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ELEMENTARY EQUIVALENCE AND DIRECT PRODUCT DECOMPOSITIONS OF PARTIALLY COMMUTATIVE GROUPS OF VARIETIES

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Abstract: Considering the partially commutative groups of the varieties that include the variety of all length 2 nilpotent groups, we study the questions of factor cancellation in direct products, coincidence of elementary theories, characterization of the group by its elementary theory, and the possibility of direct product decomposition.

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1. Introduction

The *elementary theory* of a group G is the set $\mathrm{Th}(G)$ of all propositions in the first-order language of the standard group signature which are true on G. Two groups G and H are called *elementarily equivalent* whenever $\mathrm{Th}(G) = \mathrm{Th}(H)$.

Let us recall just two of the many results on the elementary theories of groups. In 1955 Szmielew found the necessary and sufficient conditions for the elementary equivalence of abelian groups in [1]. In the middle of the last century Tarski stated a few questions no the elementary theory of free groups. In particular, he asked whether it is true that the free nonabelian groups of distinct ranks are elementarily equivalent. The positive answer was given by Myasnikov and Kharlampovich [2], as well as independently by Sela [3].

Sometimes a group G is almost uniquely characterized by the elementary theory of G. In fact, the authors established the following statement [4]:

Proposition 1. *Suppose that the elementary theory of an n-generated group G coincides with the elementary theory of the free solvable group $F_{r,m}$ of rank r and solvability length m. Then G and $F_{r,m}$ are isomorphic for $m = 2$ and an arbitrary finite r, as well as for $m > 2$ and $n = r$.*

This article deals with partially commutative groups of some varieties of groups. Recall the relevant definition. Given a variety $\mathfrak{M}$ of groups, consider its free group F with base $X = \{x_1, \dots, x_n\}$ and a finite graph Γ with vertex set X, always assumed undirected, without loops and multiple edges. Consider the normal subgroup R in F generated by the commutators $[x_i, x_j]$ for all adjacent vertices x_i and x_j, i.e. joined in the graph by an edge. The quotient group F/R, denoted below by $F(\Gamma, \mathfrak{M})$, is called the *partially commutative group* of $\mathfrak{M}$ with *defining graph* Γ.

Partially commutative rings and algebras have been studied along with partially commutative groups. As [5] established, two partially commutative associative algebras are isomorphic if and only if their defining graphs are isomorphic. Resting on this, Droms proved in [6] a similar result for the partially commutative groups of the variety of all groups. Let $\mathfrak{N}_k$ stand for the variety of nilpotent groups of nilpotency length at most k. In the course of his proof, Droms obtained the following:

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Proposition 2. *Suppose that a variety $\mathfrak{M}$ of groups contains a variety $\mathfrak{N}_2$. The groups $F(\Gamma, \mathfrak{M})$ and $F(\Delta, \mathfrak{M})$ are isomorphic if and only if the graphs Γ and Δ are isomorphic.*

As [7] demonstrated, the coincidence of elementary theories of two partially commutative metabelian groups implies the isomorphism of their defining graphs. Along the demonstration, the following was established:

Proposition 3. *Two partially commutative groups of $\mathfrak{N}_2$ have the same elementary theories if and only if they are isomorphic.*

Studying the problem of classifying nilpotent groups in accordance with their elementary properties, Oger proved in [8] the following proposition important for applications:

Proposition 4. *Suppose that G and H are finitely generated groups, and, furthermore, each is an extension of a finite group by a nilpotent group. The elementary theories of G and H coincide if and only if $G \times J \cong H \times J$, where J is an infinite cyclic group.*

Suppose that, under the assumptions of Proposition 4 for groups G and H in some class, the isomorphism $G \times J \cong H \times J$ implies the isomorphism $G \cong H$, i.e., the group J in the formula $G \times J \cong H \times J$ cancels. Then G is uniquely characterized in the appropriate class of groups by the elementary theory of G. Hirshon studied expressly the questions of cancellation in direct products; see [9] for instance.

