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ON SOLVABILITY OF ONE CLASS OF QUASIELLIPTIC SYSTEMS

L. N. Bondar and G. V. Demidenko

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Abstract: We study the class of systems of differential equations defined by one class of matrix quasielliptic operators and establish solvability conditions for the systems and boundary value problems on $\mathbb{R}^n_+$ in the special scales of weighted Sobolev spaces $W^l_{p,\sigma}$. We construct the integral representations of solutions and obtain estimates for the solutions.

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1. Introduction

We consider the class of quasielliptic systems in the whole space

$$\mathscr{L}(D_x)U = F(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

and the boundary value problems for them in the half-space

$$\begin{cases} \mathscr{L}(D_x)U = F(x), \quad x \in \mathbb{R}^n_+, \\ \mathscr{B}(D_x)U|_{x_n=0} = 0. \end{cases} \tag{1.2}$$

These systems are determined by some class of matrix quasielliptic operators $\mathscr{L}(D_x)$.

Let us specify the conditions on the operators $\mathscr{L}(D_x)$ and $\mathscr{B}(D_x)$. Denote by $l_{j,r}(i\xi)$ and $b_{j,r}(i\xi)$ the entries of the matrices $\mathscr{L}(i\xi)$ and $\mathscr{B}(i\xi)$ which are the symbols of the corresponding differential operators.

Condition 1. *Let $\mathscr{L}(i\xi)$ be a $(\nu \times \nu)$-matrix. Suppose that there are vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\mathbf{t} = (t_1, \dots, t_\nu)$, with $t_r > 0$ and $t_r/\alpha_j \in \mathbb{N}$ such that*

$$l_{j,r}(c^\alpha i\xi) = c^{t_r}\, l_{j,r}(i\xi), \quad j, r = 1, \dots, \nu,$$

for all $c > 0$.

Condition 2. $\det \mathscr{L}(i\xi) = 0$, $\xi \in \mathbb{R}^n$, *if and only if* $\xi = 0$.

The matrix operators $\mathscr{L}(D_x)$ satisfying Conditions 1 and 2 belong to the class of *quasielliptic* operators which was introduced by Volevich [1]. In particular, the following operators belong to the class of operators under consideration: homogeneous elliptic operators, Petrovskii's elliptic and parabolic operators, Eidelman's parabolic operators, backwards parabolic operators, and homogeneous quasielliptic operators (see [2]).

Condition 2 implies that the equation

$$\det \mathscr{L}(is, i\lambda) = 0, \quad s \in \mathbb{R}^{n-1} \backslash \{0\}, \tag{1.3}$$

has no real roots in λ. Denote the number of roots in the upper half-plane by μ. We assume that μ is independent of $s \in \mathbb{R}^{n-1} \backslash \{0\}$.

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Condition 3. *Let $\mathscr{B}(i\xi)$ be a $(\mu\times\nu)$-matrix. Suppose that there is a vector $(m_1,\dots,m_\mu)$ satisfying $t_r - t_{\min} \le t_r - m_j \le t_r - \alpha_n$, $t_{\min} = \min(t_1,\dots,t_\nu)$ and such that*

$$b_{j,r}(c^{\alpha} i\xi) = c^{t_r - m_j} b_{j,r}(i\xi), \quad c>0,\ j=1,\dots,\mu,\ r=1,\dots,\nu.$$

Condition 4. *Boundary value problem (1.2) satisfies the Lopatinskii condition; i.e., the boundary value problem on the half-axis*

$$\begin{cases} \mathscr{L}(is, D_{x_n})v = 0, \quad x_n>0, \\ \mathscr{B}(is, D_{x_n})v|_{x_n=0} = \varphi, \\ \sup\limits_{x_n>0} |v| < \infty \end{cases} \tag{1.4}$$

is uniquely solvable for all $s\in\mathbb{R}^{n-1}\backslash\{0\}$ and φ.

The goal of this article is to obtain solvability conditions and construct solutions to the quasielliptic systems (1.1) in $\mathbb{R}^n$ and to their boundary value problems (1.2) on $\mathbb{R}^n_+$. We search solutions to the problems in the special scales of weighted Sobolev spaces $W^l_{p,\sigma}$ which were introduced in [3].

Recall the definition of $W^{\mathbf{k}/\alpha}_{p,\sigma}(\mathbb{R}^n)$, with $\mathbf{k}/\alpha = (k/\alpha_1,\dots,k/\alpha_n)$, $k/\alpha_i\in\mathbb{N}$, $1<p<\infty$, and $\sigma\ge 0$.

Definition. A locally summable function $u(x)$ *belongs to* $W^{\mathbf{k}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ if $u(x)$ has the generalized derivatives $D^\beta_x u(x)$ on $\mathbb{R}^n$ for $\beta\alpha\le k$, and

$$\big\|(1+\langle x\rangle)^{-\sigma(k-\beta\alpha)} D^\beta_x u(x), L_p(\mathbb{R}^n)\big\| < \infty, \quad \langle x\rangle^2 = \sum_{i=1}^n x_i^{2/\alpha_i}.$$

The norm on $W^{\mathbf{k}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ is defined as follows:

$$\big\|u(x), W^{\mathbf{k}/\alpha}_{p,\sigma}(\mathbb{R}^n)\big\| = \sum_{0\le\beta\alpha\le k} \big\|(1+\langle x\rangle)^{-\sigma(k-\beta\alpha)} D^\beta_x u(x), L_p(\mathbb{R}^n)\big\|. \tag{1.5}$$

In the isotropic case when $k/\alpha_1 = \dots = k/\alpha_n = \bar{l}$, (1.5) is equivalent to

$$\sum_{0\le|\beta|\le\bar{l}} \big\|(1+|x|)^{-\sigma(\bar{l}-|\beta|)} D^\beta_x u(x), L_p(\mathbb{R}^n)\big\|.$$

In this case, the spaces under consideration coincide for $\sigma=1$ with the spaces often called the Nirenberg–Walker–Cantor spaces $M^p_{\bar{l},m}(\mathbb{R}^n)$, with $m=-\bar{l}$ (for instance, see [4–6]).

We consider the general case when the entries k/α_i of the vector of smoothness can differ from one another.

Recall that $C^\infty_0(\mathbb{R}^n)$ is everywhere dense in $W^{\mathbf{k}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ for $\sigma\le 1$ (see [3]). Henceforth, we assume that $1\ge\sigma\ge 0$.

Introduce the weighted Sobolev space $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ for vector-functions, where the parameters of smoothness are determined by the vectors $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\mathbf{t} = (t_1,\dots,t_\nu)$ from Condition 1.

We say that a vector-function $U(x) = (U^1(x),\dots,U^\nu(x))^T$ *belongs to*

$$\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n) = \prod_{k=1}^{\nu} W^{\mathbf{t_k}/\alpha}_{p,\sigma}(\mathbb{R}^n), \quad \mathbf{t_k}/\alpha = (t_k/\alpha_1,\dots,t_k/\alpha_n),\ 1<p<\infty,\ \sigma\ge 0,$$

if each of the entries $U^k(x)$ belongs to $W^{\mathbf{t_k}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ and we set

$$\big\|U(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)\big\| = \sum_{k=1}^{\nu} \big\|U^k(x), W^{\mathbf{t_k}/\alpha}_{p,\sigma}(\mathbb{R}^n)\big\|.$$

We also write

$$\mathbf{L}_p(\mathbb{R}^n) = \prod_{r=1}^{\nu} L_p(\mathbb{R}^n).$$

Definition. Denote by $L_{p,\gamma}(\mathbb{R}^n)$ the weighted space with the norm

$$\|f(x), L_{p,\gamma}(\mathbb{R}^n)\| = \|(1+\langle x\rangle)^{-\gamma} f(x), L_p(\mathbb{R}^n)\|.$$

We say that a vector-function $F(x) = (F^1(x), \dots, F^\nu(x))^T$ *belongs to*

$$\mathbf{L}_{p,\gamma}(\mathbb{R}^n) = \prod_{r=1}^{\nu} L_{p,\gamma}(\mathbb{R}^n)$$

if each of the entries $F^k(x)$ belongs to $L_{p,\gamma}(\mathbb{R}^n)$, and we set

$$\|F(x), \mathbf{L}_{p,\gamma}(\mathbb{R}^n)\| = \sum_{r=1}^{\nu} \|F^r(x), L_{p,\gamma}(\mathbb{R}^n)\|.$$

Similarly, we define $\mathbf{L}_p(\mathbb{R}^n_+)$, $\mathbf{L}_{p,\gamma}(\mathbb{R}^n_+)$, $W^{\mathbf{t_k}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$, and $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$.

2. Statement of the Main Results

We recall now that the first theorem about isomorphism for the class of quasielliptic operators $\mathscr{L}(D_x)$ in the case when $t_1 = \dots = t_\nu = 1$ was proven in [7] by using the property of $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)$. The unique solvability of "homogeneous" quasielliptic systems ensues from the theorem. Properties of quasielliptic operators for $t_1 = \dots = t_\nu = 1$ were further studied in [8]. Some theorems of isomorphism for wider classes of quasielliptic operators were proved in [2, 9–11].

Observe that the Sobolev spaces with power weights (for instance, see [5, 6, 12–15]) are also used in the theorems about isomorphic properties of elliptic operators.

Let us recall the theorem of isomorphism for the class of quasielliptic operators under consideration which was established in [2]. Put

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad t_{\max} = \max\{t_1, \dots, t_\nu\}.$$

Theorem 1. *Let an operator $\mathscr{L}(D_x)$ meet Conditions* 1 *and* 2. *If $|\alpha|/p > t_{\max}$, then*

$$\mathscr{L}(D_x) : \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n) \longrightarrow \mathbf{L}_p(\mathbb{R}^n), \quad 1 < p < \infty,\ \sigma = 1,$$

is an isomorphism.

