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TO THE SPECTRAL THEORY OF PARTIALLY ORDERED SETS. II

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Abstract: We characterize the topological spaces that are homeomorphic to the spectra of posets with certain properties.

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§ 1. Introduction

This article continues [1] and is devoted to the study of the notion of spectral space and similar notions. In particular, we elaborate and specify some results from the papers [2–4] by the first author; also see the monograph [5].

In [1], almost sober spaces were characterized as spaces homeomorphic to spectra of distributive lattices with topology (distributive posets with topology, respectively); and sober spaces, as spaces homeomorphic to the spectra of distributive lattices with topology which contain a bottom with respect to the defined order (distributive posets with topology which contains a bottom with respect to the defined order, respectively). Here we give a characterization of the topological spaces homeomorphic to the spectra of distributive posets with discrete topology which possess certain order properties, and thus generalizing some available results. More precisely, spectral spaces are characterized in Theorem 4 as inverse limits of spectra of finite distributive posets, as well as spectra of distributive meet-semilattices with a bottom and a top. Further, the notions of semispectral and almost (semi)spectral spaces are introduced. In Theorems 7, 8, 11 and Corollaries 1–4, the characterization of those spaces is given as spectra of distributive posets with some properties. In particular, the equivalence of statements (ii) and (iv) in Corollary 1 is a well-known characterization of spectra of distributive join-semilattices with a bottom; see, for example, § 5 of Chapter II in Grätzer [6, 7]. In Corollary 5, it is established that the class of (semi)spectral spaces is closed under Cartesian products and closed subspaces; and in Corollary 6, under compact open subspaces. Our definition of ideal of a poset corresponds to [8].

§ 2. Spectra of Posets

Lemma 1. *If a family $\mathscr{K}$ of compact open sets in $\mathbb{X}$ forms a base for the topology then the join-semilattices $\langle \mathscr{K}; \cup, \varnothing\rangle$ and $\big\langle \{U \in \mathscr{K} \mid U \neq \varnothing\}; \cup\big\rangle$ are distributive.*

Proof. Let $U \subseteq U_0 \cup U_1$ for some nonempty $U, U_0, U_1 \in \mathscr{K}$. Then $U \cap U_i \in \mathscr{T}(\mathbb{X})$ for each $i < 2$, whence there are families $\mathscr{V}_i \subseteq \mathscr{K}$ with the property that $U \cap U_i = \bigcup \mathscr{V}_i$, $i < 2$. Therefore $U = U \cap (U_0 \cup U_1) = (U \cap U_0) \cup (U \cap U_1) = \bigcup \mathscr{V}_0 \cup \bigcup \mathscr{V}_1$. By compactness of U, we have $U = \bigcup \mathscr{W}_0 \cup \bigcup \mathscr{W}_1$ for some finite families $\mathscr{W}_i \subseteq \mathscr{V}_i$, $i < 2$. It is clear that $\bigcup \mathscr{W}_0 = U \cap U_0 \subseteq U_0$, $\bigcup \mathscr{W}_1 = U \cap U_1 \subseteq U_1$, and $\bigcup \mathscr{W}_0, \bigcup \mathscr{W}_1 \in \mathscr{K}$. □

The next definition is a particular case of Definition 1 of [1].

Definition 1. Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq\rangle$. The space $\mathbb{S}\mathrm{pec}_\varphi S = \mathbb{S}\mathrm{pec}_\varphi\langle S, \leq, \mathscr{T}_\omega\rangle$, with $\mathscr{T}_\omega$ the discrete topology, is called the *φ-spectrum* of $\langle S; \leq\rangle$. The space $\mathbb{S}\mathrm{pec}\, L = \mathbb{S}\mathrm{pec}_\psi\langle L, \vee, \mathscr{T}_\omega\rangle$ is called the *spectrum* of $\langle L; \vee\rangle$. The *spectrum* of $\langle L; \vee, \wedge\rangle$ is the spectrum of its reduct $\langle L; \vee\rangle$.

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Remark. If an algebraic closure operator φ define a completion of a poset $\langle S; \leq \rangle$ other than a singleton and $\langle S; \leq \rangle$ is φ-distributive, then $\operatorname{Spec} S \neq \varnothing$ by Theorem 3.3 of [8]. Speaking of spectra of distributive (semi)lattices (of posets), in what follows we mean that the (semi)lattice L under consideration (the poset under consideration, respectively) is endowed with the discrete topology, and the algebraic closure operator ψ on L assigns to each subset of L the (semi)lattice ideal of L generated by the subset.

Lemma 2. *Let a closure operator φ define a completion of a poset $\langle S; \leq \rangle$ so that $\langle S; \leq \rangle$ is φ-distributive. Then $\mathbb{S}\mathrm{pec}_\varphi \langle S; \leq \rangle \cong \mathbb{S}\mathrm{pec}\,\mathrm{Id}(S, \varphi)$.*

Proof. By [1, Lemma 10], I is a $\cap$-irreducible element of $\mathrm{Id}(S, \varphi)$ for each $I \in \operatorname{Spec}\mathrm{Id}(S, \varphi)$. Moreover, since $\mathrm{Id}(S, \varphi)$ is finite and distributive, $J \in \operatorname{Spec}\mathrm{Id}(S, \varphi)$ if and only if $J = {\downarrow} I = \{X \in \mathrm{Id}(S, \varphi) \mid X \subseteq I\}$ for some $\cap$-irreducible element $I \in \mathrm{Id}(S, \varphi)$. Therefore, the mapping

$$\xi : \operatorname{Spec}_\varphi \langle S; \leq \rangle \to \operatorname{Spec}\mathrm{Id}(S, \varphi); \quad \xi : I \mapsto {\downarrow} I$$

is well defined and surjective. Furthermore, ξ is obviously one-to-one; i.e., ξ is a bijection. We put

$$V_s = \{I \in \operatorname{Spec}_\varphi \langle S; \leq \rangle \mid s \notin I\}, \quad s \in S;$$

$$W_X = \{I \in \operatorname{Spec}_\varphi \langle S; \leq \rangle \mid X \notin \xi(I)\}, \quad X \in \mathrm{Id}(S, \varphi).$$

