ISSN 0037–4466, Siberian Mathematical Journal, 2019, Vol. 60, No. 5, pp. 774–804. © Pleiades Publishing, Inc., 2019.
Russian Text © The Author(s), 2019, published in Sibirskii Matematicheskii Zhurnal, 2019, Vol. 60, No. 5, pp. 996–1034.

ISOMORPHISMS OF SOBOLEV SPACES ON RIEMANNIAN MANIFOLDS AND QUASICONFORMAL MAPPINGS

S. K. Vodopyanov UDC 517.518:517.54

Abstract: We prove that a measurable mapping of domains in a complete Riemannian manifold induces an isomorphism of Sobolev spaces with the first generalized derivatives whose summability exponent equals the (Hausdorff) dimension of the manifold if and only if the mapping coincides with some quasiconformal mapping almost everywhere.

DOI: 10.1134/S0037446619050033

Keywords: Riemannian manifold, quasiconformal mapping, Sobolev space, composition operator

Introduction

This article can be regarded as a natural continuation of [1–10]. There are various proofs of the theorem that a measurable mapping on the Euclidean space $\mathbb{R}^n$ [1–5] or a Carnot group $\mathbb{G}$ [6–8], which induces the isomorphism of some spaces of differentiable functions by the change-of-variables formula, coincides with a quasiconformal mapping almost everywhere.

The study of a similar problem was started in [9] for the measurable mappings of domains of Riemannian manifolds inducing the isomorphism of Sobolev classes with the first generalized derivatives. A complete solution of the problem discussed in [9] appears in the several articles combined: In [10] this problem is solved for the Sobolev spaces of functions whose summability exponent differs from the topological dimension of the manifold. This article includes a complete solution of the problem in the case that the summability exponent of functions in the Sobolev space coincides with the topological dimension of the Riemannian manifold.

The method of the present article is a substantial modification of the arguments of [5] which is based on the results of [4, 9, 10]. The main objects of study, the class IL^1_p of mappings of Riemannian manifolds, was introduced in [10].

Definition 1. Take two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$ in two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same topological dimension $n \geq 2$. Say that a measurable mapping $\varphi : D \to D'$ defined a.e. in D *is of class* IL^1_p with $p \in [1, \infty)$ whenever φ induces the composition operator in Sobolev spaces,

$$\varphi^* : L^1_p(D') \cap C^\infty(D') \to L^1_p(D), \quad \varphi^*(f) = f \circ \varphi, \quad f \in L^1_p(D') \cap C^\infty(D'), \tag{1}$$

so that

(1) $K^{-1}\|f \mid L^1_p(D')\| \leq \|\varphi^*(f) \mid L^1_p(D)\| \leq K\|f \mid L^1_p(D')\|$ for all $f \in L^1_p(D') \cap C^\infty(D')$, where K is a constant independent of the choice of f;

(2) $\varphi^*\left(L^1_p(D') \cap C^\infty(D')\right)$ is dense in $L^1_p(D)$.

As shown in [6, 10], item (2) of Definition 1 is independent of item (1).

This article gives a full description of the mappings of class IL^1_n, where n is the topological dimension of $\mathbb{M}$ and $\mathbb{M}'$; i.e., we obtain a full description of the measurable mappings of domains on Riemannian manifolds which induce, in the sense of Definition 1, isomorphisms of the Sobolev spaces L^1_n. The case $p \neq n$ is studied in [10], and the general scheme is explained in [9]. The following theorem is the main result of this article. The definitions of the main concepts reside after its statement.

The author was supported by the Ministry of Education and Science (Contract No. 02.a03.21.0008) and the Russian Foundation for Basic Research (Grant 17–01–00801).

[†)] To Yu. G. Reshetnyak on the occasion of his 90th birthday.

Original article submitted March 27, 2019; revised March 27, 2019; accepted May 15, 2019.

Theorem 2 [9]. *Take two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$ in two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same topological dimension $n \geq 2$. Assume that $\mathbb{M}'$ is a parabolic manifold. A measurable mapping $\varphi : D \to D'$ is of class IL^1_n if and only if φ coincides a.e. with some quasiconformal mapping $\Phi : D \setminus \{x_{\mathrm{inv}}\} \to \mathbb{M}$ for which the domains $\Phi(D \setminus x_{\mathrm{inv}}\})$ and D' are $(1, n)$-equivalent. Here $x_{\mathrm{inv}} \in D$ is some point of D.*

Definition 3. A Riemannian manifold $\mathbb{M}'$ is called *parabolic* whenever $\operatorname{cap}\big(K; L^1_n(\mathbb{M}')\big) = 0$ for every compact set $K \subset M'$ and *hyperbolic* otherwise.

Recall that the *capacity* of a compact set $K \subset M'$ in $L^1_n(\mathbb{M}')$ is the quantity

$$\operatorname{cap}\big(K; L^1_n(\mathbb{M}')\big) = \inf\big\{\|\nabla u \mid L_n(\mathbb{M}')\|^n : u \in C_0^\infty(\mathbb{M}') \cap L^1_n(\mathbb{M}') \text{ and } u \geq 1 \text{ on } K\big\}.$$

Definition 4. A homeomorphism $\Phi : D \to D'$ of class $W^1_{n,\mathrm{loc}}$ is called *quasiconformal* whenever there exists a constant K such that $|D\Phi(x)|^n \leq K|J(x, \Phi)|$ a.e. in D, where $D\Phi(x)$ is the approximative differential [11] of Φ, while $J(x, \Phi) = \det D\Phi(x)$.

Definition 5. Two open sets D_1 and D_2 are called $(1,p)$-*equivalent* whenever the restriction operators $r_i : L^1_p(D_1 \cup D_2) \to L^1_p(D_i)$ with $r_i(f) = f|_{D_i}$ for $f \in L^1_p(D_1 \cup D_2)$ are isomorphisms.

This definition coincides with the definition of $(1, p)$-equivalence in [10] and is equivalent to the definition in [12].

Definition 6 (see [12]). Two open sets D_1 and D_2 are called $(1,p)$-*equivalent* whenever the restriction operators $r_i : L^1_p(D_i) \to L^1_p(D_1 \cap D_2)$ with $r_i(f) = f|_{D_1 \cap D_2}$ for $f \in L^1_p(D_i)$ are such that $r_2^{-1} \circ r_1$ and $r_1^{-1} \circ r_2$ are isomorphisms.

In the Euclidean space, the theorem similar to Theorem 2 is proved in [1] on assuming that D' is a bounded domain. The properties of $(1, p)$-equivalent domains are studied in Euclidean spaces in [12], and on Carnot groups in [13, 14].

The proof of Theorem 2 obtained in this article relies largely on the method of [5] with substantial extensions, which are unavoidable in the current setup, because [5] deals with the Euclidean space $\mathbb{R}^n$ as the domains D and D' and with a suitable normed function space.

The classes IL^1_p of mappings with $p \neq n$ are thoroughly studied in [10], which also presents a detailed history of this question and a comprehensive bibliography. For comparison with Theorem 2, let us state the main result of [10].

Theorem 7 [10, Theorem 1]. *Take two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$ in two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same topological dimension $n \geq 2$. For $p \geq 1$ with $p \neq n$, a measurable mapping $\varphi : D \to D'$ is of class IL^1_p if and only if φ coincides a.e. with some quasi-isometry $\Phi : D \to \Phi(D)$ for which the domains $\Phi(D)$ and D' are $(1, p)$-equivalent.*

1. Preliminaries

1.1. Sobolev spaces on Riemannian manifolds. Fix a connected complete Riemannian manifold $\mathbb{M} = (\mathbf{M}, g)$, meaning a smooth manifold $\mathbb{M}$ with a Euclidean metric g_x chosen in each tangent space $T_x\mathbb{M}$ and varying smoothly from point to point.

The length of each absolutely continuous piecewise smooth curve $\gamma : [a, b] \to \mathbb{M}$ is expressed as $l(\gamma) = \int_a^b |\dot\gamma(t)|\,dt$, where $|\dot\gamma(t)| = \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}$ is the length of the tangent vector $\dot\gamma(t)$ in the Euclidean space $T_{\gamma(t)}\mathbb{M}$ with inner product $g_{\gamma(t)}$.

The metric $d(x, y)$ on the Riemannian manifold $\mathbb{M}$ is defined as the greatest lower bound of the lengths of piecewise smooth curves with endpoints x and y.

Take a domain D, meaning a connected open set, in $\mathbb{M}$. Define $L_p(D)$ as the space of functions summable to power $p \in [1, \infty)$ as the collection of Lebesgue measurable functions with finite norm

$$\|f \mid L_p(D)\| = \bigg(\int\limits_D |f(x)|^p \, d\omega\bigg)^{1/p} < \infty.$$

Here $d\omega$ is the standard volume element on $\mathbb{M}$. If a measurable function $u : D \to \mathbb{R}$ is summable on each compact subset of D then u is called *locally summable.*

A locally summable function $v : D \to \mathbb{R}$ is called the *generalized derivative* of a locally summable function $f : D \to \mathbb{R}$ along a vector field X on D and denoted by $v = Xf$ whenever

$$\int\limits_D v\psi\, d\omega = -\int\limits_D fX^*\psi\, d\omega$$

for every compactly supported function $\psi \in C_0^\infty(D)$. Here X^* is the differential operator adjoint to the differential operator X.

The homogeneous Sobolev space $L_p^1(D)$ consists of locally integrable functions $f : D \to \mathbb{R}$ with generalized gradient $\nabla f \in L_p(D)$. The seminorm in $L_p^1(D)$ is defined as

$$\left\| f \mid L_p^1(D) \right\| = \| \nabla f \mid L_p(D) \| = \left(\int\limits_D |\nabla f(x)|^p\, d\omega \right)^{\frac{1}{p}},$$

where $d\omega$ is the n-dimensional volume element, $\nabla f(x)$ is the generalized gradient of f at $x \in D$, and $|\nabla f(x)|$ is the length of $\nabla f(x)$ in the Euclidean space $T_x\mathbb{M}$ with inner product g_x.

The Sobolev space $W_p^1(D)$ consists of locally summable functions with finite norm

$$\left\| f \mid W_p^1(D) \right\| = \| f \mid L_p(D) \| + \| \nabla f \mid L_p(D) \|.$$

Say that f *is of class* $W_{p,\mathrm{loc}}^1(D)$, if $f \in W_p^1(V)$ for every bounded subdomain $V \subset D$ satisfying $\overline{V} \subset D$.

Reshetnyak suggested an approach to Sobolev classes of functions with values in a metric space of [15]. Consider a complete metric space $(\mathbb{X}, r)$ with a metric r on $\mathbb{X}$ and a domain D in a Riemannian manifold $\mathbb{M}$. Say that $\varphi : D \to \mathbb{X}$ *is of class* $W_{p,\mathrm{loc}}^1(D; \mathbb{X})$ if the following conditions are met:

(A) for every $z \in \mathbb{X}$ the function $[\varphi]_z : x \in D \mapsto r(\varphi(x), z)$ is of class $W_{p,\mathrm{loc}}^1(D)$;

(B) the family of gradients $(\nabla[\varphi]_z)_{z\in\mathbb{X}}$ has a majorant in $L_{p,\mathrm{loc}}(D)$; i.e., there exists $g \in L_{p,\mathrm{loc}}(D)$ independent of z such that $|\nabla[\varphi]_z(x)| \le g(x)$ for almost all $x \in D$.

If $\mathbb{X} = \mathbb{M}'$ is another Riemannian manifold with distance d' then we obtain a definition of a mapping of Sobolev class between different Riemannian manifolds and denote this class by $W_{p,\mathrm{loc}}^1(D; \mathbb{M}')$. In this case it is convenient to use an equivalent description of a mapping of Sobolev class; see [16, 17] for instance. A mapping $\varphi : D \mapsto \mathbb{M}'$ belongs to $W_{p,\mathrm{loc}}^1(D; \mathbb{M}')$ if and only if we can change φ on a negligible set so that

(a) $D \ni x \mapsto [\varphi]_z(x) = d'(\varphi(x), z)$ belongs to $L_{p,\mathrm{loc}}(D)$ for every point $z \in \mathbb{M}'$;

(b) $\varphi : D \to \mathbb{M}'$ is absolutely continuous on the integral lines of the basis vector fields; i.e., for every open bounded set U with $\overline{U} \subset D$, every tuple X_j, for $j = 1, \dots, n$, of basis vector fields on U, and the foliation Γ_k of U determined by X_k, the mapping φ is absolutely continuous on $\gamma \cap U \in \Gamma_k$ with respect to the one-dimensional Hausdorff measure for $d\tau$-almost all curves $\gamma \in \Gamma_k$, for $k = 1, \dots, n$, where γ is the integral line $\exp tX_k(x)$ of X_k beginning at $x \in U$, while the measure[1)] $d\tau$ on the foliation Γ_k equals the contraction $i(X_k)$ of the vector field X_k with the volume form ω;

(c) the derivative $X_k\varphi(x) = \frac{\partial}{\partial t}\varphi(\exp tX_k(x))|_{t=0}$ exists and belongs to $T_{\varphi(x)}\mathbb{M}'$ a.e. on some open set U with $\overline{U} \subset D$ and, moreover, $|X_k\varphi| \in L_p(U)$ for all $k = 1, \dots, n$.