The unique solvability of (1.1) is straightforward from Theorem 1. Namely, if $|\alpha|/p > t_{\max}$ then for every $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$ there is a unique solution $U(x) \in \mathbf{W}^{\mathbf{t}/\alpha}_{p,1}(\mathbb{R}^n)$ to (1.1) and $\|U(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,1}(\mathbb{R}^n)\| \le c\|F(x), \mathbf{L}_p(\mathbb{R}^n)\|$ with a constant $c > 0$ independent of $F(x)$.

The condition $|\alpha|/p > t_{\max}$ is essential for the unique solvability of (1.1) in $\mathbf{W}^{\mathbf{t}/\alpha}_{p,1}(\mathbb{R}^n)$. However, the condition imposes rather stringent constraints on the degree of summability p and $\sigma = 1$. In the following theorem, we establish unique solvability conditions for system (1.1) in the whole scale of weighted Sobolev spaces $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ with significantly weaker constraints on the parameters p and σ.

Theorem 2. *Let an operator $\mathscr{L}(D_x)$ meet Conditions 1 and 2. If*

$$|\alpha|/p > \sigma t_{\max}, \quad |\alpha|/p' > (1-\sigma)t_{\max}, \quad 1/p + 1/p' = 1, \tag{2.1}$$

then for every vector-function $F(x) \in \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)$ there is a unique solution $U(x) \in \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ to (1.1); moreover,

$$\|U(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)\| \le C\|F(x), \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)\| \tag{2.2}$$

with a constant $C > 0$ independent of $F(x)$.

Theorem 3. *Let the conditions of Theorem 2 be met. Then*

$$\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)}D_x^\beta U^k(x), L_p(\mathbb{R}^n)\big\| \le c\|\langle x\rangle^{(1-\sigma)(t_k-\beta\alpha)}\mathscr{L}(D_x)U(x), \mathbf{L}_p(\mathbb{R}^n)\|, \tag{2.3}$$

$$0 \le \beta\alpha \le t_k, \quad k = 1, \dots, \nu,$$

for all vector-functions $U(x) = (U^1(x), \dots, U^\nu(x))^T \in \mathbf{C}_0^\infty(\mathbb{R}^n)$, *with a constant* $c > 0$ *independent of* $U(x)$.

Remark 1. Theorem 2 generalizes the unique solvability theorem for "homogeneous" quasielliptic systems ($t_1 = \dots = t_\nu = 1$) from [7, 8].

Remark 2. As is demonstrated by examples, in some cases conditions (2.1) are necessary and sufficient for the unique solvability of systems (1.1) in $\mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}^n)$. For instance, it is easily seen for the polyharmonic equation

$$\Delta^m u = f(x), \quad x \in \mathbb{R}^n,\ n > 2m,$$

by using the results from [16, Chapter 12]. Observe that, for solvability of the equation in $W_p^{2m}(\mathbb{R}^n)$ for $p \le \frac{n}{n-2m}$, i.e., $|\alpha|/p' \le 1$, $\sigma = 0$, the right-hand side $f(x)$ needs to satisfy the orthogonality conditions of the form

$$\int\limits_{\mathbb{R}^n} f(x)x^\beta dx = 0, \quad |\beta| \le s(n, m, p).$$

Remark 3. The inequality of the form (2.3) for $\sigma = 1$ is obtained in [4] for elliptic operators; and (2.3) is established in [7] for "homogeneous" quasielliptic operators.

Let state some theorem of unique solvability for (1.2).

Theorem 4. *Let* $\mathscr{L}(D_x)$ *and* $\mathscr{B}(D_x)$ *meet Conditions 1–4. If (2.1) hold, then (1.2) has a unique solution* $U(x) \in \mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}_+^n)$ *for every vector-function* $F(x) \in \mathbf{L}_p(\mathbb{R}_+^n) \cap \mathbf{L}_{1,-\sigma t_{\max}}(\mathbb{R}_+^n)$; *moreover,*

$$\big\|U(x), \mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}_+^n)\big\| \le c(\|F(x), \mathbf{L}_p(\mathbb{R}_+^n)\| + \|F(x), \mathbf{L}_{1,-\sigma t_{\max}}(\mathbb{R}_+^n)\|) \tag{2.4}$$

with a constant $c > 0$ *independent of* $F(x)$.

Remark 4. Theorem 4 generalizes the solvability theorems for boundary value problems for quasielliptic equations in [17, 18]; and for systems in [19–21].

Remark 5. Conditions (2.1) are important for the unique solvability of boundary value problems on $\mathbb{R}_+^n$ for quasielliptic equations and systems. In particular, for $|\alpha|/p < \sigma t_{\max}$, the homogeneous problem can have a nontrivial solution. If $|\alpha|/p' \le (1-\sigma)t_{\max}$, then there exist boundary value problems without solutions in $\mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}_+^n)$ even for compactly-supported infinitely differentiable $F(x)$ (see [18, 22–24]). For instance, if

$$\int\limits_{\Gamma(s)} \mathscr{B}(is, i\lambda)\mathscr{L}^{-1}(is, i\lambda)\, d\lambda \ne 0, \quad s \in \mathbb{R}^{n-1} \setminus \{0\},$$

for $\sigma = 0$, $t_1 = \dots = t_\nu = 1$, where $\Gamma(s)$ is a contour in the complex plane encircling all roots of equation (1.3); then the condition

$$\int\limits_{\mathbb{R}_+^n} F(x)\, dx = 0$$

is necessary for solvability of (1.2) in $\mathbf{W}_p^{\mathbf{1}/\alpha}(\mathbb{R}_+^n)$ for $1 \ge |\alpha|/p' > 1 - \alpha_{\min}$, $p \in (1, 2]$.

3. Proof of Solvability of (1.1)

To prove Theorem 2, we follow the scheme of [2, 7]. Let us sketch the scheme of the proof and inspect the essential differences.

As in [2, 7], to prove the solvability of (1.1), we use the method of constructing approximate solutions which was described in [25] and based on Uspenskii's integral representation [26] for summable functions (see also [25, Chapter 1]).

We suppose first that the entries $F^j(x)$ of $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$ on the right-hand side of (1.1) have compact supports.

Consider the family of integral operators $P_{k,h}$, $k = 1, \dots, \nu$, $0 < h < 1$, of the following form:

$$P_{k,h}F(x) = (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|/t_k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(i \frac{x-y}{v^{\alpha/t_k}} \xi\right) G_k(\xi) \left(\sum_{r=1}^{\nu} l^{k,r}(\xi) F^r(y) \right) d\xi dy dv, \tag{3.1}$$

where $l^{k,r}(\xi)$ are the entries of the inverse matrix $(\mathscr{L}(i\xi))^{-1}$ and

$$G_k(\xi) = 2m\langle\xi\rangle_k^{2m} \exp\bigl(-\langle\xi\rangle_k^{2m}\bigr), \quad \langle\xi\rangle_k^2 = \sum_{i=1}^{n} \xi_i^{2t_k/\alpha_i}, \ m \in \mathbb{N}. \tag{3.2}$$

It is demonstrated in [2] that the vector-function

$$U_h(x) = P_h F(x) = (P_{1,h}F(x), \dots, P_{\nu,h}F(x))^T \tag{3.3}$$

is an approximate solution to (1.1).

Note that (3.1) and Conditions 1 and 2 imply that $U_h^k(x) = P_{k,h}F(x)$ are infinitely differentiable and, obviously, we can find a natural m_1 such that $U_h^k(x)$ in (3.2) are summable to every power $p \geq 1$ for $m \geq m_1$. Henceforth, we assume that $m \geq m_1$ in (3.2).

It ensues from the lemmas below that, under conditions (2.1),

$$\bigl\|U_h(x),\ \mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}^n)\bigr\| \leq C\|F(x),\ \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)\| \tag{3.4}$$

with a constant $C > 0$ independent of $F(x)$ and h; moreover,

$$\bigl\|U_{h_1}(x) - U_{h_2}(x),\ \mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}^n)\bigr\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{3.5}$$

First, we recall the estimates of [2] of the leading derivatives of the entries of (3.3).

Lemma 3.1. *Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha = t_k$, $k = 1, \dots, \nu$. Then*

$$\bigl\|D_x^\beta U_h^k(x),\ L_p(\mathbb{R}^n)\bigr\| \leq C_\beta \|F(x),\ \mathbf{L}_p(\mathbb{R}^n)\| \tag{3.6}$$

with a constant $C_\beta > 0$ independent of $F(x)$ and h; moreover,

$$\bigl\|D_x^\beta U_{h_1}^k(x) - D_x^\beta U_{h_2}^k(x),\ L_p(\mathbb{R}^n)\bigr\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{3.7}$$

To estimate the norms of the derivatives

$$D_x^\beta U_h^k(x), \quad 0 \leq \beta\alpha < t_k,\ k = 1, \dots, \nu, \tag{3.8}$$

we need some estimates for the integrals

$$\mathscr{K}_{\beta,h}^{k,r}(x) = \int_h^{h^{-1}} v^{-|\alpha|/t_k - \beta\alpha/t_k} \int_{\mathbb{R}^n} \exp\left(i \frac{x\xi}{v^{\alpha/t_k}}\right) G_k(\xi)(i\xi)^\beta l^{k,r}(\xi)\, d\xi dv, \tag{3.9}$$

$$x \in \mathbb{R}^n, \quad 0 < h < 1.$$

The following analog of Lemma 3.3 from [9] takes place:

Lemma 3.2. *Let $|\alpha| + \beta\alpha > t_k$. Then there is m_2 such that for $m \ge m_2$ in (3.2) we have*

$$\langle x\rangle^{|\alpha|+\beta\alpha-t_k}\big|\mathscr{K}^{k,r}_{\beta,h}(x)\big| \le c, \quad x \in \mathbb{R}^n, \tag{3.10}$$

with a constant $c > 0$ independent of h and x.

