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A CLASS OF SECOND ORDER TANGENT SETS

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Abstract: Under consideration are the construction and properties of some special class of second other tangent sets on using the technique of nonstandard analysis.

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Let X be a real vector space. Assume that we are given some almost vector topology σ with the zero neighborhood filter $\mathscr{N}_\sigma := \sigma(0)$ as well as some almost vector topology τ with the filter $\mathscr{N}_\tau := \tau(0)$.

Recall that every almost vector topology σ on X is characterized by the two properties: Firstly, multiplication by each scalar is continuous; and, secondly, addition is jointly continuous. It is clear that X admits an almost vector topology σ such that $\sigma(0)$ coincides with a fixed filter $\mathscr{N}$ if and only if the monad $\mu(\mathscr{N})$ is an external vector space over the external field of standard scalars.

In the sequel, σ will be a vector topology, unless stated otherwise explicitly. It is comfortable to work in the assumption of standard environment within Nelson internal set theory IST (see [1]). Recall that the *monad* $\mu(\mathscr{F})$ of a standard filter $\mathscr{F}$ is the external intersection of the standard elements of $\mathscr{F}$. As usual, introduce the *infinite proximity* that is associated with the appropriate uniformity in X; i. e., $x_1 \approx_\sigma x_2 \leftrightarrow x_1 - x_2 \in \mu(\mathscr{N}_\sigma)$. Note that the monad $\mu_\sigma(x)$ of the neighborhood filter $\sigma(x)$ of the topology σ is as follows: $\mu_\sigma(x) := x + \mu(\mathscr{N}_\sigma)$. Let $\approx$ stand for the infinite proximity on the reals $\mathbb{R}$.

Recall that if given are some subset F of X and some point $\bar{x}$ in X, then subdifferential calculus (see [2]) deals in particular with the *Hadamard, Clarke*, and *Bouligand* cones

$$\mathrm{Ha}(F,\bar{x}) := \bigcup_{\substack{U\in\sigma(\bar{x})\\ \lambda>0}} \mathrm{int}_\tau\Bigg(\bigcap_{\substack{x'\in F\cap U\\ 0<\lambda'\le\lambda}} \frac{F-x'}{\lambda'}\Bigg);$$

$$\mathrm{Cl}(F,\bar{x}) := \bigcap_{V\in\mathscr{N}_\tau} \bigcup_{\substack{U\in\sigma(\bar{x})\\ \lambda>0}} \bigcap_{\substack{x'\in F\cap U\\ 0<\lambda'\le\lambda}} \Bigg(\frac{F-x'}{\lambda'}+V\Bigg);$$

$$\mathrm{Bo}(F,\bar{x}) := \bigcap_{\substack{U\in\sigma(x')\\ \lambda>0}} \mathrm{cl}_\tau\Bigg(\bigcup_{\substack{x\in F\cap U\\ 0<\lambda'\le\lambda}} \frac{F-x'}{\lambda'}\Bigg),$$

where, as usual, $\sigma(\bar{x}) := \bar{x} + \mathscr{N}_\sigma$. If $h \in \mathrm{Ha}(F,\bar{x})$ then F is often called *epilipshitzian* at $\bar{x}$ with respect to h. It is obvious that

$$\mathrm{Ha}(F,\bar{x}) \subset \mathrm{Cl}(F,\bar{x}) \subset \mathrm{Bo}\,(F,\bar{x}).$$

Considering an extended real function $f : X \to \mathbb{R}\cup\{+\infty\}$, the author of [3] defined the *second order upper subderivative* at $\bar{x} \in X$ along directions $\bar{v}$, $\bar{v}_1$, and $\bar{v}_2$ as follows:

$$f^{(2)}_{\bar{x}}(\bar{v},\bar{v}_1,\bar{v}_2) := \limsup_{\substack{(x,\alpha)\to\bar{x}\\ \lambda\downarrow 0,\mu\downarrow 0}} \inf_{\substack{v\to\bar{v}\\ v_1\to\bar{v}_1\\ v_2\to\bar{v}_2}} O^2_f(x,\alpha,\lambda,\mu,v,v_1,v_2),$$

