv. Math., vol. 14, no. 4, 78–125 (2004).
48. Rickman S., *Quasiregular Mappings*, Springer, Berlin etc. (1993) (Universitext).
49. Federer H., *Geometric Measure Theory*, Springer, New York (1960).
50. Evans L. C. and Gariepy R. F., *Measure Theory and Fine Properties of Functions*, CRC, Boca Raton (1992).
51. Mazya V. G., *Sobolev Spaces*, Springer, Berlin (2011).
52. Vodopyanov S. K., "Moduli inequalities for $W^1_{n-1,\mathrm{loc}}$-mappings with weighted bounded (q,p)-distortion," Complex Variables and Elliptic Equations, vol. 65 (2020). doi 10.1080/17476933.2020.1825396
53. Hajłasz P., "Change of variables formula under minimal assumptions," Colloq. Math., vol. LXIV, no. 1, 93–101 (1993).
54. Reshetnyak Yu. G., "Some geometrical properties of functions and mappings with generalized derivatives," Sib. Math. J., vol. 7, no. 4, 704–732 (1966).
55. Martio O. and Malý J., "Lusin's condition (N) and mappings of the class $W^{1,n}$," J. Reine Angew. Math., vol. 48, no. 5, 19–36 (1995).
56. Vodopyanov S. K., "Differentiability of maps of Carnot groups of Sobolev classes," Sb. Math., vol. 194, no. 6, 857–877 (2003).

57. Shvartsman P. A., *A Continuation Theorem for Function Spaces Defined by Local approximation* [Russian]. Extended Abstract of Cand. Sci. Dissertation, Yaroslavsk. Univ., Yaroslavl (1983).
58. Martio O., Rickman S., and Väisälä J., “Definitions for quasiregular mappings,” Ann. Acad. Sci. Fenn. Math., vol. 448, 1–40 (1969).
59. Kruglikov V. I., “Capacity of condensers and spatial mappings quasiconformal in the mean,” Math. USSR-Sb., vol. 58, no. 1, 185–205 (1987).
60. Brakalova M. A. and Jenkins J. A., “On solutions of the Beltrami equation,” J. d’Anal. Math., vol. 76, 67–92 (1998).
61. Salimov R. R., “ACL and differentiability of a generalization of quasi-conformal maps,” Izv. Math., vol. 72, no. 5, 977–984 (2008).
62. Salimov R. and Sevost’yanov E., “ACL and differentiability of open discrete ring (p, Q)-mappings,” Mat. Studii, vol. 35, 28–36 (2011).
63. Trokhimchuk Yu. Yu., *Continuous Mappings and Conditions of Monogeneity*, Isr. Program for Scientific Translations, Jerusalem (1964).
64. Afanas’eva O. S., Ryazanov V. I., and Salimov R. R., “Toward the theory of the Sobolev classes with critical exponent,” Dopov. Nac. Akad. Nauk Ukr., vol. 8, 3–8 (2019).
65. Tengvall V., “Differentiability in the Sobolev space $W^{1,n-1}$,” Ann. Acad. Sci. Fenn. Ser. AI, vol. 51, no. 12, 381–399 (2014).
66. Mostow G. D., “Quasi-conformal mappings in n -space and the rigidity of hyperbolic space forms,” Inst. Hautes Études Sci. Publ. Math., vol. 34, 53–104 (1968).
67. Korányi A. and Reimann H. M., “Foundations for the theory of quasiconformal mappings on the Heisenberg group,” Adv. Math., vol. 111, 1–87 (1995).
68. Menshoff D., “Sur les conditions suftisantes pour qu’une fonction univalente soit holomorphe,” Mat. Sb., vol. 40, no. 1, 3–23 (1933).
69. Pansu P., “Métriques de Carnot–Carathéodory et quasiisométries des espacies symétriques de rang un,” Ann. Math., vol. 129, no. 2, 1–60 (1989).
70. Folland G. B. and Stein E. M., *Hardy Spaces on Homogeneous Groups*, Princeton Univ., Princeton (1982) (Math. Notes; V. 28).
71. Bonfiglioli A., Lanconelli E., and Uguzzoni F., *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer, Berlin and Heidelberg (2007) (Universitext).
72. Stein E. M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ., Princeton (1993) (Universitext).
73. Vodop’yanov S. K., “ -Differentiability on Carnot groups in different topologies and related topics,” in: *Proceedings on Analysis and Geometry*, Sobolev Institute, Novosibirsk (2000), 603–670.
74. De Guzmán M., *Differentiation of Integrals in* $\mathbb{R}^n$, Springer, Berlin (1975) (Lecture Notes in Math., V. 481).
75. Karmanova M. and Vodopyanov S., “A coarea formula for smooth contact mappings of Carnot–Carathéodory spaces,” Acta Appl. Math., vol. 128, 67–111 (2013).
76. Vodopyanov S. K. and Greshnov A. V., “Analytic properties of quasiconformal mappings on Carnot groups,” Sib. Math. J., vol. 36, no. 6, 1142–1151 (1995).

S. K. Vodopyanov
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: vodopis@math.nsc.ru