Proof. Consider the function

$$J(x) = \langle x\rangle^{|\alpha|+\beta\alpha-t_k}\mathscr{K}^{k,r}_{\beta,h}(x), \quad x \in \mathbb{R}^n.$$

Making the change $v = \omega\langle x\rangle^{t_k}$ for $x \ne 0$ in (3.9), rewrite this function as

$$J(x) = \int\limits_{h/\langle x\rangle^{t_k}}^{h^{-1}/\langle x\rangle^{t_k}} \omega^{-|\alpha|/t_k-\beta\alpha/t_k}\int\limits_{\mathbb{R}^n}\exp\left(i\frac{x\xi}{\langle x\rangle^{\alpha}\omega^{\alpha/t_k}}\right)G_k(\xi)(i\xi)^{\beta}l^{k,r}(\xi)\,d\xi d\omega.$$

By Conditions 1 and 2, $\mu(\xi) = (i\xi)^{\beta}l^{k,r}(\xi)$ is quasihomogeneous in α. Therefore, by the definition of (3.2), we can find $m_2 \in \mathbb{N}$ such that

$$\left|\int\limits_{\mathbb{R}^n} e^{iz\xi}G_k(\xi)(i\xi)^{\beta}l^{k,r}(\xi)\,d\xi\right| \le c'(1+\langle z\rangle)^{-|\alpha|}, \quad z \in \mathbb{R}^n,$$

with a constant independent of z. Hence,

$$|J(x)| \le c'\int\limits_0^{\infty}\omega^{-|\alpha|/t_k-\beta\alpha/t_k}(1+\omega^{-1/t_k})^{-|\alpha|}\,d\omega.$$

Thus, since $|\alpha| + \beta\alpha > t_k$, $0 \le \beta\alpha < t_k$, we obtain $|J(x)| \le c < \infty$; i.e., (3.10).

The lemma is proved.

Henceforth, we assume that $m \ge \max\{m_1, m_2\}$ and turn to estimating the norms of (3.8).

Lemma 3.3. *Let $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta\alpha < t_k$ and $|\alpha|/p > \sigma(t_k - \beta\alpha) > t_k - \beta\alpha - |\alpha|/p'$, $1/p + 1/p' = 1$. Then*

$$\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)}D^{\beta}_x U^k_h(x), L_p(\mathbb{R}^n)\big\| \le c\|\langle x\rangle^{(1-\sigma)(t_k-\beta\alpha)}F(x), \mathbf{L}_p(\mathbb{R}^n)\|, \tag{3.11}$$

with a constant $c > 0$ independent of $F(x)$ and h.

Proof. By (3.1), (3.2), and (3.9), $D^{\beta}_x U^k_h(x)$ can be written as

$$D^{\beta}_x U^k_h(x) = \sum_{r=1}^{\nu}(2\pi)^{-n}\int\limits_{\mathbb{R}^n}\mathscr{K}^{k,r}_{\beta,h}(x-y)F^r(y)\,dy.$$

By Lemma 3.2,

$$\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)}D^{\beta}_x U^k_h(x), L_p(\mathbb{R}^n)\big\|$$
$$\le c\sum_{r=1}^{\nu}\left\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)}\int\limits_{\mathbb{R}^n}\langle x-y\rangle^{-(|\alpha|+\beta\alpha-t_k)}|F^r(y)|\,dy, L_p(\mathbb{R}^n)\right\|.$$

The conditions imply that $|\alpha| + \beta\alpha - t_k > 0$; therefore,

$$\begin{aligned}&\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)} D_x^\beta U_h^k(x), L_p(\mathbb{R}^n)\big\| \\ &\le c_1 \sum_{r=1}^{\nu} \left\| \langle x\rangle^{-\sigma(t_k-\beta\alpha)} \int\limits_{\mathbb{R}^n} \prod_{i=1}^{n} |x_i - y_i|^{(t_k-\beta\alpha)/|\alpha|-1} |F^r(y)|\, dy, L_p(\mathbb{R}^n)\right\|.\end{aligned}$$

Since $t_k - \beta\alpha > 0$, $1 \ge \sigma \ge 0$, we obtain

$$\begin{aligned}&\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)} D_x^\beta U_h^k(x), L_p(\mathbb{R}^n)\big\| \\ &\le c_2 \sum_{r=1}^{\nu} \Bigg\| \int\limits_{\mathbb{R}^n} \prod_{i=1}^{n} |x_i|^{\sigma(\beta\alpha-t_k)/|\alpha|}\, |x_i - y_i|^{(t_k-\beta\alpha)/|\alpha|-1} |y_i|^{(1-\sigma)(\beta\alpha-t_k)/|\alpha|} \\ &\qquad \times \langle y\rangle^{(1-\sigma)(t_k-\beta\alpha)} |F^r(y)| dy,\ L_p(\mathbb{R}^n)\Bigg\|.\end{aligned}$$

Put $a = \sigma(t_k - \beta\alpha)/|\alpha|$ and $b = (1-\sigma)(t_k - \beta\alpha)/|\alpha|$. By hypotheses, $a < 1/p$, $b < 1/p'$, $a + b > 0$. Applying the Hardy–Littlewood inequality [27], we arrive at (3.11).

The lemma is proved.

Lemma 3.4. *Let the conditions of Theorem 2 be met and* $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta\alpha < t_k$. *If* $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$ *is compactly-supported then*

$$\big\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)}\big(D_x^\beta U_{h_1}^k(x) - D_x^\beta U_{h_2}^k(x)\big), L_p(\mathbb{R}^n)\big\| \to 0 \quad \text{as } h_1, h_2 \to 0,\ k = 1, \dots, \nu. \tag{3.12}$$

Proof. Put

$$K_\beta^{k,r}(x) = \int\limits_{\mathbb{R}^n} \exp(ix\xi) G_k(\xi)(i\xi)^\beta l^{k,r}(\xi)\, d\xi. \tag{3.13}$$

From (3.1), (3.3), and (3.9) it follows that

$$D_x^\beta U_h^k(x) = \sum_{r=1}^{\nu} (2\pi)^{-n} \int\limits_{h}^{h^{-1}} v^{-|\alpha|/t_k-\beta\alpha/t_k} \int\limits_{\mathbb{R}^n} K_\beta^{k,r}\left(\frac{x-y}{v^{\alpha/t_k}}\right) F^r(y)\, dy dv.$$

Using this representation and Minkowski's inequality, for $h_2 > h_1 > 0$ we obtain

$$\begin{aligned}&\big\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)}\big(D_x^\beta U_{h_1}^k(x) - D_x^\beta U_{h_2}^k(x)\big), L_p(\mathbb{R}^n)\big\| \\ &\le \sum_{r=1}^{\nu} \int\limits_{h_1}^{h_2} v^{-|\alpha|/t_k-\beta\alpha/t_k} \left\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)} \int\limits_{\mathbb{R}^n} K_\beta^{k,r}\left(\frac{x-y}{v^{\alpha/t_k}}\right) F^r(y)\, dy, L_p(\mathbb{R}^n)\right\| dv \\ &+ \sum_{r=1}^{\nu} \int\limits_{h_2^{-1}}^{h_1^{-1}} v^{-|\alpha|/t_k-\beta\alpha/t_k} \left\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)} \int\limits_{\mathbb{R}^n} K_\beta^{k,r}\left(\frac{x-y}{v^{\alpha/t_k}}\right) F^r(y)\, dy, L_p(\mathbb{R}^n)\right\| dv \\ &= \sum_{r=1}^{\nu} J_r^1(h_1, h_2) + \sum_{r=1}^{\nu} J_r^2(h_1, h_2).\end{aligned}$$

First, estimate $J_r^1(h_1,h_2)$. Applying Young's inequality, we obviously get

$$J_r^1(h_1,h_2) \le \int_{h_1}^{h_2} v^{-|\alpha|/t_k-\beta\alpha/t_k} \left\| K_\beta^{k,r}\left(\frac{x}{v^{\alpha/t_k}}\right), L_1(\mathbb{R}^n)\right\| \|F^r(y), L_p(\mathbb{R}^n)\| dv$$
$$= \int_{h_1}^{h_2} v^{-\beta\alpha/t_k}\, dv \|K_\beta^{k,r}(z), L_1(\mathbb{R}^n)\| \|F^r(y),\ L_p(\mathbb{R}^n)\|.$$

Hence, using (3.13) and the condition $\beta\alpha < t_k$, it follows that

$$J_r^1(h_1,h_2) \to 0 \quad \text{as } h_1,h_2 \to 0. \tag{3.14}$$

Estimate $J_r^2(h_1,h_2)$. Using the inequality

$$(1+\langle x\rangle)^{-1}\langle x-y\rangle \le a(1+\langle y\rangle), \quad a = \text{const}, \tag{3.15}$$

we infer

$$J_r^2(h_1,h_2) \le \int_{h_2^{-1}}^{h_1^{-1}} v^{-|\alpha|/t_k-\beta\alpha/t_k}$$
$$\times \left\| \int_{\mathbb{R}^n} \langle x-y\rangle^{-\sigma(t_k-\beta\alpha)} K_\beta^{k,r}\left(\frac{x-y}{v^{\alpha/t_k}}\right)(1+\langle y\rangle)^{\sigma(t_k-\beta\alpha)} F^r(y)\, dy, L_p(\mathbb{R}^n)\right\| dv.$$