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where we use the limit construction that is attributed to Painlevé, Kuratowski, and Rockafellar (for instance, see [1, Section 5.3], [4], or [5]) while putting

$$O_f^2(x,\alpha,\lambda,\mu,v,v_1,v_2)$$
$$:= \lambda^{-1}\mu^{-1}(f(x+\lambda v_1+\mu v_2+\lambda\mu v)-f(x+\lambda v_1)-f(x+\mu v_2)+\alpha).$$

Here $(x,\alpha)\to\bar{x}$ stands for the convergence to $(\bar{x},f(\bar{x}))$ in the induced topology of the epigraph of f. The article [6] contains some approach to the explicit description of the tangent sets that are determined by similar constructions. The description uses the tools of IST.

As we will see soon, it is convenient to slightly modify the above construction by inserting the multiplier 4 but retaining the previous notations:

$$f_{\bar{x}}^{(2)}(\bar{v},\bar{v}_1,\bar{v}_2) := \limsup_{\substack{(x,\alpha)\to\bar{x}\\ \lambda\downarrow 0,\,\mu\downarrow 0}} \inf_{\substack{v\to\bar{v}\\ v_1\to\bar{v}_1\\ v_2\to\bar{v}_2}} O_f^2(x,\alpha,\lambda,\mu,v,v_1,v_2),$$

where we now put

$$O_f^2(x,\alpha,\lambda,\mu,v,v_1,v_2)$$
$$:= \lambda^{-1}\mu^{-1}(f(x+\lambda v_1+\mu v_2+4\lambda\mu v)-f(x+\lambda v_1)-f(x+\mu v_2)+\alpha).$$

Given $F\subset X$, denote the *indicator function* of F by δ_F; i. e., $\delta_F(x):=0$ at $x\in F$ and $\delta_F(x):=\infty$ at $x\notin F$. Introduce the set $\mathrm{Cl}^{(2)}(F,\bar{x})(v_1,v_2)$ as follows:

$$v\in \mathrm{Cl}^{(2)}(F,\bar{x})(v_1,v_2)\leftrightarrow (v,v_1,v_2)\in \mathrm{dom}(\delta_F)_{\bar{x}}^{(2)}.$$

Considering the case of a normed space with τ the norm topology and σ the discrete topology, $\mathrm{Cl}^{(2)}(F,\bar{x})(v,v)$ coincides with $\mathrm{A}^{(2)}(F,\bar{x},v)$ the *second order attainable direction set* to F at $(\bar{x},v)$ provided that $\bar{x}+\mathrm{cnt}(v)$ lies in F. As usual, $\mathrm{cnt}(v):=\{\lambda v:\lambda>0,\lambda\approx 0\}$ is the *conatus* of v (see [1, Subsection 5.1.2]). Recall (see, for instance, [7]) that

$$\mathrm{A}^{(2)}(F,\bar{x},v):=\Big\{h\in X:(\forall\lambda_n\downarrow 0)(\exists h_n\to h)\ \bar{x}+\lambda_n v+\frac{1}{2}\lambda_n^2 h_n\in F\Big\}.$$

To simplify bulky formulas we will assume that f is continuous at $\bar{x}\in F$ with respect to the topology τ on X.

Theorem 1. *The following holds:*

$$\mathrm{Cl}^{(2)}(F,\bar{x})(v_1,v_2)$$
$$=\bigcap_{\substack{V\in\mathscr{N}_\sigma\\ V_1\in\sigma(v_1)\\ V_2\in\sigma(v_2)}}\ \bigcup_{\substack{U\in\tau(\bar{x})\\ \lambda_1>0\\ \lambda_2>0}}\ \bigcap_{\substack{x'\in F\cap U\\ 0<\lambda'\le\lambda_1\\ 0<\lambda''\le\lambda_2}}\ \bigcup_{\substack{v'\in\frac{F-x'}{\lambda'}\cap V_1\\ v''\in\frac{F-x'}{\lambda''}\cap V_2}}\left(\frac{F-x'-\lambda'v'-\lambda''v''}{4\lambda'\lambda''}+V\right).$$

