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ABSOLUTE CONVERGENCE OF THE DOUBLE FOURIER–FRANKLIN SERIES

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Abstract: We prove that, for every $0 < \epsilon < 1$, there exists a measurable set $E \subset T = [0,1]^2$ with measure $|E| > 1 - \epsilon$ such that, for all $f \in L^1(T)$ and $0 < \eta < 1$, we can find $\tilde{f} \in L^1(T)$ with $\iint_T |f(x,y) - \tilde{f}(x,y)|\, dxdy \le \eta$ coinciding with $f(x,y)$ on E whose double Fourier–Franklin series converges absolutely to f almost everywhere on T.

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§ 1. Introduction

The article is devoted to absolute convergence almost everywhere of the series in the double Franklin system from the point of view of the classical Luzin [1] and Menshov [2] Theorems of "functions improvement."

The Franklin system [3] is one of the popular systems of functions, and many articles are devoted to its study. One of its principal properties is the fact that it constitutes an orthogonal basis for $C[0,1]$ and an unconditional basis for $L^p[0,1]$, $p \in (1,\infty)$, where $C[0,1]$ is the space of all continuous functions on $[0,1]$ ($\|f\|_C = \max_{x\in[0,1]} |f(x)|$), while $L^p[0,1]$ ($p > 0$) is the space of measurable functions on $[0,1]$ for which $\int_{[0,1]} |f(x)|^p\, dx < \infty$. We denote by $|E|$ the Lebesgue measure of a set $E \subseteq [0,1]$ ($E \subseteq T = [0,1]^2$).

Many articles address the convergence of the Fourier series in the classical systems after changing the values of the function on a set of small measure.

The following result is well known:

Theorem A [2]. *Let $f(x)$ be a measurable function finite almost everywhere on $[0,2\pi]$. For every $\epsilon > 0$, we can define a continuous function $g(x)$ coinciding with $f(x)$ on some set E with measure $|E| > 2\pi - \epsilon$ whose Fourier series in the trigonometric system converges uniformly on $[0,2\pi]$.*

Many interesting results have been obtained in this area. We describe those relevant to the results of this article.

Theorem B [4]. *Given an almost everywhere measurable and finite function $f(x)$ on $[0,1]$ and a real $0 < \epsilon < 1$, we can find $\tilde{f} \in L^2[0,1)$ with $|\{x : f(x) \ne \tilde{f}(x)\}| < \epsilon$ whose Fourier series in the Haar system converges absolutely and uniformly on $[0,1]$.*

Theorem C [5]. *Given $f \in C[0,1]$ and $0 < \epsilon < 1$, we can find $\tilde{f} \in C[0,1]$ with $|\{x : f(x) \ne \tilde{f}(x)\}| < \epsilon$ whose Fourier series in the Franklin system converges absolutely and uniformly on $[0,1]$.*

We should observe that Katsnelson proved in [6] that it is impossible in the Menshov Theorem to achieve absolute convergence (i.e., Theorem C is false for the trigonometric system).

Theorem D [7]. *For every $0 < \epsilon < 1$, there exists a measurable set $E \subset [0,1]$ with measure $|E| > 1 - \epsilon$ such that for each $f \in L^1[0,1)$ we can find $\tilde{f} \in L^1[0,1)$ coinciding with f on E whose Fourier*

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series in the Haar system converges absolutely almost everywhere on $[0,1]$ *and all nonzero terms in the sequence of Fourier coefficients of* $\tilde{f}$ *in the Haar system are placed in descending order.*

Observe that the "exceptional" set e, on which the change of $f(x)$ happens, is universal in Theorem D (serves the entire function class), whereas e essentially depends on the improved function $f(x)$ in Theorems B and C; in those theorems, it is impossible to choose e independently of $f(x)$. The following theorem of [8] yields the same fact for the Franklin system:

Theorem E. *For every set* E *with positive measure and for each density point* x_0 *of* E*, there exists* $f_0 \in C[0,1]$ *such that the Fourier–Franklin series of* $f(x)$ *diverges absolutely at* x_0 *for every bounded function* $f(x)$ *coinciding with* $f_0(x)$ *on* E.

Theorem E is also valid for the Haar system. Note that this "bad" property is not common for all bases for $C[0,1]$; in particular, it is proven in [9] that the Faber–Schauder system does not possess this property. Namely, the following holds:

Theorem F. *For every* $0<\epsilon<1$*, there exists a measurable set* $E\subset[0,1]$ *with measure* $|E|>1-\epsilon$ *such that, for every* $f\in C[0,1]$*, we can find* $g\in C[0,1]$ *coinciding with* f *on* E *whose expansion* $\sum_{k=0}^{\infty} A_k(g)\varphi_k(x)$ *in the Faber–Schauder system converges absolutely and uniformly on* $[0,1]$ *and*

$$\left\|\sum_{n=1}^{\infty} |A_n(g)|\varphi_n\right\|_C \le \|g\|_C < 2\|f\|_C.$$

This leads immediately to the question whose answer is still unknown.

Question 1. *Is there an orthogonal basis for* $C[0,1]$ *for which Theorem* F *is valid?*

Theorem E implies that it is impossible, by changing the values of each continuous function $f(x)$ on the given set, to obtain $g(x)\in C[0,1]$ whose Fourier–Franklin series converges absolutely and uniformly on $[0,1]$. However, the problem becomes solvable if we require that, after the change of $f(x)\in L^1[0,1]$ on the given set, we obtain $g(x)$ whose Fourier–Franklin series converges absolutely and almost everywhere on $[0,1]$, while $f(x)$ itself is only summable. Moreover, the following is proved in [8]:

Theorem G. *For every* $0<\epsilon<1$*, there exists a measurable set* $E\subset[0,1]$ *with measure* $|E|>1-\epsilon$ *such that, for each* $f\in L^1[0,1)$*, we can find* $\tilde{f}\in L^1[0,1)$ *coinciding with* f *on* E *whose Fourier–Franklin series converges absolutely to* $\tilde{f}$ *almost everywhere on* $[0,1]$ *and the sequence of the Fourier coefficients of* $\tilde{f}$ *in the Franklin system* $\{f_n(x)\}_{n=0}^{\infty}$ *lies in all* l^r*,* $r>2$*, i.e.,*

$$\sum_{n=0}^{\infty} |c_n(\tilde{f})|^r < \infty \ \forall r>2, \quad \text{where } c_n(\tilde{f}) = \int_0^1 \tilde{f}(x) f_n(x)\,dx.$$

In this article we investigate whether we can obtain similar results for double Franklin series.