By applying Young's inequality, we obviously get

$$J_r^2(h_1,h_2) \le \int_{h_2^{-1}}^{h_1^{-1}} v^{-|\alpha|/t_k-\beta\alpha/t_k} \left\| \langle x\rangle^{-\sigma(t_k-\beta\alpha)} K_\beta^{k,r}\left(\frac{x}{v^{\alpha/t_k}}\right), L_p(\mathbb{R}^n)\right\| dv$$
$$\times \|(1+\langle y\rangle)^{\sigma(t_k-\beta\alpha)} F^r(y), L_1(\mathbb{R}^n)\|$$
$$= \int_{h_2^{-1}}^{h_1^{-1}} v^{-|\alpha|/p' t_k-\beta\alpha/t_k-\sigma(1-\beta\alpha/t_k)}\, dv \|\langle z\rangle^{-\sigma(t_k-\beta\alpha)} K_\beta^{k,r}(z), L_p(\mathbb{R}^n)\|$$
$$\times \|(1+\langle y\rangle)^{\sigma(t_k-\beta\alpha)} F^r(y),\ L_1(\mathbb{R}^n)\|.$$

Since $|\alpha|/p > \sigma t_{\max}$, $t_k > \beta\alpha$, and $F^r(y)$ is compactly-supported; using (3.13), we infer that

$$J_r^2(h_1,h_2) \to 0 \quad \text{as } h_1,h_2 \to 0. \tag{3.16}$$

Recalling (3.14) and (3.16), we arrive at (3.12). The lemma is proved.

Proof of Theorem 2. Estimates (3.6) and (3.11) yield (3.4), while (3.7) and (3.12) imply (3.5). By completeness of $\mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}^n)$, it follows from (3.4) and (3.5) that there is a continuous linear operator

$$P : \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n) \longrightarrow \mathbf{W}_{p,\sigma}^{\mathbf{t}/\alpha}(\mathbb{R}^n), \quad 1 < p < \infty,\ 0 \le \sigma \le 1,$$

defined on compactly-supported vector-functions $F(x)$ by the formula

$$PF(x) = \lim_{h\to 0} P_h F(x);$$

moreover, the vector-function $U(x) = PF(x) \in \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ is a solution to (1.1). Owing to the density of the set of compactly-supported vector-functions in $\mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)$, the operator P can be uniquely extended to the whole space $\mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)$ with the same norm. We preserve the notation P for the extended operator.

By (3.4), the linear operators

$$P_h : \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n) \longrightarrow \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n), \quad 1 < p < \infty,\ 0 \le \sigma \le 1,$$

are continuous; and the collection of their norms is bounded: $\|P_h\| \le C$. Consequently, by the Banach–Steinhaus Theorem,

$$\big\|P_h F(x) - PF(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)\big\| \to 0 \quad \text{as } h \to 0$$

for all $F(x) \in \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)$.

By the above arguments, some solution $U(x) \in \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n)$ to (1.1) exists for any right-hand side $F(x) \in \mathbf{L}_{p,(\sigma-1)t_{\max}}(\mathbb{R}^n)$; moreover, (2.2) holds.

The uniqueness of the solution to system (1.1) in the space under consideration is proven by analogy with [9].

Theorem 2 is proved.

Proof of Theorem 3. Estimate (2.3) is straightforward from Theorem 2 for $\beta\alpha = t_k$, $k = 1, \dots, \nu$. Assume that $\beta\alpha < t_k$. Putting

$$U^k_h(x) = P_{k,h}F(x), \quad F(x) = \mathscr{L}(D_x)U(x),$$

we derive

$$\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)} D^\beta_x P_{k,h}(\mathscr{L}(D)U)(x), L_p(\mathbb{R}^n)\big\| \le c\|\langle x\rangle^{(1-\sigma)(t_k-\beta\alpha)}\mathscr{L}(D_x)U(x),\ \mathbf{L}_p(\mathbb{R}^n)\|$$

from Lemma 3.3 with a constant $c > 0$ independent of $U(x)$ and h. Then, for every $\varepsilon > 0$ we have

$$\begin{aligned} J_{k,\varepsilon} &= \big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)} D^\beta_x U^k(x), L_p(\{\langle x\rangle \ge \varepsilon\})\big\| \\ &\le \big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)} D^\beta_x P_{k,h}(\mathscr{L}(D)U)(x),\ L_p(\{\langle x\rangle \ge \varepsilon\})\big\| \\ &+\big\|\langle x\rangle^{-\sigma(t_k-\beta\alpha)} D^\beta_x (P_{k,h}(\mathscr{L}(D)U)(x) - U^k(x)),\ L_p(\{\langle x\rangle \ge \varepsilon\})\big\| \\ &\le c\|\langle x\rangle^{(1-\sigma)(t_k-\beta\alpha)}\mathscr{L}(D_x)U(x),\ \mathbf{L}_p(\mathbb{R}^n)\| \\ &+c(\varepsilon)\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)} D^\beta_x (P_{k,h}(\mathscr{L}(D)U)(x) - U^k(x)), L_p(\{\langle x\rangle \ge \varepsilon\})\|, \end{aligned} \tag{3.17}$$

where $c(\varepsilon) = (1 + 1/\varepsilon)^{\sigma(t_k-\beta\alpha)}$. By the proof of Theorem 2,

$$\big\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)} D^\beta_x (P_{k,h}(\mathscr{L}(D)U)(x) - U^k(x)),\ L_p(\mathbb{R}^n)\big\| \to 0 \quad \text{as } h \to 0.$$

Passing to the limit in (3.17) as $h \to 0$ yields

$$J_{k,\varepsilon} \le c\|\langle x\rangle^{(1-\sigma)(t_k-\beta\alpha)}\mathscr{L}(D_x)U(x),\ \mathbf{L}_p(\mathbb{R}^n)\|.$$

Hence, passing to the limit as $\varepsilon \to 0$ implies (2.3).

Theorem 3 is proved.

4. Construction of Approximate Solutions to (1.2)

The proof of Theorem 4 is based on the construction of approximate solutions to boundary value problems for systems with constant coefficients (see [19]). We present the construction in this section.

We suppose that $F(x) \in \mathbf{C}^\infty(\overline{\mathbb{R}^n_+})$ and $F(x) \equiv 0$ for $|x| \gg 1$.

Consider the boundary value problem of the form (1.4) on the half-axis for the system of ordinary differential equations with parameter $s \in \mathbb{R}^{n-1}\backslash\{0\}$:

$$\begin{cases} \mathscr{L}(is, D_{x_n})\omega = \widetilde{F}(s, x_n), \quad x_n > 0, \\ \mathscr{B}(is, D_{x_n})\omega|_{x_n=0} = 0, \\ \sup\limits_{x_n>0} |\omega| < \infty, \end{cases} \tag{4.1}$$

where $\widetilde{F}(s, x_n)$ is the Fourier transform of the vector-function $F(x', x_n)$ with respect to x'.

By the Lopatinskii condition, (4.1) is uniquely solvable. The solution of (4.1) can be represented as

$$\omega(s, x_n) = \omega_0(s, x_n) + v(s, x_n), \tag{4.2}$$

where $\omega_0(s, x_n)$ is a bounded solution to the system

$$\mathscr{L}(is, D_{x_n})\omega = \widetilde{F}(s, x_n), \quad x_n > 0,$$

and the vector-function $v(s, x_n)$ is a solution to (1.4) with

$$\varphi(s) = -\mathscr{B}(is, D_{x_n})\omega_0(s, x_n)|_{x_n=0}.$$

Put $a(is, i\lambda) = \det \mathscr{L}(is, i\lambda)$. Denote by $\widetilde{\mathscr{L}}(is, i\lambda)$ the adjugate matrix to $\mathscr{L}(is, i\lambda)$. The identity

$$\mathscr{L}(is, D_{x_n})\widetilde{\mathscr{L}}(is, D_{x_n})\omega(x_n) \equiv a(is, D_{x_n})\omega(x_n)$$

is obviously valid for sufficiently smooth vector-functions $\omega(x_n)$. Equation (1.3) has no real roots; therefore, the boundary value problem on the axis

$$\begin{cases} a(is, D_{x_n})u = g(x_n), \quad -\infty < x_n < \infty, \\ \sup\limits_{-\infty<x_n<\infty} |u| < \infty \end{cases} \tag{4.3}$$

has a unique solution for $s \in \mathbb{R}^{n-1}\backslash\{0\}$ and every bounded $g(x_n) \in C(\mathbb{R})$. Consequently, using the formula of the solution to (4.3) (for instance, see [25, Chapter 1]), as $\omega_0(s, x_n)$, we can take the bounded vector-function

$$\omega_0(s, x_n) = \widetilde{\mathscr{L}}(is, D_{x_n})R\widetilde{F}(s, x_n) \tag{4.4}$$

where

$$R\widetilde{F}(s, x_n) = \int\limits_0^{x_n} J_+(s, x_n - y_n)\widetilde{F}(s, y_n)\,dy_n + \int\limits_{x_n}^{\infty} J_-(s, x_n - y_n)\widetilde{F}(s, y_n)\,dy_n,$$

$$J_+(s, x_n) = \frac{1}{2\pi}\int\limits_{\Gamma^+} \frac{\exp(ix_n\lambda)}{a(is, i\lambda)}\,d\lambda, \quad J_-(s, x_n) = -\frac{1}{2\pi}\int\limits_{\Gamma^-} \frac{\exp(ix_n\lambda)}{a(is, i\lambda)}\,d\lambda, \tag{4.5}$$

and the contour $\Gamma^+ = \Gamma^+(s)$ encircles all roots of (1.3) in the upper half-plane, while $\Gamma^- = \Gamma^-(s)$ encircles the roots in the lower half-plane. Denote by $\{\omega_1(s, x_n), \dots, \omega_\mu(s, x_n)\}$ the *canonical basis* of (1.4); i.e., each vector-function $\omega_j(s, x_n)$ is a solution to (1.4) with the unit boundary vector $\varphi = e_j$ whose jth entry is 1. Then the vector-function $v(s, x_n)$ from (4.2) can be represented as

$$v(s, x_n) = \sum_{j=1}^{\mu} \varphi^j(s)\omega_j(s, x_n), \tag{4.6}$$

where $\varphi(s) = -\mathscr{B}(is, D_{y_n})\omega_0(s, y_n)|_{y_n=0}$.

By (4.2), (4.4), and (4.6), we derive the representation of the solution to (4.1) in the form

$$\omega(s, x_n) = \widetilde{\mathscr{L}}(is, D_{x_n})\, R\, \widetilde{F}(s, x_n) + \sum_{j=1}^{\mu} \varphi^j(s)\omega_j(s, x_n). \tag{4.7}$$

Construct a solution to (1.2). By applying the inverse Fourier transform to (4.7) with respect to s, we can obtain a formal solution to (1.2). However, the contour integrals (4.5) and the components of the canonical basis of (1.4), in general, have nonintegrable singularities for $s = 0$. Therefore, to obtain a formula for a solution to (1.2) we need to regularize the inverse Fourier transform. To this end, use Uspenskii's integral representation for $f(x') \in L_p(\mathbb{R}^{n-1})$ (see [26]):

$$f(x') = \lim_{h\to 0}(2\pi)^{1-n} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \exp\left(i(x'-y')s\right) G(sv^{\alpha'}) f(y')\, ds dy' dv, \tag{4.8}$$

where

$$G(s) = 2m\langle s\rangle^{2m} \exp(-\langle s\rangle^{2m}), \quad \langle s\rangle^2 = \sum_{i=1}^{n-1} s_i^{2/\alpha_i}, \tag{4.9}$$

and the limit is understood in the sense of convergence in $L_p(\mathbb{R}^{n-1})$ (see [25, Chapter 1]). The natural m can be taken however large. By analogy with [19], introduce the vector-functions

$$U_h(x) = (2\pi)^{(1-n)/2} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{\mathbb{R}^{n-1}} \exp\left(ix's\right) G(sv^{\alpha'})\omega(s, x_n)\, ds dv, \tag{4.10}$$

where $\omega(s, x_n)$ is defined in (4.7). It follows from the above that

$$\begin{cases} \mathscr{L}(D_x)U_h(x) = F_h(x), \quad x \in \mathbb{R}^n_+, \\ \mathscr{B}(D_x)U_h(x', x_n)|_{x_n=0} = 0, \end{cases}$$

where

$$F_h(x) = (2\pi)^{1-n} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{\mathbb{R}^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \exp\left(i(x'-y')s\right) G(sv^{\alpha'}) F(y', x_n)\, ds dy' dv.$$

By (4.8), $U_h(x)$ can be considered as an approximate solution to (1.2), and the existence of a solution to (1.2) reduces to proving the convergence of $\{U_h(x)\}$ in $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$.

To obtain L_p-estimates for $U_h(x)$, we will use

Lemma 4.1. *For all $x_n > 0$, $s \in \mathbb{R}^{n-1}\backslash\{0\}$, γ_n, and $\kappa = (\kappa_1, \dots, \kappa_{n-1})$, the following estimates are valid:*

$$\left|D_s^\kappa\left(D_{x_n}^{\gamma_n}\widetilde{\mathscr{L}}_{r,l}(is, D_{x_n})J_+(s, x_n)\right)\right| \le c\langle s\rangle^{(\gamma_n+1)\alpha_n - \kappa\alpha' - t_r} \exp\left(-\delta x_n\langle s\rangle^{\alpha_n}\right),$$

$$\left|D_s^\kappa\left(D_{x_n}^{\gamma_n}\widetilde{\mathscr{L}}_{r,l}(is, D_{x_n})J_-(s, -x_n)\right)\right| \le c\langle s\rangle^{(\gamma_n+1)\alpha_n - \kappa\alpha' - t_r} \exp\left(-\delta x_n\langle s\rangle^{\alpha_n}\right),$$

$$\left|D_{x_n}^{\gamma_n}D_s^\kappa\omega_j^r(s, x_n)\right| \le c\langle s\rangle^{\gamma_n\alpha_n - \kappa\alpha' + m_j - t_r} \exp\left(-\delta x_n\langle s\rangle^{\alpha_n}\right),$$

$$r, l = 1, \dots, \nu, \quad j = 1, \dots, \mu,$$

where c and $\delta > 0$ are constant, $\widetilde{\mathscr{L}}_{r,l}(is, i\xi_n)$ are the entries of $\widetilde{\mathscr{L}}(is, i\xi_n)$, and $\omega_j^r(s, x_n)$ are the entries of $\omega_j(s, x_n)$.

Proof. This repeats the proof of Lemma 4.2 from [25, Chapter 4].

5. Solvability of (1.2)

Consider the vector-function $U_h(x)$ from (4.10) and represent it, in accord with [19], as

$$U_h(x) = U_{0,h}(x) + \sum_{j=1}^{\mu} U_{j,h}(x), \tag{5.1}$$

where

$$U_{0,h}(x) = \int\limits_{\mathbb{R}^{n-1}} \exp(ix's)\mathscr{G}(s,h)\widetilde{\mathscr{L}}(is, D_{x_n})R\widetilde{F}(s,x_n)\,ds, \tag{5.2}$$

$$U_{j,h}(x) = \int\limits_{\mathbb{R}^{n-1}} \exp(ix's)\mathscr{G}(s,h)\varphi^j(s)\omega_j(s,x_n)\,ds, \tag{5.3}$$

$$\mathscr{G}(s,h) = (2\pi)^{(1-n)/2}\int\limits_{h}^{h^{-1}} v^{-1}G(sv^{\alpha'})\,dv, \tag{5.4}$$

$$\varphi^j(s) = -\mathscr{B}_j(is, D_{y_n})\widetilde{\mathscr{L}}(is, D_{y_n})\,R\widetilde{F}(s,y_n)|_{y_n=0},$$

$$\mathscr{B}_j(is, D_{y_n}) = (b_{j,1}(is, D_{y_n}), \dots, b_{j,\nu}(is, D_{y_n})).$$

Denote the kth entry of $U_{0,h}(x)$ by $U^k_{0,h}(x)$.

Lemma 5.1. *Let $\beta = (\beta_1, \dots, \beta_n)$ and $\beta\alpha = t_k$, $k = 1, \dots, \nu$. Then*

$$\left\|D^\beta_x U^k_{0,h}(x),\, L_p(\mathbb{R}^n_+)\right\| \le c\left\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\right\|, \quad 0 < h < 1, \tag{5.5}$$

with $c > 0$ independent of h and $F(x)$; moreover,

$$\left\|D^\beta_x U^k_{0,h_1}(x) - D^\beta_x U^k_{0,h_2}(x), L_p(\mathbb{R}^n_+)\right\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{5.6}$$

Proof. This repeats the proof from [19]; we focus on some changes.

Extend the vector-function $F(x', x_n)$ by zero for $x_n < 0$. Denote the Fourier transform of $F(x)$ in $x = (x', x_n)$ by $\widehat{F}(\xi)$. As demonstrated in [19],

$$\widetilde{\mathscr{L}}(is, D_{x_n})R\widetilde{F}(s,x_n) \equiv (2\pi)^{-1/2}\int\limits_{-\infty}^{\infty} e^{ix_n\xi_n}(\mathscr{L}(is, i\xi_n))^{-1}\widehat{F}(s,\xi_n)\,d\xi_n \tag{5.7}$$

for $s \in \mathbb{R}^{n-1}\backslash\{0\}$. By (5.7), for $\beta\alpha = t_k$ we obtain

$$D^\beta_x U^k_{0,h}(x) = (2\pi)^{-n/2}\int\limits_{h}^{h^{-1}} v^{-1}\int\limits_{\mathbb{R}^n} \exp(ix\xi)G(sv^{\alpha'})(i\xi)^\beta((\mathscr{L}(i\xi))^{-1})_k\widehat{F}(\xi)\,d\xi dv, \quad \xi = (s, \xi_n),$$

where $((\mathscr{L}(i\xi))^{-1})_k$ is the kth row of $(\mathscr{L}(i\xi))^{-1}$. By the Lizorkin Theorem [28] and Conditions 1 and 2, the elements of the row $\xi^\beta((\mathscr{L}(i\xi))^{-1})_k$, $\beta\alpha = t_k$, are multipliers. Using this and the properties of Uspenskii's representation [25], we come to (5.5) and (5.6).