Proof. By transfer it suffices to check the case of standard parameters. Theorem 5.3.11 of [1] yields

$$v\in\mathrm{Cl}^{(2)}(F,\bar{x})(v_1,v_2)$$
$$\leftrightarrow(\forall x'\approx_\tau\bar{x},x'\in F)(\forall\lambda'\approx 0,\lambda''\approx 0,\lambda'>0,\lambda''>0)$$
$$(\exists v_1'\approx_\sigma v_1)(\exists v_2'\approx_\sigma v_2)(\exists v'\approx_\sigma v)$$
$$x'+\lambda'v_1'\in F\wedge x'+\lambda''v_2'\in F\wedge x'+\lambda'v_1'+\lambda''v_2'+4\lambda'\lambda''v'\in F.$$

Denote the set on the right-hand side of the claim by A. Take $v\in\mathrm{Cl}^{(2)}(F,\bar{x})(v_1,v_2)$ and some standard neighborhoods $V\in\mathscr{N}_\sigma$, $V_1\in\sigma(v_1)$, and $V_2\in\sigma(v_2)$. If λ_1 and λ_2 are strictly positive infinitesimal while U is an infinitesimal τ-neighborhood of $\bar{x}$; i. e., $U \subset \mu_\tau(\bar{x})$; then there are some $v_1' \approx_\sigma v_1$, $v_2'' \approx_\sigma v_2$, and $v' \approx_\sigma v$ such that $x' + \lambda' v_1' \in F$, $x' + \lambda'' v_2' \in F$, and $x' + \lambda' v_1' + \lambda'' v_2' + 4\lambda'\lambda'' v' \in F$ for all $x' \in F \cap U$, $0 < \lambda' \leq \lambda_1$, and $0 < \lambda'' \leq \lambda_2$. In other words, there exist $v_1' \in (F - x')/\lambda' \cap V_1$, $v_2' \in (F - x')/\lambda'' \cap V_2$, and $v' \in v + V$ satisfying the needed properties. Since the parameters are standard, conclude that $v \in A$.

Assume now that $v \in A$. Take some standard neighborhoods $V \in \mathscr{N}_\sigma$, $V_1 \in \sigma(v_1)$, and $V_2 \in \sigma(v_2)$ once again. By transfer there are $U \in \tau(\bar{x})$, $\lambda_1 > 0$, and $\lambda_2 > 0$ such that $x' + x' v_1' + \lambda'' v_2' + 4\lambda'\lambda'' v' \in F$ for all $x' \in F \cap U$, $0 < \lambda' \leq \lambda_1$, and with some $v_1' \in (F - x')/\lambda' \cap V_1$, $v_2' \in (F - x')/\lambda'' \cap V_2$, and $v' \in v + V$. Recalling that $x' \in U$ if $x' \approx_\tau \bar{x}$ and appealing to the properties of infinitesimals, we infer by idealization that $v \in \operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$. The proof of Theorem 1 is complete.

Theorem 2. *The following hold:*
(1) *If* $\operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2) \neq \varnothing$ *then* v_1 *and* v_2 *belong to the Clarke cone* $\operatorname{Cl}(F, \bar{x})$.
(2) *If* $v_1, v_2 \in \operatorname{Ha}(F, \bar{x})$ *then* $\operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$ *is a closed semigroup in the topology* σ.

Proof. Claim (1) becomes obvious on recalling that in the standard environment we have

$$h \in \operatorname{Cl}(F, \bar{x}) \leftrightarrow (\forall x' \approx_\sigma \bar{x}, x' \in F)(\forall \alpha' > 0, \alpha' \approx 0)(\exists h' \approx_\tau h) x' + \alpha' h' \in F.$$

Take $u_1, u_2 \in \operatorname{Ha}(F, \bar{x})$. Without loss of generality, we will proceed in the standard environment. Therefore,

$$v_1 \in \operatorname{Ha}(F, \bar{x}) \leftrightarrow (\forall x' \approx_\sigma \bar{x}, x' \in F)(\forall \alpha > 0, \alpha \approx 0)(\forall v_1' \approx_\tau v_1) x' + \alpha' v_1' \in F;$$

$$v_2 \in \operatorname{Ha}(F, \bar{x}) \leftrightarrow (\forall x' \approx_\sigma \bar{x}, x' \in F)(\forall \alpha > 0, \alpha \approx 0)(\forall v_2' \approx_\tau v_2) x' + \alpha' v_2' \in F.$$