For the partially commutative groups of varieties, we study the questions of coincidence of elementary theories, characterization of the group by its elementary theory, and the possibility of direct product decomposition. Let us state the main results:

Theorem 1. *Given a variety $\mathfrak{M}$ of groups which includes $\mathfrak{N}_2$, suppose that a group $G = F(\Gamma, \mathfrak{M})$ decomposes as the direct product $H \times A$, where A is an abelian group. Then $H \cong F(\Delta, \mathfrak{M})$, where Δ is a subgraph of Γ induced by some set of vertices which includes $X \backslash X^{\perp}$; here $X^{\perp}$ stands for the set of vertices joined in the graph by edges to all other vertices.*

Theorem 2. *Suppose that the variety of nilpotent groups $\mathfrak{N}$ contains $\mathfrak{N}_2$ and $G = F(\Gamma, \mathfrak{N})$. If a finitely generated group H has the same elementary theory as G then $G \cong H$.*

Theorem 2 and Proposition 2 imply the following

Corollary. *Given a variety $\mathfrak{N}$ of nilpotent groups which includes $\mathfrak{N}_2$, consider $G = F(\Gamma, \mathfrak{N})$ and $H = F(\Delta, \mathfrak{N})$. The groups G and H have the same elementary theory if and only if $\Gamma \cong \Delta$.*

Theorem 3. *Suppose that $\mathfrak{M}$ is a variety of solvable groups which includes $\mathfrak{N}_2$. If the graph Γ is disconnected then the group $G = F(\Gamma, \mathfrak{M})$ is not decomposable as a direct product.*

2. Proof of Theorem 1

Lemma 1. (1) *If a group G is generated by $\{x_i \mid i \in I\}$, while $c_i \in G' \cap \mathscr{Z}(G)$, where $\mathscr{Z}(G)$ is the center of G and G' is its commutant, then G is generated by $\{x_i c_i \mid i \in I\}$.*

(2) *Under the same hypotheses, if G/G' is the free abelian group with base $\{x_i G' \mid i \in I\}$ then the mapping $x_i \mapsto x_i c_i$ for $i \in I$ determines an automorphism of G.*

Proof. The commutant of the group generated by $\{x_i c_i \mid i \in I\}$ coincides with G' because, as a normal subgroup, it is generated by the commutators $[x_i c_i, x_j c_j] = [x_i, x_j]$ for $i, j \in I$. This implies the first claim.

To prove the second claim, take the free group $F = \langle y_1, \dots, y_n \rangle$ and $v(y_1, \dots, y_n) \in F$. If in G we have either $v(x_1, \dots, x_n) = 1$ or $v(x_1 c_1, \dots, x_n c_n) = 1$ then $v \in F'$. However, for $v \in F'$ the values of $v(x_1, \dots, x_n)$ and $v(x_1 c_1, \dots, x_n c_n)$ coincide. Thus, $v(x_1, \dots, x_n) = 1 \Leftrightarrow v(x_1 c_1, \dots, x_n c_n) = 1$. $\square$

Proceed to proving Theorem 1. Put $X^{\perp} = \{x_1, \dots, x_m\}$. Then G/G' is the free abelian group with base $\{a_1 = x_1 G', \dots, a_n = x_n G'\}$. Given $a \in A$, suppose that $a \equiv x_1^{\alpha_1} \dots x_n^{\alpha_n} \pmod{G'}$. For $i > m$ there is a vertex x_j with $j > m$ not adjacent to x_i. Suppose that $\alpha_i \neq 0$. Then the inequality

$\left[x_1^{\alpha_1}\dots x_n^{\alpha_n}, x_j\right] \neq 1$ is satisfied in the group $F(\Gamma, \mathfrak{N}_2)$ which is a homomorphic image of G. This contradicts the commutation of a and x_j in G. Therefore, $A \leq \langle x_1, \dots, x_m\rangle G'$, so that the homomorphism $G \to G/G' = \langle a_1, \dots, a_n\rangle$ embeds A into the subgroup $\langle a_1, \dots, a_m\rangle$, where A is a direct factor. Note that in Γ we can replace the set of vertices $X^{\perp} = \{x_1, \dots, x_m\}$ by an arbitrary set constituting a base for the free abelian group $\langle x_1, \dots, x_m\rangle$, ending up with an isomorphic graph. Hence, without loss of generality we may assume that $A = \langle x_1c_1, \dots, x_kc_k\rangle$, where $k \leq m$ and $c_i \in G' \cap \mathscr{Z}(G)$. By Lemma 1, the mapping

$$x_1 \to x_1c_1, \dots, x_k \to x_kc_k, x_{k+1} \to x_{k+1}, \dots, x_n \to x_n$$

determines an automorphism of G. Basing on this, we can assert that $H \cong G/A \cong \langle x_{k+1}, \dots, x_n\rangle$. The last group is of the required form $F(\Delta, \mathfrak{M})$. $\square$

3. Proof of Theorem 2

Given some variety $\mathfrak{M}$ of groups, assign to each finite graph Γ the partially commutative group $F(\Gamma, \mathfrak{M})$. Call the variety $\mathfrak{M}$ *faithful* if the correspondence between the graph and the partially commutative groups of this variety is bijective; i.e., $F(\Gamma, \mathfrak{M}) \cong F(\Delta, \mathfrak{M})$ if and only if $\Gamma \cong \Delta$.

Proposition 2 means that every variety of groups which includes $\mathfrak{N}_2$ is faithful.

Given finite graphs Γ and Δ on disjoint vertex sets, put $G = F(\Gamma, \mathfrak{M})$ and $H = F(\Delta, \mathfrak{M})$. Clearly, the direct product $G \times H$ is also a partially commutative group of the same variety $\mathfrak{M}$. The defining graph of $G \times H$ is the graph called the *complete connection* of Γ and Δ and denoted by $\Gamma + \Delta$; in it every vertex of one graph is joined by edges to all vertices of the other.

Lemma 2. *If $\mathfrak{M}$ is a faithful variety of groups then for all finite graphs A, B, and D we have*

$$F(A, \mathfrak{M}) \times F(B, \mathfrak{M}) \cong F(A, \mathfrak{M}) \times F(D, \mathfrak{M}) \Leftrightarrow B \cong D.$$

Proof. Proceed by verifying the following properties of the connection of graphs. Suppose that there are two representations of Γ as connections, $\Gamma = A + B = C + D$, of pairwise disjoint graphs A and B, as well as C and D. Then $A \cong C$ would imply $B \cong D$.

As usual, $\overline{\Delta}$ stands for the complement graph to Δ. Clearly, two graphs are isomorphic if and only if so are their complement graphs. It is easy to verify that we can express the connection $\Delta_1 + \Delta_2$ of graphs on disjoint vertex sets as $\overline{\overline{\Delta_1} \sqcup \overline{\Delta_2}}$. Hence, it suffices to establish the following property: If Δ is represented as the disjoint union $\Delta = X \sqcup Y = U \sqcup V$ and $X \cong U$ then $Y \cong V$.

Take the tuple $\Delta_1, \dots, \Delta_n$ of all connected components of Δ. Then

$$X = \Delta_{i_1} \sqcup \dots \sqcup \Delta_{i_m}, \quad Y = \Delta_{i_{m+1}} \sqcup \dots \sqcup \Delta_{i_n}.$$

We can find similar decompositions for U and V. By assumption, $X \cong U$; hence, the numbers of connected components in X and U are the same and the connected components of X are isomorphic to the corresponding connected components of U. Obviously, then the corresponding connected components of Y and V are also isomorphic. $\square$

Let us prove Theorem 2. By hypotheses, the group H must be nilpotent. Then Proposition 4 yields $G \times A \cong H \times A$, where A is an infinite cyclic group. The group $G \times A$ is a partially commutative group of $\mathfrak{N}$. Thus, $G \times H$ decomposes as the direct product $H \times A$. Then Theorem 1 shows that H is of the form $F(\Delta, \mathfrak{N})$. By Proposition 2, the variety $\mathfrak{N}$ is faithful. Lemma 2 implies that $\Gamma \cong \Delta$, and so $G \cong H$. $\square$

4. Proof of Theorem 3

Lemma 3. *Suppose that G is a length 2 nilpotent product of nontrivial free abelian groups $A = \langle a_1, \dots, a_m\rangle$ and $B = \langle b_1, \dots, b_n\rangle$. Then the center $\mathscr{Z}(G)$ coincides with G', while the centralizer of $g \in G\backslash G'$ either equals AG' if $g \in AG'$ or BG' if $g \in BG'$ or is cyclic modulo G'. In particular, in all cases $C_G(g)$ is an abelian group.*

Proof. It is easy to see that G' is a free abelian group with base $\{[a_i, b_j] \mid 1 \le i \le m,\ 1 \le j \le n\}$. Without loss of generality we may assume that $g \in G \backslash G'$, whose centralizer we consider, satisfies $g \equiv a_1^{\alpha} b_1^{\beta}$, where $0 \ne \alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}$. Take $h \in C_G(g)$ with $h \equiv a_1^{\alpha_1} \dots a_m^{\alpha_m} b_1^{\beta_1} \dots b_n^{\beta_n} \pmod{G'}$. The condition $[g, h] = 1$ and the form of the base of G' imply the following: For $\beta = 0$ all $\beta_1, \dots, \beta_n$ are also equal to zero and $C_G(g) = AG'$. For $\beta \ne 0$ all $\alpha_2, \dots, \alpha_m, \beta_2, \dots, \beta_n$ are equal to zero, while the rows (α, β) and $\alpha_1, \beta_1)$ are collinear. Taking the greatest common divisor d of α and β, put $\alpha' = \alpha/d$ and $\beta' = \beta/d$. In this case $C_G(g)$ modulo G' is generated by $a_1^{\alpha'} b_1^{\beta'}$. Since $C_g(a_1) \cap C_G(b_1) = G'$, it follows that $\mathscr{Z}(G) = 1$. □

Proceed to proving Theorem 3. Assume on the contrary that there is a nontrivial decomposition $G = A \times B$. Since both A and B are solvable groups, the quotient groups $A/\gamma_3(A)$ and $B/\gamma_3(B)$ are nontrivial. Replacing G with $G/\gamma_3(G) = A/\gamma_3(A) \times B/\gamma_3(B)$ reduces the problem to the case $\mathfrak{M} = \mathfrak{N}_2$. We see that in this case we can express G as the product of subgroups AG' and BG', and furthermore $[AG', BG'] = 1$. Clearly, the theorem follows from the next statement:

Lemma 4. *If Γ is a disconnected graph then the group $G = F(\Gamma, \mathfrak{N}_2)$ cannot be expressed as the product AB of subgroups A and B with $A > G'$, $B > G'$, and $[A, B] = 1$.*

Proof. Assume on the contrary that this expression is possible. Partition the graph Γ into its connected components $\Gamma_1 \sqcup \dots \sqcup \Gamma_n$. By assumption, $n \ge 2$. Put $\Delta = \Delta_1 \sqcup \Delta_2$, where Δ_1 is the complete graph on the vertices of Γ_1, while Δ_2 is the complete graph on the vertices of $\Gamma_2 \sqcup \dots \sqcup \Gamma_n$. The group $F(\Delta, \mathfrak{N}_2)$ is isomorphic to the quotient group of G by the normal subgroup included into G', and we can assert that the images of A and B in $F(\Delta, \mathfrak{N}_2)$ satisfy the hypotheses of the lemma. This reduces the argument to the case of $\Gamma = \Gamma_1 \sqcup \Gamma_2$, where Γ_1 and Γ_2 are nontrivial complete graphs. In this case G is a length 2 nilpotent product of two free abelian groups of finite rank. If A turns out abelian then it lies in $\mathscr{Z}(G)$, while Lemma 3 would yield $\mathscr{Z}(G) = G' < A$, which is a contradiction. Therefore, A is a nonabelian group. Take $b \in B \backslash G'$. Then $C_G(b)$ includes A and, consequently, it is nonabelian. We again arrive at a contradiction with Lemma 3. This justifies Lemma 4, and Theorem 3 along with it. □

In closing, let us state the open question:

Question. *Suppose that $\mathfrak{M}$ includes $\mathfrak{N}_2$. Is it true that the group $F(\Gamma, \mathfrak{M})$ has a nontrivial direct product decomposition $F(\Gamma, \mathfrak{M}) = G \times H$ if and only if G and H are partially commutative groups of $\mathfrak{M}$?*

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