Let $T=[0,1]^2$, $p\in[1,\infty)$, and $f\in L^p(T)$. The Fourier coefficients of $f\in L^p(T)$ in the double Franklin system $\{f_k(x)f_s(y)\}_{k,s=0}^{\infty}$ are denoted by

$$c_{k,s}(f) = \iint_T f(t,\tau) f_k(t) f_s(\tau)\,dt d\tau, \quad k,s\in\mathbb{N}\cup\{0\}. \tag{1.1}$$

Put

$$\Lambda(f) := \operatorname{spec}\{c_{k,s}(f)\} = \operatorname{spec}(f) = \{(k,s), c_{k,s}(f)\} \ne 0, \quad k,s\in\mathbb{N}\cup\{0\}. \tag{1.2}$$

The rectangular and spherical partial sums of the double Fourier–Franklin series are determined as follows:

$$S_{N,M}(x,y,f) := \sum_{k=0}^{N}\sum_{s=0}^{M} c_{k,s}(f) f_k(x) f_s(y), \tag{1.3}$$

$$S_R(x,y,f) := \sum_{k^2+s^2\le R^2} c_{k,s}(f) f_k(x) f_s(y). \tag{1.4}$$

Observe that some results are impossible to transfer from the one-dimensional case to the two-dimensional; even particular (spherical, rectangular, or square) partial sums differ strikingly from each other in their properties with respect to convergence in L^p, $p \geq 1$, and convergence almost everywhere.

In particular, the following result justifies the above-mentioned fact: There exists a summable function $f_0(x, y)$ on T whose rectangular partial sums of the double Fourier–Haar series [10] diverge almost everywhere on T (we do not know whether such result is valid for double Fourier–Franklin series).

Note that in the one-dimensional case the Fourier–Franklin series of every $f \in L^1[0, 1]$ converges almost everywhere on $[0, 1]$.

The question arises naturally: Is there a measurable set e with arbitrarily small measure such that after changing the values of each $g \in L^1(T)$ on e the Fourier series in the double Franklin system $\{f_k(x)f_s(y)\}_{k,s=0}^{\infty}$ of the resulting function converges almost everywhere on T by rectangles or spheres? It turns out that the answer to this question is in the affirmative.

Theorem 1. *For every $0 < \epsilon < 1$, there exists a measurable set $E \subset T = [0, 1]^2$ with measure $|E| > 1 - \epsilon$ such that, for each $f \in L^1(T)$, we can find $\tilde{f} \in L^1(T)$ coinciding with $f(x, y)$ on E and such that both rectangular and spherical partial sums of the double Fourier–Franklin series of $\tilde{f}$ converge to $\tilde{f}$ almost everywhere on T.*

Moreover, in this article we prove

Theorem 2. *For every $0 < \epsilon < 1$, there exists a measurable set $E \subset T = [0, 1]^2$ with measure $|E| > 1-\epsilon$ such that, for all $f \in L^1(T)$ and $0 < \eta < 1$, we can find $\tilde{f} \in L^1(T)$ with $\iint_T |f(x, y)-\tilde{f}(x, y)|\, dxdy \leq \eta$ coinciding with $f(x, y)$ on E whose double Fourier–Franklin series converges absolutely to $\tilde{f}$ almost everywhere on T.*

This theorem follows from the stronger result:

Theorem 3. *There exists a series in the double Franklin system of the form*

$$\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} d_{k,s} f_k(x) f_s(y), \quad \sum_{k,s=0}^{\infty} |d_{k,s}|^r < \infty, \quad r > 2, \tag{1.5}$$

such that, for every $0 < \epsilon < 1$, there exists a measurable set $E \subset T$ with measure $|E| > 1 - \epsilon$ such that, for all $f \in L^1(T)$ and $0 < \eta < 1$, we can find $\tilde{f} \in L^1(T)$ with $\iint_T |f(x, y) - \tilde{f}(x, y)|\, dxdy \leq \eta$ coinciding with $f(x, y)$ on E whose Fourier–Franklin series converges absolutely to $\tilde{f}$ almost everywhere on T and

$$c_{k,s}(\tilde{f}) = d_{k,s}, \quad (k, s) \in \Lambda(\tilde{f}) = \operatorname{spec}(\tilde{f}).$$

Question 2. *Is it possible to choose as a (1.5) series the Fourier series in the double Franklin system for some $g \in L^1(T)$?*

§ 2. Proof of the Main Lemma

Recall the definition of the Franklin system [3]. Let $\pi_1 = \{0, 1\}$ and

$$\pi_n = \{t_s\}_{s=0}^{n}, \quad \text{where } t_s = t_s(n) = \begin{cases} \frac{s}{2^{k+1}} & \text{if } s = 0, 1, \dots, 2i, \\ \frac{s-i}{2^k} & \text{if } s = 2i + 1, \dots, n, \end{cases}$$

for $n = 2^k + i$, $k = 0, 1, \dots$, $i = 1, 2, \dots, 2^k$.

Denote by S_n the space of functions continuous on $[0, 1]$ and piecewise linear with nodes from π_n. Observe that π_n is obtained by adding the point $z_n = t_{2i-1}(n) = \frac{2i-1}{2^{k+1}}$ to π_{n-1}.

The system of the Franklin functions $F = \{f_n(x)\}$ is determined on $[0, 1]$ as follows:

$$f_0(x) = 1, \quad f_1(x) = \sqrt{3}(2x - 1), \quad x \in [0, 1],$$
$$f_n(x) \in S_n, \quad f_n \perp S_{n-1}, \quad \|f_n\|_{L^2} = 1, \quad f_n(t_{2i-1}(n)) > 0, \quad n \geq 2.$$

Divide the interval $[0,1]$ into 2^q equal parts: $\Delta_q^{(j)} = \left[\frac{j-1}{2^q}, \frac{j}{2^q}\right]$, $1 \le i \le 2^q$, which we call *binary intervals*.

REMARK 1. Given $f \in L^1(T)$ and a positive real ξ, there exists a polynomial $Q(x,y)$ in the double Franklin system $\{f_k(x)f_s(y)\}_{k,s=0}^{\infty}$ such that

$$\frac{3}{4}\xi \le \iint\limits_T |f(x,y) - Q(x,y)|\,dxdy \le \frac{5}{4}\xi. \tag{2.1}$$

Indeed, it is easy to see that we can choose a polynomial $Q(x,y)$ in the double Franklin system $\{f_k(x)f_s(y)\}_{k,s=0}^{\infty}$ such that

$$\iint\limits_T |(f(x,y) - \xi) - Q(x,y)|\,dxdy \le \frac{\xi}{4}.$$

Hence,

$$\iint\limits_T |f(x,y) - Q(x,y)|\,dxdy \le \frac{\xi}{4} + \xi,$$

$$\iint\limits_T |f(x,y) - Q(x,y)|\,dxdy \ge \iint\limits_T \xi\,dxdy - \iint\limits_T |[f(x,y) - \xi] - Q(x,y)|\,dxdy \ge \xi - \frac{\xi}{4}.$$

Lemma 1. *Given $f \in L^1(T)$ and a sequence of positive reals $\{\xi_k\}_{k=1}^{\infty}$ with*

$$0 < \xi_{k+1} \le \frac{\xi_k}{5}, \quad k \ge 1, \tag{2.2}$$

we can find a sequence of polynomials $\{\Pi_k(x,y)\}_{k=1}^{\infty}$ in the double Franklin system $\{f_n(x)f_s(y)\}_{n,s=0}^{\infty}$ with rational coefficients such that

$$\lim_{N\to\infty} \iint\limits_T \left|\sum_{k=1}^{N} \Pi_k(x,y) - f(x,y)\right| dxdy = 0, \quad \lim_{N\to\infty} \sum_{k=1}^{N} \Pi_k(x,y) = f(x,y) \quad \text{a.e. on } T,$$

$$\frac{\xi_k}{2} \le \iint\limits_T |\Pi_k(x,y)|\,dxdy \le \frac{3}{2}\xi_k, \quad k \ge 2.$$

PROOF. It is easily seen that, by Remark 1, we can choose a sequence of polynomials $\{Q_k(x,y)\}_{k=1}^{\infty}$ in the double Franklin system $\{f_n(x)f_s(y)\}_{n,s=0}^{\infty}$ with rational coefficients such that

$$\frac{3}{4}\xi_{k+1} \le \iint\limits_T |f(x,y) - Q_k(x,y)|\,dxdy \le \frac{5}{4}\xi_{k+1}, \quad k \ge 1. \tag{2.3}$$

Put

$$\Pi_k(x,y) = Q_k(x,y) - Q_{k-1}(x,y), \quad k \ge 1,\ Q_0(x,y) = 0. \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$\iint\limits_T \left|f(x,y) - \sum_{k=1}^{N} \Pi_k(x,y)\right| dxdy = \iint\limits_T |f(x,y) - Q_N(x,y)|\,dxdy \le \frac{5}{4}\xi_N, \quad N \ge 1. \tag{2.5}$$

By (2.2)–(2.4), derive

$$\iint_T |\Pi_k(x,y)|\,dxdy \le \iint_T |f(x,y)-Q_k(x,y)|\,dxdy$$

$$+\iint_T |f(x,y)-Q_{k-1}(x,y)|\,dxdy \le \frac{3}{2}\xi_k, \quad k\ge 2,$$

$$\iint_T |\Pi_k(x,y)|\,dxdy \ge \iint_T |f(x,y)-Q_{k-1}(x,y)|\,dxdy$$

$$-\iint_T |f(x,y)-Q_k(x,y)|\,dxdy \ge \frac{\xi_k}{2}, \quad k\ge 2.$$

Putting

$$B := \bigcup_{q=1}^{\infty}\bigcap_{N=q}^{\infty}\left\{(x,y)\in T : |f(x,y)-\sum_{k=1}^{N}\Pi_k(x,y)| < \sqrt{\xi_N}\right\}$$

and using (2.2) and (2.5), we get $|B|=1$.

It is clear that $\left|f(x,y)-\sum_{k=1}^{N}\Pi_k(x,y)\right|\to 0$ as $\to\infty$ on B; consequently,

$$\lim_{N\to\infty}\sum_{k=1}^{N}\Pi_k(x,y)=f(x,y) \quad \text{a.e. on } T.$$

Lemma 1 is proven.

Below we will use the following lemma (see [8, Lemma 6]):

Lemma 2. *Assume given ε_0, δ_0, λ_0, θ_0, τ_0, $\sigma_0\in(0,1)$, some $N_0\in\mathbb{N}$, $\varepsilon_0<\delta_0$, and the binary interval $\Delta=[a,b]$. Then there exist a polynomial in the Franklin system of the form $Q(t)=\sum_{n=N_0}^{M}a_nf_n(t)$ and $G\subset E\subset[a,b]$ such that*

(1) $$|E|>(1-\varepsilon_0)(b-a), \quad |G|>(1-\delta_0)(b-a),$$

(2) $$Q(t)=0 \quad \text{for all } t\notin[a,b],$$

(3) $$Q(t)=1 \quad \text{for all } x\in E,$$

(4) $$\int_{[a,b]}|Q(t)|\,dt<2(b-a),$$

(5) $$\sum_{n=N}^{M}|a_nf_n(t)|<\theta_0 \quad \text{for all } t\notin(a-\lambda_0,b+\lambda_0),$$

(6) $$\left(\sum_{n=N_0}^{M}|a_n|^{2+\sigma_0}\right)^{\frac{1}{2+\sigma_0}}<\tau_0,$$

(7) $$\sum_{n=N_0}^{M}|a_nf_n(t)|<\frac{A\left(\log\frac{1}{\delta_0}\right)}{\delta_0} \quad \text{for all } t\in G, \text{ where } A \text{ is constant.}$$

Lemma 3. *Assume given $0 < \eta < \varepsilon < \delta < 1$, $r > 2$, some $N \in \mathbb{N}$, and $f(x,y) \in L^1(T)$ with $\iint_T |f(x,y)|\,dxdy > 0$. Then there exist $G \subset E \subset T$, $g(x,y) \in L^1(T)$, and the polynomial in the double Franklin system of the form*

$$Q(x,y) = \sum_{k,s=N}^{M} c_{k,s} f_k(x) f_s(y)$$

such that

(1) $$|E| > 1 - \varepsilon, \quad |G| > 1 - \delta,$$

(2) $$g(x,y) = f(x,y) \quad \text{on } E,$$

(3) $$\iint_T |g(x,y) - Q(x,y)|\,dxdy \le \eta,$$

(4) $$\iint_T |g(x,y)|\,dxdy \le 5 \iint_T |f(x,y)|\,dxdy,$$

(5) $$\sum_{k,s=N}^{M} |c_{k,s} f_n(x) f_s(y)| \le \frac{B\left(\log \frac{1}{\delta}\right)^2 |f(x,y)|}{\delta^2} + \eta, \quad (x,y) \in G, \text{ where } B \text{ is constant},$$

(6) $$\left(\sum_{k,s=N}^{M} |c_{k,s}|^r\right)^{\frac{1}{r}} \le \eta.$$

Proof of Lemma 3. Take the step-function

$$\varphi(x,y) = \sum_{l,j=1}^{2^q} \gamma_{l,j} \chi_{\Delta_{l,,j}}(x,y), \tag{2.6}$$

where

$$\Delta_{l,,j} = \Delta'_l \times \Delta''_j = [\alpha_{l-1}, \alpha_l] \times [\alpha_{j-1}, \alpha_j], \tag{2.7}$$

$$\alpha_j = \frac{j}{2^q}, \quad j = 0, 1, \dots, 2^q, \tag{2.8}$$

such that

$$\iint_T |f(x,y) - \varphi(x,y)|\,dxdy < \min\left[\frac{\eta\delta^3}{128A^2\left(\log\frac{1}{\delta}\right)^2}; \frac{1}{2}\iint_T |f(x,y)|\,dxdy\right]. \tag{2.9}$$

Let

$$E_0 = \left\{(x,y) \in T : |f(x,y) - \varphi(x,y)| < \frac{\eta\delta^2}{32A^2\left(\log\frac{1}{\delta}\right)^2}\right\}. \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$|E_0| > \left(1 - \frac{\delta}{4}\right). \tag{2.11}$$

Put

$$\partial = \frac{\delta\eta}{\Big(4A\sum\limits_{l,j=1}^{2^q}|\gamma_{l,j}|+1\Big)\big(\log\frac{1}{\delta}\big)}. \tag{2.12}$$

Apply Lemma 2 (for each $l \in [1, 2^q]$), putting

$$\Delta = \Delta'_l, \quad \varepsilon_0 = \frac{\varepsilon}{4}, \quad \delta_0 = \frac{\delta}{4}, \quad N_0 = N_l, \quad \tau_0 = \Bigg(\sum_{l,j=1}^{2^q}|\gamma_{l,j}|\Bigg)^{-1},$$

$$\sigma_0 = r-2, \quad \lambda_0 = \frac{\delta}{2^{q+4}}, \quad \theta_0 = \partial.$$

Then the measurable set $G'_l \subset E'_l \subset \Delta'_l$ and polynomials in the Franklin system

$$Q'_l(x) = \sum_{n=N_l+1}^{N_{l+1}} a_k^{(l)} f_n(x) \tag{2.13}$$

are determined for each $l \in [1, 2^q]$ and satisfy the conditions

$$|E'_l| > |\Delta'_l|\bigg(1-\frac{\varepsilon}{4}\bigg), \quad |G'_l| > |\Delta'_l|\bigg(1-\frac{\delta}{4}\bigg), \tag{2.14}$$

$$Q'_l(x) = \begin{cases} 1 & \text{for } x \in E'_l, \\ 0 & \text{for } x \notin \Delta'_l, \end{cases} \tag{2.15}$$

$$\int\limits_{\Delta'_l} |Q_l(x)|\,dx < 2|\Delta'_l|, \tag{2.16}$$

$$\sum_{k=N_l+1}^{N_{l+1}} \big|a_k^{(l)} f_k(x)\big| < \partial, \quad x \notin \bigg[\alpha_l - \frac{\delta}{2^{q+4}}, \alpha_{l+1} + \frac{\delta}{2^{q+4}}\bigg], \tag{2.17}$$

$$\sum_{k=N_l+1}^{N_{l+1}} \big|a_k^{(l)} f_k(x)\big| < \frac{4A\big(\log\frac{1}{\delta}\big)}{\delta}, \quad x \in G'_l, \tag{2.18}$$

$$\Bigg(\sum_{k=N_l+1}^{N_{l+1}} \big|a_k^{(l)}\big|^r\Bigg)^{\frac{1}{r}} < \Bigg(\sum_{l,j=1}^{2^q}|\gamma_{l,j}|\Bigg)^{-1}. \tag{2.19}$$

Again, apply Lemma 2 (for each $j \in [1, 2^q]$), putting

$$\Delta = \Delta''_j, \quad \varepsilon_0 = \frac{\varepsilon}{4}, \quad \delta_0 = \frac{\delta}{4}, \quad N_0 = N_{2^q+1},$$

$$\tau_0 = \eta, \quad \sigma_0 = r-2, \quad \lambda_0 = \frac{\delta}{2^{q+4}}, \quad \theta_0 = \partial.$$

Then the measurable sets $G''_j \subset E''_j \subset \Delta''_j$ and the polynomial in the Franklin system

$$Q''_j(y) = \sum_{s=M_j+1}^{M_{j+1}} b_s^{(j)} f_s(y), \quad M_1 = N_{2^q}, \tag{2.20}$$

are determined for each $j \in [1, 2^q]$ and satisfy the conditions

$$|E_j''| > |\Delta_j''|\left(1 - \frac{\varepsilon}{4}\right), \quad |G_j''| > |\Delta_j''|\left(1 - \frac{\delta}{4}\right), \tag{2.21}$$

$$Q_j''(y) = \begin{cases} 1 & \text{for } y \in E_j'', \\ 0 & \text{for } x \notin \Delta_j'', \end{cases} \tag{2.22}$$

$$\int\limits_{\Delta_j''} |Q_j''(y)|\, dy < 2|\Delta_j''|, \tag{2.23}$$

$$\sum_{s=M_j+1}^{M_{j+1}} \left|b_s^{(j)} f_s(y)\right| < \partial, \quad y \notin \left[\beta_j - \frac{\delta}{2^{q+4}}, \beta_{j+1} + \frac{\delta}{2^{q+4}}\right], \tag{2.24}$$

$$\sum_{s=M_j+1}^{M_{j+1}} \left|b_s^{(j)} f_s(y)\right| < \frac{A\left(\log \frac{1}{\delta}\right)}{\delta}, \quad x \in G_j'', \tag{2.25}$$

$$\left(\sum_{s=M_j+1}^{M_{j+1}} \left|b_s^{(j)}\right|^r\right)^{\frac{1}{r}} < \eta. \tag{2.26}$$

Define $Q(x, y)$, $g(x, y)$, E, and G as follows:

$$Q(x, y) = \sum_{l,j=1}^{2^q} \gamma_{l,j} Q_l'(x) Q_j''(y) = \sum_{l,j=1}^{2^q} \gamma_{l,j} \sum_{k=N_l+1}^{N_{l+1}} a_k^{(l)} f_k(x) \sum_{s=M_j+1}^{M_{j+1}} b_s^{(j)} f_s(y)$$

$$= \sum_{k,s=N}^{M} c_{k,s} f_k(x) f_s(y), \quad M = M_{2^q+1}, \tag{2.27}$$

$$c_{k,s} = \begin{cases} \gamma_{l,j} a_k^{(l)} b_s^{(j)}, & (k, s) \in \Omega_{l,j} := (N_l, N_{l+1}] \times (M_j, M_{j+1}],\ l, j \in [1, 2^q], \\ 0, & (k, s) \notin \bigcup\limits_{lj=1}^{2^q} \Omega_{l,j}, \end{cases} \tag{2.28}$$

$$g(x, y) = f(x, y) - [\varphi(x, y) - Q(x, y)], \tag{2.29}$$

$$E = \bigcup_{l=1}^{2^q} \bigcup_{j=1}^{2^q} (E_l' \times E_j''), \tag{2.30}$$

$$G = E_0 \cap \left[\left(\bigcup_{l=1}^{2^q} G_l'\right) \times \left(\bigcup_{j=1}^{2^q} G_j''\right)\right] \setminus E_0 \cap [(A_q \times [0, 1]) \cup ([0, 1] \times A_q)], \tag{2.31}$$

where $A_q = \bigcup_{l=1}^{2^q} \left[\alpha_j - \frac{\delta}{2^{q+4}}, \alpha_j + \frac{\delta}{2^{q+4}}\right]$. From (2.6), (2.11), (2.14), (2.15), (2.21), (2.22), (2.27), and (2.29)–(2.31) it follows that

$$|E| > 1 - \varepsilon, \quad |G| > 1 - \delta, \quad g(x, y) = f(x, y) \text{ on } E, \quad \iint\limits_T |g(x, y) - Q(x, y)|\, dxdy \leq \eta.$$

By (2.6), (2.15), (2.16), (2.22), (2.23), and (2.27), we obtain

$$\iint\limits_T |Q(x, y)|\, dxdy = \sum_{l,j=1}^{q} |\gamma_{l,j}| \int\limits_{\Delta_l'} |Q_l(x)|\, dx \int\limits_{\Delta_j''} |Q_j''(y)|\, dy \leq 4 \sum_{l,j=1}^{q} |\gamma_{l,j}||\Delta_l'||\Delta_j''| = 4 \iint\limits_T |\varphi(x, y)|\, dxdy.$$

Hence, (2.9) and (2.29) imply that

$$\iint\limits_T |g(x,y)|\,dxdy \le \iint\limits_T |f(x,y)-\varphi(x,y)|\,dxdy + \iint\limits_T |Q(x,y)|\,dxdy \le 5\iint\limits_T |f(x,y)|\,dxdy.$$

Verify the claim (5) of Lemma 3.

If $(x,y)\in G$, then $(x,y)\in G'_{l_0}\times G''_{j_0}$ for some l_0 and j_0. Using (2.6), (2.12), (2.17), (2.18), (2.24), (2.25), (2.28), and (2.31), we obtain

$$\begin{aligned}
\sum_{k,s=N}^{M} |c_{k,s} f_n(x) f_s(y)| &\le \sum_{l,j=1}^{2^q} |\gamma_{l,,j}| \Bigg(\sum_{n=N_l+1}^{N_{l+1}} \big|a_k^{(l)} f_k(x)\big|\Bigg)\Bigg(\sum_{s=M_j+1}^{M_{j+1}} \big|b_s^{(j)} f_s(y)\big|\Bigg) \\
&= |\gamma_{l_0,j_0}| \Bigg(\sum_{n=N_{l_0}+1}^{N_{l_0+1}} \big|a_k^{(l_0)} f_k(x)\big|\Bigg)\Bigg(\sum_{s=M_{j_0}+1}^{M_{j_0+1}} \big|b_s^{(j_0)} f_s(y)\big|\Bigg) \\
&+ \sum_{(l,j)\in[1,2^q]^2\setminus(l_0,j_0)} |\gamma_{l,j}| \Bigg(\sum_{n=N_l+1}^{N_{l+1}} \big|a_k^{(l)} f_k(x)\big|\Bigg)\Bigg(\sum_{s=M_j+1}^{M_{j+1}} \big|b_s^{(j)} f_s(y)\big|\Bigg) \\
&\le \frac{16A^2\big(\log\frac{1}{\delta}\big)^2 |\gamma_{l_0,j_0}|}{\delta^2} + 2\Bigg(\sum_{l,j=1}^{2^q} |\gamma_{l,j}|\Bigg)\Bigg(\frac{A\log\frac{1}{\delta}}{\delta}\partial + \partial^2\Bigg) \\
&\le \frac{B\big(\log\frac{1}{\delta}\big)^2 |\varphi(x,y)|}{\delta^2} + \frac{\eta}{2}, \quad \text{where } B = 16A^2.
\end{aligned} \tag{2.32}$$

The inequality (see (2.9) and (2.31))

$$|\varphi(x,y)| < |f(x,y)| + \frac{\eta\delta^2}{32A^2\big(\log\frac{1}{\delta}\big)^2}, \quad (x,y)\in G\subset E_0,$$

together with (2.32) implies

$$\sum_{k,s=N}^{M} |c_{k,s} f_k(x) f_s(y)| \le \frac{B\big(\log\frac{1}{\delta}\big)^2 |f(x,y)|}{\delta^2} + \eta.$$

Owing to (2.19), (2.26), and (2.28) we have

$$\begin{aligned}
\Bigg(\sum_{k,s=N}^{M} |c_{k,s}|^r\Bigg)^{\frac{1}{r}} &= \Bigg(\sum_{l,j=1}^{2^q}\sum_{n=N_l+1}^{N_{l+1}}\sum_{s=M_j+1}^{M_{j+1}} \big|\gamma_{l,j} a_k^{(l)} b_s^{(j)}\big|^r\Bigg)^{\frac{1}{r}} \\
&= \Bigg(\sum_{l,j=1}^{2^q} |\gamma_{l,j}|^r \sum_{n=N_l+1}^{N_{l+1}} \big|a_k^{(l)}\big|^r \sum_{s=M_j+1}^{M_{j+1}} \big|b_s^{(j)}\big|^r\Bigg)^{\frac{1}{r}} \\
&\le \sum_{l,j=1}^{2^q} |\gamma_{l,j}| \Bigg(\sum_{k=N}^{M} \big|a_k^{(l)}\big|^r\Bigg)^{\frac{1}{r}} \Bigg(\sum_{s=N}^{M} \big|b_k^{(j)}\big|^r\Bigg)^{\frac{1}{r}} \le \eta.
\end{aligned}$$

Lemma 3 is proven.

§ 3. Proof of Theorem 3

Let $\epsilon > 0$. Denote by

$$\{\phi_n(x,y)\}_{n=1}^{\infty} \tag{3.1}$$

the sequence of polynomials in the double Franklin system $\{f_k(x)f_s(y)\}_{k,s=0}^{\infty}$ with rational coefficients and put

$$\delta_n = \min\left\{\frac{1}{2};\ \sqrt[4]{\iint_T |\phi_n(x,y)|\,dxdy}\right\}. \tag{3.2}$$

Applying Lemma 3, we can find the sequences of sets $\{G_n\}$ and $\{E_n\}$, the functions $\{g_n(x,y)\}$, and the polynomials in the double Franklin system

$$Q_n(x,y) = \sum_{k,s=M_n+1}^{M_{n+1}} c_{k,s}^{(n)} f_k(x) f_s(y), \tag{3.3}$$

which for all $n \geq 1$ satisfy the conditions:

$$E_n, G_n \subset T, \quad |E_n| > 1 - \epsilon 2^{-(n+2)}, \quad |G_n| > 1 - \delta_n, \tag{3.4}$$

$$g_n(x,y) = \phi_n(x,y) \quad \text{on } E_n, \tag{3.5}$$

$$\iint_T |g_n(x,y) - Q_n(x,y)|\,dxdy \leq 2^{-5(n+3)}, \tag{3.6}$$

$$\iint_T |g_n(x,y)|\,dxdy \leq 5 \iint_T |\phi_n(x,y)|\,dxdy, \tag{3.7}$$

$$\sum_{k,s=M_n+1}^{M_{n+1}} \left|c_{k,s}^{(n)} f_k(x) f_s(y)\right| \leq \frac{B\left(\log\frac{1}{\delta_n}\right)^2 |\phi_n(x,y)|}{\delta_n^2} + 2^{-5(n+3)}, \quad (x,y) \in G_n, \tag{3.8}$$

$$\left(\sum_{k,s=M_n+1}^{M_{n+1}} \left|c_{k,s}^{(n)}\right|^{2+2^{-n}}\right)^{\frac{1}{2+2^{-n}}} \leq 2^{-5(n+3)}. \tag{3.9}$$

Put

$$E = \bigcap_{n=1}^{\infty} E_n, \tag{3.10}$$

$$\sum_{k,s=0}^{\infty} d_{k,s} f_k(x) f_s(y) = \sum_{n=1}^{\infty} Q_n(x,y) = \sum_{n=1}^{\infty} \sum_{k,s=M_n+1}^{M_{n+1}} c_{k,s}^{(n)} f_k(x) f_s(y), \tag{3.11}$$

$$d_{k,s} = \begin{cases} c_{k,s}^{(n)} & \text{for } (k,s) \in \Omega := \bigcup_{n=1}^{\infty} (M_n; M_{n+1}] \times (M_n; M_{n+1}], \\ 0 & \text{for } (k,s) \notin \Omega. \end{cases} \tag{3.12}$$

It is obvious (see (3.4), (3.9), (3.10), and (3.12)) that

$$|E| > 1 - \epsilon, \quad \sum_{k,s=0}^{\infty} |d_{k,s}|^r < \infty, \ r > 2.$$

Let $f(x,y) \in L^1(T)$ and η be an arbitrary positive real. By Lemma 1, we can choose a subsequence $\{\phi_{n_k}(x,y)\}_{n=1}^{\infty}$ from (3.1) such that

$$\lim_{N\to\infty} \iint\limits_T \left|\sum_{k=1}^{N} \phi_{n_k}(x,y) - f(x,y)\right| dxdy = 0, \tag{3.13}$$

$$\lim_{N\to\infty} \sum_{k=1}^{N} \phi_{n_k}(x,y) = f(x,y) \quad \text{a.e. on } T, \tag{3.14}$$

$$\eta 2^{-4(k+1)} \leq \iint\limits_T |\phi_{n_k}(x,y)|\, dxdy \leq 3\eta 2^{-4(k+1)}, \quad k \geq 2. \tag{3.15}$$

From (3.13) and (3.15) we obtain that

$$\iint\limits_T |f(x,y) - \phi_{n_1}(x,y)|\, dxdy < \frac{\eta}{2}. \tag{3.16}$$

Suppose that the numbers $n_1 = \nu_1 < \cdots < \nu_{q-1}$, the functions $g_1(x,y) = \phi_{n_1}(x,y)$, $g_2(x,y), \ldots, g_{q-1}(x,y)$, and the polynomials

$$Q_{\nu_n}(x,y) = \sum_{k,s=M_{\nu_n}+1}^{M_{\nu_n+1}} c_{k,s}^{(\nu_n)} f_k(x) f_s(y), \quad 1 \leq n \leq q-1,$$

are already defined so as to satisfy the following conditions:

$$g_l(x,y) = \phi_{k_l}(x,y), \quad (x,y) \in E,\ l \in [1, q-1],$$
$$\iint\limits_T |g_l(x,y)|\, dxdy < 2^{-(l+1)}, \quad l \in [1, q-1],$$
$$\iint\limits_T \left|\sum_{j=1}^{l} [g_j(x,y) - Q_{\nu_j}(x,y)]\right| dxdy < \eta 2^{-5(l+2)}, \quad l \in [1, q-1]. \tag{3.17}$$

It is easy to see that we can choose a natural $\nu_q > \nu_{q-1}$ ($\phi_{\nu_q}(x,y)$ from (3.1)) such that

$$\iint\limits_T \left|\left\{\phi_{n_q}(x,y) - \sum_{j=1}^{q-1} [g_j(x,y) - Q_{\nu_j}(x,y)]\right\} - \phi_{\nu_q}(x,y)\right| dxdy \leq \eta 2^{-6(q+3)}. \tag{3.18}$$

Show that

$$\eta 2^{-4q-5} \leq \iint\limits_T |\phi_{\nu_q}(x,y)|\, dxdy \leq \eta 2^{-4q}. \tag{3.19}$$

By (3.15), (3.17), and (3.18), we get

$$\begin{aligned}
\iint\limits_T |\phi_{\nu_q}(x,y)|\, dxdy &\geq \iint\limits_T |\phi_{n_q}(x,y)|\, dxdy - \iint\limits_T \left|\sum_{j=1}^{q-1} [g_j(x,y) - Q_{\nu_j}(x,y)]\right| dxdy \\
&- \iint\limits_T \left|\left\{\phi_{n_q}(x,y) - \sum_{j=1}^{q-1} [g_j(x,y) - Q_{\nu_j}(x,y)]\right\} - \phi_{\nu_q}(x,y)\right| dxdy \\
&\geq \eta 2^{-4(q+1)} - \eta 2^{-5(q+2)} - \eta 2^{-6(q+3)} \geq \eta 2^{-4q-5}
\end{aligned}$$

and, similarly,

$$\iint_T |\phi_{\nu_q}(x,y)|\,dxdy \le \eta 2^{-4(q+1)} + \eta 2^{-5(q+2)} + \eta 2^{-6(q+3)} \le \eta 2^{-4q}.$$

Put

$$g_q(x,y) = \phi_{n_q}(x,y) + [g_{\nu q}(x,y) - \phi_{\nu_q}(x,y)]. \tag{3.20}$$

Hence, from (3.5) and (3.10) it follows that

$$g_q(x,y) = \phi_{n_q}(x,y), \quad (x,y) \in E,\ q \ge 1. \tag{3.21}$$

Taking (3.6), (2.18), and (3.20) into account, for all $q \ge 2$, we obtain

$$\begin{aligned}
&\iint_T \left| \sum_{j=1}^{q} [g_j(x,y) - Q_{\nu_j}(x,y)] \right| dxdy \\
&\le \iint_T \left| \left\{ \phi_{n_q}(x,y) - \sum_{j=1}^{q-1} [g_j(x,y) - Q_{\nu_j}(x,y)] \right\} - \phi_{\nu_q}(x,y) \right| dxdy \\
&\quad + \iint_T |g_{\nu_q}(x,y) - Q_{\nu_q}(x,y)|\,dxdy \le \eta 2^{-5(q+2)}.
\end{aligned} \tag{3.22}$$