Since $\mathrm{Id}(S, \varphi)$ is finite, we get by Corollary 6(i) of [1] that $\xi(W_X) = \bigcup_{s \in X} \xi\big(W_{L(s)}\big) = \bigcup_{s \in X} \xi(V_s)$ for every $X \in \mathrm{Id}(S, \varphi)$. The set $\{\xi(W_X) \mid X \in \mathrm{Id}(S, \varphi)\}$ is a base for the topology of $\mathbb{S}\mathrm{pec}\,\mathrm{Id}(S, \varphi)$ by Lemma 15 of [1], whence $\{\xi(V_s) \mid s \in S\}$ is a base for the topology of $\mathbb{S}\mathrm{pec}\,\mathrm{Id}(S, \varphi)$ too. As $\{V_s \mid s \in S\}$ is a base for the topology of $\mathbb{S}\mathrm{pec}_\varphi \langle S; \leq \rangle$, we conclude that $\mathbb{S}\mathrm{pec}_\varphi \langle S; \leq \rangle \cong \mathbb{S}\mathrm{pec}\,\mathrm{Id}(S, \varphi)$. $\square$

Lemma 3. *Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq \rangle$ so that $\langle S; \leq \rangle$ is φ-distributive. Then V_a is compact for every $a \in S$. Moreover, if a is not a bottom of $\langle S; \leq \rangle$ then $V_a \neq \varnothing$.*

Proof. Suppose that $V_a \subseteq \bigcup_{b \in X} V_b$ for some nonempty set $X \subseteq S$. By (i) and (iii) of Corollary 6 from [1], $V_a \subseteq V_{\varphi(X)}$. Assume that $a \notin \varphi(X)$. Then ${\downarrow} a \cap \varphi(X) = \varnothing$, whence by Theorem 3.3 of [8], there is $I \in \operatorname{Spec}_\varphi S$ such that $a \notin I$, but $\varphi(X) \subseteq I$, which contradicts the inclusion $V_a \subseteq V_{\varphi(X)}$. Thus $a \in \varphi(X)$. By the algebraicity of φ, there is a finite set $F \subseteq X$ such that $a \in \varphi(F)$. Hence, by (i) and (iii) of Corollary 6 from [1], $V_a \subseteq V_{\varphi(F)} = \bigcup_{b \in F} V_b$. By Lemma 15 of [1], $\{V_b \mid b \in L\}$ is a base for the topology of $\mathbb{S}\mathrm{pec}_\varphi S$, whence V_a is compact.

We assume further that a is not a bottom of $\langle S; \leq \rangle$. Thus there is $b \in S$ such that $a \nleq b$. This means that $U(a) \cap L(b) = \varnothing$; moreover, $L(b)$ is a φ-ideal and $U(a)$ is a filter in $\langle S; \leq \rangle$. By Theorem 3.3 of [8], there is a prime φ-ideal $I \in \operatorname{Spec}_\varphi S$ such that $a \notin I$ and $L(b) \subseteq I$; in particular, $I \in V_a \neq \varnothing$. $\square$

Proposition 1. *Let an algebraic closure operator φ define a completion of a down-directed poset $\langle S; \leq \rangle$ so that $\langle S; \leq \rangle$ is φ-distributive. The space $\mathbb{S}\mathrm{pec}_\varphi S$ is sober if and only if $\langle S; \leq \rangle$ has a bottom.*

Proof. If $\langle S; \leq \rangle$ has a bottom, then $\mathbb{S}\mathrm{pec}_\varphi S$ is sober by Theorem 5 of [1]. Suppose that $\mathbb{S}\mathrm{pec}_\varphi S$ is a sober space but has no bottom. Since $\langle S; \leq \rangle$ is down-directed, for all $a, b \in S$ there is $c \leq a, b$. But then $V_c \subseteq V_a \cap V_b$. Since the set $\{V_a \mid a \in S\}$ forms a base for the topology of $\mathbb{S}\mathrm{pec}_\varphi S$, we obtain by Lemma 3 that the intersection of two nonempty open sets in $\mathbb{S}\mathrm{pec}_\varphi S$ is nonempty. Thus $\operatorname{Spec}_\varphi S$ is an irredundant set. According to our assumption about the sobriety of $\mathbb{S}\mathrm{pec}_\varphi S$ and Lemma 16 of [1], there is an inclusion-least prime φ-ideal $I \in \operatorname{Spec}_\varphi S$; let $a \in I$. This means in view of Lemma 3 that $a \in I \subseteq I'$ for each $I' \in V_a \neq \varnothing$, which is impossible. This contradiction demonstrates that $\langle S; \leq \rangle$ contains a bottom. $\square$

Proposition 2. *Let an algebraic closure operator φ define a completion of an up-directed poset $\langle S; \leq \rangle$ so that $\langle S; \leq \rangle$ is φ-distributive. The space $\mathbb{S}\mathrm{pec}_\varphi S$ is compact if and only if $\langle S; \leq \rangle$ has a top.*

Proof. If $\langle S; \leq \rangle$ contains a top, then $\mathbb{S}\mathrm{pec}_\varphi S$ is compact by Lemma 3. Suppose that $\operatorname{Spec}_\varphi S$ is compact. Since $\{V_a \mid a \in S\}$ is a base for the topology of $\mathbb{S}\mathrm{pec}_\varphi S$, there are $a_0, \dots, a_n \in S$ such that $\operatorname{Spec}_\varphi S = V_{a_0} \cup \cdots \cup V_{a_n}$. As $\langle S; \leq \rangle$ is up-directed, $a_0, \ldots, a_n \leq a$ for some $a \in S$. But then $\operatorname{Spec}_\varphi S = V_{a_0} \cup \cdots \cup V_{a_n} \subseteq V_a$ whence $\operatorname{Spec}_\varphi S = V_a$. If $b \nleq a$ for some $b \in S$ then $L(a) \cap U(b) = \varnothing$. By Theorem 3.3 of [8], there exists a prime φ-ideal $I \in \operatorname{Spec}_\varphi S$ such that $b \notin I$ and $a \in L(a) \subseteq I$; in particular, $I \notin V_a = \operatorname{Spec}_\varphi S$, which is impossible. We have shown therefore that a is a top of $\langle S; \leq \rangle$. $\square$

Theorem 1. *Let the set $\mathscr{K}$ of all compact open sets in a sober space $\mathbb{X}$ be a base for $\mathscr{T}(\mathbb{X})$. Then the mapping*

$$f : \mathbb{X} \to \mathbb{S}\mathrm{pec}\, \mathscr{K}, \quad f : x \mapsto \{V \in \mathscr{K} \mid x \notin V\}$$

is a homeomorphism.