Assume now that $u_1, u_2 \in \operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$. By Theorem 1 we can write that

$$(\forall x' \approx_\sigma \bar{x}, x' \in F)(\forall \lambda' > 0, \lambda' \approx 0)(\forall \lambda'' > 0, \lambda'' \approx 0)$$
$$(\exists v_1' \approx_\tau v_1)(\exists v_2' \approx_\tau v_2)(\exists u' \approx_\tau u_1) \quad x'' := x' + \lambda' v_1' + \lambda'' v_2' + 4\lambda'\lambda'' u' \in F.$$

Using the properties of the vector topology σ and its monad $\mathscr{N}_\sigma$, conclude that $x'' \approx_\sigma \bar{x}$. Recalling Theorem 1 once again, we find $v_1'' \approx_\tau v_1$, $v_2'' \approx_\tau v_2$, and $u'' \approx_\tau u_2$ satisfying $x'' \lambda' v_1'' + \lambda'' v_2'' + 4\lambda'\lambda'' u'' \in F$. Put $v' := v_1' + v_2''$, $v'' := v_2' + v_2''$, and $u := u' + u''$. Undoubtedly, $v' \approx_\tau v_1$, $v'' \approx_\tau v_2$, and $u \approx_\tau u_1 + u_2$. Furthermore, $x' + \lambda' v' \in F$ and $x' + \lambda'' v'' \in F$, since v_1 and v_2 are hypertangents, i. e., elements of $\operatorname{Ha}(F, \bar{x})$. Moreover,

$$\begin{aligned} x' + \lambda' v' + \lambda'' v'' + 4\lambda'\lambda'' u &= x' + \lambda' v_1' + \lambda' v_2' + \lambda'' v_2' + \lambda'' v_2'' + 4\lambda'\lambda'' u' + 4\lambda'\lambda'' u'' \\ &= (x' + \lambda' v_2' + \lambda'' v_2'' + 4\lambda'\lambda'' u') + \lambda' v_2' + \lambda'' v_2'' + 4\lambda'\lambda'' u'' \\ &= x'' + \lambda' v_1' + \lambda'' v_2'' + 4\lambda'\lambda'' u'' \in F. \end{aligned}$$

Consequently, $u_1 + u_2 \in \operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$.

To prove closedness, take $u_0 \in \operatorname{cl}_\sigma \operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$ and some standard neighborhoods $V, V_1, V_2 \in \mathscr{N}_\tau$ such that $V_1 + V_2 \subset V$. There is a standard vector $u \in \operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$ satisfying $u - u_0 \in V_1$. Moreover, using Theorem 1, we conclude that there are $x' \approx_\sigma \bar{x}$, $x' \in F$, $\lambda' \approx 0$, $\lambda' > 0$, $\lambda'' \approx 0$, and $\lambda'' > 0$ such that $u' \in u + V_2$, $v' \in v_1 + W_1$, and $v'' \in W_2 + v_2$ for the previously given standard neighborhoods $W_1, W_2 \in \mathscr{N}_\tau$ satisfying the containments $v' + \lambda' v' \in F$, $v' + \lambda'' v'' \in F$, and $x' + \lambda' v' + \lambda'' v'' + 4\lambda'\lambda'' u' \in F$. This implies easily that $u' \in u + V_2 \subset u_0 + V_1 + V_2 \subset u_0 + V$. By idealization we find $v' \approx_\tau v_1$, $v'' \approx_\tau v_2$, and $u_0' \approx_\tau u_0$ such that $x' + \lambda' v' \in F$, $x' + \lambda'' v' \in F$, and $x' + \lambda' v' + \lambda'' v'' + 4\lambda'\lambda'' u_0' \in F$. This means that $u_0 \in \operatorname{Cl}^{(2)}(F, \bar{x})(v_1, v_2)$.

Remark. Theorems 1 and 2 can be generalized to the case of the epiderivatives determined from some collection of infinitesimals along the lines of [4] and [8].

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