Lemma 5.2. *Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha = t_k$, $k = 1, \dots, \nu$. Then*

$$\big\| D_x^\beta U_{j,h}^k(x), L_p(\mathbb{R}_+^n) \big\| \le c \| F(x), \mathbf{L}_p(\mathbb{R}_+^n) \|, \quad 0 < h < 1, \tag{5.8}$$

with $c > 0$ independent of h and $F(x)$; moreover,

$$\big\| D_x^\beta U_{j,h_1}^k(x) - D_x^\beta U_{j,h_2}^k(x), L_p(\mathbb{R}_+^n) \big\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{5.9}$$

Proof. Consider the kth entry of the vector-function $U_{j,h}(x)$ from (5.3). Estimate $D_x^\beta U_{j,h}^k(x)$, where $\beta\alpha = t_k$. We follow the scheme of [19]. Represent $D_x^\beta U_{j,h}^k(x)$ as the sum of two summands:

$$\begin{gathered} D_x^\beta U_{j,h}^k(x) = \int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\infty \exp(ix's)(is)^{\beta'} \mathscr{G}(s,h) \\ \times B_j(is, D_{y_n}) \widetilde{\mathscr{L}}(is, D_{y_n}) R\widetilde{F}(s, y_n) D_{y_n}^{\beta_n+1} \omega_j^k(s, x_n + y_n)\, dy_n ds dv \\ + \int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\infty \exp(ix's)(is)^{\beta'} \mathscr{G}(s,h) \\ \times B_j(is, D_{y_n}) D_{y_n} \widetilde{\mathscr{L}}(is, D_{y_n}) R\widetilde{F}(s, y_n) D_{x_n}^{\beta_n} \omega_j^k(s, x_n + y_n)\, dy_n ds \\ = \Phi_{j,h,k}^1(x) + \Phi_{j,h,k}^2(x), \end{gathered} \tag{5.10}$$

where $\mathscr{G}(s,h)$ is defined in (5.4). Consider the first summand. By the property of the Fourier transform, for $x_n > 0$ we obtain

$$\begin{gathered} \Phi_{j,h,k}^1(x) = \int\limits_{\mathbb{R}^n} \exp(ix's)(is)^{\beta'} \mathscr{G}(s,h) \\ \times B_j(is, D_{y_n}) \widetilde{\mathscr{L}}(is, D_{y_n}) R\widetilde{F}(s, y_n) \theta(y_n) D_{y_n}^{\beta_n+1} \omega_j^k(s, x_n + y_n) \theta(x_n + y_n)\, dy_n ds \\ = (2\pi)^{-1} \int\limits_{\mathbb{R}^n} \exp(ix's - ix_n\xi_n) \langle s \rangle^{m_j - t_k + \beta_n \alpha_n} (is)^{\beta'} \mu_{j,k}(s, \xi_n) \\ \times \Bigg(\int\limits_R \exp(-it_n\xi_n) \mathscr{G}(s,h) B_j(is, D_{t_n}) \widetilde{\mathscr{L}}(is, D_{t_n}) R\widetilde{F}(s, t_n) \theta(t_n)\, dt_n \Bigg) ds d\xi_n, \end{gathered}$$

where

$$\mu_{j,k}(s, \xi_n) = \langle s \rangle^{t_k - m_j - \beta_n \alpha_n} \int\limits_0^\infty \exp(iy_n\xi_n) D_{y_n}^{\beta_n+1} \omega_j^k(s, y_n)\, dy_n.$$

Conditions 1–4 yield

$$\omega_j^k(s, x_n) = c^{t_k - m_j} \omega_j^k(c^{\alpha'} s, c^{-\alpha_n} x_n), \quad c > 0;$$

and so, by the Lizorkin Theorem [28], $\mu_{j,k}(s, \xi_n)$ is a multiplier. Then

$$\begin{gathered} \big\| \Phi_{j,h,k}^1(x), L_p(\mathbb{R}_+^n) \big\| \le c_1 \Bigg\| \int\limits_{\mathbb{R}^{n-1}} \exp(ix's)(is)^{\beta'} \mathscr{G}(s,h) \\ \times \langle s \rangle^{m_j - t_k + \beta_n \alpha_n} B_j(is, D_{x_n}) \widetilde{\mathscr{L}}(is, D_{x_n}) R\widetilde{F}(s, x_n) \theta(x_n)\, ds, L_p(\mathbb{R}^n) \Bigg\| \, dv. \end{gathered}$$

Since the elements

$$(\langle s\rangle^{\alpha_n}+i\xi_n)\langle s\rangle^{m_j-\alpha_n}B_j(is,i\xi_n)\mathscr{L}^{-1}(is,i\xi_n) \tag{5.11}$$

are multipliers, we obtain

$$\left\|\Phi^1_{j,h,k}(x),L_p(\mathbb{R}^n_+)\right\|\le c_2\left\|\int\limits_{\mathbb{R}^n}\exp\left(ix's+ix_n\xi_n\right)\mathscr{G}(s,h)(is)^{\beta'}\langle s\rangle^{-t_k+\beta_n\alpha_n}\right.$$
$$\left.\times\frac{\langle s\rangle^{\alpha_n}}{\langle s\rangle^{\alpha_n}+i\xi_n}\widehat{F}(s,\xi_n)\,d\xi_n ds,\mathbf{L}_p(\mathbb{R}^n_+)\right\|.$$

By applying

$$\int\limits_{\mathbb{R}}\frac{\exp\left(ix_n\xi_n\right)}{\langle s\rangle^{\alpha_n}+i\xi_n}\widehat{F}(s,\xi_n)\,d\xi_n=(2\pi)^{-1/2}\int\limits_0^{x_n}\exp\left(-(x_n-y_n)\langle s\rangle^{\alpha_n}\right)\widetilde{F}(s,y_n)\,dy_n$$

and Young's inequality, we derive

$$\left\|\Phi^1_{j,h,k}(x),L_p(\mathbb{R}^n_+)\right\|$$
$$\le c_4\left\|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^{n-1}}\exp\left(i(x'-y')s\right)\exp\left(-(x_n-y_n)\langle s\rangle^{\alpha_n}\right)\theta(x_n-y_n)\right.$$
$$\left.\times(is)^{\beta'}\langle s\rangle^{\alpha_n-t_k+\beta_n\alpha_n}\mathscr{G}(s,h)F(y)\theta(y_n)\,dsdy,\mathbf{L}_p(\mathbb{R}^n)\right\|$$
$$\le c_5\left\|\int\limits_{\mathbb{R}^{n-1}}\exp\left(ix's\right)\exp\left(-x_n\langle s\rangle^{\alpha_n}\right)\theta(x_n)\right.$$
$$\left.\times\mathscr{G}(s,h)(is)^{\beta'}\langle s\rangle^{\alpha_n-t_k+\beta_n\alpha_n}\,ds,L_1(\mathbb{R}^n)\right\|\|F(y),\mathbf{L}_p(\mathbb{R}^n_+)\|.$$

Since

$$\int\limits_h^{h^{-1}}v^{-1}G(sv^{\alpha'})\,dv\equiv\exp\left(-h^{2m}\langle s\rangle^{2m}\right)-\exp\left(-h^{-2m}\langle s\rangle^{2m}\right),$$

by Minkowski's inequality and the fact that $\beta'\alpha'+\beta_n\alpha_n=t_k$, we obtain

$$\left\|\Phi^1_{j,h,k}(x),L_p(\mathbb{R}^n_+)\right\|\le c_6\left\|\int\limits_{\mathbb{R}^{n-1}}\exp\left(ix's\right)\exp\left(-x_n\langle s\rangle^{\alpha_n}\right)\theta(x_n)\right.$$
$$\left.\times\exp\left(-\langle s\rangle^{2m}\right)(is)^{\beta'}\langle s\rangle^{\alpha_n-t_k+\beta_n\alpha_n}\,ds,L_1(\mathbb{R}^n)\right\|\|F(y),\mathbf{L}_p(\mathbb{R}^n_+)\|.$$

Choosing m in (4.9) sufficiently large, we arrive at the sought estimate for $\Phi^1_{j,h,k}(x)$. Analogous arguments for $\Phi^2_{j,h,k}(x)$ yield (5.8). Convergence (5.9) is proved similarly.

The lemma is proved.

Lemma 5.3. *Let $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta\alpha < t_k$, and $|\alpha|/p > \sigma(t_k - \beta\alpha) > t_k - \beta\alpha - |\alpha|/p'$, $1/p + 1/p' = 1$. Then*

$$\begin{aligned} &\big\|(1 + \langle x\rangle)^{-\sigma(t_k-\beta\alpha)} D_x^\beta U_{0,h}^k(x), L_p(\mathbb{R}_+^n)\big\| \\ &\le c(\|F(x), \mathbf{L}_p(\mathbb{R}_+^n)\| + \|(1 + \langle x\rangle)^{\sigma(t_k-\beta\alpha)} F(x), \mathbf{L}_1(\mathbb{R}_+^n)\|), \quad 0 < h < 1, \end{aligned} \tag{5.12}$$

with a constant $c > 0$ independent of $F(x)$ and h; moreover,

$$\big\|(1 + \langle x\rangle)^{-\sigma(t_k-\beta\alpha)} \big(D_x^\beta U_{0,h_1}^k(x) - D_x^\beta U_{0,h_2}^k(x)\big), L_p(\mathbb{R}_+^n)\big\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{5.13}$$

Proof. We will present the arguments for $|\beta| = 0$.