If $\varphi : D \mapsto \mathbb{M}'$ satisfies only conditions (a) and (b) then we say that φ *belongs to* $ACL(D)$. For this φ the derivatives $X_k\varphi \in T_{\varphi(x))}\mathbb{M}'$ along the vector fields X_k, for $k = 1, \dots, n$, exist a.e. in U; see [11, 16, 17].

---

1) More exactly, $d\tau$ is a measure on every smooth $(n-1)$-dimensional surface S transversal to the foliation Γ_k. The stated property means the following: Given two $(n-1)$-dimensional surfaces S_1 and S_2 transversal to Γ_k such that each curve of the part Γ_k' of Γ_k meets both S_1 and S_2, the $d\tau$-measure of $\Gamma_k' \cap S_1$ is zero if and only if the $d\tau$-measure of $\Gamma_k' \cap S_2$ is too. This clarification applies to [10, p. 64] as well.

The matrix whose columns are the vectors $X_k\varphi(x)$ for $j = 1, \dots, n$ determines the linear operator $D\varphi(x) : T_x\mathbb{M} \mapsto T_{\varphi(x)}\mathbb{M}'$ from the tangent space $T_x\mathbb{M}$ into the tangent space $T_{\varphi(x))}\mathbb{M}'$ for almost all x and is called the (*formal*) *differential* of φ at x. Denote by $|D\varphi|(x)$ the norm of $D\varphi(x)$. In the case $\dim\mathbb{M} = \dim\ M'$ the Jacobian $J(x,\varphi) = \det D\varphi(x)$ amounts to the determinant of the matrix $D\varphi(x)$. In this case the formal differential $D\varphi(x)$ coincides a.e. with the approximative differential of φ (see [11]).

We have the following change-of-variables formula.

Proposition 8 [18]. *Suppose that a mapping $\varphi : A \to \mathbb{M}'$ of a measurable set, where $A \subset \mathbb{M}$, has an approximative partial derivative on A. Then there exists a negligible set $\Sigma_\varphi \subset A$ such that the change-of-variables formula in the Lebesgue integral for every nonnegative measurable function $f : A \to \mathbb{R}$ is of the form*

$$\int\limits_A f(x)|J(x,\varphi)|\,d\omega(x) = \int\limits_{\mathbb{M}'} \Big(\sum_{x\in\varphi^{-1}(y)\cap(A\setminus\Sigma_\varphi)} f(x) \Big)\,d\nu(y). \tag{2}$$

If φ has the Luzin $\mathscr{N}$-property then we may assume that Σ_φ is empty.

1.2. John domains and Poincaré's inequality. In this subsection we apply the Poincaré inequality in John domains, as proved in [19] with the previous results established in [20–23]. Moreover, below we need a certain special modification of this inequality; see Lemma 12.

Definition 9 [24]. A proper domain $\Omega \subset \mathbb{M}$ is called a *John domain* of type $J_{\alpha,\beta}$ with $0 < \alpha \le \beta$, in symbols $\Omega \in J_{\alpha,\beta}$, whenever there is $x_0 \in \Omega$ such that we can connect each $x \in \Omega$ to x_0 by a rectifiable curve γ lying in Ω and satisfying the conditions: If $s \in [0,l]$ is the natural parametrization of γ with $\gamma(0) = x$ and $\gamma(l) = x_0$ then

$$l \le \beta \text{ and } \operatorname{dist}(\gamma(s), \partial\Omega) \ge \frac{\alpha s}{l} \text{ for all } s \in [0,l].$$

Lemma 10 [10, Lemma 3]. *Consider an arbitrary domain D in $\mathbb{M}$ and two balls B_0 and B_1 in D. Then there is a John domain $\Omega \Subset D$ of type $J_{\alpha,\beta}$ for suitable α and β depending on D and some ball that includes both balls.*

Remark 11. The proof of Lemma 3 of [10] yields the following property: If $\operatorname{dist}(\partial D, B_0) > 0$ and $\operatorname{dist}(\partial D, B_1) > 0$ then for a sufficiently small parameter $\lambda > 0$ we can construct a complementary John domain Ω_λ such that $\Omega \Subset \Omega_\lambda \Subset D$; namely, Ω and Ω_λ are bounded and, moreover, $\operatorname{dist}(\partial D, \Omega_\lambda) > 0$ and $\operatorname{dist}(\partial\Omega, \Omega_\lambda) > 0$. Indeed, the idea of the proof in [10] amounts to constructing a rectifiable curve Γ in D connecting the centers of B_0 and B_1. The John domain Ω is constructed as the collection of balls centered on Γ of radius at most $\frac{1}{2}\operatorname{dist}(\Gamma, \partial D)$. We can construct Ω_λ as the union of concentric balls by appropriately increasing the radii to any value in the interval $\big(\frac{1}{2}\operatorname{dist}(\Gamma, \partial D), \frac{3}{4}\operatorname{dist}(\Gamma, \partial D)\big)$.

Lemma 12 [10, Lemma 4]. *Consider a compactly embedded John domain $U \Subset M$ of type $J(\alpha,\beta)$ and a measurable subset $F \subset U$ of positive measure $|F| > 0$. If $p \le q \le \frac{np}{n-p}$ with $p < n$ then all $u(x) \in W^1_p(U)$ with $u|_F = 0$ satisfy*

$$\bigg(\int\limits_U |u(x)|^q\,d\omega\bigg)^{\frac{1}{q}} \le \frac{|U|^{\frac{1}{q}}}{|F|^{\frac{1}{q}}} C_U \left(\frac{\alpha}{\beta}\right)^n \operatorname{diam}(U)^{1-\frac{n}{p}+\frac{n}{q}} \bigg(\int\limits_U |\nabla u(x)|^p\,d\omega\bigg)^{\frac{1}{p}};$$

furthermore, $C_U > 0$ is independent of u, α, and β but depends on the constant in the doubling condition[2] *on U; for details, see Section 6 of* [19].

1.3. Properties of mappings of class IL^1_p. The following properties of mappings of class IL^1_p are established in [10]:

---

[2] Since $U \Subset \mathbb{M}$, there exist positive reals r_0 and M such that $|B(x,2r)| \le M|B(x,r)|$ for all $x \in U$ and $r \in (0,r_0)$ (see [25] for instance).

Proposition 13. (1) *We may assume that the domain of φ is $T = \bigcup_k T_k$ with $|D \setminus T| = 0$, where $\{T_k\}$ is an inclusion increasing sequence of bounded sets of positive measure consisting of points of positive density* [10, *Lemma* 13, *Remarks* 9, 10].

(2) *The mapping φ is continuous on each T_k* [10, *Lemma* 13].

(3) *On T the mapping φ enjoys the Luzin $\mathscr{N}$- and $\mathscr{N}^{-1}$-properties* [10, *Lemmas* 9 *and* 17].

(4) *$\varphi : T \to D'$ is injective* [10, *Proposition* 8].

(5) *$\varphi(T)$ is dense in D' and $|D' \setminus \varphi(T)| = 0$* [10, *Lemma* 22].

The operator φ^* of (1) extends to $L^1_p(D)$ so that the properties of φ^* are preserved.

Lemma 14 [10, Lemma 11]. *Take two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$ in two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same topological dimension $n \geq 2$. If a mapping $\varphi : D \to D'$ lies in IL^1_p with $p \in [1, \infty)$ then $\varphi^* : L^1_p(D') \cap C^\infty(D') \to L^1_p(D)$ extends by continuity to the operator $\widetilde{\varphi^*} : L^1_p(D') \to L^1_p(D)$ with the following properties:*

(1) *We can evaluate $\widetilde{\varphi^*} : L^1_p(D') \to L^1_p(D)$ on the classes $[f] \in L^1_p(D')$ as*

$$\widetilde{\varphi^*}([f]) = \begin{cases} f \circ \varphi \text{ for } p \leq n, \text{ where } f \text{ is a representative of } [f], \\ \tilde{f} \circ \varphi \text{ for } p > n, \text{ where } \tilde{f} \text{ is a continuous representative of } [f]; \end{cases}$$

(2) $K^{-1}\big\|f \mid L^1_p(D')\big\| \leq \big\|\widetilde{\varphi^*}(f) \mid L^1_p(D)\big\| \leq K\big\|f \mid L^1_p(D')\big\|$;

(3) *$\widetilde{\varphi^*} : L^1_p(D') \to L^1_p(D)$ is an isomorphism.*

2. The Space $L^1_{n,F}$

Henceforth we fix two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same topological dimension $n \geq 2$, two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$ in $\mathbb{M}$ and $\mathbb{M}'$, and a mapping $\varphi : D \to D'$ of class IL^1_n. Every mapping of this type enjoys the properties of Subsection 1.3.

Fix $k_0 \in \mathbb{N}$ and a closed set $F \subset T_{k_0}$ of positive measure without isolated points. We may assume that $F \subset B_F$, where $B_F \subset D$ is some ball. By Remark 14 of [10], we can also assume that $\varphi : F \to \varphi(F)$ is a bi-Lipschitz mapping. This choice ensures that the image of $\varphi(F)$ inherits the properties of F. Namely, $\varphi(F)$ is closed, lacks isolated points, and the measure of $\varphi(F)$ is positive.

Consider the collection of functions

$$L^1_{n,F}(D) = \big\{u \in L^1_n(D) : u(x) = 0 \text{ for almost all } x \in F\big\}.$$

Observe that $L^1_{n,F}(D)$ is a closed subspace of $L^1_n(D)$ and the norm $\big\|u \mid L^1_{n,F}(D)\big\| = \big\|u \mid L^1_n(D)\big\|$ makes $L^1_{n,F}(D)$ a normed space, which is easy to show by using Lemma 12. Consequently, $L^1_{n,F}(D)$ is a Banach space.

By analogy, define the Banach space

$$L^1_{n,\varphi(F)}(D') = \big\{v \in L^1_n(D') : v(y) = 0 \text{ for almost all } y \in \varphi(F)\big\}.$$

Using Proposition 13 and Lemma 14, we can verify that $f \in L^1_{n,\varphi(F)}(D')$ if and only if $f \circ \varphi \in L^1_{n,F}(D)$. Consequently,

$$\varphi^*_F : L^1_{n,\varphi(F)}(D') \to L^1_{n,F}(D), \quad \varphi^*_F(f) = f \circ \varphi, \quad f \in L^1_{n,\varphi(F)}(D')$$

is an isomorphism. The spaces $L^1_{n,F}$ will enable us to establish the existence of a quasicontinuous representative for φ.

Put $D_F = D \setminus F$ and $D'_F = D' \setminus \varphi(F)$.

3. Capacity

This section collects the main properties of capacity in Sobolev spaces, which help us elaborate the properties of φ.

3.1. Capacity in $L^1_{n,F}(D)$ and its properties. Let us present the concept of capacity in $L^1_{n,F}(D)$ and the properties of capacity we require below. The properties of capacity stated in Subsections 3.1 and 3.2 are similar to the properties of capacity in [3, 26–28; 5, § 6; 13, § 6; 14], where they are justified in other spaces of functions. For the reader's convenience, we cite the articles that contain statements similar in meaning to the formulas in this article. The formulas claimed here can be proved by analogy with the proofs of their prototypes.

Refer as the *capacity* $\operatorname{Cap}(K; L^1_{n,F}(D))$ of a compact set $K \subset D_F$ in $L^1_{n,F}(D)$ to

$$\operatorname{Cap}(K; L^1_{n,F}(D)) = \inf \|g \mid L^1_{n,F}(D)\|^n, \tag{3}$$

where the greatest lower bound is taken over all continuous functions $g \in L^1_{n,F}(D)$ with $g \geq 1$ on K.

Remark 15. The greatest lower bound in (3) remains the same when we consider nonnegative continuous functions in $L^1_{n,F}(D)$ with $g > 1$ on K.

Given $E \subset D_F$, the *inner capacity* of E equals

$$\underline{\operatorname{Cap}}(E; L^1_{n,F}(D)) = \sup\{\operatorname{Cap}(K; L^1_{n,F}(D)) : K \subset E \text{ with } K \text{ compact}\},$$

while the *outer capacity* of E equals

$$\overline{\operatorname{Cap}}(E; L^1_{n,F}(D)) = \inf\{\underline{\operatorname{Cap}}(U; L^1_{n,F}(D)) : E \subset U,\ U \subset D_F \text{ is open}\}.$$

The following lemma states the main properties of capacity.

Lemma 16 (cf. [5, Lemma 6.1; 13, Theorem 6.1; 14]). *Capacity in $L^1_{n,F}(D)$ enjoys the properties:*

(1) *If $K \subset D_F$ is a compact set then for every $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset D_F$ such that $K \subset U_\varepsilon$ and for every compact set $K' \subset U_\varepsilon$*

$$\operatorname{Cap}(K'; L^1_{n,F}(D)) \leq \operatorname{Cap}(K; L^1_{n,F}(D)) + \varepsilon.$$

(2) *If $E \subset E'$ then*

$$\underline{\operatorname{Cap}}(E; L^1_{n,F}(D)) \leq \underline{\operatorname{Cap}}(E'; L^1_{n,F}(D)), \quad \overline{\operatorname{Cap}}(E; L^1_{n,F}(D)) \leq \overline{\operatorname{Cap}}(E'; L^1_{n,F}(D)).$$

(3) *If $K_1, K_2 \subset D_F$ are two compact sets then*

$$\operatorname{Cap}(K_1 \cup K_2; L^1_{n,F}(D)) + \operatorname{Cap}(K_1 \cap K_2; L^1_{n,F}(D)) \leq \operatorname{Cap}(K_1; L^1_{n,F}(D)) + \operatorname{Cap}(K_2; L^1_{n,F}(D)).$$

(4) *Take $E_1, \dots, E_k \subset D_F$, $F_i \subset E_i$, $\overline{\operatorname{Cap}}\Big(\bigcup_{i=1}^k F_i; L^1_{n,F}(D)\Big) < \infty$. Then*

$$\overline{\operatorname{Cap}}\Big(\bigcup_{i=1}^k E_i; L^1_{n,F}(D)\Big) - \overline{\operatorname{Cap}}\Big(\bigcup_{i=1}^k F_i; L^1_{n,F}(D)\Big) \leq \sum_{i=1}^k \big(\overline{\operatorname{Cap}}(E_i; L^1_{n,F}(D)) - \overline{\operatorname{Cap}}(F_i; L^1_{n,F}(D))\big).$$

(5) *For every increasing sequence $E_1 \subset E_2 \subset \dots \subset E_k \subset \dots \subset D_F$ we have*

$$\overline{\operatorname{Cap}}\Big(\bigcup_{k=1}^\infty E_k; L^1_{n,F}(D)\Big) = \lim_{k\to\infty} \overline{\operatorname{Cap}}(E_k; L^1_{n,F}(D)).$$

(6) *Take a sequence* $\{E_k\} \subset D_F$, $k \in \mathbb{N}$, *and put* $E = \bigcup_{k=1}^{\infty} E_k$. *Then*

$$\overline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big) \le \sum_{k=1}^{\infty} \overline{\mathrm{Cap}}\big(E_k; L^1_{n,F}(D)\big).$$

A set E is called *capacitable* whenever

$$\underline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big) = \overline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big).$$

By Lemma 16, the capacity of $L^1_{n,F}(D)$ is a Choquet capacity [29]. This implies that all analytic and, in particular, Borel sets are capacitable [29].

Say that some property holds *quasieverywhere* or for quasiall points of a set whenever it holds everywhere but a subset of capacity zero.

Definition 17. A function $f \in L^1_{n,F}(D)$ is called *refined* whenever there exists a sequence $\{f_s\}$, $s \in \mathbb{N}$, of functions in $L^1_{n,F}(D) \cap C(D)$ such that

(1) $\|f - f_s \mid L^1_{n,F}(D)\| \to 0$ as $s \to \infty$;

(2) for every positive $\varepsilon > 0$ there is an open set $U_\varepsilon \subset D_F$ with $\overline{\mathrm{Cap}}(U_\varepsilon) < \varepsilon$ and f_s converges to f uniformly on $D_F \setminus U_\varepsilon$.

Remark 18. (1) Each element of $L^1_{n,F}(D)$ includes a refined function; see Corollary 6.4 of [5].

(2) Each sequence of refined functions converging in $L^1_{n,F}(D)$ to a refined function f includes a subsequence converging to f quasieverywhere; see Corollary 6.7 of [5].

Lemma 19 (cf. [5, Lemma 6.5; 13, Lemma 6.4]). *Given* $E \subset D_F$ *and a refined* $f \in L^1_{n,F}(D)$ *with* $|f(x)| \ge \alpha > 0$ *quasieverywhere on* E, *we have*

$$\overline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big) \le \frac{\|f \mid L^1_{n,F}(D)\|^n}{\alpha^n}.$$

Corollary 20. *Two refined functions belonging to the same element of* $L^1_{n,F}(D)$ *coincide quasieverywhere on* D_F.

Proof. Take two refined functions f and g belonging to the same element of $L^1_{n,F}(D)$. In particular,

$$\|f - g \mid L^1_{n,F}(D)\| = 0. \tag{4}$$

Put $\Sigma = \{x \in D_F : f(x) \ne g(x)\}$ and $\Sigma_k = \{x \in D_F : |f(x) - g(x)| > 2^{-k}\}$. Then

$$\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k.$$

By Lemma 19 and (4), $\overline{\mathrm{Cap}}\big(\Sigma_k; L^1_{n,F}(D)\big) = 0$ for every $k \in \mathbb{N}$. Since outer capacity is countably semiadditive, see Lemma 16, we infer that $\overline{\mathrm{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) = 0$. □

Definition 21. Given $E \subset D_F$, put

$$A(E) = \big\{f \in L^1_{n,F}(D) : \text{ the refined representative of } \tilde{f}(x) \text{ is at least 1 quasieverywhere on } E\big\}.$$

Each function $f \in A(E)$ is called *admissible* for the set E.

Lemma 22 (cf. [14, Lemma 6.5]). *For all* $E \subset D_F$ *the set* $A(E)$ *of admissible functions is weakly closed and convex in* $L^1_{n,F}(D)$.

Lemma 22 implies the next statement.

Corollary 23. *If $E \in D_F$ and $A(E) \neq \varnothing$ then there exists a unique element $f_E \in A(E)$ such that*

$$\big\| f_E \mid L^1_{n,F}(D) \big\| = \inf\big\{ \| f \mid L^1_{n,F}(D) \| : f \in A(E) \big\}.$$

Proof. Denote the right-hand side by I and take a sequence $\{f_k\}_{k\in\mathbb{N}} \subset A(E)$ with $I = \lim_{k\to\infty} \big\| f_k \mid L^1_{n,F}(D) \big\|$. Extract from $\{f_k\}_{k\in\mathbb{N}}$ a weakly converging subsequence f_{k_j} and denote by f_E its weak limit: $f_E \underset{w}{=} \lim_{j\to\infty} f_{k_j}$. Lemma 22 yields $f_E \in A(E)$. The uniqueness follows by the standard method from the uniform convexity of the norm in $L^1_{n,F}(D)$. □

Corollary 24 (cf. [13, Corollary 6.5]). *Given an increasing sequence $\{E_m\}_{m\in\mathbb{N}}$ with $A(E_m) \neq \varnothing$ for all m, put $E = \bigcup_{m=1}^{\infty} E_m$. Then*

$$A(E) = \bigcap_{m=1}^{\infty} A(E_m), \quad \lim_{m\to\infty} \big\| f_{E_m} \mid L^1_{n,F}(D) \big\| = \inf\big\{ \big\| f \mid L^1_{n,F}(D) \big\| : f \in A(E) \big\}.$$

Theorem 25 (cf. [5, Theorem 6.11; 13, Theorem 6.4]). *For every $E \subset D_F$ we have*

$$\overline{\operatorname{Cap}}\big(E; L^1_{n,F}(D)\big) = \inf\big\{ \big\| f \mid L^1_{n,F}(D) \big\|^n : f \in A(E) \big\}.$$

If $A(E) \neq \varnothing$ then there is f_E such that

$$\overline{\operatorname{Cap}}\big(E; L^1_{n,F}(D)\big) = \big\| f_E \mid L^1_{n,F}(D) \big\|^n.$$

The function f_E in Theorem 25 is called a *capacity function* for E.

Lemma 26. *If $f \in L^1_{n,F}(D)$ is a refined function then*

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(z)\, d\omega(z) \tag{5}$$

for quasiall $x \in D_F$.

Proof. Since the result is local, we can work in an arbitrary ball $2B \subset D_F$ within some coordinate neighborhood. Using Lemma 12, we can conclude that the refined function $f \in L^1_{n,F}(D)$ lies in $W^1_n(2B)$. Since the Euclidean metric is equivalent to the Riemannian metric in the ball $2B$, Poincaré's inequality holds in $2B$. Hence, we can apply the conclusion of [30] to obtain the pointwise estimate

$$|f(x) - f(y)| \le c d(x,y)(g(x) + g(y)) \tag{6}$$

for all $x, y \in B \setminus \Sigma$, where $\Sigma \subset B$ is a negligible set, while $g \in L_n(B)$. Theorem 4.5 of [31] shows that for every refined function satisfying (6) we have (5) for quasiall $x \in B$. □

Definition 27. A function f defined quasieverywhere on D_F is called *quasicontinuous* whenever for every $\varepsilon > 0$ we can find an open set $U_\varepsilon \subset D_F$ such that $\overline{\operatorname{Cap}}\big(U_\varepsilon; L^1_{n,F}(D)\big) < \varepsilon$ and the restriction of f to $D_F \setminus U_\varepsilon$ is continuous.

Remark 28. Proposition 34 shows that a function of class $L^1_{n,F}(D)$ is quasicontinuous if and only if it is refined.

Definition 29. Given a measurable set $E \subset D_F$, call $x \in D_F$ a *nonzero density point* for E whenever

$$\varlimsup_{r\to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} > 0.$$

Denote the collection of all points $x \in D_F$ of nonzero density for E by $\widetilde{E}$.

Lemma 30 (cf. [5, Proposition 6.16; 13, Theorem 6.5]). *Consider a set $E \subset D_F$ of positive measure. If $f \in L^1_{n,F}(D)$ is quasicontinuous and $f(x) \geq g(x)$ for almost all $x \in E$, where $g : E \cup \widetilde{E} \to \mathbb{R}$ is a lower semicontinuous function, then $f(x) \geq g(x)$ for quasiall $x \in \widetilde{E}$.*

Proof. Since f is quasicontinuous, we see that for every $\varepsilon > 0$ there exists an open set U_ε such that $\overline{\mathrm{Cap}}\big(U_\varepsilon; L^1_{n,F}(D)\big) < \varepsilon$ and f is continuous on $D_F \setminus U_\varepsilon$. Take a nonnegative capacity function f_m for $U_{\frac{1}{m}}$. Since $\big\|f_m \mid L^1_{n,F}(D)\big\| \to 0$ as $m \to \infty$, by passing to a subsequence we may assume that

$$\lim_{m\to\infty} f_m(x) = 0 \quad \text{for quasiall } x \in D_F. \tag{7}$$

By Lemma 26, for every m we have

$$f_m(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f_m(z)\, d\omega(z) \quad \text{for quasiall } x \in D_F. \tag{8}$$

Thus, (7) and (8) hold for quasiall $x \in \widetilde{E}$. Take $x \in \widetilde{E}$ so that (7) and (8) hold. Since x is a point of positive density for E, there is a real $\rho_0 > 0$ such that $\frac{|E\cap B(x,\rho)|}{|B(x,\rho)|} > \delta > 0$ for all $\rho \in (0, \rho_0)$. Verify that for all sufficiently large m we have

$$|U_{\frac{1}{m}} \cap E \cap B(x,\rho)| < |E \cap B(x,\rho)|$$

for ρ sufficiently small.

Indeed, since $\lim_{m\to\infty} f_m(x) = 0$ for m sufficiently large; therefore, $f_m(x) < \delta$. Furthermore,

$$\begin{aligned}\overline{\lim_{\rho\to 0}} \frac{|U_{\frac{1}{m}} \cap E \cap B(x,\rho)|}{|E \cap B(x,\rho)|} &\leq \overline{\lim_{\rho\to 0}} \frac{|B(x,\rho)|}{|E\cap B(x,\rho)|}\, \overline{\lim_{\rho\to 0}} \frac{1}{|B(x,r)|} \int\limits_{U_{\frac{1}{m}}\cap E\cap B(x,\rho)} f_m(y)\, d\omega(y) \\ &\leq \frac{1}{\delta} \lim_{\rho\to 0} \frac{1}{|B(x,\rho)|} \int\limits_{B(x,\rho)} f_m(y)\, d\omega(y) = \delta^{-1} f_m(x) < 1.\end{aligned}$$

Thus, there are $m(x) \in \mathbb{N}$ and $\rho(x) > 0$ such that for all $m > m(x)$ and $\rho \in (0, \rho(x))$ the measure of $V_\rho = (E \cap B(x,\rho)) \setminus U_{\frac{1}{m}}$ is positive.

Appreciating the continuity of f on $D_F \setminus U_{\frac{1}{m}}$ and the property that $f(y) \geq g(y)$ for almost all $y \in E$, we obtain

$$f(x) = \lim_{\rho\to 0} \frac{1}{|V_\rho|} \int\limits_{V_\rho} f(y)\, dy \geq \underline{\lim_{\rho\to 0}} \frac{1}{|V_\rho|} \int\limits_{V_\rho} g(y)\, dy \geq g(x). \quad \square$$

Corollary 31 (cf. [5, Corollary 6.17; 13, Corollary 6.7]). *Consider a measurable set $E \subset D_F$ of positive measure. If two quasicontinuous functions $f_1, f_2 \in L^1_{n,F}(D)$ coincide almost everywhere on E then they coincide quasieverywhere on $\widetilde{E}$.*

Proof. Indeed, putting $f(x) = f_1(x) - f_2(x)$ or $f(x) = f_2(x) - f_1(x)$ in Lemma 30 with $g \equiv 0$ yields $f_1(x) - f_2(x) \geq 0$ or $f_2(x) - f_1(x) \geq 0$ quasieverywhere on $\widetilde{E}$. Consequently, $f_1(x) = f_2(x)$ quasieverywhere on $\widetilde{E}$. $\square$

Corollary 32 (cf. [5, Corollary 6.19]). $\overline{\mathrm{Cap}}\big(E \cup \widetilde{E}; L^1_{n,F}(D)\big) = \overline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big)$ *for every $E \subset D_F$.*

Proof. Claim (2) of Lemma 16 yields

$$\overline{\mathrm{Cap}}\big(E \cup \widetilde{E}; L^1_{n,F}(D)\big) \geq \overline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big). \tag{9}$$

If $\overline{\mathrm{Cap}}\big(E; L^1_{n,F}(D)\big) = \infty$ then the equality is obvious.

Assume that $\overline{\mathrm{Cap}}(E; L^1_{n,F}(D)) < \infty$, Then, by Theorem 25 and Remark 18, there exists a refined function f_E such that $\overline{\mathrm{Cap}}(E; L^1_{n,F}(D)) = \|f_E \mid L^1_{n,F}(D)\|^n$ and $f(x) \geq 1$ quasieverywhere on E. Lemma 30 shows that $f(x) \geq 1$ quasieverywhere on $\widetilde{E}$. Therefore, $f_E \in A(E \cup \widetilde{E})$. Consequently,

$$\overline{\mathrm{Cap}}(E \cup \widetilde{E}; L^1_{n,F}(D)) \leq \|f_E \mid L^1_{n,F}(D)\|^n = \overline{\mathrm{Cap}}(E; L^1_{n,F}(D)),$$

which together with (9) implies the required equality. □

Corollary 33 [5, Corollary 6.20]. *Under the hypotheses of Lemma* 30*, if $f(x) = g(x)$ almost everywhere on some set $E \subset D_F$, where f is a quasicontinuous function on D_F, while g is a continuous function on $E \cup \widetilde{E}$, then $f(x) = g(x)$ for quasiall $x \in \widetilde{E}$.*

Proof. The claim follows immediately from Lemma 30 because g is in particular lower semicontinuous on $E \cup \widetilde{E}$. □

Proposition 34. *Definitions* 17 *and* 27 *are equivalent: every refined function is quasicontinuous; and, conversely, every quasicontinuous function of class $L^1_{n,F}(D)$ is refined.*

Proof. Indeed, if f is a refined function then by condition (2) of Definition 17 for every $\varepsilon > 0$ there is an open set U_ε of capacity less than $\varepsilon > 0$ such that the sequence of continuous functions $f_n \in L^1_{n,F}(D)$ converges uniformly on the complement $D_F \setminus U_\varepsilon$. Consequently, f is continuous on $D_F \setminus U_\varepsilon$.

Assume that $f \in L^1_{n,F}(D)$ is quasicontinuous. Then by Remark 18 there exists a refined function $\tilde{f}$ coinciding with f almost everywhere in D_F. By the above argument, $\tilde{f}$ is quasicontinuous, and so Corollary 31 implies that f and $\tilde{f}$ coincide quasieverywhere. It remains to observe that each function coinciding quasieverywhere with a refined function is refined itself. □

3.2. Capacity in the space of potentials. Consider an open connected set Ω in $\mathbb{R}^n$. Refer as the *capacity* of a compact set $K \subset \Omega$ in $W^1_n(\Omega)$ to

$$\mathrm{cap}(K; W^1_n(\Omega)) = \inf \|g \mid W^1_n(\Omega)\|^n,$$

where the greatest lower bound is taken over all continuous functions $g \in W^1_n(\Omega)$ with $g \geq 1$ on K. Given $E \subset \Omega$, the *inner capacity* of E equals

$$\underline{\mathrm{cap}}(E; L^1_n(\Omega)) = \sup\{\mathrm{cap}(K; W^1_n(\Omega)) : K \subset E \text{ with } K \text{ compact}\},$$

while the *outer capacity* of E equals

$$\overline{\mathrm{cap}}(E; W^1_n(\Omega)) = \inf\{\underline{\mathrm{cap}}(U; W^1_n(\Omega)) : E \subset U \text{ with } U \text{ open}\}.$$

The properties of capacity in $W^1_n(\Omega)$ are similar to the properties of capacity in $L^1_{n,F}(D)$ established above (see [5] for instance).

The *space of Bessel potentials* in the Euclidean space $\mathbb{R}^n$ is the space $S^\alpha_p(\mathbb{R}^n)$ of functions of the form

$$g(x) = J_\alpha * f(x) = \int\limits_{\mathbb{R}^n} J_\alpha(x-y) f(y)\, dy,$$

where $f \in L_p(\mathbb{R}^n)$ with $p \in (1, \infty)$ and J_α is the Bessel kernel [25] on $\mathbb{R}^n$ with $\alpha \in (0, \infty)$. Define the norm in the space of potentials as

$$\|g \mid S^\alpha_p(\mathbb{R}^n)\| = \|f \mid L_p(\mathbb{R}^n)\|.$$

If $\alpha = k$ is a positive integer then $S^k_p(\mathbb{R}^n)$ coincides with the Sobolev space $W^k_p(\mathbb{R}^n)$ [25]. Henceforth we are interested in the case $\alpha = 1$ because $S^1_p(\mathbb{R}^n)$ coincides with $W^1_p(\mathbb{R}^n)$.

The *Bessel capacity* of an arbitrary subset $E \subset \mathbb{R}^n$ is defined as

$$\mathrm{cap}(E; S^1_p(\mathbb{R}^n)) = \inf\Big\{\int\limits_{\mathbb{M}} f(y)^p\, dy : J_1 * f(x) \geq 1 \text{ at } x \in E\Big\}.$$

For more details see [26] which shows that the capacity on $S^1_p(\mathbb{R}^n)$ is an outer capacity.

Proposition 35 [26, Corollary 2]. *For $x \in \mathbb{R}^n$ and $r < 1$ the Bessel capacity of balls satisfies the equivalence* $\operatorname{cap}\big(B(x,r); S^1_n(\mathbb{R}^n)\big) \sim \big(\log \frac{2}{r}\big)^{1-n}$.

Remark 36. By the equivalence [25] of the norms of $S^1_n(\mathbb{R}^n)$ and $W^1_n(\mathbb{R}^n)$, $\operatorname{cap}\big(E; S^1_n(\mathbb{R}^n)\big)$ and $\overline{\operatorname{cap}}\big(E; W^1_n(\mathbb{R}^n)\big)$ are also comparable; i.e., there exist constants m and M with

$$m\,\overline{\operatorname{cap}}\big(E; W^1_n(\mathbb{R}^n)\big) \le \operatorname{cap}\big(E; S^1_n(\mathbb{R}^n)\big) \le M\,\overline{\operatorname{cap}}\big(E; W^1_n(\mathbb{R}^n)\big).$$

In particular, $\overline{\operatorname{cap}}\big(E; W^1_n(\mathbb{R}^n)\big) = 0$ if and only if $\operatorname{cap}\big(E; S^1_n(\mathbb{R}^n)\big) = 0$.

Lemma 37. *For $\Sigma \subset D_F$ the following two properties are equivalent:*

$$\overline{\operatorname{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) = 0 \quad \textit{and} \quad \overline{\operatorname{cap}}\big(\Sigma; W^1_n(\mathbb{M})\big) = 0.$$

Proof. Suppose that $\overline{\operatorname{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) = 0$. Since capacity is countably semiadditive, we may assume that Σ lies in some ball $B_\Sigma \subset D$, while F lies in some ball $B_F \subset D$. Furthermore, we have $\operatorname{dist}(B_F, B_\Sigma) > 0$, $\operatorname{dist}(\partial D, B_F) > 0$, and $\operatorname{dist}(\partial D, B_\Sigma) > 0$.

Since $\overline{\operatorname{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) = 0$, there is a nested sequence of open sets $\{U_k\}$ such that

$$B_\Sigma \supset U_1 \supset U_2 \supset \cdots \supset \Sigma \quad \text{and} \quad \overline{\operatorname{Cap}}\big(U_k; L^1_{n,F}(D)\big) \le \frac{1}{2^k}.$$

By Theorem 25, there is a sequence of functions $h_k \in L^1_{n,F}(D)$ such that $h_k \ge 1$ quasieverywhere on U_k and $\big\|h_k \mid L^1_{n,F}(D)\big\| \le 1/2^k$. Passing to the cutoff $\min(1, h_k)$, we may assume that $h_k = 1$ everywhere on U_k.

Consider a John domain $\Omega \subset D$ that includes the balls B_Σ and B_F with $\operatorname{dist}(\partial\Omega, \partial D) > 0$; see Lemma 10. Given $\delta > 0$ sufficiently small, choose an additional John domain $\Omega_\delta \supset \Omega$ with $\operatorname{dist}(\partial\Omega, \partial\Omega_\delta) \ge \delta$ and $\operatorname{dist}(\partial D, \partial\Omega_\delta) \ge \delta$ (see Remark 11). Poincaré's inequality yields $\|h_k \mid L_n(\Omega_\delta)\| \le C\big\|h_k \mid L^1_n(\Omega_\delta)\big\|$ (see Lemma 12). Therefore, passing to a subsequence, we may assume that $h_k \to 0$ a.e. on Ω_δ and $\nabla h_k \to 0$ a.e. on Ω_δ. Choose a cutoff $\eta \in C^\infty_0(\mathbb{M})$ such that $\eta = 1$ on Ω and $\eta = 0$ on $\mathbb{M} \setminus \Omega_\delta$. Then the product $\eta h_k \in W^1_n(\mathbb{M})$ satisfies

$$\eta h_k(x) = \begin{cases} h_k(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{M} \setminus \Omega_\delta. \end{cases}$$

Furthermore, $|\nabla(\eta h_k)| \le |(\nabla\eta)h_k| + |\eta\nabla h_k|$ and $\big\|\eta h_k \mid W^1_n(\mathbb{M})\big\| \to 0$ as $k \to \infty$. Consequently, $\operatorname{cap}\big(U_k; W^1_n(\mathbb{M})\big) \to 0$ as $k \to \infty$, whence

$$\overline{\operatorname{cap}}\big(\Sigma; W^1_n(\mathbb{M})\big) = 0. \tag{10}$$

Suppose that (10) holds. Then there is an inclusion-decreasing sequence $\{W_k\} \subset B_\Sigma$ of open sets, which include Σ, such that $\operatorname{cap}\big(\Sigma; W^1_n(\mathbb{M})\big) \le 1/2^{k+1}$, and a sequence of $u_k \in W^1_n(\mathbb{M})$ such that $u_k = 1$ quasieverywhere on W_k and $\big\|u_k \mid W^1_n(\mathbb{M})\big\|^n \le 1/2^k$.

Define the cutoff $\eta' \in C^\infty_0(\mathbb{M})$ such that $\eta' = 1$ on B_Σ and $\eta' = 0$ on $\mathbb{M} \setminus \lambda B_\Sigma$, with $\lambda > 1$, for which $D \supset \lambda B_\Sigma \supset B_\Sigma$ and $\lambda B_\Sigma \cap B_F = \varnothing$. Then the functions $\eta' \cdot u_k = f_k \in L^1_{n,F}(D)$ satisfy $f_k = 1$ on W_k and $\big\|f_k \mid L^1_{n,F}(D)\big\|^n \le c/2^k$, where c is a constant independent of k. This yields $\operatorname{Cap}\big(\Sigma; L^1_{n,F}(\mathbb{M})\big) = 0$. $\square$

Remark 38. The method of proof of Lemma 37 applies to a more general statement. Consider a sequence $\{U_k\}_1^\infty \subset D_F$ of open sets included into some ball $B(0,R)$ such that $\operatorname{dist}(U_k, \partial D_F) \ge \eta > 0$ for all $k \in \mathbb{N}$. Then the following two equalities are equivalent:

$$\lim_{k\to\infty} \overline{\operatorname{Cap}}\big(U_k; L^1_{n,F}(D)\big) = 0, \quad \lim_{k\to\infty} \overline{\operatorname{cap}}\big(U_k; W^1_n(\mathbb{M})\big) = 0.$$

Moreover, the last property is independent of the choice of F. In particular, Proposition 35 implies an estimate for the capacity of $B(x,r) \subset D_F$:

$$\overline{\operatorname{Cap}}\big(B(x,r); L^1_{n,F}(D)\big) = O\left(\left(\log\frac{2}{r}\right)^{1-n}\right) = o(1) \quad \text{as } r \to 0.$$

The next lemma describes a characteristic property of capacity zero sets.

Lemma 39. *A set $\Sigma \subset D_F$ has outer capacity zero if and only if there is a lower semicontinuous function $u \in L^1_{n,F}(D)$ such that $u = \infty$ on Σ. The norm of u can be chosen arbitrarily small.*

Proof. Necessity. Step 1. Firstly, consider the special location of Σ; i.e., $\Sigma \subset B_\Sigma \Subset W \subset D_F$, where W is a coordinate neighborhood, while B_Σ is a ball such that $\rho B_\Sigma \subset W$ for some real $\rho > 1$. Passing into a coordinate neighborhood, we may assume that $\rho B_\Sigma \subset W \subset \mathbb{R}^n$. By Lemma 37, we infer that

$$\begin{aligned}\overline{\mathrm{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) &= \overline{\mathrm{cap}}\big(\Sigma; W^1_n(\mathbb{M})\big) = \overline{\mathrm{cap}}\big(\Sigma; W^1_n(W)\big) \\ &= \mathrm{cap}\big(\Sigma; W^1_n(\mathbb{R}^n)\big) = \mathrm{cap}\big(\Sigma; S^1_n(\mathbb{R}^n)\big) = 0.\end{aligned}$$

Then there is a sequence of nonnegative functions $f_k \in L_n(\mathbb{R}^n)$ such that $\|f_k \mid L_n(\mathbb{R}^n)\| \le 2^{-k}$ and $f_k * J_1(x) \ge 1$ at all $x \in \Sigma$. The function

$$f = \sum_{k=1}^{\infty} f_k \tag{11}$$

is nonnegative, lies in $L_n(\mathbb{R}^n)$, and $f * J_1(x) = \infty$ at all $x \in \Sigma$. Since the kernel $J_1(z)$ is nonnegative on Σ and continuous everywhere but one point $z = 0$, the convolution $f * J_1(x)$ is lower semicontinuous by Fatou's Lemma.

Consider the Lipschitz function $\eta : W \to [0, 1]$ such that

$$\eta(x) = \begin{cases} 0 & \text{if } x \notin \rho B_\Sigma, \\ 1 & \text{if } x \in B_\Sigma. \end{cases}$$

Since $W^1_n(\mathbb{R}^n) = S^1_n(\mathbb{R}^n)$, the restriction of the product $\eta(x) \cdot f * J_1(x)$ to W: namely, $u(x) = \eta(x) \cdot (f * J_1)(x)|_W$, lies in $L^1_{n,F}(D)$ and satisfies all claims of the lemma.

Observe that we can make the norm of $u(x)$ arbitrarily small because the property of $f * J_1(x)$ to be equal to ∞ at the points of Σ is independent of the number of terms in (11). Therefore, removing finitely many terms of the series in (11) if need be and using the absolute convergence of the series, we can make

$$\big\|f * J_1 \mid W^1_n(W)\big\| = \|f * J_1 \mid L_n(W)\| + \big\|f * J_1 \mid L^1_n(W)\big\| \le \big\|f * J_1 \mid W^1_n(W)\big\|$$

arbitrarily small. Since

$$|\nabla(\eta \cdot f * J_1)(x)| \le |\nabla\eta(x)| \cdot |f * J_1(x)| + |\eta(x)| \cdot |\nabla(f * J_1)(x)|,$$

we can also make $\big\|u \mid L^1_{n,F}(D)\big\|$ arbitrarily small.

Step 2. Assume that the set $\Sigma \subset D_F$ of outer capacity zero is located arbitrarily.

By Corollary 2 of [32], there exists a locally finite covering[3)] $\{B_k\}_{k\ge1}$ of the open set $D \setminus F$ by balls $B_k \subset D \setminus F$ and a partition of unity $\{\psi_k\}_{k\ge1}$ subordinate to this covering; we may assume that each ball of B_k lies in some coordinate neighborhood. Moreover, there exists a vanishing monotone sequence $\{\rho_k\}$ of positive reals such that the sequence of balls $\{(1+\rho_k)B_k\}$ constitutes a finite covering of $D \setminus F$ locally.

The intersection $\Sigma \cap B_k$ has outer capacity zero and satisfies the hypotheses of the first step. Consequently, there exists a nonnegative lower semicontinuous function $u_k \in L^1_{n,F}(D)$ such that $u_k = \infty$ on $\Sigma \cap B_k$ and $\big\|u_k \mid L^1_{n,F}(D)\big\| \le \frac{\varepsilon}{2^k}$, where ε is an arbitrary real specified aforehand.

The function $u(x) = \sum_{k=1}^{\infty} u_k(x)$ lies in $L^1_{n,F}(D)$, is lower semicontinuous, $u = \infty$ on Σ, and

$$\big\|u \mid L^1_{n,F}(D)\big\| \le \sum_{k=1}^{\infty} \big\|u_k \mid L^1_{n,F}(D)\big\| \le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \le \varepsilon.$$

---

3) This means that for each $x \in D \setminus F$ there is a neighborhood $U \subset D \setminus F$ intersecting only finitely many balls of the covering $\{B_k\}$.

Sufficiency. Assume that there exists a lower semicontinuous function $u \in L^1_{n,F}(D)$ such that $u = \infty$ on some $\Sigma \subset D_F$. For each $\lambda > 0$ the set $U_\lambda = \{x \in D_F : u(x) > \lambda\}$ is open and includes Σ, while the refined function $\frac{u(x)}{\lambda}$ lies in the class $A(U_\lambda)$ of admissible functions for the capacity estimate

$$\overline{\mathrm{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) \le \overline{\mathrm{Cap}}\big(U_\lambda; L^1_{n,F}(D)\big) \le \frac{\big\|u \mid L^1_{n,F}(D)\big\|^n}{\lambda^n}.$$

In the last step we used Lemma 19. Since λ is arbitrary, $\overline{\mathrm{Cap}}\big(\Sigma; L^1_{n,F}(D)\big) = 0$. $\square$

Theorem 40 [27, Theorem 9]. *Consider a compact set $K \subset \mathbb{R}^n$ and a nondecreasing continuous function $h(\rho)$ with $h(0) = 0$. Suppose that*

$$\int_0^1 h(\rho)^{\frac{1}{n-1}} \frac{d\rho}{\rho} < \infty.$$

Then there exists a constant A such that $H^\infty_h(K) \le A\,\mathrm{cap}\big(K; W^1_n(\mathbb{R}^n)\big)$. Therefore, $H^\infty_h(K) = 0$ provided that $\mathrm{cap}\big(K; W^1_n(\mathbb{R}^n)\big) = 0$, where $H^\infty_h(K)$ stands for the Hausdorff capacity.

The next statement is analogous to Lemma 7.19 of [5].

Lemma 41. *Consider a coordinate neighborhood $W \subset D_F$ and a ball B_γ with $\lambda B_\gamma \subset W$ for some $\lambda > 1$. A sequence $\{\gamma_m\}_{m\in\mathbb{N}}$ of continua included into the closed ball $\overline{B}_\gamma \subset D_F$ satisfies*

$$\lim_{m\to\infty} \overline{\mathrm{Cap}}\big(\gamma_m; L^1_{n,F}(D)\big) = 0$$

if and only if $\lim_{m\to\infty} \mathrm{diam}\,\gamma_m = 0$.

Proof. Assume that $\lim_{m\to\infty} \overline{\mathrm{Cap}}\big(\gamma_m; L^1_{n,F}(D)\big) = 0$. There is a sequence $f_m \in L^1_{n,F}(D)$ of continuous functions with $f_m = 1$ on γ_m and $\lim_{m\to\infty}\big\|f_m \mid L^1_{n,F}(D)\big\| = 0$. We have $\mathrm{dist}(B_\gamma, F) > 0$ and $\lambda B_\gamma \subset D_F$. Define a cutoff $\eta \in C^\infty_0(\mathbb{M})$ such that $\eta = 1$ on B_γ and $\eta = 0$ on $\mathbb{M} \setminus \lambda B_\gamma$. Then $\eta \cdot f_m = u_m \in W^1_n(\mathbb{M})$ and Poincaré's inequality (Lemma 12) shows that $\lim_{m\to\infty}\big\|f_m \mid W^1_n(\mathbb{M})\big\| = 0$. Hence,

$$\lim_{m\to\infty} \mathrm{cap}\big(\gamma_m; W^1_n(\mathbb{M})\big) = \lim_{m\to\infty} \mathrm{cap}\big(\gamma_m; W^1_n(W)\big) = 0, \quad \lim_{m\to\infty} \mathrm{cap}\big(\gamma_m; W^1_n(\mathbb{R}^n)\big) = 0.$$

Putting $h(\rho) = \rho$ in Theorem 40, we infer that $\lim_{m\to\infty} H^\infty_1(\gamma_m) = 0$. It remains to observe that $H^\infty_1(E) = \mathrm{diam}(E)$.

The converse is obvious. $\square$

3.3. Generalized Teichmüller capacity.

Definition 42. Refer as the *generalized Teichmüller capacity* of the annulus $D_{r,R}(x_0) = \{x \in \mathbb{R}^n : r < d(x_0, x) < R\}$ to

$$CT(r, R) = \inf_u \int_{D_{r,R}} |\nabla u|^n \, dx,$$

where the greatest lower bound is taken over all quasicontinuous functions $u \in W^1_n(D_{r,R}(x_0))$ such that $\min u|_{S(0,t)(x_0)} \le 0$ and $\max u|_{S(0,t)(x_0)} \ge 1$ for almost all $t \in (r, R)$.

Each quasicontinuous function is continuous on almost all spheres (see Proposition 56). Note that the maximum and minimum in Definition 42 apply to precisely those spheres.

Proposition 43 [33, Proposition 7]. *The generalized Teichmüller capacity $CT(r, R)$ is strictly positive for all $0 < r < R < \infty$.*

Corollary 44 [33, Corollary 4]. *For the generalized Teichmüller capacity we have the lower bound*

$$CT(r,R) \geq \gamma_1 \log \frac{R}{r}.$$

Proposition 45. *Consider a domain U in $\mathbb{M}'$ and two connected sets $\gamma_0, \gamma_1 \subset U$ of positive diameter. If γ_0 and γ_1 share a limit point in U then no quasicontinuous function $v \in L^1_n(U)$ with $v|_{\gamma_0} = 0$ and $v|_{\gamma_1} = 1$ can exist.*

Proof. Assume on the contrary that some function with these properties exists. Consider the annulus $D_{r,R} \subset U$ centered at a common limit point $x \in U$ such that $B(x,R)$ lies in some coordinate neighborhood. Then the definition of generalized Teichmüller capacity, see Definition 42, and Corollary 44 yield

$$\int\limits_U |\nabla v|^n \, dx \geq \gamma_2 \int\limits_{D_{r,R}} |\nabla v|^n \, dx \geq \gamma_2 CT(r,R) \geq \gamma_1\gamma_2 \log \frac{R}{r}, \tag{12}$$

where γ_2 is a constant depending on the geometry in the neighborhood W. As $r \to 0$, we infer from this that $\|v \mid L^1_n(U)\|^n = \infty$, which contradicts the membership of v in $L^1_n(U)$. □

4. Properties of the Mapping φ

Continue studying $\varphi : D \to D'$ of class IL^1_n. All these mappings enjoy the properties that are stated in Subsection 1.3.

4.1. Construction of a quasicontinuous representative for φ. In this subsection we construct a quasicontinuous mapping ψ which coincides with φ almost everywhere on D_F.

Lemma 46. *Consider a set $E \subset D_F$ of positive measure such that φ is continuous on E and a lower semicontinuous function $f \in L^1_{n,\varphi(F)}(D')$. If $g = \varphi^* f$ is a refined function in $L^1_{n,F}(D)$ then $g(x) \geq f \circ \varphi(x)$ quasieverywhere on $E \cap \widetilde{E}$.*

Proof. Since φ is continuous on E, the function $f \circ \varphi$ is lower semicontinuous on E. The properties of φ^* in Lemma 14 show that $g = f \circ \varphi$ a.e. on D; and, in particular, $g(x) \geq f \circ \varphi(x)$ for almost all $x \in E$. By Lemma 30, we see that $g(x) \geq f \circ \varphi(x)$ for quasiall $x \in E \cap \widetilde{E}$. □

From of Lemma 46 we obtain the next

Corollary 47. *Consider a set $E \subset D_F$ of positive measure consisting of positive density points such that φ is continuous on E and a lower semicontinuous function $f \in L^1_{n,\varphi(F)}(D')$. If $g = \varphi^* f$ is a refined function in $L^1_{n,F}(D)$ then $g(x) \geq f \circ \varphi(x)$ quasieverywhere on E.*

Lemma 48. *Consider a set $E \subset D_F$ of positive measure consisting of positive density points such that φ is continuous on E. If $\Sigma \subset D'_F$ is a set of outer capacity zero then $\varphi^{-1}(\Sigma) \cap E$ is of capacity zero.*

Proof. Take the lower semicontinuous function $f \in L^1_{n,\varphi(F)}(D')$ constructed in Lemma 39. By Corollary 47, the refined function $g = \varphi^* f$ is at least $f \circ \varphi$ quasieverywhere on E. In particular, $g(x) = \infty$ for quasiall $x \in \varphi^{-1}(\Sigma) \cap E$. Lemma 19 yields $\overline{\operatorname{Cap}}\big(\varphi^{-1}(\Sigma) \cap E; L^1_{n,F}(D)\big) = 0$. □

Lemma 48 implies the next statement.

Lemma 49. *Consider some sequence $f_k \in L^1_{n,\varphi(F)}(D')$ of refined functions converging quasieverywhere to $f \in L^1_{n,\varphi(F)}(D')$. Then the sequence $f_k \circ \varphi$ converges to $f \circ \varphi \in L^1_{n,F}(D)$ almost everywhere on D and quasieverywhere on $T \cap D_F$, where T is the set from Proposition 13.*

Proof. Recall that $T = \bigcup_k T_k$ and $|D \setminus T| = 0$, where $\{T_k\}$ is a sequence of bounded sets of positive measure, increasing with respect to inclusion and consisting of positive density points. The mapping φ is continuous on each T_k.

Take a set $S \subset D'_F$ of outer capacity zero on which there is no convergence. By Lemma 48 $\varphi^{-1}(S) \cap T_k \cap D_F$ is of capacity zero for each k. Consequently, by claim (5) of Lemma 16, $\varphi^{-1}(S) \cap T \cap D_F$ is of capacity zero. Hence, we obtain the convergence of the sequence $f_k \circ \varphi$ to $f \circ \varphi \in L^1_{n,F}(D)$ quasieverywhere on $T \cap D_F$. The convergence of the sequence $f_k \circ \varphi$ to $f \circ \varphi \in L^1_{n,F}(D)$ almost everywhere on D is obvious: $|D \setminus T| = 0$. $\square$

Lemma 50. *Consider a set $E \subset D_F$ of positive measure consisting of positive density points such that φ is continuous on E and a refined function $f \in L^1_{n,\varphi(F)}(D')$. If $g = \varphi^* f$ is a refined function in $L^1_{n,F}(D)$ then $g|_E$ coincides quasieverywhere with $f \circ \varphi|_E$.*

Proof. Take a sequence $f_k \in L^1_{n,\varphi(F)}(D')$ of continuous functions converging to f everywhere but some set Σ of outer capacity zero. Using Remark 18, we may assume that the sequence of refined functions $g_k = \varphi^* f_k$ converges quasieverywhere to $\varphi^* f$. According to Corollary 33, $g_k = \varphi^* f_k$ coincides quasieverywhere on E with $f_k \circ \varphi|_E$. Therefore, g coincides quasieverywhere on E with $f \circ \varphi$. $\square$

Corollary 51. *Consider the set T of Proposition 13 and a refined function $f \in L^1_{n,\varphi(F)}(D')$. If $g = \varphi^* f$ is a refined function in $L^1_{n,F}(D)$ then $g|_{T \cap D_F}$ coincides quasieverywhere with $f \circ \varphi|_{T \cap D_F}$.*

Proof. Recall that $T = \bigcup_k T_k$ and $|D \setminus T| = 0$, where $\{T_k\}$ is a sequence of bounded sets of positive measure increasing by inclusion and consisting of positive density points. The mapping φ is continuous on each of T_k. Put $E = T_k \cap D_F$ in Lemma 50. Then $g = \varphi^* f$, refined in $L^1_{n,F}(D)$, coincides with $f \circ \varphi$ quasieverywhere on $T_k \cap D_F$. Hence, the claim follows because k is an arbitrary positive integer. $\square$

We continue studying the properties of φ.

Lemma 52. *Consider two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same topological dimension $n \geq 2$ with two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$. Assume that $\mathbb{M}'$ is a parabolic manifold. If a measurable mapping $\varphi : D \to D'$ is of class IL^1_n then there exist $S_\varphi \subset D$ of capacity zero and a quasicontinuous mapping $\psi : D_F \setminus S_\varphi \to \overline{D'_F}$ such that $\psi(x) = \varphi(x)$ a.e. on D_F.*

Proof. By Corollary 31, it suffices, given an arbitrary open ball $Q \Subset D_F$, to construct a quasicontinuous mapping $\bar{\varphi} : Q \to \mathbb{M}'$ coinciding with φ a.e. on Q.

Take a continuous function $f \in L^1_{n,F}(D)$ with $f \geq 1$ on Q. There exists a refined $g \in L^1_{n,\varphi(F)}(D')$ with $f = \varphi^* g$. By Corollary 51, $f|_{T \cap Q}$ and $g \circ \varphi|_{T \cap Q}$ coincide quasieverywhere. Consider the set $S_Q \subset T \cap Q$ of capacity zero on which the values of $f|_{T \cap Q}$ and $g \circ \varphi|_{T \cap Q}$ differ. The mapping $\varphi : T \cap Q \to \mathbb{M}'$ satisfies the hypotheses of Lemmas 46, 48–50, and their corollaries. Moreover, $g(y) = g(\varphi(x)) = f(x) \geq 1$ for all $y \in \varphi(T \cap Q \setminus S_Q)$. Therefore, the capacity of $\varphi(T \cap Q \setminus S_Q)$ is finite. Henceforth, consider φ only on $T \cap Q \setminus S_Q$, assuming that φ is undefined on $Q \setminus (T \cap Q \setminus S_Q)$. By the image of $V \subset Q$ we should understand $\varphi(V \cap (T \cap Q \setminus S_Q))$.

Put $P_k = \varphi(Q) \cap B(0, k)$ and $CP_k = \varphi(Q) \setminus P_k = \varphi(Q) \setminus B(x_0, k)$, where $x_0 \in T \cap Q \setminus S_Q$ is an arbitrary point, for $k \in \mathbb{N}$. Verify that

$$\overline{\mathrm{Cap}}\big(CP_k; L^1_{n,\varphi(F)}(D')\big) \to 0 \quad \text{as } k \to \infty. \tag{13}$$

Indeed, fix $k_0 \in \mathbb{N}$ and $0 < r < k_0 - 1$ such that $\varphi(F) \subset B(x_0, r)$ and

$$CP_k \subset D' \setminus B(x_0, k-1) \subset \mathbb{M}' \setminus B(x_0, k-1)$$

for $k > k_0$. This implies directly that

$$\overline{\mathrm{Cap}}\big(CP_k; L^1_{n,\varphi(F)}(D')\big) \leq \mathrm{Cap}\big(\mathbb{M}' \setminus B(x_0, k-1); L^1_{n,\varphi(F)}(\mathbb{M}')\big).$$

The capacity on the right vanishes as $k \to \infty$. This property follows since $\mathbb{M}'$ is parabolic, because $\varphi(F)$ is a compact set in D', while the support of an arbitrary function admissible for capacity in Definition 3 lies in $\mathbb{M}' \cap B(x_0, k-1)$ for some k. Thus, we obtain (13) as $k \to \infty$.

Take a sequence of functions $g_k \in A(CP_k)$ such that

$$\big\|g_k \mid L^1_{n,\varphi(F)}(D')\big\|^n = \overline{\mathrm{Cap}}\big(CP_k; L^1_{n,\varphi(F)}(D')\big).$$

By Corollary 51, the refined function $f_k = \varphi^* g_k$ coincides quasieverywhere on $T \cap D_F$ with $g_k \circ \varphi$. Thus, $f_k \in A(\varphi^{-1}(CP_k))$.

Denote by CF_k the subset of the ball Q consisting of the points in $\varphi^{-1}(CP_k)$ and all nonzero density points of $\varphi^{-1}(CP_k)$. Corollary 32 shows that

$$\overline{\mathrm{Cap}}\big(CF_k; L^1_{n,F}(D)\big) = \overline{\mathrm{Cap}}\big(\varphi^{-1}(CP_k); L^1_{n,F}(D)\big). \tag{14}$$

Furthermore,

$$\overline{\mathrm{Cap}}\big(\varphi^{-1}(CP_k); L^1_{n,F}(D)\big) \le \big\|f_k \mid L^1_{n,F}(D)\big\|^n$$
$$\le K^n \big\|g_k \mid L^1_{n,\varphi(F)}(D'))\big\|^n = K^n \overline{\mathrm{Cap}}\big(CP_k; L^1_{n,\varphi(F)}(D')\big),$$

where K is the norm of φ^*. From (13) and (14) we deduce that

$$\lim_{k\to\infty} \overline{\mathrm{Cap}}\big(CF_k; L^1_{n,F}(D)\big) = 0. \tag{15}$$

Put $F_k = Q \setminus CF_k$. Observe that $F_k \supset S_Q$ and $F_k \supset (D \setminus T) \cap Q$. If $x \in F_k \cap T \setminus S_Q$ then $\varphi(x) \in P_k$, and for all points $x \in F_k$ we have

$$\lim_{r\to 0} \frac{|F_k \cap B(x,r)|}{|B(x,r)|} = 1. \tag{16}$$

Indeed, for r sufficiently small and satisfying $B(x,r) \subset Q$ we obtain $|B(x,r)| = |F_k \cap B(x,r)| + |CF_k \cap B(x,r)|$ or

$$1 = \frac{|F_k \cap B(x,r)|}{|B(x,r)|} + \frac{|CF_k \cap B(x,r)|}{|B(x,r)|}. \tag{17}$$

By the construction of CF_k, for $x \notin CF_k$, which means $x \in F_k$, we see that

$$|CF_k \cap B(x,r)|/|B(x,r)| \to 0 \quad \text{as } r \to 0;$$

i.e., (17) implies (16).

Take the cutoff $\eta_k \in C_0^\infty(\mathbb{M}')$ such that $\eta_k(x) = 1$ at $x \in B(x_0,k)$ and $\eta_k(x) = 0$ for $x \notin B(x_0,k+1)$, for $k \in \mathbb{N}$. Consider a refined function $\psi_{i,k} \in L^1_{n,F}(D)$ such that $\psi_{i,k} = \varphi^*(y_i \cdot \eta_k)$. Here $y_i(\cdot)$ are the coordinate functions.[4)] Corollary 51 yields $\psi_{i,k}(x) = (y_i \cdot \eta_k)(\varphi(x))$ for quasiall $x \in F_k \cap T$. Therefore,

$$(\varphi^*(y_i\eta_k))(x) = (y_i \cdot \eta_k)(\varphi(x)) = y_i(\varphi(x)) = \varphi_i(x)$$

for quasiall points $x \in F_k \cap T$. Thus, almost everywhere on F_k the coordinate function φ_i coincides with the refined function $\psi_{i,k}$.

Put

$$\bar\varphi_{i,k}(x) = \begin{cases} \psi_{i,k}(x) & \text{if } x \in F_k \setminus S_Q, \\ \varphi_i(x) & \text{if } x \in Q \setminus F_k. \end{cases}$$

Since φ_i changes on a negligible set, for every $k \in \mathbb{N}$ the equality $\varphi_i(x) = \bar\varphi_{i,k}(x)$ holds a.e. on Q.

Assume that $k < m$. By the construction of F_k, we have $F_k \subset F_m$; therefore, on $F_k \cap T$ the functions $\varphi^*(y_i\eta_k)$ and $\varphi^*(y_i\eta_m)$ coincide quasieverywhere with φ_i.

[4)] We consider an arbitrary isometric embedding $i : \mathbb{M}' \to \mathbb{R}^m$ into the Euclidean space $\mathbb{R}^m$ of sufficiently large dimension. The coordinate functions $y_i(\cdot)$ for $i = 1,\dots,m$ are the coordinate functions of $i \circ f : \mathbb{M} \to \mathbb{R}^m$. It is known, see [34] for instance, that $\varphi \in W^1_{p,\mathrm{loc}}(D,\mathbb{M}')$ if and only if $y_i \in W^1_{p,\mathrm{loc}}(D,\mathbb{R})$ for all $i = 1,\dots,m$.

Since by construction all points of F_k are of density 1, the refined functions $\psi_{i,m}$ and $\psi_{i,k}$ coincide quasieverywhere on F_k; see Corollary 31. This enables us to define the function

$$\bar{\varphi}_{iQ}(x) = \begin{cases} \psi_{i,k}(x) & \text{if } x \in F_k \setminus S_Q, \\ \varphi_i(x) & \text{if } x \in Q \setminus \bigcup\limits_k^{\infty} F_k \end{cases}$$

quasieverywhere on Q. Since $Q \setminus \bigcup_k^\infty F_k = \bigcap_k^\infty CF_k$, by (15) the function $\bar{\varphi}_{iQ}$ is defined quasieverywhere on the ball Q.

Verify that $\bar{\varphi}_{iQ}$ is quasicontinuous on Q. Fix $\varepsilon > 0$. Then there are open sets U_1, U_2, and U_3 such that

(1) there exists an index k such that $CF_k \subset U_1$ and $\overline{\mathrm{Cap}}(U_1) < \varepsilon/3$ by (15) and Lemma 16;

(2) $\psi_{i,k}$ is continuous on $D_F \setminus U_2$ and $\overline{\mathrm{Cap}}(U_2) < \varepsilon/3$ because $\psi_{i,k}$ is a quasicontinuous function;

(3) U_3 contains all points of the capacity zero set $Q \setminus U_1$ at which the values of $\bar{\varphi}_{iQ}$ and $\psi_{i,k}$ differ, and $\overline{\mathrm{Cap}}(U_3) < \varepsilon/3$ because $\bar{\varphi}_{iQ}$ and $\psi_{i,k}$ coincide quasieverywhere on F_k outside CF_k.

The set $U = U_1 \cup U_2 \cup U_3$ is of capacity $\overline{\mathrm{Cap}}(U) < \varepsilon$, while $\bar{\varphi}_{iQ}$ is continuous on $Q \setminus U$. Since $\varepsilon > 0$ is arbitrary, $\bar{\varphi}_{iQ}$ is a quasicontinuous function. Therefore, $\bar{\varphi}_Q : Q \setminus S_Q \to \overline{D'_F}$ is quasicontinuous, where $S_Q \subset Q$ is a capacity zero set.

Covering the domain D_F by a countable collection of open balls Q_j with finite multiplicity and repeating the above procedure on each ball Q_j, we construct the quasicontinuous mapping

$$\psi(x) = \bar{\varphi}_{Q_j}(x) \quad \text{if } x \in Q_j.$$

The mapping $\psi(x)$ is well defined in view of the following properties: For two disjoint balls Q_j and Q_i we have $\varphi_{Q_j}(x) = \varphi_{Q_i}(x)$ for all $x \in Q_i \cap Q_j$ with the exception of some set $\Sigma_{ij} \subset Q_i \cap Q_j$ of capacity zero; see Corollary 31. Remove from D_F the set

$$S_\varphi = \bigcup_{i \neq j} \Sigma_{ij} \cup \bigcup_j S_{Q_j} \tag{18}$$

of capacity zero.

Then ψ is well-defined on $D_F \setminus S_\varphi$. Furthermore, $\psi(x) = \varphi(x)$ for almost all $x \in D_F$. □

Assume that the image of $V \subset D_F$ is $\psi(V \setminus S_\varphi)$.

Remark 53. The mapping ψ enjoys the following property: $\psi(x) = \varphi(x)$ for all $x \in T \setminus Z$, where Z is a capacity zero set.

4.2. Construction of the mapping φ_0. All subsequent statements rely on the mapping $\psi : D_F \setminus S_\varphi \to \overline{D'_F}$ constructed in Lemma 52 on assuming that the Riemannian manifold $\mathbb{M}'$ is parabolic. Thus, henceforth we assume tacitly that $\mathbb{M}'$ is parabolic.

In this subsection we construct φ_0 such that $\varphi_0 = \psi$ quasieverywhere and the equivalent estimates on the capacity of the image and preimage are satisfied; see Lemma 55.

The following lemma describes the properties of ψ and strengthens Lemmas 46, 48, and 50.

Lemma 54. (1) *Take a lower semicontinuous function $f \in L^1_{n,\varphi(F)}(D')$. If $g = \varphi^* f$ is a refined function in $L^1_{n,F}(D)$ then $g(x) \geq f \circ \psi(x)$ quasieverywhere on $D_F \cap \psi^{-1}(D'_F)$.*

(2) *For $\Sigma \subset D'_F$, if $\overline{\mathrm{Cap}}\big(\Sigma; L^1_{n,\varphi(F)}(D')\big) = 0$ then $\overline{\mathrm{Cap}}\big(\psi^{-1}(\Sigma) \cap D_F; L^1_{n,F}(D)\big) = 0$.*

(3) *If $f \in L^1_{n,\varphi(F)}(D') \cap C(D')$ then the refined function $\varphi^* f$ coincides with $f \circ \psi$ quasieverywhere on $D_F \cap \psi^{-1}(D'_F)$.*

(4) *Suppose that*

$$f(y) = \sum_{k=1}^\infty f_k(y), \quad \text{where } f_k \in L^1_{n,\varphi(F)}(D') \cap C(D') \text{ and } \sum_{k=1}^\infty \big\| f_k \mid L^1_{n,\varphi(F)}(D') \big\| < \infty$$

holds quasieverywhere in D'_F. *Then the refined function* $\varphi^* f$ *coincides with* $\sum_{k=1}^{\infty}(f\circ\psi)(x)$ *quasieverywhere on* $D_F\cap\psi^{-1}(D'_F)$.

(5) *For every refined function* $f\in L^1_{n,\varphi(F)}(D')$ *the refined* $g=\varphi^* f$ *coincides with* $f\circ\psi$ *quasieverywhere on* $D_F\cap\psi^{-1}(D'_F)$.

(6) *For all* $A\subset D_F$ *and* $B\subset D'_F$ *such that* $\psi(A)\subset D'_F$ *we have*

$$\overline{\mathrm{Cap}}\big(\psi^{-1}(B);L^1_{n,F}(D)\big)\le K^n\overline{\mathrm{Cap}}\big(B;L^1_{n,\varphi(F)}(D')\big), \tag{19}$$

$$\overline{\mathrm{Cap}}\big(A\cap\psi^{-1}(D'_F);L^1_{n,F}(D)\big)\le K^n\overline{\mathrm{Cap}}\big(\psi(A)\cap D'_F;L^1_{n,\varphi(F)}(D')\big), \tag{20}$$

where $K=\max(\|\varphi^*\|,\|\varphi^{*-1}\|)$.

Proof. (1) Given $\varepsilon>0$, choose an open set $U_\varepsilon\subset D_F$ such that ψ is continuous on $D_F\setminus U_\varepsilon$ and $\mathrm{Cap}\big(U_\varepsilon;L^1_{n,F}(D)\big)<\varepsilon$. Lemma 30 also yields $\overline{\mathrm{Cap}}\big(\widetilde{U}_\varepsilon;L^1_{n,F}(D)\big)<\varepsilon$. Observe that all points of $D_F\setminus\widetilde{U}_\varepsilon$ are positive density points. Therefore, so are all points of $(D_F\cap\psi^{-1}(D'_F))\setminus\widetilde{U}_\varepsilon$ because $|D_F\setminus\psi^{-1}(D'_F)|=0$.

The composition $f\circ\psi(x)$ is lower semicontinuous at all points of $(D_F\cap\psi^{-1}(D'_F))\setminus U_\varepsilon$. Lemma 46 shows that $g(x)\ge f\circ\psi(x)$ quasieverywhere on $(D_F\cap\psi^{-1}(D'_F))\setminus\widetilde{U}_\varepsilon$. Since ε is arbitrary, it follows that $g(x)\ge f\circ\psi(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

(2) Take the lower semicontinuous function $f\in L^1_{n,\varphi(F)}(D')$ constructed in Lemma 39. By claim 1, the refined function $g=\varphi^* f$ is at least $f\circ\varphi$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. In particular, $g(x)=\infty$ for quasiall $x\in\varphi^{-1}(\Sigma)\cap D_F\cap\psi^{-1}(D'_F)=\varphi^{-1}(\Sigma)\cap D_F$. Therefore, Lemma 19 yields $\mathrm{Cap}\big(\varphi^{-1}(\Sigma)\cap D_F;L^1_{n,F}(D)\big)=0$.

(3) If $g=\varphi^* f$ is a refined function in $L^1_{n,F}(D)$ then by Proposition 1 we simultaneously have $g(x)\ge f\circ\psi(x)$ and $-g(x)\ge -f\circ\psi(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. Hence, $g(x)=f\circ\psi(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

(4) Suppose that $f_k\in L^1_{n,\varphi(F)}(D')\cap C(D')$ with $f(y)=\sum_{k=1}^{\infty}f_k(y)$ quasieverywhere on D'_F and

$$\sum_{k=1}^{\infty}\big\|f_k\mid L^1_{n,\varphi(F)}(D')\big\|<\infty.$$

By Proposition 2, $f\circ\psi(x)=\sum_{k=1}^{\infty}f_k\circ\psi(x)$ converges quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. Moreover,

$$\sum_{k=1}^{\infty}\big\|f_k\circ\psi\mid L^1_{n,\varphi(F)}(D')\big\|\le K\sum_{k=1}^{\infty}\big\|f_k\mid L^1_{n,\varphi(F)}(D')\big\|<\infty. \tag{21}$$

By an available method (see [5] for instance) we can deduce from (21) that $\sum_{k=1}^{\infty}f_k\circ\psi(x)$ converges uniformly on D_F outside some open set of however small capacity. Therefore, $f\circ\psi$ is refined, and so $\varphi^* f(x)=\sum_{k=1}^{\infty}(f\circ\psi)(x)$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

(5) Take a sequence $f_k\in L^1_{n,\varphi(F)}(D')$ of continuous functions converging to f everywhere but a set Σ of outer capacity zero. By claims 2 and 4, the sequence of refined functions $g_k=\varphi^* f_k$ converges quasieverywhere to the refined function $g=\varphi^* f$. According to claim (3), the refined functions $g_k=\varphi^* f_k$ coincide with $f_k\circ\varphi|_{D_F\cap\psi^{-1}(D'_F)}$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$. Therefore, the refined function $g=\varphi^* f$ coincides with $f\circ\psi$ quasieverywhere on $D_F\cap\psi^{-1}(D'_F)$.

(6) Consider the capacity function $f_B\in L^1_{n,\varphi(F)}(D')$ for B (see Proposition 13); i.e.,

$$\overline{\mathrm{Cap}}\big(B;L^1_{n,\varphi(F)}(D')\big)=\big\|f_B\mid L^1_{n,\varphi(F)}(D')\big\|^n.$$

By the definition of capacity function, $\{y\in B\mid f_B(y)<1\}$ has capacity zero. Therefore, the refined function $g=\varphi^* f_B$ satisfies $g(x)\ge 1$ quasieverywhere on $\psi^{-1}(B)$. Since

$$\big\|g\mid L^1_{n,F}(D)\big\|\le K\big\|f_B\mid L^1_{n,\varphi(F)}(D')\big\|;$$

we deduce the chain of inequalities

$$\overline{\mathrm{Cap}}\big(\psi^{-1}(B); L^1_{n,F}(D)\big) \le \big\|g \mid L^1_{n,F}(D)\big\|^n$$
$$\le K^n \big\|f_B \mid L^1_{n,\varphi(F)}(D')\big\|^n = K^n \overline{\mathrm{Cap}}\big(B; L^1_{n,\varphi(F)}(D')\big),$$

which implies (19).

Consider the capacity function $f_{\psi(A)} \in L^1_{n,\varphi(F)}(D')$ for $\psi(A) \cap D'_F$. The set $\{y \in \psi(A) \mid f_{\psi(A)}(y) < 1\}$ is of capacity zero. Hence, the refined function $g = \varphi^* f_{\psi(A)}$ satisfies $g(x) \ge 1$ quasieverywhere on $A \cap \psi^{-1}(D'_F)$. From $\|g \mid L^1_{n,F}(D)\| \le K \|f_{\psi(A)} \mid L^1_{n,\varphi(F)}(D')\|$ we deduce that

$$\overline{\mathrm{Cap}}\big(A \cap \psi^{-1}(D'_F); L^1_{n,F}(D)\big) \le \big\|g \mid L^1_{n,F}(D)\big\|^n$$
$$\le K^n \big\|f_{\psi(A)} \mid L^1_{n,\varphi(F)}(D')\big\|^n = K^n \overline{\mathrm{Cap}}\big(\psi(A) \cap D'_F; L^1_{n,\varphi(F)}(D')\big),$$

which implies (20). □

Now, fix some countable system

$$\mathscr{B} = \{B_j\} \tag{22}$$

of balls in D_F constituting a base for the open sets $U \subset D_F$. Assume that the balls involved enjoy the properties:

(1) $B_j \Subset D_F$ for all $j \in \mathbb{N}$;

(2) together with each ball $B_j = B_j(x_j, r_j)$ the system $\mathscr{B}$ also contains the countable collection of balls centered at x_j of radius of the form $2^{-k}\,\mathrm{dist}(x_j, D_F)$ for $k \in \mathbb{N}$.

Lemma 55. *There exist a set $S_\psi \subset D_F$ of capacity zero and a mapping $\varphi_0 : D_F \setminus S_\psi \to \overline{D'_F}$ such that $\varphi_0(x)$ coincides with $\psi(x)$ for quasiall $x \in D_F$. For the mapping φ_0 all claims of Lemma 54 hold, as well as the estimate*

$$\overline{\mathrm{Cap}}\big(\varphi_0(B_j) \cap D'_F; L^1_{n,\varphi(F)}(D')\big) \le K^{-n} \overline{\mathrm{Cap}}\big(B_j; L^1_{n,F}(D)\big) \tag{23}$$

for every ball B_j of (22).

Proof. Take some ball $B \in \mathscr{B}$ in the countable base of neighborhoods (22) and the capacity function $g_B \in A(B)$ of B; see Theorem 25. Since φ^* is an isomorphism (see Lemma 14), there exists a refined function $f_B \in L^1_{n,\varphi(F)}(D')$ such that $g_B(x) = f_B \circ \psi(x)$ for quasiall $x \in D_F \cap \psi^{-1}(D'_F)$ by claim (3) of Lemma 54.

Consider $\Sigma_B = \{x \in B \cap \psi^{-1}(D'_F) : f_B(\psi(x)) < 1\}$. Then, since $f_B(\psi(x)) = g_B(x) \ge 1$ for quasiall $x \in B \cap \psi^{-1}(D'_F)$, it follows that $\overline{\mathrm{Cap}}\big(\Sigma_B; L^1_{n,F}(D)\big) = 0$.

Since $\psi((B \cap \psi^{-1}(D'_F)) \setminus \Sigma_B) = (\psi(B) \cap D'_F) \setminus \psi(\Sigma_B) \subset D'_F$, the function f_B is admissible for the set $(\psi(B) \cap D'_F) \setminus \psi(\Sigma_B) \subset D'_F$; i.e., $f \in A((\psi(B) \cap D'_F) \setminus \psi(\Sigma_B))$. Therefore,

$$\overline{\mathrm{Cap}}\big(\psi((B \cap \psi^{-1}(D'_F)) \setminus \Sigma_B); L^1_{n,\varphi(F)}(D')\big) \le \big\|f_B \mid L^1_{n,\varphi(F)}(D')\big\|^n$$
$$\le K^{-n} \big\|g_B \mid L^1_{n,F}(D)\big\|^n = K^{-n} \overline{\mathrm{Cap}}\big(B; L^1_{n,F}(D)\big).$$

Assume that $\varphi_0(x)$ equals $\psi(x)$ on $D_F \setminus \bigcup_{B_j \in \mathscr{B}} \Sigma_{B_j}$ and is undefined on $\bigcup_{B_j \in \mathscr{B}} \Sigma_{B_j}$. Claim 6 of Lemma 16 shows that $\overline{\mathrm{Cap}}\big(\bigcup_{B_j \in \mathscr{B}} \Sigma_B; L^1_{n,F}(D)\big) = 0$. Hence, φ_0 and ψ coincide quasieverywhere on D_F.

Thus, φ_0 is now defined on $D_F \setminus S_\psi$, where

$$S_\psi = S_\varphi \cup \bigcup_{B_j \in \mathscr{B}} \Sigma_{B_j},$$

while S_φ is defined in (18). The validity of all claims of Lemma 54 for φ_0 can be verified directly. □

By the image $\varphi_0(V)$ of an arbitrary set $V \subset D_F$ we understand $\psi\big(V \setminus S_\psi\big)$.

4.3. Topological properties of the mapping φ_0. Continue studying the properties of the quasicontinuous mapping φ_0. Note that we consider the balls $B(x, r)$ and the spheres $S(x, r)$ in the Riemannian metric.

Proposition 56 [33, Proposition 5]. (1) *The mapping φ_0 is defined and continuous at all points of the sphere $S(x,r)$ for almost all $r \in (0, \operatorname{dist}(x, \partial D_F))$.*

(2) *The mapping φ_0 is continuous on almost all integral lines of the basis vector fields: For each ball $B(x,r) \subset D_F$ and almost all integral lines $\gamma \subset B(x,r)$ of the basis vector field X_i, for $i = 1, \dots, n$. The mapping φ_0 is defined and continuous at all points of γ.*

Proposition 57. *There exists a negligible set $\Sigma \subset D_F$ such that arbitrarily small neighborhoods of $x_1, x_2 \in B \setminus \Sigma$, where $B \subset D_F$ is an open ball, contain points that can be connected by a curve $\gamma \subset B$ on which φ_0 is continuous.*

Proof. Step 1. Fix a countable system

$$\mathscr{B} = \{B_j\} \tag{24}$$

of balls in D_F constituting a base for open sets $U \subset D_F$. Assume that the balls involved enjoy the properties:

(1) $B_j \Subset D_F$ for all $j \in \mathbb{N}$;

(2) together with each ball $B_j = B_j(x_j, r_j)$, the system $\mathscr{B}$ also contains the countable collection of balls centered at x_j of radii $2^{-k} r_j$;

(3) each ball $B_j \in \mathscr{B}$ lies in some coordinate neighborhood.

In order to prove Proposition 57, it clearly suffices to find in each ball B_j a negligible set $\Sigma_{B_j} \subset B_j$ such that arbitrarily small neighborhoods of $x_1, x_2 \in B_j \setminus \Sigma_{B_j}$ contain points that can be connected by a curve $\gamma \subset B_j$ on which φ_0 is continuous.

Indeed, if this is established then as $\Sigma \subset D_F$ we can take the union $\bigcup_j \Sigma_{B_j}$, which has measure zero. Two arbitrary points $x_1, x_2 \in B \setminus \Sigma$, where $B \subset D_F$ is some open ball, can be connected by a continuous curve $\gamma \subset B$. On this curve we can fix finitely many points $x_1 = y_1, y_2, \dots, y_{l+1} = x_2$ so that two adjacent points y_j and y_{j+1} lie in some ball $\{B_j\}$, and the balls may repeat. Since the distance from γ to ∂D_F is positive in arbitrarily small neighborhoods of $y_j, y_{j+1} \in B_j \setminus \Sigma_{B_j}$, there are $y'_j, y'_{j+1} \in B_j \setminus \Sigma_{B_j}$ which we can connect by a curve $\gamma_j \subset B_j$ and on which φ_0 is continuous. It remains to connect y_j and y'_j by a curve on which φ_0 is continuous. We can take y_j and y'_j in some cube (see Steps 2 and 3 of the proof) which guarantees that the required curve exists.

Step 2. Take a ball $B_j \in \mathscr{B}$ included into some coordinate neighborhood and put $B = B_j$. In a coordinate neighborhood $W \supset B$ consider the constant vector fields $\frac{\partial}{\partial x_i}$, for $i = 1, 2, \dots, n$. Take a cube $Q \subset B$ with sides parallel to the coordinate axes. We can connect two arbitrary points $x_1, x_2 \in Q$ by a broken line $\sigma = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_k$, where $k \le n$, consisting of segments of the integral lines of $\frac{\partial}{\partial x_i}$.

Step 3. As a negligible set $\Sigma_B \subset B$, take the collection of all points in B outside the union of all integral lines of $\frac{\partial}{\partial x_i}$ for $i = 1, \dots, n$, on each of which φ_0 is continuous; see Proposition 56.

Step 4. Assume that $x_1, x_2 \in B \setminus \Sigma_B$ and take a continuous curve $\Gamma \subset B$ connecting x_1 and x_2. Put

$$R_\Gamma = \frac{\operatorname{dist}(\Gamma, \partial B)}{\Lambda},$$

where we choose a positive real Λ so that for every point $x \in \Gamma$ the cube $Q(x, R_\Gamma)$ lies in B, with the sides of $Q(x, R_\Gamma)$ parallel to the coordinate axes, while the edge length f $Q(x, R_\Gamma)$ equals $2R_\Gamma$.

Cover Γ with finitely many cubes $Q_j \subset B$ of the above form with edge length $2R_\Gamma$ and choose points $x_1 = y_1, y_2, \dots, y_{l+1} = x_2$ on this curve so that two adjacent points y_j and y_{j+1} lie in $\{Q_j\}$, where the cubes may repeat. By the choice of a suitable cube $\{Q_j\}$, we can connect y_j and y_{j+1} by a curve $\gamma_j \subset Q_j$ constructed in Step 1 of the proof. The compound curve $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_l$ need not yet be the required one because some points $y_2, \dots, y_l$ may belong to Σ.

Step 5. In order to construct the required curve, take a ball $B(y_1, \varepsilon)$ of sufficiently small radius such that the tubular neighborhood $\bigcup_t B(\gamma(t), \varepsilon)$ of γ is inside B. For every $\varepsilon > 0$ this tubular neighborhood contains a curve that is composed of segments of integral lines of the vector fields $\frac{\partial}{\partial x_i}$, for $i = 1, \dots, n$, on each of which φ_0 is continuous. The initial point of this curve lies in $B(x_1, \varepsilon)$, while the terminal one, in $B(x_2, \varepsilon)$.

Since we can take ε arbitrarily small, the proof of Proposition 57 is complete. $\square$

Given $x \in T \cap D_F$, put

$$\widehat{B}(x, r) = \Big\{ \bigcup_{\rho \in (0,r)} S(x, \rho) \mid \ \varphi_0 : S(x, \rho) \to \mathbb{M}' \text{ is continuous} \Big\} \subset D_F.$$

Therefore, $\widehat{B}(x, r)$ differs from $B(x, r)$ only in that we have removed from $B(x, r)$ all spheres $S(x, \rho)$ with $\rho \in \sigma_{x,r} \subset (0, r)$ on which φ_0 is discontinuous and, furthermore, the collection $\sigma_{x,r}$ of these radii is of measure zero on $(0, r)$.

Lemma 58. *Given a sequence $\{r_k\}$ of positive reals converging to 0 as $k \to \infty$, a point $x \in D_F$, and a sequence $u_k \in \widehat{B}(x, r_k) \cap D_F$ with $\varphi_0(u_k) \to y \in D'_F$ as $k \to \infty$, where y is some point, the images $\varphi_0(\widehat{B}(x, r_k))$ tend to $y \subset D'_F$ as $k \to \infty$; i.e.,*

$$\{y\} = \bigcap_{k \in \mathbb{N}} \overline{\varphi_0(\widehat{B}(x, r_k))} \in D'_F. \tag{25}$$

Proof. Obviously, (25) is equivalent to

$$\sup_{z \in \widehat{B}(x, r_k) \cap D_F} d(\varphi_0(z), y) \to 0 \tag{26}$$

as $k \to \infty$. Assume on the contrary that (26) is false. Then there exist $\vartheta > 0$ and a sequence $\varkappa_k \in (0, r_k) \setminus \sigma_{x, r_k}$ of radii satisfying

$$\operatorname{diam}(\{y\} \cup \varphi_0(S(x, \varkappa_k)) = \sup_{z \in S(x, \varkappa_k) \cap D_F} d(\varphi_0(z), y) \geq \vartheta, \quad k \in \mathbb{N}. \tag{27}$$

Since $u_k \in \widehat{B}(x, r_k) \cap D_F$, it follows that $u_k \in S(x, \tau_k)$, where $\tau_k \in (0, r_k) \setminus \sigma_{x, r_k}$ and $\tau_k \to 0$ as $k \to \infty$.

Clearly, for k sufficiently large each $B(x, r_k)$ lies in some ball $B_k = B(x_{j_k}, \rho_{j_k})$ of the collection (24) such that $x_{j_k} \in B(x, r_k)$ and $\rho_{j_k} > 2r_k$; the latter ensures that $B(x, r_k) \subset B_k$. Furthermore, $\rho_{j_k} \to 0$ as $k \to \infty$, i.e., as r_k vanishes, so does ρ_{j_k}; see the description of the collection (24) above.

Given $k \in \mathbb{N}$, consider the curve $\gamma_k \subset B_k$ of Proposition 57 with endpoints in $B(x, \min(\tau_k, \varkappa_k))$ and $B_k \setminus B(x, r_k)$, at whose points φ_0 is defined and continuous.

Denote the compact set $S(x, \tau_k) \cup S(x, \varkappa_k) \cup \gamma_k$ by K_k. Then $K_k \subset B_k$, the set K_k is connected, and $\varphi_0 : K_k \to \overline{D'_F}$ is continuous. Using this choice of K_k and $B_k = B(x_{j_k}, \rho_{j_k})$ and appreciating (23), we obtain the chain of inequalities

$$\begin{aligned} \overline{\operatorname{Cap}}\big(\varphi_0(K_k) \cap D'_F; L^1_{n,\varphi(F)}(D')\big) &\leq \overline{\operatorname{Cap}}\big(\varphi_0(\widehat{B}(x, r_k)) \cap D'_F; L^1_{n,\varphi(F)}(D')\big) \\ &\leq \overline{\operatorname{Cap}}\big(\varphi_0(B_k) \cap D'_F; L^1_{n,\varphi(F)}(D')\big) \\ &\leq K^{-n} \overline{\operatorname{Cap}}\big(B_k; L^1_{n,F}(D)\big) = O\left(\left(\log \frac{2}{\rho_{j_k}}\right)^{1-n}\right) = o(1) \end{aligned} \tag{28}$$

as $k \to \infty$. The last row here follows from the condition $u_k \to x \in D_F$ as $k \to \infty$ and Remark 38.

From (28) we infer that $\overline{\operatorname{Cap}}\big(\varphi_0(K_k) \cap D'_F; L^1_{n,\varphi(F)}(D')\big) \to 0$ as $k \to \infty$. Applying Theorem 40 to the compact set $\varphi_0(K_k)$ and taking into account the condition $\varphi_0(u_k) \to y \in D'_F$ as $k \to \infty$, we find that $\operatorname{diam} \varphi_0(K_k) \to 0$ as $k \to \infty$. Consequently, $\operatorname{diam}(\{y\} \cap \varphi_0(K_k)) \to 0$ as $k \to \infty$. This contradicts (27) because $S(x, \varkappa_k) \subset K_k$. $\square$

The next statement shows that the images of concentric spheres on which φ_0 is continuous contract to a point as the radius vanishes.

Corollary 59. *If $x \in T \cap D_F$ then*

$$\sup_{y \in \varphi_0(\widehat{B}(x,r)) \cap D_F} d(y, \varphi_0(x)) \to 0 \quad \text{as } r \to 0. \tag{29}$$

Proof. Fix k with $x \in T_k \cap D_F$. Assume on the contrary that (29) is false. Then there exist $\vartheta > 0$ and a sequence $r_l \to 0$ as $l \to \infty$ such that

$$\sup_{y \in \varphi_0(\widehat{B}(x,r_l)) \cap D_F} d(y, \varphi_0(x)) \geq 2\vartheta$$

for all $l \in \mathbb{N}$. