ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 5, pp. 902–910. © *Pleiades Publishing, Inc., 2019.*
Russian Text © *The Author(s), 2019, published in Sibirskii Matematicheskii Zhurnal, 2019, Vol. 60, No. 5, pp. 1153–1164.*

TWO APPLICATIONS OF BOOLEAN VALUED ANALYSIS

A. G. Kusraev and S. S. Kutateladze UDC 517.11:517.98

Abstract: The paper contains two main results that are obtained by using Boolean valued analysis. The first asserts that a universally complete vector lattice without locally one-dimensional bands can be decomposed into a direct sum of two vector sublattices that are laterally complete and invariant under all band projections and there exists a band preserving linear isomorphism of each of these sublattices onto the original lattice. The second result establishes a counterpart of the Ando Theorem on the joint characterization of AL^p and $c_0(\Gamma)$ for the class of the so-called $\mathbb{B}$-cyclic Banach lattices, using the Boolean valued transfer for injective Banach lattices.

DOI: 10.1134/S0037446619050124

Keywords: universally complete vector lattice, injective Banach lattice, M-projection, Maharam operator, AL^p-space, Boolean valued representation

§ 1. Introduction

We use the Boolean valued approach to the two problems of the theory of vector and Banach lattices. The prerequisites of Boolean valued analysis are given in Section 2; see details in [1, 2]. The theory of vector and Banach lattices is presented in [3, 4].

In [5, Problem B], Abramovich and Kitover asked whether vector lattices X and Y are lattice isomorphic if there is a linear invertible operator $T : X \to Y$ such that T and T^{-1} are disjointness preserving. The negative answer is given in the same memoir (see [5, Theorem 13.4]). Some strengthening of this result is presented in Section 3.

The famous Ando Theorem states that a Banach lattice X of dimension ≥ 3 is isometrically lattice isomorphic to $L^p(\Omega, \Sigma, \mu)$ for some $1 \leq p \in \mathbb{R}$ and measure space (Ω, Σ, μ) or to $c_0(\Gamma)$ for some nonempty set Γ if and only if each closed sublattice of X is the image of a contractive positive projection (see, for example, [4, Theorem 2.7.13] or [6, Theorem 1.b.8]). The analogous result in the class of $\mathbb{B}$-cyclic Banach lattices is established in Section 6. The preparatory material about Boolean valued AL^p-spaces and Boolean valued Banach lattices of the form $c_0(\Gamma)$ is given in Sections 4 and 5 respectively.

In what follows, we use the notations: $\mathbb{B}$ is a complete Boolean algebra with unity $\mathbb{1}$, zero $\mathbb{0}$, join $\vee$, meet $\wedge$, and complement $(\cdot)^*$, moreover, $\mathbb{1} \neq \mathbb{0}$; while $\mathbb{P}(X)$ is the Boolean algebra of all band projections in a vector lattice X. By a *partition of unity* in $\mathbb{B}$ we mean a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{0}$ for $\xi \neq \eta$. The symbol $:=$ is used for assignation by definition, and $\mathbb{N}$ and $\mathbb{R}$ are the naturals and the reals, respectively.

§ 2. Preliminaries

Applying the Transfer and Maximum Principles to the ZFC-theorem of existence of the reals, find $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$, called the *reals within* $\mathbb{V}^{(\mathbb{B})}$, satisfying $[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}$. The following result by Gordon states that the interpretation of the reals in the model $\mathbb{V}^{(\mathbb{B})}$ is a universally complete vector lattice.

Theorem 2.1 [7]. *Let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order) is a universally complete vector lattice. Moreover, there is a Boolean isomorphism $\chi : \mathbb{B} \to \mathbb{P}(\mathscr{R}{\downarrow})$*

The authors were supported by the Program of Basic Scientific Research of the Siberian Branch of the Russian Academy of Sciences (Grant No. I.1.2, Project 0314–2019–0005).

†) To Yu. G. Reshetnyak on the occasion of his 90th birthday.

Original article submitted March 4, 2019; revised March 11, 2019; accepted March 12, 2019.

such that

$$\chi(b)x = \chi(b)y \iff b \leq [\![x = y]\!], \quad \chi(b)x \leq \chi(b)y \iff b \leq [\![x \leq y]\!] \tag{1}$$

for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$.

PROOF. See [2, Theorem 2.2.4; 3, Theorem 8.1.2]. $\square$

Consider the reals $\mathbb{R} \in \mathbb{V}$. The standard name $\mathbb{R}^{\wedge}$ is a field within $\mathbb{V}^{(\mathbb{B})}$, and we may assume that $[\![\mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{R}]\!] = \mathbb{1}$ (see [2, Subsections 2.2.2 and 2.2.3]). In some questions, it is important to know when $\mathbb{R}^{\wedge} = \mathscr{R}$ (see, for example, [8]). The following result by Gutman answers this question in terms of the equivalent algebraic properties of $\mathbb{B}$ and $\mathscr{R}{\downarrow}$:

Theorem 2.2 [9]. *Let $\mathbb{B}$ be a complete Boolean algebra and let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. Then the following are equivalent:*

(1) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^{\wedge}$.

(2) $\mathbb{B}$ *is σ-distributive;*

(3) $\mathscr{R}{\downarrow}$ *is locally one-dimensional.*

PROOF. See [2, Theorems 4.4.9 and 4.7.6; 3, Theorem 5.1.6]. $\square$

Recall the notion of $\mathbb{B}$-cyclic Banach lattice (see [1, 2]).

DEFINITION 2.3. A band projection π in a Banach lattice X is an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^* x\|\}$ for all $x \in X$, where $\pi^* := I_X - \pi$. The set of all M-projections in X is denoted by $\mathbb{M}(X)$. A *complete Boolean algebra of M-projections* in X is a subset $\mathscr{B} \subset \mathbb{M}(X)$ that is an order closed subalgebra in the complete Boolean algebra $\mathbb{P}(X)$.

Note that $\mathbb{M}(X)$ is always a subalgebra of $\mathbb{P}(X)$ but can fail to be order complete (see [10, Theorem 1.10]).

DEFINITION 2.4. If $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathscr{B}$ and $(x_\xi)_{\xi\in\Xi}$ is a family in X then $x \in X$ satisfying $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$ is called the *mixing* of (x_ξ) by (b_ξ). A Banach lattice X is called $\mathbb{B}$-*cyclic* if $\mathbb{B}$ is a complete Boolean algebra isomorphic to a complete Boolean algebra $\mathscr{B}$ of M-projections in X and the mixing of every family in the unit ball of X by every partition of unity in $\mathscr{B}$ (with the same index set) exists and belongs to the unit ball.

In what follows, we identify $\mathbb{B}$ and $\mathscr{B}$ and assume that $\mathbb{B} \subset \mathbb{P}(X)$ (see [2, Definition 5.7.13; 11, Definition 2.5]). We say that $\mathbb{B}$-cyclic Banach lattices X and Y are $\mathbb{B}$-*isometric* and write $X \simeq_{\mathbb{B}} Y$ if there exists an isometric lattice isomorphism between X and Y commuting with the elements of $\mathbb{B}$.

Thus, a Banach lattice is $\mathbb{B}$-cyclic if it is a $\mathbb{B}$-cyclic Banach space with respect to some Boolean algebra of M-projections $\mathbb{B} \subset \mathbb{P}(X)$ (cf. [2, Definition 5.8.8] and [3, Definitions 7.3.1 and 7.3.3]).

DEFINITION 2.5. Denote by $\Lambda = \mathscr{R}{\Downarrow}$ the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$; i.e., Λ is the order dense ideal in $\mathscr{R}{\downarrow}$ generated by the order unity $\mathbb{1} := 1^{\wedge} \in \mathscr{R}{\downarrow}$. Take a nonzero Banach space $\mathscr{X} = (\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ within $\mathbb{V}^{(\mathbb{B})}$ and put

$$\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : \lvert x \rvert \in \Lambda\}, \quad |\!|\!|x|\!|\!| := \|\,\lvert x\rvert\,\|_\infty \quad (x \in X),$$
$$\|\lambda\|_\infty := \inf\{\alpha > 0 : |\lambda| \leq \alpha\mathbb{1}\}, \quad \lambda \in \Lambda,$$

where $\lvert\cdot\rvert$ is the descent of $\|\cdot\|_{\mathscr{X}}$, i.e., $[\![\lvert x\rvert = \|x\|_{\mathscr{X}}]\!] = \mathbb{1}$ for all $(x \in \mathscr{X}{\downarrow})$. Then $\mathscr{X}{\Downarrow} := (\mathscr{X}{\Downarrow}, |\!|\!|\cdot|\!|\!|)$ is called the *bounded descent* of $\mathscr{X}$.

Since Λ is an order complete *AM*-space with unity, $\mathscr{X}{\Downarrow}$ is a Banach space with mixed norm over Λ and so a $\mathbb{B}$-cyclic Banach space (see [3, 7.3.3]). The following result states that the concept of $\mathbb{B}$-cyclic Banach lattice is nothing but interpretation of the notion of Banach lattice in a Boolean valued model.

Theorem 2.6 [11]. *A Banach lattice X is $\mathbb{B}$-cyclic if and only if $\mathbb{V}^{(\mathbb{B})}$ contains a Banach lattice $\mathscr{X}$ unique up to a lattice isometry whose bounded descent is $\mathbb{B}$-isometric to X. Moreover, $\pi \mapsto \pi{\Downarrow} := \pi{\downarrow}|_X$ is an isomorphism of the Boolean algebras $\mathbb{M}(\mathscr{X}){\downarrow}$ and $\mathbb{M}(X)$ (in symbols: $\mathbb{M}(\mathscr{X}){\downarrow} \simeq \mathbb{M}(\mathscr{X}{\Downarrow})$).*

PROOF. See [2, Theorem 5.9.1] or [11, Theorem 2.1]. $\square$

Definition 2.7. The Banach lattice $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ of Theorem 2.6 is called the *Boolean valued representation* of a $\mathbb{B}$-cyclic Banach lattice X.

Remark 2.8. The bounded descent of Definition 2.5 appeared firstly under another name in the articles [12, 13] by Takeuti in his study of von Neumann algebras and C^*-algebras by using Boolean valued models (also see [14]).

We will need the following result on the structure of Boolean valued cardinals:

Proposition 2.9. *A member $x \in \mathbb{V}^{(\mathbb{B})}$ is a cardinal within $\mathbb{V}^{(\mathbb{B})}$ if and only if there are a nonempty set of cardinals $\Gamma \in \mathbb{V}$ and a partition of unity $(b_\gamma)_{\gamma\in\Gamma} \subset \mathbb{B}$ such that $x = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$ and $\gamma^\wedge$ is a cardinal within $\mathbb{V}^{(\mathbb{B}_\gamma)}$, where $\mathbb{B}_\gamma := [\mathbb{0}, b_\gamma]$ and $b_\gamma \neq 0$ for all $\gamma \in \Gamma$.*

Proof. See [1, Theorem 9.1.3]; also [2, 1.9.11]. □

§ 3. Band Preserving Linear Isomorphisms

In [5, Sections 6 and 13], some approach was proposed to constructing counterexamples to the above-mentioned Problem B that uses the concept of d-basis. In this section, we show that, up to passage to a suitable Boolean-valued model, this is equivalent to the application of a classical Hamel basis.

Definition 3.1. Suppose that X is a vector lattice and $u \in X$. An element $v \in X$ is called a *fragment* or *component* of u if $|v| \wedge |u - v| = 0$. The set of all components of u is denoted by $\mathbb{C}(u)$. A subset X_0 is called *componentwise closed* if $\mathbb{C}(u)$ lies in X_0 for all $u \in X_0$ (see [5, Proposition 4.9]).

Definition 3.2. A sublattice $X_0 \subset X$ is called *laterally complete* if each disjoint set of positive elements in X_0 has a supremum; and $\mathbb{P}(X)$-*invariant*, if $\pi(X_0) \subset X_0$ for all $\pi \in \mathbb{P}(X)$. An operator $T : X_0 \to X$ is called *nonexpanding* or *band preserving* if $T(B \cap X_0) \subset B$ for every band $B \subset X$ (see [2, Definition 4.1.2]).

Lemma 3.3. *Let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. Consider the universally complete vector lattice $X := \mathscr{R}{\downarrow}$. The following are equivalent for a sublattice $X_0 \subset X$:*

(1) *X_0 is order dense, laterally complete, and componentwise closed;*

(2) *X_0 is order dense, laterally complete, and $\mathbb{P}(X)$-invariant;*

(3) *X_0 is laterally complete, $\mathbb{P}(X)$-invariant, and $X_0^{\perp\perp} = X$;*

(4) *$X_0 = \mathscr{X}_0{\downarrow}$ for some vector sublattice $\mathscr{X}_0$ of the field $\mathscr{R}$ regarded as a vector lattice over the subfield $\mathbb{R}^\wedge$.*

Proof. This is immediate from [2, Theorem 2.5.1]. □

Lemma 3.4. *Let $\mathbb{P}$ be a proper subfield of $\mathbb{R}$. Then there exist $\mathbb{P}$-linear subspaces $\mathscr{X}_1$ and $\mathscr{X}_2$ in $\mathbb{R}$ such that $\mathscr{X}_k$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$ but not isomorphic as ordered vector spaces over $\mathbb{P}$. Moreover, $\mathbb{R} = \mathscr{X}_1 \oplus \mathscr{X}_2$ and the corresponding projections $p_1 : \mathbb{R} \to \mathscr{X}_1$ and $p_2 : \mathbb{R} \to \mathscr{X}_2$ are not order bounded.*

Proof. The reals $\mathbb{R}$ are a finite extension of no proper subfield $\mathbb{P} \subset \mathbb{R}$ (see, for example, [15, Lemma 17]). Consequently, $\mathbb{R}$ is an infinite-dimensional vector space over $\mathbb{P}$. Let $\mathscr{E}$ be a Hamel basis for the $\mathbb{P}$-vector space $\mathbb{R}$ and let $|\mathscr{E}|$ be the cardinality of $\mathscr{E}$. Since the cardinal $|\mathscr{E}|$ is infinite, we can choose a proper subset $\mathscr{E}_1 \subsetneq \mathscr{E}$ so that $\mathscr{E}_1$ and $\mathscr{E}_2 := \mathscr{E} \setminus \mathscr{E}_1$ have the same cardinality equal to $|\mathscr{E}|$, i.e., $|\mathscr{E}| = |\mathscr{E}_1| = |\mathscr{E}_2|$. Let $\mathscr{X}_k$ denote the $\mathbb{P}$-linear subspace in $\mathbb{R}$ generated by $\mathscr{E}_k$, where $k = 1, 2$. Then $\{0\} \subsetneq \mathscr{X}_k \subsetneq \mathbb{R}$, where $\mathscr{X}_k$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$ since $|\mathscr{E}| = |\mathscr{E}_k|$. If $\mathscr{X}_k$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$ then $\mathscr{X}_k$ would be order complete and so we would obtain the contradictory equality $\mathscr{X}_k = \mathbb{R}$. By the choice of $\mathscr{E}_1$ and $\mathscr{E}_2$, we have $\mathbb{R} = \mathscr{X}_1 \oplus \mathscr{X}_2$, and so there are projections $p_k : \mathbb{R} \to \mathscr{X}_k$, where $p_1 + p_2 = I_{\mathscr{R}}$. If $p_k : \mathbb{R} \to \mathscr{X}_k$ is order bounded then p_k is continuous as an additive function on $\mathbb{R}$; consequently, $p_k(x) = c_k x$ $(x \in \mathbb{R})$ for some $c_k \in \mathbb{R}$. But $c_k^2 = 1$; i.e., either $c_k = 1$ and then $\mathscr{X}_k = \mathbb{R}$ or $c_k = 0$ and then $\mathscr{X}_k = \{0\}$; in both cases, we get a contradiction. □

Let us now prove the first main result of the article which is the interpretation of Lemma 3.4 in an arbitrary Boolean valued model.

Theorem 3.5. *Let X be a universally complete vector lattice not containing nonzero locally one-dimensional bands. Then there are componentwise closed laterally complete vector sublattices $X_1 \subset X$ and $X_2 \subset X$ and linear bijections $T_1 : X_1 \to X$ and $T_2 : X_2 \to X$ such that*

(1) *$X = X_1 \oplus X_2$ and $X = X_1^{\perp\perp} = X_2^{\perp\perp}$;*

(2) *T_k and T_k^{-1} are band preserving $(k = 1, 2)$;*

(3) *the canonical projections $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$ are band preserving;*

(4) *none of the sublattices X_1 and X_2 is order complete and so neither is lattice isomorphic to X.*

Proof. By the Gordon Theorem 2.1, we may assume without loss of generality that $X = \mathscr{R}{\downarrow}$. Since X contains no locally one-dimensional bands, by the Gutman Theorem 2.2, $[\![\mathscr{R} \neq \mathbb{R}^{\wedge}]\!] = \mathbb{1}$. The Transfer Principle enables us to apply Lemma 3.4 within $\mathbb{V}^{(\mathbb{B})}$; therefore, there are $\mathbb{R}^{\wedge}$-linear subspaces $\mathscr{X}_1$ and $\mathscr{X}_2$ in $\mathscr{R}$ such that $\mathscr{R} = \mathscr{X}_1 \oplus \mathscr{X}_2$ as well as $\mathbb{R}^{\wedge}$-linear isomorphisms τ_k in $\mathscr{X}_k$ on $\mathscr{R}$; moreover, $\mathscr{X}_k$ and $\mathscr{R}$ are nonisomorphic as ordered vector spaces over $\mathbb{R}^{\wedge}$ $(k = 1, 2)$. Let $p_k : \mathscr{R} \to \mathscr{X}_k$ $(k = 1, 2)$ be the projections corresponding to the decomposition $\mathscr{R} = \mathscr{X}_1 \oplus \mathscr{X}_2$. Put $X_k := \mathscr{X}_k{\downarrow}$, $T_k := \tau{\downarrow}$, $S_k := \tau_k^{-1}{\downarrow}$, and $P_k := p_k{\downarrow}$. Since $\mathscr{X}_k$ is linearly isomorphic to $\mathscr{R}$, $[\![\mathscr{X}_k \neq \{0\}]\!] = \mathbb{1}$. Hence, $X_k^{\perp\perp} = X$. By Lemma 3.3, X_1 and X_2 are componentwise closed and laterally complete. Moreover, $X = X_1 \oplus X_2$. The operators S_k, T_k, and P_k are band preserving and $\mathbb{R}$-linear by [2, Theorem 4.3.4]. Furthermore, $S_k = (\tau_k{\downarrow})^{-1} = T_k^{-1}$ and P_1 and P_2 are the projections corresponding to the decomposition $X = X_1 \oplus X_2$. It remains to observe that X_k and X are lattice isomorphic if and only if $\mathscr{X}_k$ and $\mathscr{R}$ are isomorphic as ordered vector spaces over $\mathbb{R}^{\wedge}$ and P_1 and P_2 are order bounded if and only if p_1 and p_1 are order bounded within $\mathbb{V}^{(\mathbb{B})}$. $\square$

Remark 3.6. Lemma 3.4 and, hence, Theorem 3.5 are based on the absorption property of infinite cardinals: $\varkappa + \varkappa = \varkappa$. However, this property holds also in the case when the number of summands is infinite but does not exceed $\varkappa$; i.e., $\varkappa = \sum_{\alpha \in \mathrm{A}} \varkappa_\alpha$ if $\varkappa_\alpha = \varkappa$ for all $\alpha \in \mathrm{A}$ and $|\mathrm{A}| \leq \varkappa$. Consequently, we arrive at the version of Theorem 3.5 with infinitely many summands of sublattices in the direct sum.

§ 4. Boolean Valued AL^p-Spaces

In this section, we introduce the class of $\mathbb{B}$-cyclic Banach lattices that are classical AL^p-spaces in Boolean valued models. Start from the key notion of injective Banach lattice.

Definition 4.1. A real Banach lattice X is called *injective* if for each Banach lattice Y, each closed vector sublattice $Y_0 \subset Y$, and each positive linear operator $T_0 : Y_0 \to X$, there is a positive linear extension $T : Y \to X$ of T_0 such that $\|T_0\| = \|T\|$. In other words, the injective Banach lattices are injective objects in the category of Banach lattices with contractive positive operators as morphisms (see [2, § 5.10; 11]).

Recall some facts on the structure of injective Banach lattices. If $X \neq \{0\}$ is an injective Banach lattice then $\mathbb{M}(X)$ is an order closed subalgebra in $\mathbb{P}(X)$ (see Definition 2.3). In this case, X is a $\mathbb{B}$-cyclic Banach lattice for any order closed subalgebra $\mathbb{B} \subset \mathbb{M}(X)$. This circumstance makes it possible to establish a version of the Transfer Principle for injective Banach lattices.

Lemma 4.2. *For every injective Banach lattice X, there is $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B} = \mathbb{M}(X)$, such that $[\![\mathscr{X}$ is an AL-space$]\!] = \mathbb{1}$ and X is isometrically lattice $\mathbb{B}$-isomorphic to the bounded descent $\mathscr{X}{\Downarrow}$.*

Proof. See [11, Theorem 4.4]. $\square$

Definition 4.3. A positive operator T from a vector lattice X into a vector lattice Y is said to possess the *Levy property* whenever $Y = T(X)^{\perp\perp}$ and $\sup x_\alpha$ exists in X for an increasing net $(x_\alpha) \subset X_+$ whenever (Tx_α) is order bounded in Y. A *Maharam operator* is an order continuous interval preserving operator; i.e., $T([0, x]) = [0, Tx]$ for all $x \in X_+$ (see [3, 3.4.1]). We say that T is *strictly positive* if T is positive and $T(|x|) = 0$ implies $x = 0$.

Denote by $(\Lambda(\mathbb{B}), \|\cdot\|_\infty)$ an order complete *AM*-space with unity $\mathbb{1}$ such that the Boolean algebras $\mathbb{B}$ and $\mathbb{P}(\Lambda(\mathbb{B}))$ are isomorphic. The norm $\|\cdot\|_\infty$ is introduced in the same way as in Definition 2.5: $\|x\|_\infty = \inf\{\alpha > 0 : |x| \leq \alpha\mathbb{1}\}$.

Lemma 4.4. *For an arbitrary injective Banach lattice $(X, \|\cdot\|)$ with Boolean algebra of M-projections $\mathbb{B} = \mathbb{M}(X)$, there exists a strictly positive Maharam operator $\Phi : X \to \Lambda(\mathbb{B})$ with the Levy property such that $\|x\| = \|\Phi(|x|)\|_\infty$ for all $x \in X$. Moreover, there exists a Boolean isomorphism h from $\mathbb{P}(\Lambda(\mathbb{B}))$ onto $\mathbb{M}(X)$ such that $\pi \circ \Phi = \Phi \circ h(\pi)$ for all $\pi \in \mathbb{P}(\Lambda(\mathbb{B}))$.*

Proof. See [11, Corollary 4.5]. $\square$

If X and Φ are the same as in Lemma 4.4 then we will write $X = L^1(\Phi)$. Assume that $X = L^1(\Phi)$, $\mathbb{B} = \mathbb{M}(X)$, and $\Lambda := \Lambda(\mathbb{B})$. Let X^u and Λ^u be the universal completions of X and Λ, respectively. Take $\mathbb{1} \leq p \in \Lambda^u$ and put

$$X^p := L^p(\Phi) := \{x \in X^u : |x|^p \in X,\ |x|_p := (\Phi(|x|^p))^{\frac{1}{p}} \in \Lambda\}, \quad \|x\|_p := \|\, |x|_p\|_\infty \quad (x \in X^p).$$

Note that if $\mathbb{1} \leq p \in \Lambda^u$ then p^{-1} exists and belongs to Λ. Moreover, we may assume that Λ^u is a sublattice in X^u with the same order unity as X^u. Then X^u is an f-algebra and Λ^u is an f-subalgebra of X^u; therefore, the mappings $x \mapsto |x|^p$ and $\lambda \mapsto |\lambda|^{1/p}$ are well defined on X^u and Λ^u, respectively.

Lemma 4.5. *Let X be an injective Banach lattice and let $\mathscr{X}$ be the representation of X in the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B} = \mathbb{M}(X)$, and $\mathbb{1} \leq p \in \Lambda^u$. Then $[\![\mathscr{X}^p$ is an AL^p-space$]\!] = \mathbb{1}$ and X^p is a $\mathbb{B}$-cyclic Banach lattice, where $\mathscr{X}^p{\Downarrow}$ and X^p are isometrically lattice $\mathbb{B}$-isomorphic.*

Proof. By the Gordon Theorem 2.1 and Lemma 4.2, we may assume that $\Lambda^u = \mathscr{R}{\downarrow}$ and $X = \mathscr{X}{\Downarrow}$. By Lemma 4.4, there exists a strictly positive Maharam operator $\Phi : X \to \Lambda(\mathbb{B})$ with the Levy property such that $\|x\| = \|\Phi(|x|)\|_\infty$ for all $x \in X$. Put $\varphi := \Phi{\uparrow}$ and note that $[\![\, \varphi : \mathscr{X} \to \mathscr{R}$ is an order continuous strictly positive functional with the Levy property $]\!] = \mathbb{1}$; moreover, X is an order ideal in $\mathscr{X}{\downarrow}$ and the restriction of $\varphi{\downarrow}$ to $\mathscr{X}{\Downarrow}$ coincides with Φ (see [2, Theorem 5.2.8]). If $\mathscr{X}^u$ is the universal completion of $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ then the vector lattices $\mathscr{X}^u{\downarrow}$ and X^u can be identified since they are lattice isomorphic by [2, Theorem 2.11.8]. Moreover, the multiplication in X^u coincides with the descent of the multiplication in $\mathscr{X}^u$; hence, $x \mapsto |x|^p$ $(x \in X)$ is the descent of the mapping $x \mapsto |x|^p$ $(x \in \mathscr{X})$. Since $p \in \mathscr{R}{\downarrow}$, by the Transfer Principle, $[\![\mathscr{X}^p$ is the AL^p-space$]\!] = \mathbb{1}$. By definition, the norm $|\!|\!|\cdot|\!|\!|_p \in \mathbb{V}^{(\mathbb{B})}$ of the AL^p-space $\mathscr{X}^p$ has the form $|\!|\!|x|\!|\!|_p = (\varphi(|x|^p))^{1/p}$ $(x \in \mathscr{X}^p)$. Using the rules for the descent of the composition of operators and the inverse operator (see [2, 1.5.5]) as well as the equality $\Phi = \varphi{\downarrow}|_X$, we infer that $|\cdot|_p = |\!|\!|\cdot|\!|\!|_p{\downarrow}$. It remains to notice that the inclusions $x \in X^p$ and $|x|_p \in \Lambda$ are equivalent for all $x \in \mathscr{X}^u{\downarrow}$. $\square$

Theorem 4.6. *For every injective Banach lattice $X = L^1(\Phi)$ and every $\mathbb{1} \leq p \in \Lambda^u$, the space $X^p = L^p(\Phi)$ is a $\mathbb{B}$-cyclic Banach lattice, and the Boolean algebras $\mathbb{B} = \mathbb{M}(X^p)$ and $\mathbb{M}(X)$ are isomorphic.*

Proof. This is immediate from Theorem 2.6 and Lemma 4.5. $\square$

Remark 4.7. Applying the Gutman Theorem on the Banach–Kantorovich representation (see [3, Theorem 2.4.10]), we can show that $L^p(\Phi)$ is isometrically and lattice isomorphic to the Banach lattice of continuous sections of a continuous Banach bundle of AL^p-spaces (hence, of the classical Lebesgue spaces) $(L^{p(\omega)}(\varphi_\omega))_{\omega \in \Omega}$, where Ω is the Stone representation space of $\mathbb{B}$. Details can be found in [3, § 4 and § 5]. A similar result on the representation of injective Banach lattices by means of a continuous bundle of AL-spaces was obtained by Haydon in [16, Theorem 7B].

Remark 4.8. Suppose that X is an injective Banach lattice, $\mathbb{B} = \mathbb{M}(X)$, and $\Lambda = \Lambda(\mathbb{B})$; while Q and P are the Stone representation spaces of the Boolean algebras $\mathbb{P}(X)$ and $\mathbb{B}$ respectively. The inclusion $\mathbb{M}(X) \subset \mathbb{P}(X)$ induces the continuous epimorphism $\tau : Q \to P$. Then there exists a modular Maharam measure $\mu : \mathscr{B} \to \Lambda = C(P)$, where $\mathscr{B} := \mathscr{B}(Q)$ is the Borel σ-algebra of Q, such that X is isometrically lattice $\mathbb{B}$-isomorphic to $L^1(\mu) := L^1(Q, \mathscr{B}, \mu)$ (see [16, Theorem 6H]). Moreover, to the Maharam operator Φ of Lemma 4.4 there corresponds the integration operator $f \mapsto \int_Q f\, d\mu$ $(f \in L^1(\mu))$. We can now define $L^p(\mu)$ as the space of functions μ-integrable with a variable exponent $p \in C_\infty(P)$:

$$L^p(\mu) = \left\{ f \in L^0(\mu) : \int_Q |f(q)|^{p(\tau(q))}\, d\mu(q) \in \Lambda \right\}.$$

The order and metric properties of classes of functions integrable with variable exponent with respect to a vector measure deserve independent study.

§ 5. The Banach Lattices $c_0(\Gamma)$ and $C_\#(Q, c_0(\Gamma))$

Recall that $l^\infty(\Gamma)$ stands for the Banach lattice of all bounded functions $x : \Gamma \to \mathbb{R}$ with the norm $\|x\|_\infty := \sup_{\gamma\in\Gamma} |x(\gamma)|$, and the Banach lattice $c_0(\Gamma)$ is defined as the closure in $l^\infty(\Gamma)$ of the sublattice $c_{00}(\Gamma)$ consisting of functions with finite support $\operatorname{supp}(x) := \{\gamma \in \Gamma : x(\gamma) \neq 0\}$.

Lemma 5.1. *Let Γ be an arbitrary nonempty set. Then $c_0(\Gamma^\wedge)$ is the completion within $\mathbb{V}^{(\mathbb{B})}$ of the metric space $c_0(\Gamma)^\wedge$.*

Proof. Denote by $\mathsf{P}_{\rm fin}(\Gamma)$ the set of all finite subsets of a set Γ. We will use the formula

$$\mathbb{V}^{(\mathbb{B})} \models \mathsf{P}_{\rm fin}(X^\wedge) = \mathsf{P}_{\rm fin}(X)^\wedge \tag{2}$$

(see [1, 5.1.9]). Let $c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge)$ be the subset of $c_0(\Gamma^\wedge)$ consisting of functions $x : \Gamma^\wedge \to \mathbb{R}^\wedge$ with finite support $\operatorname{supp}(x)$. Since $c_{00}(\Gamma)$ is dense in $c_0(\Gamma)$ and $[\![c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge)$ is dense in $c_0(\Gamma^\wedge)]\!] = \mathbb{1}$, it suffices to show that $[\![c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge) \subset c_{00}(\Gamma)^\wedge]\!] = \mathbb{1}$. By the properties of descents (see [2, 1.5.2]), the last equation can be rewritten as

$$[\![(\forall x \in c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge))\, x \in c_{00}(\Gamma)^\wedge]\!] = \bigwedge\{[\![x \in c_{00}(\Gamma)^\wedge]\!] : [\![x \in c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge)]\!] = \mathbb{1}\} = \mathbb{1}.$$

Take $x \in c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge)\!\downarrow$ and consider the modified descent $x\!\downarrow : \Gamma \to \mathbb{R}^\wedge\!\downarrow$ (see [2, 1.5.8]). Verify that $[\![x \in c_{00}(\Gamma)^\wedge]\!] = \mathbb{1}$. Involving (2), we infer

$$\begin{aligned}[\![x \in c_{00}(\Gamma^\wedge, \mathbb{R}^\wedge)]\!] &= [\![(\exists \theta \in \mathsf{P}_{\rm fin}(\Gamma^\wedge))\, (\forall \gamma \in \Gamma^\wedge)\, (\gamma \notin \theta \to x(\gamma) = 0)]\!] \\ &= \bigvee_{\theta\in\mathsf{P}_{\rm fin}(\Gamma)} [\![(\forall \gamma \in \Gamma^\wedge)\, (\gamma \notin \theta^\wedge \to x(\gamma) = 0)]\!].\end{aligned}$$

By the Exhaustion Principle (see [2, 1.2.8]), there exists a partition of unity $(b_\theta)_{\theta\in\Theta}$ in $\mathbb{B}$, where $\Theta := \mathsf{P}_{\rm fin}(\Gamma)$, such that for all $\theta \in \Theta$ we have

$$\begin{aligned}b_\theta &\leq [\![(\forall \gamma \in \Gamma^\wedge)\, (\gamma \notin \theta^\wedge \to x(\gamma) = 0)]\!] \\ &= \bigwedge_{\gamma\in\Gamma} [\![\gamma^\wedge \notin \theta^\wedge]\!] \Rightarrow [\![x(\gamma^\wedge) = 0)]\!] = \bigwedge_{\gamma\in\Gamma\setminus\theta} [\![x\!\downarrow\!(\gamma) = 0)]\!].\end{aligned}$$

Since $\{x\!\downarrow\!(\gamma) : \gamma \in \theta\}$ is a finite subset of $\mathbb{R}^\wedge\!\downarrow$, there are a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(t_{\xi,\gamma})_{\xi\in\Xi}$ in $\mathbb{R}$ such that $b_\theta x\!\downarrow\!(\gamma) = \operatorname{mix}_{\xi\in\Xi} \pi_\xi t^\wedge_{\xi,\gamma}$ for $\gamma \in \theta$. Define $x_\xi \in c_{00}(\Gamma)$ by setting $x_\xi(\gamma) = t_{\xi,\gamma}$ if $\gamma \in \theta$ and $x_\xi(\gamma) = 0$ otherwise. Then $[\![x^\wedge_\xi \in c_{00}(\Gamma)^\wedge]\!] = \mathbb{1}$ and

$$\begin{aligned}[\![x \in c_{00}(\Gamma)^\wedge]\!] &\geq [\![x^\wedge_\xi \in c_{00}(\Gamma)^\wedge]\!] \wedge [\![x^\wedge_\xi = x]\!] = [\![x^\wedge_\xi = x]\!] \geq \bigwedge_{\gamma\in\Gamma} [\![x^\wedge_\xi(\gamma^\wedge) = x(\gamma^\wedge)]\!] \\ &= \bigwedge_{\gamma\in\theta} [\![t^\wedge_{\xi,\gamma} = x\!\downarrow\!(\gamma)]\!] \wedge \bigwedge_{\gamma\notin\theta} [\![0 = x\!\downarrow\!(\gamma)]\!] \geq \pi_\xi \wedge b_\theta\end{aligned}$$

for all $\xi \in \Xi$ and $\theta \in \Theta$; whence $[\![x \in c_{00}(\Gamma)^\wedge]\!] \geq \bigvee_{\xi\in\Xi} b_\theta \wedge \pi_\xi = b_\theta$. Consequently, $[\![x \in c_{00}(\Gamma)^\wedge]\!] \geq \bigvee_{\theta\in\Theta} = \mathbb{1}$; q.e.d. □

Definition 5.2. Suppose that Q is an extremally disconnected compact space and X is a Banach lattice. Denote by $C_{\#}(Q,X)$ the set of cosets of continuous vector-functions $u : \operatorname{dom}(u) \subset Q \to X$ such that $Q \setminus \operatorname{dom}(u)$ is a meager set in Q and the continuous extension $|u|$ of the pointwise norm $q \mapsto \|u(q)\|$ to the whole of Q belongs to the Banach lattice $C(Q)$ of continuous functions. Vector-functions u and v are equivalent if $u(q) = v(q)$ for all $q \in \operatorname{dom}(u) \cap \operatorname{dom}(v)$. If $\tilde{u}$ is the coset of u then we put $|\tilde{u}| := |u|$. The set $C_{\#}(Q,X)$ is naturally endowed with the structure of a module over the f-algebra $C(Q)$ and the norm $\|u\| := \|\,|u|\,\|_\infty$ (cf. [3, 2.3.3]).

Lemma 5.3. *Let $X \in \mathbb{V}$ be a Banach lattice and let $\mathscr{X}$ be the completion of the metric space $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Then $[\![\,\mathscr{X}$ is the Banach lattice $]\!] = \mathbb{1}$ and $\mathscr{X}{\Downarrow}$ is $\mathbb{B}$-cyclic Banach lattice isometrically and lattice $\mathbb{B}$-isomorphic to $C_{\#}(Q,X)$, where Q is the Stone representation space of the Boolean algebra $\mathbb{B}$.*

Proof. We must only apply the description of the descent $\mathscr{X}{\downarrow}$ of [3, Theorem 8.3.4(1)] and carry out the obvious passage to the bounded parts. □

Corollary 5.4. *$C_{\#}(Q,X)$ is a $\mathbb{B}$-cyclic Banach lattice, where $\mathbb{B}$ is isomorphic to the Boolean algebra of clopen subsets in Q. Moreover, the M-projection in $C_{\#}(Q,X)$ corresponding to a clopen set $U \subset Q$ is the multiplication by the characteristic function χ_U.*

Corollary 5.5. *Let Γ be a nonempty set and let Q be the Stone representation space of a complete Boolean algebra $\mathbb{B}$. Then the $\mathbb{B}$-cyclic Banach lattices $c_0(\Gamma^\wedge){\Downarrow}$ and $C_{\#}(Q, c_0(\Gamma))$ are isometrically lattice $\mathbb{B}$-isomorphic.*

Proof. This is immediate from Lemmas 5.1 and 5.3. □

Remark 5.6. Lemma 5.3 is a particular case of the general result on the functional representation of the construction of the Boolean extension which was obtained by Gordon and Lyubetsky (see [17, 18]).

Remark 5.7. Important information on the structure of a Banach lattice is given by the possibility of embedding into it the classical sequence spaces c_0, l^1, and l^∞ (see, for example, [4, § 2.3 and § 2.4]). Corollary 5.5 shows that the analogous role in the theory of $\mathbb{B}$-cyclic Banach lattices must belong to the $\mathbb{B}$-embeddability of the Banach lattices $C_{\#}(Q, c_0)$, $C_{\#}(Q, l^1)$, and $C_{\#}(Q, l^\infty)$ (see also Remarks 6.5 and 6.6 below).

§ 6. The Boolean Valued Ando Theorem

We begin with formulating the Ando Theorem whose Boolean valued interpretation leads us to an analogous result for $\mathbb{B}$-cyclic Banach lattices.

Theorem 6.1 [19]. *Let X be a Banach lattice of dimension ≥ 3. Then X is isometrically isomorphic to $L^p(\Omega, \Sigma, \mu)$ for some $1 \leq p \in \mathbb{R}$ and measure space (Ω, Σ, μ) or to $c_0(\Gamma)$ for some nonempty set Γ if and only if each closed vector sublattice in X is the image of a contractive positive projection.*

Recall that a *contractive positive projection* in a Banach lattice is a positive linear operator $P : X \to X$ with $P^2 = P$ and $\|P\| \leq 1$.

Definition 6.2. A subspace X_0 in a $\mathbb{B}$-cyclic Banach lattice X is called $\mathbb{B}$-*complete* if X_0 contains the mixings of all families from X_0 by all partitions of unity in $\mathbb{B}$. We say that the Boolean dimension $\mathbb{B}\text{-}\dim(X)$ is at least 3 if we can choose three elements in X so that the least closed $\mathbb{B}$-complete subspace containing them is $\mathbb{B}$-isomorphic to $\Lambda(\mathbb{B})^3$.

Lemma 6.3. *If $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is the Boolean valued representation of a $\mathbb{B}$-cyclic Banach lattice X then*

$$\mathbb{B}\text{-}\dim(X) \geq 3 \iff [\![\dim(\mathscr{X}) \geq 3^\wedge]\!] = \mathbb{1}.$$

Proof. From [2, Proposition 5.8.5 and Theorem 5.8.11] it follows that a $\mathbb{B}$-cyclic Banach lattice is an f-module over the ring $\Lambda := \Lambda(\mathbb{B})$ (see [2, 2.11.1] for the definition of f-module). Take $e_1, e_2, e_3 \in X$ and consider the mapping $\varphi : (\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ from Λ^3 into X. It is easy to see that if e_1, e_2, and e_3 are Λ-linearly independent then $X_0 := \varphi(\Lambda^3)$ is a $\mathbb{B}$-complete subspace in X. At the same time, φ is injective if and only if e_1, e_2, and e_3 are Λ-linearly independent. It remains to observe that the Λ-linear independence of e_1, e_2, and e_3 is equivalent to the equality $[\![\dim(\mathscr{X}) \geq 3^\wedge]\!] = \mathbb{1}$. $\square$

We have all ingredients necessary for formulating and proving the second main result of this article.

Theorem 6.4. *Let $\mathbb{B}$ be a complete Boolean algebra and let Q be the Stone representation space of $\mathbb{B}$. The following are equivalent for a $\mathbb{B}$-cyclic Banach lattice X satisfying the condition $\mathbb{B}$-$\dim(X) \geq 3$:*

(1) *there is a contractive positive projection onto any $\mathbb{B}$-complete closed sublattice in X that commutes with the projections from $\mathbb{B}$;*

(2) *there is a partition of unity $(\pi_\gamma)_{\Gamma\cup\{0\}}$ in $\mathbb{B}$, with Γ a nonempty set of cardinals, such that $\pi_0 X \simeq_{\pi_0\mathbb{B}} L^p(\Phi)$ for some $\mathbb{1} \leq p \in \Lambda(\mathbb{B})^u$ and an injective Banach lattice $L := L^1(\Phi)$ with $\mathbb{M}(L) \simeq \pi_0\mathbb{B}$, and $\pi_\gamma X \simeq_{\pi_\gamma\mathbb{B}} C_\#(Q_\gamma, c_0(\gamma))$ for all $\gamma \in \Gamma$, where Q_γ is a clopen subset of Q corresponding to the projection π_γ.*

Proof. (1) $\Longrightarrow$ (2): In accordance with Theorem 2.6, we may assume that $X = \mathscr{X}{\Downarrow}$, where $\mathscr{X}$ is a Banach lattice within $\mathbb{V}^{(\mathbb{B})}$; moreover, the relations $\mathbb{B}$-$\dim(X) \geq 3$ and $[\![\dim(\mathscr{X}) \geq 3^\wedge]\!] = \mathbb{1}$ are equivalent by Lemma 6.3. It is easy to check that $\mathscr{X}_0$ is a closed sublattice in $\mathscr{X}$ if and only if $X_0 := \mathscr{X}_0{\Downarrow}$ is a $\mathbb{B}$-complete norm closed sublattice in X. The operator $P : X \to X$ has the form $P = \pi{\Downarrow}$ for an operator $\pi : \mathscr{X} \to \mathscr{X}$ if and only if P is extensional (see [2, 1.5.6]), and by (1) the last means that P commutes with the projections from $\mathbb{B}$. Moreover, P is a contractive positive projection only if $[\![\,\pi$ is a contractive positive projection$\,]\!] = \mathbb{1}$. Thus, the assertion 6.4(1) is equivalent to $[\![\,$each closed sublattice in $\mathscr{X}$ admits a contractive positive projection$\,]\!] = \mathbb{1}$. By the above-mentioned theorem, 6.4(1) is equivalent also to the fact that $\mathscr{X}$ is isometrically lattice isomorphic to $L^p(\mu)$ for some $1 \leq p \in \mathscr{R}$ and some measure space (Ω, Σ, μ) or to $c_0(S)$ for some nonempty set S within $\mathbb{V}^{(\mathbb{B})}$. Write this down in symbolic form:

$$[\![(\exists\, 1 \leq p \in \mathscr{R})(\exists\, \varphi \in \mathscr{X}_n')\mathscr{X} \simeq L^p(\varphi) \vee (\exists\, S)\mathscr{X} \simeq c_0(S)]\!] = \mathbb{1}.$$

This implies the existence of two pairwise complementary elements $\pi_0, \pi_0^* \in \mathbb{B}$ such that if we put $\mathbb{B}_1 := [0, \pi_0]$ and $\mathbb{B}_2 := [0, \pi_0^*]$ then we can choose $p, \varphi \in \mathbb{V}^{(\mathbb{B}_1)}$ and $S \in \mathbb{V}^{(\mathbb{B}_2)}$ for which the following two assertions hold:

($*$) within the model $\mathbb{V}^{(\mathbb{B}_1)}$, the real $1 \leq p \in \mathscr{R}$, the injective Banach lattice L, and the strictly positive order continuous functional $\varphi : L \to \mathscr{R}$ with the Levy property satisfy the estimate $\pi_0 \leq [\![\mathscr{X} \simeq L^p(\varphi)]\!]$;

($**$) within the model $\mathbb{V}^{(\mathbb{B}_2)}$, the set S satisfies the estimate $\pi_0^* \leq [\![\mathscr{X} \simeq c_0(S)]\!]$.

Put $\Phi := \varphi{\Downarrow}$ and observe that, by Lemma 4.5, $L^p(\Phi) \simeq_{\pi_0\mathbb{B}} L^p(\varphi){\Downarrow}$. Now, assertion ($*$) shows that $\pi_0 X \simeq_{\pi_0\mathbb{B}} L^p(\Phi)$.

Let $\varkappa \in \mathbb{V}^{(\mathbb{B}_2)}$ be the cardinality of S. In view of Proposition 2.9, the Boolean valued cardinal $\varkappa$ is a mixing of standard cardinals; i.e., $\varkappa = \mathrm{mix}_{\gamma\in\Gamma}\, b_\gamma \gamma^\wedge$, where Γ is a nonempty set of cardinals, $(b_\gamma)_{\gamma\in\Gamma}$ is a partition of unity in $\mathbb{B}_2$, while $\gamma^\wedge$ is a cardinal within $\mathbb{V}^{(\mathbb{B}_\gamma)}$ and $\mathbb{B}_\gamma := [\mathbb{0}, b_\gamma]$ for all $\gamma \in \Gamma$. Passing to the model $\mathbb{V}^{(\mathbb{B}_\gamma)}$ (by the scheme of [2, 1.3.7]) and involving assertion ($**$), we see that $\mathscr{X} \simeq c_0(\gamma^\wedge)$. Applying Lemmas 5.1, 5.3, and Definition 5.2, we see that $b_\gamma X \simeq_{b_\gamma\mathbb{B}} C_\#(Q_\gamma, c_0(\gamma))$.

(2) $\Longrightarrow$ (1) As in Lemma 3.3, the mapping $\mathscr{X}_0 \mapsto \mathscr{X}_0{\Downarrow}$ performs a bijection between the norm-closed sublattices of $\mathscr{X}$ and the $\mathbb{B}$-complete norm-closed sublattices of X. Similarly, the contractive positive projections in $\mathscr{X}$ are in a one-to-one correspondence with the contractive positive projections in $\mathscr{X}{\Downarrow}$ commuting with projections in $\mathbb{B}$. We are left with noting that if assertion (2) holds then the Banach lattice $\mathscr{X}$ is either an AL^p-space or $c_0(\Gamma)$, and so any closed sublattice of $\mathscr{X}$ admits a contractive positive projection by the Transfer Principle. $\square$

Remark 6.5. A nonzero element x in a $\mathbb{B}$-cyclic Banach lattice X is called a $\mathbb{B}$-*atom* if for every pair of disjoint elements $z, y \in X_+$, $y + z \leq x$, there exists a projection $\pi \in \mathbb{B}$ such that $\pi y = 0$ and $\pi^* z = 0$. The lattice X is called $\mathbb{B}$-*atomic* if the zero element is the only element in X disjoint from every $\mathbb{B}$-atom in X. If X is the same as in Theorem 6.4(2) then there exists an M-projection $\rho \leq \pi_0$ such that $\rho^* X$

has no $\mathbb{B}$-atoms and ρX admits the representation

$$\rho X \simeq_{\rho\mathbb{B}} \left(\sum_{\gamma\in\Delta}^{\oplus} C_{\#}(P_\gamma, l^{p(\gamma)}(\gamma)) \right)_{l_\infty},$$

where Δ is a set of cardinals and $(P_\gamma)_{\gamma\in\Delta}$ is a family of extremally disconnected compact spaces. This claim can be deduced from [20, Theorem 5.4].

REMARK 6.6. If a $\mathbb{B}$-cyclic Banach lattice X satisfies the conditions of Theorem 6.4(1) then, as in 6.4(2), $\pi_0 X \simeq_{\pi_0\mathbb{B}} L^p(\Phi)$, and if $\pi := \pi_0^* = I_X - \pi_0$ then we have the representation

$$\pi X \simeq_{\pi\mathbb{B}} \left(\sum_{\gamma\in\Gamma}^{\oplus} C_{\#}(Q_\gamma, c_0(\gamma)) \right)_{l_\infty}.$$

Moreover, the family of extremally disconnected compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ (as in Remark 6.5) is nonunique in general. The reason is the phenomenon of the *cardinal collapsing* in a Boolean valued model. Overcoming this difficulty and obtaining a unique representation is possible in the same way as in [20], by involving the notions of pure $\mathbb{B}$-*atomicity* and *stability* [20, Definitions 3.4 and 3.6].

The authors express their gratitude to the referee for removing numerous typos and inaccuracies.

References

1. Kusraev A. G. and Kutateladze S. S., *Introduction to Boolean Valued Analysis* [Russian], Nauka, Moscow (2005).
2. Kusraev A. G. and Kutateladze S. S., *Boolean Valued Analysis: Selected Topics* (Trends Sci. South Russia), Vladikavkaz Sci. Center Press, Vladikavkaz (2014).
3. Kusraev A. G., *Dominated Operators*, Kluwer Academic Publishers, Dordrecht (2001).
4. Meyer-Nieberg P., *Banach Lattices*, Springer, Berlin etc. (1991).
5. Abramovich Y. A. and Kitover A. K., *Inverses of Disjointness Preserving Operators*, Amer. Math. Soc., Providence (2000) (Mem. Amer. Math. Soc.; No. 679).
6. Lindenstrauss J. and Tzafriri L., *Classical Banach Spaces. Vol. 2. Function Spaces*, Springer-Verlag, Berlin etc. (1979).
7. Gordon E. I., "Real numbers in Boolean-valued models of set theory and K-spaces," Dokl. Akad. Nauk SSSR, vol. 237, no. 4, 773–775 (1977).
8. Gutman A. E., Kusraev A. G., and Kutateladze S. S., "The Wickstead problem," Sib. Èlektron. Mat. Izv., vol. 5, 293–333 (2008).
9. Gutman A. E., "Locally one-dimensional K-spaces and σ-distributive Boolean algebras," Siberian Adv. Math., vol. 5, no. 2, 99–121 (1995).
10. Harmand P., Werner D., and Wener W., *M-Ideals in Banach Spaces and Banach Algebras*, Springer-Verlag, Berlin etc. (1993) (Lecture Notes Math.; V. 1547).
11. Kusraev A. G., "The Boolean transfer principle for injective Banach lattices," Sib. Math. J., vol. 25, no. 1, 888–900 (2015).
12. Takeuti G., "Von Neumann algebras and Boolean valued analysis," J. Math. Soc. Japan., vol. 35, no. 1, 1–21 (1983).
13. Takeuti G., "C^*-algebras and Boolean valued analysis," Japan J. Math., vol. 9, no. 2, 207–246 (1983).
14. Ozawa M., "Boolean valued interpretation of Banach space theory and module structure of von Neumann algebras," Nagoya Math. J., vol. 117, 1–36 (1990).
15. Coppel W. A., *Foundations of Convex Geometry*, Cambridge University Press, Cambridge (1988).
16. Haydon R., "Injective Banach lattices," Math. Z., Bd 156, 19–47 (1977).
17. Gordon E. I. and Lyubetsky V. A., "Some applications of nonstandard analysis in the theory of Boolean valued measures," Dokl. Akad. Nauk SSSR, vol. 256, no. 5, 1037–1041 (1981).
18. Gordon E. I. and Lyubetsky V. A., "Boolean completion of uniformities," in: *Studies on Nonclassical Logics and Formal Systems* [Russian], Nauka, Moscow, 1983, 82–153.
19. Ando T., "Banachverbände und positive Projektionen," Math. Z., Bd 109, 121–130 (1969).
20. Kusraev A. G., "Atomicity in injective Banach lattices," Vladikavkaz. Mat. Zh., vol. 17, no. 3, 34–42 (2015).

KUSRAEV A. G.
SOUTHERN MATHEMATICAL INSTITUTE
VLADIKAVKAZ SCIENTIFIC CENTER OF THE RUSSIAN ACADEMY OF SCIENCES
NORTH OSSETIAN STATE UNIVERSITY, VLADIKAVKAZ, RUSSIA
E-mail address: kusraev@smath.ru

KUTATELADZE S. S.
SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, RUSSIA
E-mail address: sskut@math.nsc.ru