Consider the kth entry of $U_{0,h}(x)$ from (5.2) and represent it as the sum of two summands:

$$U_{0,h}^k(x) = U_{0,h,1}^k(x) + U_{0,h,2}^k(x), \tag{5.14}$$

where

$$U_{0,h,1}^k(x) = (2\pi)^{(1-n)/2} \int_h^1 v^{-1} \int_{\mathbb{R}^{n-1}} \exp(ix's) G(sv^{\alpha'}) \widetilde{\mathscr{L}}_k(is, D_{x_n}) R\widetilde{F}(s, x_n)\, ds dv,$$

$$U_{0,h,2}^k(x) = (2\pi)^{(1-n)/2} \int_1^{h^{-1}} v^{-1} \int_{\mathbb{R}^{n-1}} \exp(ix's) G(sv^{\alpha'}) \widetilde{\mathscr{L}}_k(is, D_{x_n}) R\widetilde{F}(s, x_n)\, ds dv,$$

with $\widetilde{\mathscr{L}_k}(is, i\xi_n)$ the kth row of $\widetilde{\mathscr{L}}(is, i\xi_n)$. Using the homogeneity of the entries of $\mathscr{L}(is, i\xi_n)$ and the definition of $R\widetilde{F}(s, x_n)$, we derive

$$\begin{aligned} U_{0,h,1}^k(x) &= (2\pi)^{1-n} \int_h^1 v^{-1} \int_{\mathbb{R}^n} K_{0,k}(v, x' - y', x_n - y_n) \theta(y_n) F(y)\, dy dv, \\ U_{0,h,2}^k(x) &= (2\pi)^{1-n} \int_1^{h^{-1}} v^{-1} \int_{\mathbb{R}^n} K_{0,k}(v, x' - y', x_n - y_n) \theta(y_n) F(y)\, dy dv, \end{aligned} \tag{5.15}$$

where

$$K_{0,k}(v, x', x_n) = \int_{\mathbb{R}^{n-1}} \exp(ix's) G(sv^{\alpha'}) \widetilde{\mathscr{L}_k}(is, D_{x_n}) [J_+(s, x_n)\theta(x_n) + J_-(s, x_n)\theta(-x_n)]\, ds$$

and $\theta(x_n)$ is the Heaviside function.

Consider $U_{0,h,1}^k(x)$. Since $\sigma t_k \ge 0$; therefore,

$$\big\|(1 + \langle x\rangle)^{-\sigma t_k} U_{0,h,1}^k(x), L_p(\mathbb{R}_+^n)\big\| \le \big\|U_{0,h,1}^k(x), L_p(\mathbb{R}_+^n)\big\|.$$

Conditions 1 and 2 imply that $K_{0,k}(v, x', x_n) = v^{t_k - |\alpha|} K_{0,k}(1, x'v^{-\alpha'}, x_n v^{-\alpha_n})$, and, after the change $z_j = x_j v^{-\alpha_j}$, $j = 1, \dots, n$, we obtain

$$\big\|(1 + \langle x\rangle)^{-\sigma t_k} U_{0,h,1}^k(x), L_p(\mathbb{R}_+^n)\big\| \le c_1 \int_h^1 v^{-1+t_k}\, dv \|K_{0,k}(1, z', z_n), \mathbf{L}_1(\mathbb{R}^n)\| \|F(x), \mathbf{L}_p(\mathbb{R}_+^n)\|.$$

In line with [17, 18], we can show that choosing a sufficiently large number m in (4.9) in the definition of $G(s)$ yields

$$\|K_{0,k}(1, z', z_n), \mathbf{L}_1(\mathbb{R}^n)\| \le c < \infty.$$

Then

$$\big\|(1+\langle x\rangle)^{-\sigma t_k} U^k_{0,h,1}(x), L_p(\mathbb{R}^n_+)\big\| \le c\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\|. \tag{5.16}$$

Consider the function $U^k_{0,h,2}(x)$ from (5.15). Applying (3.15), as well as Minkowski's and Young's inequalities, we obtain

$$\begin{aligned}&\big\|(1+\langle x\rangle)^{-\sigma t_k} U^k_{0,h,2}(x), L_p(\mathbb{R}^n_+)\big\|\\ &\le c\int\limits_1^{h^{-1}} v^{-1}\|\langle x\rangle^{-\sigma t_k} K_{0,k}(v, x', x_n), L_p(\mathbb{R}^n)\|\,dv\|(1+\langle y\rangle)^{\sigma t_k} F(y), \mathbf{L}_1(\mathbb{R}^n_+)\|.\end{aligned}$$

By the arguments similar to estimation of $U^k_{0,h,1}(x)$, since $|\alpha|/p > \sigma t_{\max}$, we derive

$$\begin{aligned}&\big\|(1+\langle x\rangle)^{-\sigma t_k} U^k_{0,h,2}(x), L_p(\mathbb{R}^n_+)\big\|\\ &\le c_1\int\limits_1^{h^{-1}} v^{-1+(1-\sigma)t_k-|\alpha|/p'}\,dv\|(1+\langle y\rangle)^{\sigma t_k} F(y), \mathbf{L}_1(\mathbb{R}^n_+)\|.\end{aligned}$$

Hence, the condition $|\alpha|/p' > (1-\sigma)t_k$ implies

$$\big\|(1+\langle x\rangle)^{-\sigma t_k} U^k_{0,h,2}(x), L_p(\mathbb{R}^n_+)\big\| \le c\|(1+\langle y\rangle)^{\sigma t_k} F(y), \mathbf{L}_1(\mathbb{R}^n_+)\|, \tag{5.17}$$

with a constant $c > 0$ independent of $F(x)$ and h.

We obtain (5.12) for $|\beta| = 0$ from (5.14), (5.16), and (5.17). The proof of (5.12) for $t_k > \beta\alpha > 0$, as well as (5.13), is carried out in exactly the same way.

The lemma is proved.

Lemma 5.4. *Let $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta\alpha < t_k$, and $|\alpha|/p > \sigma(t_k - \beta\alpha) > t_k - \beta\alpha - |\alpha|/p'$, $1/p + 1/p' = 1$. Then*

$$\begin{aligned}&\big\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)} D^\beta_x U^k_{j,h}(x), L_p(\mathbb{R}^n_+)\big\|\\ &\le c(\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\| + \|(1+\langle x\rangle)^{\sigma(t_k-\beta\alpha)} F(x), \mathbf{L}_1(\mathbb{R}^n_+)\|), \quad 0 < h < 1,\end{aligned} \tag{5.18}$$

with a constant $c > 0$ independent of $F(x)$ and h; moreover,

$$\big\|(1+\langle x\rangle)^{-\sigma(t_k-\beta\alpha)}\big(D^\beta_x U^k_{j,h_1}(x) - D^\beta_x U^k_{j,h_2}(x)\big), L_p(\mathbb{R}^n_+)\big\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{5.19}$$

Proof. The main difficulty in establishing (5.18) is the case that $|\beta| = 0$. To settle it, consider the kth entry of the vector-function $U_{j,h}(x)$ from (5.3) and represent the latter, in line with [19], as

$$U^k_{j,h}(x) = U^k_{j,h,1}(x) + U^k_{j,h,2}(x), \tag{5.20}$$

where

$$U^k_{j,h,1}(x) = (2\pi)^{(1-n)/2} \int\limits_h^1 v^{-1} \int\limits_{\mathbb{R}^{n-1}} \exp(ix's)G(sv^{\alpha'}) \int\limits_0^\infty \exp(ix's)G(sv^{\alpha'})$$

$$\times D_{y_n}(B_j(is, D_{y_n})\widetilde{\mathscr{L}}(is, D_{y_n})R\widetilde{F}(s, y_n)\omega^k_j(s, x_n + y_n))\, dy_n ds dv,$$

$$U^k_{j,h,2}(x) = (2\pi)^{(1-n)/2} \int\limits_1^{h^{-1}} v^{-1} \int\limits_{\mathbb{R}^{n-1}} \exp(ix's)G(sv^{\alpha'}) \int\limits_0^\infty W_j(s, z_n)\widetilde{F}(s, z_n)\, dz_n \omega^k_j(s, x_n)\, ds dv,$$

$$W_j(s, x_n) = -\mathscr{B}_j(is, D_{y_n})\widetilde{\mathscr{L}}(is, D_{y_n})J_-(s, y_n - x_n)|_{y_n=0}.$$

Since $\sigma t_k \geq 0$; therefore,

$$\big\|(1 + \langle x\rangle)^{-t_k\sigma}U^k_{j,h,1}(x), L_p(\mathbb{R}^n_+)\big\| \leq \big\|U^k_{j,h,1}(x), L_p(\mathbb{R}^n_+)\big\|. \tag{5.21}$$

Estimate the first summand $U^k_{j,h,1}(x)$ from (5.20). By analogy with [19], represent this function as

$$U^k_{j,h,1}(x) = (2\pi)^{(1-n)/2} \int\limits_h^1 v^{-1} \int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\infty \exp(ix's)G(sv^{\alpha'})$$

$$\times B_j(is, D_{y_n})\widetilde{\mathscr{L}}(is, D_{y_n})R\widetilde{F}(s, y_n)D_{y_n}\omega^k_j(s, x_n + y_n)\, dy_n ds dv$$

$$+(2\pi)^{(1-n)/2} \int\limits_h^1 v^{-1} \int\limits_{\mathbb{R}^{n-1}} \int\limits_0^\infty \exp(ix's)G(sv^{\alpha'})$$

$$\times B_j(is, D_{y_n})D_{y_n}\widetilde{\mathscr{L}}(is, D_{y_n})R\widetilde{F}(s, y_n)\omega^k_j(s, x_n + y_n)\, dy_n ds dv$$

$$= \Phi^1_{j,h,k}(x) + \Phi^2_{j,h,k}(x). \tag{5.22}$$

Arguing similarly to the estimation of $\Phi^1_{j,h,k}(x)$ in Lemma 5.2 and using the fact that the function

$$\langle s\rangle^{t_k - m_j} \int\limits_0^\infty \exp(iy_n\xi_n)D_{y_n}\omega^k_j(s, y_n)\, dy_n$$

and the elements of (5.11) are multipliers, we obtain

$$\big\|\Phi^1_{j,h,k}(x), L_p(\mathbb{R}^n_+)\big\| \leq c_5 \int\limits_h^1 v^{-1}\big\|K^{\alpha_n}_k(v, x', x_n), L_1(\mathbb{R}^n)\big\|\, dv\|F(y), \mathbf{L}_p(\mathbb{R}^n_+)\|,$$

where

$$K^{\alpha_n}_k(v, x', x_n) = \int\limits_{\mathbb{R}^{n-1}} \exp(ix's)\exp(-x_n\langle s\rangle^{\alpha_n})\theta(x_n)G(sv^{\alpha'})\langle s\rangle^{\alpha_n - t_k}\, ds.$$