Proof. According to the definition and Lemma 1, $\langle \mathscr{K}; \cup \rangle$ is a distributive join-semilattice. Moreover, $\varnothing \in \mathscr{K}$; whence $\langle \mathscr{K}; \cup \rangle$ has a bottom. To simplify our notation, we write $\mathscr{T}$ instead of $\mathscr{T}(\mathbb{S}\mathrm{pec}\, \mathscr{K})$. We note that the specialization order in $\mathbb{S}\mathrm{pec}\, \mathscr{K}$ coincides with the relation $\supseteq$ of reverse inclusion.

Claim 1. *f is a well-defined mapping.*

Proof. It is not hard to see that $f(x)$ is an ideal of the join-semilattice $\mathscr{K}$ for every $x \in X$. We show that $f(x)$ is prime. Indeed, $f(x) \neq \varnothing$, as $\varnothing \in f(x)$. On the other hand, since $\mathscr{K}$ is a base for $\mathscr{T}(\mathbb{X})$ and $x \in X$, there exists $U \in \mathscr{K}$ such that $x \in U$. Then $U \notin f(x)$, whence $f(x) \neq \mathscr{K}$. Let $V_0, V_1 \in \mathscr{K}$ be such that $V \in f(x)$ for each $V \in \mathscr{K}$ satisfying $V \subseteq V_0 \cap V_1$. Then $x \notin V_0 \cap V_1$, whence $x \notin V_i$ for some $i < 2$. Therefore $V_i \in f(x)$, which was required. $\square$

Claim 2. *f is a one-to-one onto mapping.*

Proof. Indeed, consider an arbitrary prime ideal $\mathscr{B} \subseteq \mathscr{K}$. Let $U = \bigcup_{V \in \mathscr{B}} V$; then $U \in \mathscr{T}(\mathbb{X})$ and $F = X \backslash U$ is closed in $\mathbb{X}$. Moreover, $F \neq \varnothing$, as we would otherwise have $V \in \mathscr{B}$ for an arbitrary set $V \in \mathscr{K}$ in contradiction to the assumption that $\mathscr{B}$ is prime. We show that F is an irreducible set. Indeed, let $U_0 \cap F \neq \varnothing$ and $U_1 \cap F \neq \varnothing$ for some $U_0, U_1 \in \mathscr{T}(\mathbb{X})$. This means by assumption that $V_0 \cap F \neq \varnothing$ and $V_1 \cap F = \varnothing$ for some $V_0, V_1 \in \mathscr{K}$ with the property that $V_i \subseteq U_i$, $i < 2$. Thus $V_0, V_1 \nsubseteq U$ whence $V_0, V_1 \notin \mathscr{B}$. In view of the primality of $\mathscr{B}$, this means that $V \notin \mathscr{B}$ for some set $V \in \mathscr{K}$ such that $V \subseteq V_0 \cap V_1$. Since the set V is compact, $V \nsubseteq U$. Hence, $U_0 \cap U_1 \cap F \supseteq V_0 \cap V_1 \cap F \supseteq V \cap F \neq \varnothing$, which was required.

In view of the sobriety of $\mathbb{X}$, there is $x \in X$ such that $F = {\downarrow} x$. We prove that $f(x) = \mathscr{B}$. Indeed, for an arbitrary $V \in \mathscr{K}$ the condition $V \notin f(x)$ is equivalent to the condition $x \in V$. The latter is satisfied if and only if $V \cap F \neq \varnothing$, which is equivalent to the condition $V \nsubseteq U$ and, in turn, to the condition $V \notin \mathscr{B}$.

Further, let $x \nleq y$ in $\mathbb{X}$. This means that $x \in U$ and $y \notin U$ for some $U \in \mathscr{K}$, i.e., $U \in f(y) \backslash f(x)$. Thus, f is one-to-one. $\square$

Claim 3. *f is a continuous and open mapping.*

Proof. Given $U \in \mathscr{K}$, consider

$$V_U = \{\mathfrak{I} \in \operatorname{Spec} \mathscr{K} \mid U \notin \mathfrak{I}\} \in \mathscr{T}.$$

Then $x \in U$ if and only if $U \notin f(x)$, which is equivalent to the condition $f(x) \in V_U$ in view of Claim 1. Since $\mathscr{K}$ is a base for $\mathscr{T}(\mathbb{X})$, this proves the continuity and openness of f. $\square$

By Claims 1–3, f is a homeomorphism. The proof is complete. $\square$

Theorem 2. *Let a space $\mathbb{X}$ be almost sober but not sober and let the set $\mathscr{K}_0$ of all nonempty compact open sets in $\mathbb{X}$ form a base for $\mathscr{T}(\mathbb{X})$. Then*

$$f : \mathbb{X} \to \mathbb{S}\mathrm{pec}\, \mathscr{K}_0, \quad f : x \mapsto \{V \in \mathscr{K}_0 \mid x \notin V\}$$

is a homeomorphism.

Proof. By assumption, X is irreducible, whence the intersection of every two nonempty open sets is again a nonempty open set. Therefore, $\langle \mathscr{K}_0; \cup \rangle$ is a down-directed distributive join-semilattice. We note also that $\varnothing = \bigcup \varnothing$.

Claim 1. *$f(x) = \{V \in \mathscr{K}_0 \mid x \notin V\}$ is nonempty for each $x \in X$.*

Proof. Suppose that $f(x) = \varnothing$ for some $x \in X$. This means that $x \in V$ for each $V \in \mathscr{K}_0$. Since $\mathscr{K}_0$ is a base for $\mathscr{T}(\mathbb{X})$; therefore, $x \in U$ for each nonempty set $U \in \mathscr{T}(\mathbb{X})$. But this implies that x is a top of $\mathbb{X}$ with respect to the specialization order. Thus, $\mathbb{X}$ is sober, which contradicts our assumption. □

It follows from Claim 1 that the join-semilattice $\langle \mathscr{K}_0; \cup \rangle$ has no bottom.

Claim 2. *f is a well-defined mapping.*

Proof. It is not hard to see that $f(x)$ is an ideal of the join-semilattice $\mathscr{K}_0$ for each $x \in X$. We show that $f(x)$ is prime. Indeed, $f(x) \neq \varnothing$ by Claim 1. On the other hand, since $\mathscr{K}_0$ is a base for $\mathscr{T}(\mathbb{X})$ and $x \in X$, there is $U \in \mathscr{K}_0$ such that $x \in U$. Then $U \notin f(x)$ whence $f(x) \neq \mathscr{K}_0$. Let $V_0, V_1 \in \mathscr{K}_0$ be such that $V \in f(x)$ for each $V \in \mathscr{K}_0$ satisfying $V \subseteq V_0 \cap V_1$. Then $x \notin V_0 \cap V_1$ whence $x \notin V_i$ for some $i < 2$. Thus, $V_i \in f(x)$ for some $i < 2$ which was desired. □

Then the proof repeats that of Theorem 1. □

Proposition 3 [5, Proposition 13.1.3]. *Let the family $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ be a base for $\mathscr{T}(\mathbb{X})$ of a topological T_0-space $\mathbb{X}$. The following are equivalent:*

(i) *Every family $\mathscr{W} \cup \{F\}$ with $\mathscr{W} \subseteq \mathscr{K}_0$ down-directed and $F \subseteq X$ closed has nonempty intersection.*

(ii) *Every family $\mathscr{W} \cup \{F\}$ with $\mathscr{W} \subseteq \mathscr{K}_0$ down-directed and $F \subseteq X$ closed and irreducible has nonempty intersection.*

(iii) *$\mathbb{X}$ is sober.*

If $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, then the above conditions are equivalent to the following:

(iv) *Every family $\mathscr{W} \cup \{F\}$, having the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and $F \subseteq X$ closed has nonempty intersection.*

(v) *Every family $\mathscr{W} \cup \{F\}$, having the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and $F \subseteq X$ closed and irreducible has nonempty intersection.*

The proof of the following proposition is similar to that of Proposition 3. Nonetheless, we provide the demonstration for the sake of completeness.

Proposition 4. *Let the family $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ be a base for $\mathscr{T}(\mathbb{X})$ in a topological T_0-space $\mathbb{X}$. The following are equivalent:*

(i) *Every family $\mathscr{W} \cup \{F\}$ with $\mathscr{W} \subseteq \mathscr{K}_0$ down-directed and $F \subset X$ closed has nonempty intersection.*

(ii) *Every family $\mathscr{W} \cup \{F\}$ with $\mathscr{W} \subseteq \mathscr{K}_0$ down-directed and $F \subset X$ closed and irreducible has nonempty intersection.*

(iii) *$\mathbb{X}$ is almost sober.*

If $\mathscr{K}$ forms a multiplicative base for $\mathscr{T}(\mathbb{X})$, then the above conditions are equivalent to the following:

(iv) *Every family $\mathscr{W} \cup \{F\}$, having the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and $F \subset X$ closed has nonempty intersection.*

(v) *Every family $\mathscr{W} \cup \{F\}$, having the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and $F \subset X$ closed and irreducible has nonempty intersection.*

Proof. It is clear that (i) implies (ii). Show that (ii) implies (iii). Indeed, consider an arbitrary proper closed set $F \subseteq X$. Put $\mathscr{W} = \{U \in \mathscr{K} \mid U \cap F \neq \varnothing\}$. Since F is irreducible and $\mathscr{K}$ is a base for the topology of $\mathbb{X}$, the family $\mathscr{W}$ is down-directed and so $\mathscr{W}$ has the finite intersection property. But then the family $\mathscr{W} \cup \{F\}$ has the finite intersection property too. In view of (ii), there exists $x \in F \cap \bigcap \mathscr{W}$. We claim that $F = {\downarrow} x$. Indeed, consider an arbitrary $y \in F$. If $y \in U \in \mathscr{K}$ then $U \in \mathscr{W}$ whence $x \in F \cap \bigcap \mathscr{W} \subseteq U$. Since $\mathscr{K}$ is a base for $\mathscr{T}(\mathbb{X})$; therefore, $y \leq_{\mathbb{X}} x$, as was required.

Show that (iii) implies (i). Indeed, consider an arbitrary proper closed set $F \subseteq X$ and a nontrivial down-directed family $\mathscr{W} \subseteq \mathscr{K}$ such that $\mathscr{W} \cup \{F\}$ has the finite intersection property. Then $\mathscr{I} = \{U \in \mathscr{K} \mid U \subseteq X \backslash F\}$ is an ideal of the join-semilattice $\langle \mathscr{K}; \cup \rangle$. As $\varnothing \neq X \backslash F \in \mathscr{T}(\mathbb{X})$ and $\mathscr{K}$ is a base for $\mathscr{T}(\mathbb{X})$, there is a nonempty $U \in \mathscr{I}$. Consider

$$\mathscr{F} = \{U \in \mathscr{K} \mid U \supseteq U' \text{ for some } U' \in \mathscr{W}\}.$$

Then $\mathscr{W} \subseteq \mathscr{F}$ and $U \cap F \neq \varnothing$ for each $U \in \mathscr{F}$; in particular, $\mathscr{I} \cap \mathscr{F} = \varnothing$. It is not hard to see that $\mathscr{F}$ is an upper cone with respect to set-theoretic inclusion. Suppose that $V, V' \in \mathscr{F}$. Then $U \subseteq V$ and $U' \subseteq V'$ for some $U, U' \in \mathscr{W}$. Since the family $\mathscr{W}$ is down-directed, there is $W \in \mathscr{W}$ such that $W \subseteq U \cap U' \subseteq V \cap V'$. As $W \in \mathscr{F}$, we conclude that $\mathscr{F}$ is a nontrivial filter of $\langle \mathscr{K}; \cup \rangle$. According to Lemma 1 and [8, Theorem 3.3], there is a prime ideal $\mathscr{P}$ of $\langle \mathscr{K}; \cup \rangle$ such that $\mathscr{I} \subseteq \mathscr{P}$ and $\mathscr{F} \cap \mathscr{P} = \varnothing$. We put $G = X \backslash \bigcup \mathscr{P}$. We note that if $G \cap U = \varnothing$ for some $U \in \mathscr{K}$, then $U \in \mathscr{P}$ in view of the compactness of U. Since $\bigcup \mathscr{P} \notin \{\varnothing, X\}$, we conclude that G is a proper closed set in $\mathbb{X}$. We show that G is irreducible in $\mathbb{X}$. Indeed, let $\varnothing \notin \{G \cap V_0, G \cap V_1\}$ for some $V_0, V_1 \in \mathscr{T}(\mathbb{X})$. Then there are sets $U_0, U_1 \in \mathscr{K}$ such that $U_i \subseteq V_i$, $i < 2$, and $\varnothing \notin \{G \cap U_0, G \cap U_1\}$. This means that $U_0, U_1 \notin \mathscr{P}$. As $\mathscr{P}$ is a prime ideal, we conclude that $U \notin \mathscr{P}$ for some $U \in \mathscr{K}$ such that $U \subseteq U_0 \cap U_1$. This implies that $\varnothing \neq U \cap G \subseteq (U_0 \cap U_1) \cap G \subseteq (V_0 \cap V_1) \cap G$. We demonstrated therefore that G is irreducible in $\mathbb{X}$. In view of (iii), there is an element $x \in X$ such that $G = {\downarrow} x$. If $U \in \mathscr{W}$ then $U \in \mathscr{F}$ whence $U \notin \mathscr{P}$. But then $U \cap G \neq \varnothing$, as otherwise using the compactness of U, we would find that $U \in \mathscr{P}$. It follows that $x \in U$ for each $U \in \mathscr{W}$. Since $\mathscr{K}$ is a base for $\mathscr{T}(\mathbb{X})$; therefore, $X \backslash F = \bigcup \mathscr{I} \subseteq \bigcup \mathscr{P}$. Thus, $x \in G \subseteq F$ and so $x \in F \cap \bigcap \mathscr{W}$. Hence, claims (i)–(iii) are equivalent.

Assume now that $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$. It is obvious that (iv) implies (v) and (v) implies (ii). We show that (i) implies (iv). Indeed, consider an arbitrary proper closed set $F \subseteq X$ and a nontrivial family $\mathscr{W} \subseteq \mathscr{K}$ such that $U_0 \cap \dots \cap U_n \cap F \neq \varnothing$ for each $U_0, \dots, U_n \in \mathscr{W}$. By Zorn's Lemma, there is a family $\mathscr{W}' \subseteq \mathscr{K}$ maximal with respect to the following conditions:

$$\mathscr{W} \subseteq \mathscr{W}' \text{ and } \mathscr{W}' \cup \{F\} \text{ has the finite intersection property.}$$

Then $\mathscr{W}' \cup \{F, U_0 \cap \dots \cap U_n\}$ has the finite intersection property for all $U_0, \dots, U_n \in \mathscr{W}$. Since $\mathscr{W}'$ is maximal and $U_0 \cap \dots \cap U_n \in \mathscr{K}$, we conclude that $U_0 \cap \dots \cap U_n \in \mathscr{W}'$, i.e., the family $\mathscr{W}'$ is down-directed. In view of (i), we have $\varnothing \neq F \cap \bigcap \mathscr{W}' \subseteq F \cap \bigcap \mathscr{W}$. $\square$

§ 3. Spectral Spaces

Definition 2. A topological T_0-space $\mathbb{X}$ is called *spectral*, if $\mathbb{X}$ is a compact sober space, and the compact open sets of $\mathbb{X}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$.

A topological T_0-space $\mathbb{X}$ is called *almost spectral*, if $\mathbb{X}$ is an almost sober space, and the compact open sets of $\mathbb{X}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$.

Spectral spaces were characterized in [5] as follows:

Theorem 3 [5, Theorem 13.1.5]. *Let $\mathbb{X}$ be a topological T_0-space, let $\mathscr{K}$ stand for the set of all compact open sets in $\mathbb{X}$, and let $\mathscr{F}$ be the set of all nonempty closed sets in $\mathbb{X}$. The following are equivalent:*

(i) $\mathbb{X}$ *is profinite.*

(ii) $\mathbb{X}$ *is homeomorphic to the inverse limit of spectra of finite distributive lattices with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice with a bottom and a top which is endowed with the discrete topology.*

(iv) $\mathbb{X}$ *is compact, $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \subseteq \mathscr{K} \cup \mathscr{F}$ with the finite intersection property is nonempty.*

(v) $\mathbb{X}$ *is compact, $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ with the finite intersection property, where $\mathscr{W} \subseteq \mathscr{K}$ and $F \in \mathscr{F}$, is nonempty.*

(vi) $\mathbb{X}$ *is compact, $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ with the finite intersection property, where $\mathscr{W} \subseteq \mathscr{K}$ and $F \in \mathscr{F}$ is irreducible, is nonempty.*

(vii) $\mathbb{X}$ *is spectral.*

The following elaborates Theorem 3:

Theorem 4. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is spectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the inverse limit of spectra of finite distributive posets with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive meet-semilattice with a bottom and a top which is endowed with the discrete topology, where φ is an algebraic closure operator.*

Proof. In view of Theorem 3, (i) implies (ii) and (iii). If (ii) holds for $\mathbb{X}$ then, by Lemma 2, $\mathbb{X}$ is homeomorphic to the inverse limit of spectra of finite distributive lattices. Thus, (ii) implies (i) by Theorem 3.

Finally, if an algebraic closure operator φ defines a completion of a φ-distributive meet-semilattice $\langle S; \wedge\rangle$ with a bottom and a top (endowed with the discrete topology), then $\operatorname{Spec}_\varphi S = V_1$. By Lemma 3, the space $\mathbb{S}\mathrm{pec}_\varphi S$ is compact. For all $a, b \in S$, the equality $V_{a\wedge b} = V_a \cap V_b$ holds in an obvious way. Hence by Lemma 3, the topology on $\mathbb{S}\mathrm{pec}_\varphi S$ has a multiplicative base consisting of compact open sets. By Theorem 5 of [1], $\mathbb{S}\mathrm{pec}_\varphi S$ is a sober space. Hence, (iii) implies (i). $\square$

Theorem 5. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is compact, almost spectral, but not spectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice with a top and without a bottom which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive meet-semilattice with a top and without a bottom which is endowed with the discrete topology, where φ is an algebraic closure operator.*

Proof. Show that (i) implies (ii). Let $\mathscr{K}_0$ denote the set of all nonempty compact open sets in $\mathbb{X}$. The set X is irreducible, but $\mathbb{X}$ has no top with respect to the specialization order. In particular, the intersection of every two nonempty open sets is open, whence by Lemma 1 $\langle \mathscr{K}_0; \cup, \cap\rangle$ is a distributive lattice having a top, as $X \in \mathscr{K}_0$. By Theorem 2 $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\,\mathscr{K}_0$; moreover, the lattice $\langle \mathscr{K}_0; \cup, \cap\rangle$ has no bottom, as $\mathbb{X}$ would have a top with respect to $\leq_{\mathbb{X}}$ otherwise.

It is clear that (ii) implies (iii). Furthermore, let φ define a completion of a φ-distributive meet-semilattice $\langle S; \wedge\rangle$ with a top and without a bottom. By Lemma 3, $\operatorname{Spec}_\varphi S = V_1$ is a compact set. For all $a, b \in S$, the equality $V_{a\wedge b} = V_a \cap V_b$ holds in an obvious way. Thus by Lemma 3 $\{V_a \mid a \in S\}$ is a multiplicative base for the topology on $\mathbb{S}\mathrm{pec}_\varphi S$ which consists of nonempty compact open sets. By Theorem 5 of [1], $\mathbb{S}\mathrm{pec}_\varphi S$ is an almost sober space. By Proposition 1, $\mathbb{S}\mathrm{pec}_\varphi S$ is not sober. Therefore, $\mathbb{S}\mathrm{pec}_\varphi S$ is compact, almost spectral but not spectral; and (iii) implies (i). $\square$

Using Theorems 4 and 5 together with Proposition 4, we get

Corollary 1. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is compact and almost spectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice with a top which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive meet-semilattice with a top which is endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) $\mathbb{X}$ *is compact, the set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and the set $F \subset X$ closed (and irreducible), is nonempty.*

Theorem 6. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is sober, almost spectral, but not spectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice with a bottom and without a top which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of an up-directed φ-distributive meet-semilattice with a bottom and without a top which is endowed with the discrete topology, where φ is an algebraic closure operator.*

Proof. Let (i) hold and let $\mathscr{K}$ denote the set of all compact open sets in $\mathbb{X}$. Since $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$; therefore, $\langle \mathscr{K}; \cup, \cap \rangle$ is a distributive lattice. It is obvious that $\varnothing \in \mathscr{K}$; i.e., $\langle \mathscr{K}; \cup, \cap \rangle$ has a bottom. As $\mathbb{X}$ is sober, almost spectral, but not spectral, X is not compact, i.e. $X \notin \mathscr{K}$. Since $\mathscr{K}$ is a base for $\mathscr{T}(\mathbb{X})$, $\langle \mathscr{K}; \cup, \cap \rangle$ has no top. By Theorem 1, $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\, \mathscr{K}$, whence (i) implies (ii). It is clear that (ii) implies (iii).

Let an algebraic closure operator φ define a completion of an up-directed φ-distributive meet-semilattice $\langle S; \wedge \rangle$ with a bottom but without a top. For all $a, b \in S$, the equality $V_{a \wedge b} = V_a \cap V_b$ holds in an obvious way. By Lemma 15 of [1] and Lemma 3, the topology on $\mathbb{S}\mathrm{pec}_\varphi S$ possesses a multiplicative base consisting of compact open sets. By Theorem 5 of [1], $\mathbb{S}\mathrm{pec}_\varphi S$ is sober and almost spectral. By Proposition 2, $\mathbb{S}\mathrm{pec}_\varphi S$ is not compact. Hence, $\mathbb{S}\mathrm{pec}_\varphi S$ is not spectral, and (iii) implies (i). □

From Theorems 5, 6 and Proposition 3 we obtain

Corollary 2. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is sober and almost spectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice with a bottom which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive meet-semilattice with a bottom which is endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) *The set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and $F \subseteq X$ closed* (*and irreducible*), *is nonempty.*

Theorem 7. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is almost spectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive meet-semilattice endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) *The set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ and $F \subset X$ closed* (*and irreducible*), *is nonempty.*

Proof. Show first that (i) implies (ii). By Corollaries 1 and 2, it suffices to consider the case when $\mathbb{X}$ is neither sober nor compact. Therefore, X is irreducible, but $\mathbb{X}$ has no top with respect to the specialization order. Let $\mathscr{K}_0$ denote the set of all nonempty compact open sets in $\mathbb{X}$. In view of the irreducibility of X, the intersection of two nonempty open sets in $\mathbb{X}$ is again nonempty. Thus according to our assumption, $\mathscr{K}_0$ forms a multiplicative base for $\mathscr{T}(\mathbb{X})$, whence $\langle \mathscr{K}_0; \cup, \cap \rangle$ is a distributive lattice. By Theorem 2, $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\, \mathscr{K}_0$.