By (3.15), (3.19) and (3.20), derive

$$\iint_T |g_q(x,y)|\,dxdy \le \iint_T |\phi_{n_q}(x,y)|\,dxdy + \iint_T |g_{\nu_q}(x,y)|\,dxdy + \iint_T |\phi_{\nu_q}(x,y)|\,dxdy \le \eta 2^{-q}. \tag{3.23}$$

The sequence of functions $\{g_q(x,y)\}_{q=1}^{\infty}$ $(g_1(x,y) = \phi_{n_1}(x,y))$ and the polynomials $\{Q_{\nu_q}(x,y\}$ satisfying (3.21)–(3.23) for all $q > 1$ are determined by induction;

$$\sum_{q=1}^{\infty} \iint_T |g_q(x,y)|\,dxdy < \infty$$

by (3.23). Define $\tilde{f}(x,y)$ and the sequence $\{\varepsilon_{k,s}\}_{k,s=0}^{\infty}$ as follows:

$$\tilde{f}(x,y) = \sum_{q=1}^{\infty} g_q(x,y), \tag{3.24}$$

$$\varepsilon_{k,s} = \begin{cases} 1 & \text{for } (k,s) \in \Omega_0 := \bigcup_{q=1}^{\infty} (M_{\nu_q}, M_{\nu_q+1}] \times (M_{\nu_q}, M_{\nu_q+1}], \\ 0 & \text{for } (k,s) \notin \Omega_0. \end{cases} \tag{3.25}$$

From (3.14), (3.21), and (3.24) it follows that $\tilde{f} \in L^1(T)$ and $\tilde{f}(x,y) = f(x,y)$, $(x,y) \in E$. Since $g_1(x,y) = \phi_{n_1}(x,y)$, from (3.13), (3.16), and (3.24) we derive

$$\iint_T |f(x,y) - \tilde{f}(x,y)|\,dxdy \le \iint_T |f(x,y) - \phi_{n_1}(x,y)|\,dxdy + \sum_{q=2}^{\infty} \iint_T |g_q(x,y)|\,dxdy \le \eta.$$

By (3.3), (3.12), (3.23)–(3.25), for all $q \geq 2$ we have

$$\iint\limits_T \Big| \sum_{k,s=0}^{M_{\nu_q+1}} \varepsilon_{k,s} d_{k,s} f_k(x) f_s(y) - \tilde{f}(x,y) \Big| \, dxdy$$

$$= \iint\limits_T \Big| \sum_{j=1}^{q} \sum_{k,s=M_{\nu_j}+1}^{M_{\nu_j+1}} c_{k,s}^{(\nu_j)} f_k(x) f_s(y) - \tilde{f}(x,y) \Big| \, dxdy$$

$$\leq \iint\limits_T \Big| \sum_{j=1}^{q} [g_j(x,y) - Q_{\nu_j}(x,y)] \Big| \, dxdy + \sum_{j=q+1}^{\infty} \iint\limits_T |g_j(x,y)| \, dxdy \leq 2^{-q}. \tag{3.26}$$

Hence,

$$\varepsilon_{k,s} d_{k,s} = c_{k,s}(\tilde{f}) = \iint\limits_T \tilde{f}(t,\tau) f_k(t) f_s(\tau) \, dt d\tau. \tag{3.27}$$

Consequently, (3.24) and (3.25) yield

$$c_{k,s}(\tilde{f}) = d_{k,s}, \quad (k,s) \in \Lambda(f) = \operatorname{spec}(\tilde{f}) \subset \Omega_0.$$

Show that the double Fourier–Franklin series

$$\sum_{k,s=0}^{\infty} c_{k,s}(\tilde{f}) f_k(x) f_s(y) = \sum_{k,s=0}^{\infty} \varepsilon_{k,s} d_{k,s} f_k(x) f_s(y)$$

for $\tilde{f}(x,y)$ converges absolutely almost everywhere on T. Denoting

$$B_q = \{(x,y) \subset T = [0,1]^2; |\phi_{\nu_q}(x,y)| \, dx \leq \eta 2^{-3q}\}, \quad q \geq 2, \tag{3.28}$$

we obtain

$$\eta 2^{-3q} |T \setminus B_q| \leq \iint\limits_{[0,1) \setminus B_k} |\phi_{\nu_q}(x,y)| \, dxdy \leq \eta 2^{-4q}, \quad q \geq 2;$$

thus, $|B_q| > 1 - 2^{-q}$.

Put

$$B = \bigcup_{k=2}^{\infty} \bigcap_{q=k}^{\infty} (B_q \cap G_{\nu_q}), \tag{3.29}$$

whence $|B| = 1$ since $|B_q \cap G_{\nu_q}| > 1 - 2^{-q+4}$ (see (3.2), (3.4), (3.19), and (3.28)).

Let $(x,y) \in B$. There exists a natural q_0 such that $(x,y) \in B_q \cap G_{\nu_q}$, $q \geq q_0$.

Using (3.2), (3.8), (3.19), and (3.28), we obtain

$$\sum_{k,s=M_{\nu_q}+1}^{M_{\nu_q+1}} \big| c_{k,s}^{(\nu_q)} f_k(x) f_s(y) \big| \leq \frac{B \big(\log \frac{1}{\delta_{\nu_q}}\big)^2 |\phi_{\nu_q}(x,y)|}{\delta_{\nu_q}^2} + 2^{-\nu_q} \leq B \sqrt{\eta} q^2 2^{-q} + 2^{-q}. \tag{3.30}$$

Hence, from (3.12), (3.26), (3.27), (3.29), and (3.30), we conclude that

$$\sum_{k,s=0}^{\infty} |c_{k,s}(\tilde{f}) f_k(x) f_s(y)| < \infty$$

almost everywhere on T. It easy to verify (see (3.26), (3.27) and (3.30)) that

$$\lim_{N,M \to \infty} \sum_{k,s=0}^{N,M} c_{k,s}(\tilde{f}) f_k(x) f_s(y) = \tilde{f}(x,y)$$

almost everywhere on T. Theorem 3 is proven.

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