Since

$$K^{\alpha_n}_k(v, x', x_n) = v^{t_k - |\alpha|}K^{\alpha_n}_k(1, x'v^{-\alpha'}, x_nv^{-\alpha_n}),$$

making the change

$$z_i = x_iv^{-\alpha_i}, \quad i = 1, \ldots, n,$$

we derive

$$\left\|\Phi^1_{j,h,k}(x), L_p(\mathbb{R}^n_+)\right\| \le c_5 \int\limits_h^1 v^{-1+t_k}\,dv \left\|K_k^{\alpha_n}(1,z',z_n), L_1(\mathbb{R}^n)\right\| \|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\|.$$

By the definition of $K_k^{\alpha_n}(1,z',z_n)$, we obtain

$$\left\|\Phi^1_{j,h,k}(x), L_p(\mathbb{R}^n_+)\right\| \le c\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\|,$$

with a constant $c > 0$ independent of $F(x)$ and h. By similar arguments for the second summand in (5.22), we arrive at the estimate

$$\left\|\Phi^2_{j,h,k}(x), L_p(\mathbb{R}^n_+)\right\| \le c\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\|.$$

From (5.21), (5.22), and the above inequalities we obtain that

$$\left\|(1+\langle x\rangle)^{-t_k\sigma} U^k_{j,h,1}(x), L_p(\mathbb{R}^n_+)\right\| \le c\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\|, \tag{5.23}$$

with a constant $c > 0$ independent of $F(x)$ and h. Estimate $U^k_{j,h,2}(x)$ from (5.20). Using (3.15) and Minkowski's inequality, we derive

$$\left\|(1+\langle x\rangle)^{-t_k\sigma} U^k_{j,h,2}(x), L_p(\mathbb{R}^n_+)\right\| \le c_2 \int\limits_1^{h^{-1}} v^{-1} \Bigg\| \int\limits_{\mathbb{R}^n_+} \langle x-y\rangle^{-t_k\sigma}$$

$$\times \Bigg| \int\limits_{\mathbb{R}^{n-1}} \exp\left(i(x'-y')s\right) G(sv^{\alpha'}) W_j(s,y_n)\omega^k_j(s,x_n)\,ds \Bigg|$$

$$\times (1+\langle y\rangle)^{t_k\sigma}|F(y)|\,dy, L_p(\mathbb{R}^n_+)\Bigg\|\,dv = c_2 \int\limits_1^{h^{-1}} v^{-1} \Bigg\| \int\limits_{\mathbb{R}^n_+} |K_{j,k}(v,x'-y',x_n,y_n)|$$

$$\times ((1+\langle y\rangle)^{t_k\sigma}|F(y)|)^{(1/p+1/p')}\,dy, \mathbf{L}_p(\mathbb{R}^n_+)\Bigg\|\,dv,$$

where

$$K_{j,k}(v,x'-y',x_n,y_n) = \langle x-y\rangle^{-t_k\sigma} \int\limits_{\mathbb{R}^{n-1}} \exp\left(i(x'-y')s\right) G(sv^{\alpha'}) W_j(s,y_n)\omega^k_j(s,x_n)\,ds. \tag{5.24}$$

Applying Hölder's inequality, we obtain

$$\left\|(1+\langle x\rangle)^{-t_k\sigma} U^k_{j,h,2}(x), L_p(\mathbb{R}^n_+)\right\|$$

$$\le c_2 \int\limits_1^{h^{-1}} v^{-1} \Bigg\| \int\limits_{\mathbb{R}^n_+} |K_{j,k}(v,x'-y',x_n,y_n)|^p (1+\langle y\rangle)^{t_k\sigma}|F(y)|\,dy, L_1(\mathbb{R}^n_+)\Bigg\|^{1/p} dv$$

$$\times \|(1+\langle y\rangle)^{t_k\sigma}|F(y)|, \mathbf{L}_1(\mathbb{R}^n_+)\|^{1/p'}.$$

Using the Tonelli Theorem, rewrite this inequality as

$$\left\|(1+\langle x\rangle)^{-t_k\sigma} \Phi^2_{j,h,k}(x), L_p(\mathbb{R}^n_+)\right\|$$

$$\le c_2 \int\limits_1^{h^{-1}} v^{-1} \Bigg\| \int\limits_{\mathbb{R}^n_+} |K_{j,k}(v,x'-y',x_n,y_n)|^p\,dx (1+\langle y\rangle)^{t_k\sigma}|F(y)|, L_1(\mathbb{R}^n_+)\Bigg\|^{1/p} dv$$

$$\times \|(1+\langle y\rangle)^{t_k\sigma}F(y), \mathbf{L}_1(\mathbb{R}^n_+)\|^{1/p'}. \tag{5.25}$$

Put

$$A_{j,k}(v,y) = \int\limits_{\mathbb{R}^n} |K_{j,k}(v, x' - y', x_n, y_n)|^p \theta(x_n)\theta(y_n)\, dx.$$

Repeating the arguments from Lemma 2 in [18, § 4] and using Lemma 4.1 and the inequality $|\alpha|/p > t_k\sigma$, we derive

$$A_{j,k}(v,y) \le cv^{p((1-\sigma)t_k - |\alpha|/p')} \tag{5.26}$$

with a constant $c > 0$ independent of v and y. Inserting (5.26) in (5.25), we get

$$\begin{aligned} &\big\|(1 + \langle x\rangle)^{-t_k\sigma} U^k_{j,h,2}(x), L_p(\mathbb{R}^n_+)\big\| \\ &\le c \int\limits_1^{h^{-1}} v^{-1+(1-\sigma)t_k - |\alpha|/p'}\, dv \|(1 + \langle x\rangle)^{t_k\sigma} F(x), \mathbf{L}_1(\mathbb{R}^n_+)\|. \end{aligned}$$

By the hypotheses of the lemma, $(1-\sigma)t_k - |\alpha|/p' < 0$, and so

$$\big\|(1 + \langle x\rangle)^{-t_k\sigma} U^k_{j,h,2}(x), L_p(\mathbb{R}^n_+)\big\| \le c\|(1 + \langle x\rangle)^{t_k\sigma} F(x), \mathbf{L}_1(\mathbb{R}^n_+)\|, \tag{5.27}$$

with a constant $c > 0$ independent of $F(x)$ and h.

Since

$$U^k_{j,h}(x) = U^k_{j,h,1}(x) + U^k_{j,h,2}(x),$$

by (5.23) and (5.27), we arrive at (5.18) for $|\beta| = 0$. Obtaining (5.18) for $t_k > \beta\alpha > 0$, as well as (5.19), is carried out similarly.

The lemma is proved.

Proof of Theorem 4. This follows from Lemmas 5.1–5.4.

Indeed, by the above lemmas and under conditions (2.1), the vector-function $U_h(x)$ defined in (4.10) and (5.1) belongs to $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$ for every $F(x) \in \mathbf{C}^\infty(\overline{\mathbb{R}^n_+})$ such that $F(x) \equiv 0$ for $|x| \gg 1$ and the following estimate holds:

$$\big\|U_h(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)\big\| \le c(\|F(x), \mathbf{L}_p(\mathbb{R}^n_+)\| + \|F(x), \mathbf{L}_{1,-\sigma t_{\max}}(\mathbb{R}^n_+)\|)$$

with a constant $c > 0$ independent of $F(x)$ and h; moreover,

$$\big\|U_{h_1}(x) - U_{h_2}(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)\big\| \to 0 \quad \text{as } h_1, h_2 \to 0.$$

By completeness of $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$, there exists $U(x) \in \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$ such that

$$\big\|U_h(x) - U(x), \mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)\big\| \to 0 \quad \text{as } h \to 0.$$

U_x is a solution to (1.2) and estimate (2.4) holds for it. Since the set of the functions in $C^\infty(\overline{\mathbb{R}^n_+})$ vanishing for large $|x|$ is everywhere dense in $L_p(\mathbb{R}^n_+) \cap L_{1,-\sigma t_{\max}}(\mathbb{R}^n_+)$, boundary value problem (1.2) is solvable in $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$ for every vector-function $F(x) \in \mathbf{L}_p(\mathbb{R}^n_+) \cap \mathbf{L}_{1,-\sigma t_{\max}}(\mathbb{R}^n_+)$, and (2.4) holds for the solution.

The proof of uniqueness of the solution to (1.2) in $\mathbf{W}^{\mathbf{t}/\alpha}_{p,\sigma}(\mathbb{R}^n_+)$, $\sigma \in [0,1]$, follows the proof of uniqueness of solutions to the boundary value problem for quasielliptic equations (see [18]).

Theorem 4 is proved.

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L. N. Bondar; G. V. Demidenko
Sobolev Institute of Mathematics, Novosibirsk, Russia
E-mail address: `b_lina@ngs.ru`; `demidenk@math.nsc.ru`