It is obvious that (ii) implies (iii). Let an algebraic closure operator φ define a completion of a φ-distributive meet-semilattice $\langle S; \wedge \rangle$. For any $a, b \in S$, the equality $V_{a \wedge b} = V_a \cap V_b$ holds in an obvious way. By Lemma 15 of [1] and Lemma 3, the topology on $\mathbb{S}\mathrm{pec}_\varphi S$ possesses a multiplicative base consisting of compact open sets. By Theorem 5 of [1], $\mathbb{S}\mathrm{pec}_\varphi S$ is almost sober. Hence, $\mathbb{S}\mathrm{pec}_\varphi S$ is almost spectral, and (iii) implies (i).

Conditions (i) and (iv) are equivalent by Proposition 4. □

§ 4. Semispectral Spaces

Definition 3. A topological T_0-space $\mathbb{X}$ is called *semispectral*, if $\mathbb{X}$ is a compact sober space, and the compact open sets of $\mathbb{X}$ comprise a base for $\mathscr{T}(\mathbb{X})$.

A topological T_0-space $\mathbb{X}$ is called *almost semispectral*, if $\mathbb{X}$ is an almost sober space, and its compact open sets form a base for $\mathscr{T}(\mathbb{X})$.

Theorem 8. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is semispectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive join-semilattice with a bottom and a top which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive poset with a bottom and a top which is endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) $\mathbb{X}$ *is compact, the set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ forms a base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ down-directed and $F \subseteq X$ closed (and irreducible), is nonempty.*

Proof. According to Proposition 3, (i) and (iv) are equivalent. Let $\langle \mathscr{K}; \cup \rangle$ denote the join-semilattice of all compact open sets in $\mathbb{X}$. By Lemma 1, $\langle \mathscr{K}; \cup \rangle$ is distributive. Obviously, $\varnothing \in \mathscr{K}$; i.e., $\langle \mathscr{K}; \cup \rangle$ contains a bottom. Since X is compact, $X \in \mathscr{K}$; whence $\langle \mathscr{K}; \cup \rangle$ contains a top. By Theorem 1, $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\, \mathscr{K}$, whence (i) implies (ii). It is clear that (ii) implies (iii).

Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq \rangle$ with a bottom and a top so that $\langle S; \leq \rangle$ is φ-distributive. As $\mathrm{Spec}_\varphi S = V_1$, the space $\mathbb{S}\mathrm{pec}_\varphi S$ is compact by Lemma 3. According to Lemma 15 of [1] and Lemma 3, $\{V_a \mid a \in S\}$ forms a basis for the topology on $\mathbb{S}\mathrm{pec}_\varphi S$ which consists of compact open sets. By Theorem 5 of [1], $\mathbb{S}\mathrm{pec}_\varphi S$ is sober. Therefore (iii) implies (i). $\square$

Theorem 9. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is compact, almost semispectral, but not semispectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a down-directed distributive join-semilattice with a top and without a bottom which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a down-directed φ-distributive poset with a top and without a bottom which is endowed with the discrete topology, where φ is an algebraic closure operator.*

Proof. Show that (i) implies (ii). Let $\mathscr{K}_0$ denote the set of all nonempty compact open sets in $\mathbb{X}$. According to our assumption, $X \in \mathscr{K}_0$ whence the join-semilattice $\langle \mathscr{K}_0; \cup \rangle$ has a top. Moreover, X is irreducible, but $\mathbb{X}$ has no top with respect to the specialization order. In particular, the intersection of every two nonempty open sets is again nonempty, whence $\langle \mathscr{K}_0; \cup \rangle$ is a down-directed distributive join-semilattice. According to Theorem 2, $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\, \mathscr{K}_0$; moreover, the join-semilattice $\langle \mathscr{K}_0; \cup \rangle$ has no bottom, as $\mathbb{X}$ would contain a top with respect to $\leq_{\mathbb{X}}$ otherwise.

It is obvious that (ii) implies (iii). Furthermore, let an algebraic closure operator φ define a completion of a down-directed φ-distributive poset $\langle S; \leq \rangle$ with a top and without a bottom. By Lemma 3, the set $\mathrm{Spec}_\varphi S = V_1$ is compact. By Theorem 5 of [1], $\mathbb{S}\mathrm{pec}_\varphi S$ is almost sober, and by Lemma 15 of [1] and Lemma 3, $\{V_a \mid a \in S\}$ is a basis for the topology which consists of compact open sets. Therefore, $\mathbb{S}\mathrm{pec}_\varphi S$ is almost semispectral. According to Proposition 1, $\mathbb{S}\mathrm{pec}_\varphi S$ is not a sober space. Thus, $\mathbb{S}\mathrm{pec}_\varphi S$ is not semispectral, and (iii) implies (i). $\square$

From Theorems 8, 9 and Proposition 4 we obtain

Corollary 3. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is compact and almost semispectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive join-semilattice with a top which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive poset with a top which is endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) $\mathbb{X}$ *is compact, the set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ forms a base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ down-directed and $F \subset X$ closed (and irreducible) is nonempty.*

Theorem 10. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is sober, almost semispectral, but not semispectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive join-semilattice with a bottom and without a top which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of an up-directed φ-distributive poset with a bottom and without a top which endowed with the discrete topology, where φ is an algebraic closure operator.*

Proof. Let $\mathscr{K}$ denote the set of all compact open sets in $\mathbb{X}$. By Lemma 1, the join-semilattice $\langle \mathscr{K}; \cup\rangle$ is distributive. It is obvious that $\varnothing \in \mathscr{K}$, i.e., $\langle \mathscr{K}; \cup\rangle$ has a bottom. Since $\mathbb{X}$ is sober, almost semispectral, but not semispectral, X is not compact; i.e., $X \notin \mathscr{K}$. Since $\mathscr{K}$ forms a base for $\mathscr{T}(\mathbb{X})$; therefore, $\langle \mathscr{K}; \cup\rangle$ has no top. According to Theorem 1, $\mathbb{X} \cong \operatorname{Spec} \mathscr{K}$ whence (i) implies (ii). Obviously, (ii) implies (iii) by Lemma 13 of [1].

Let an algebraic closure operator φ define a completion of an up-directed poset $\langle S; \leq\rangle$ with a bottom and without a top so that $\langle S; \leq\rangle$ is φ-distributive. By Lemma 15 of [1] and Lemma 3, $\{V_a \mid a \in S\}$ forms a base for the topology on $\operatorname{Spec}_\varphi S$ which consists of compact open sets. By Theorem 5 of [1], $\operatorname{Spec}_\varphi S$ is sober. It follows that $\operatorname{Spec}_\varphi S$ is a sober almost semispectral space. By Proposition 2, $\operatorname{Spec}_\varphi S$ is not compact. Hence, $\operatorname{Spec}_\varphi S$ is not a semispectral space, and (iii) implies (i). □

From Theorems 8, 10 and Proposition 3 we obtain

Corollary 4. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is sober and almost semispectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive join-semilattice with a bottom which is endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive poset with a bottom which is endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) *The set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ forms a base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ down-directed and $F \subseteq X$ closed (and irreducible), is nonempty.*

Theorem 11. *For an arbitrary topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is almost semispectral.*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive join-semilattice endowed with the discrete topology.*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of a φ-distributive poset endowed with the discrete topology, where φ is an algebraic closure operator.*

(iv) *The set $\mathscr{K}$ of all compact open sets in $\mathbb{X}$ forms a base for $\mathscr{T}(\mathbb{X})$, and the intersection of each family $\mathscr{W} \cup \{F\}$ possessing the finite intersection property, with $\mathscr{W} \subseteq \mathscr{K}$ down-directed and $F \subset X$ closed (and irreducible) is nonempty.*

Proof. Show first that (i) implies (ii). Taking into account Corollaries 3 and 4, it suffices to consider the case when $\mathbb{X}$ is neither sober nor compact. It follows that X is irreducible, but $\mathbb{X}$ has no top with respect to the specialization order. Let $\mathscr{K}_0$ denote the set of all nonempty compact open sets in $\mathbb{X}$. By Lemma 1, the join-semilattice $\langle \mathscr{K}_0; \cup\rangle$ is distributive. Moreover, since X is irreducible, the intersection of every two nonempty sets open in $\mathbb{X}$ is again nonempty, whence the join-semilattice $\langle \mathscr{K}_0; \cup\rangle$ is down-directed. By Theorem 2, $\mathbb{X} \cong \operatorname{Spec} \mathscr{K}_0$.

It is obvious that (ii) implies (iii). Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq\rangle$ so that $\langle S; \leq\rangle$ is φ-distributive. By Lemma 15 of [1] and Lemma 3, $\{V_a \mid a \in S\}$ is a base for the topology on $\operatorname{Spec}_\varphi S$ which consists of compact open sets. By Theorem 5 of [1], $\operatorname{Spec}_\varphi S$ is an almost sober space. It follows that $\operatorname{Spec}_\varphi S$ is an almost semispectral space, and (iii) implies (i).

Conditions (i) and (iv) are equivalent by Proposition 4. □

From Tychonoff's Theorem, Theorem 5.3.7, and Corollaries 1.12.8, 1.12.10, 5.3.6(ii) of [5] we obtain

Corollary 5. *Consider the next properties of topological spaces:*

(1) *to be a (semi)spectral space;*

(2) *to be a compact almost (semi)spectral space;*

(3) *to be an almost (semi)spectral space;*

(4) *to be a sober almost (semi)spectral space.*

Then the following hold:

(i) *If topological T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$ possess one of the properties* (1)–(4), *then $\mathbb{X} \times \mathbb{Y}$ possesses the same property.*

(ii) *If a topological T_0-space $\mathbb{X}_i$ possesses one of the properties* (1)–(2) *for all $i \in I$, then the space $\prod_{i\in I} \mathbb{X}_i$ possesses the same property.*

(iii) *A closed subspace of a T_0-space with one of the properties* (1)–(4) *possesses the same property.*

Corollary 6. *Let $\mathfrak{P}$ denote one of the following properties:*

(i) *to be a spectral space;*

(ii) *to be an almost spectral space;*

(iii) *to be a compact almost spectral space;*

(iv) *to be a sober almost spectral space.*

If $\mathbb{X}$ possesses property $\mathfrak{P}$ and $U \subseteq X$ is a nonempty compact open set in $\mathbb{X}$, then the space $\mathbb{U}$ also possesses property $\mathfrak{P}$ and is a compact space.

Proof. Let $\mathscr{K}$ denote the set of all compact open subsets in $\mathbb{X}$; and $\mathscr{K}_{\mathbb{U}}$, the set of all compact open subsets in $\mathbb{U}$. Then $\mathscr{K}_{\mathbb{U}} \subseteq \mathscr{K}$. Moreover, since $\mathscr{K}$ is a multiplicative base for $\mathscr{T}(\mathbb{X})$; therefore, $\mathscr{K}_{\mathbb{U}}$ is a multiplicative base for $\mathscr{T}(\mathbb{U})$. It is obvious that $\mathbb{U}$ is a compact space. Suppose that a family $\mathscr{W} \subseteq \mathscr{K}_{\mathbb{U}} \subseteq \mathscr{K}$ and a (proper) set $F \subseteq U$ closed in $\mathbb{U}$ are such that the family $\mathscr{W} \cup \{F\}$ has the finite intersection property. It is not hard to see that there exists a (proper) set $G \subseteq X$ closed in $\mathbb{X}$ such that $F = G \cap U$. Hence, the family $\mathscr{W}' = \{U' \in \mathscr{K}_{\mathbb{U}} \mid U' \in \mathscr{W}\} \cup \{G\}$ has the finite intersection property too. We use one of the following statements: Theorem 3, Corollaries 1 and 2, and Theorem 7. According to this statement, $\varnothing \neq \bigcap \mathscr{W}' = \bigcap \mathscr{W}$. It remains to refer to the same statement again. $\square$

§ 5. A Remark

Theorem 7 of [1] should be read as follows:

Theorem 12. *Let $\mathbb{S} = \langle S, \vee, \mathscr{T}\rangle$ be a join-semilattice with topology and let the topology $\mathscr{T} = \mathscr{T}_\psi^\pi$ be T_0-separable. Then $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\psi \mathbb{S})$ is a biggest essential extension of $\mathbb{S}$.*

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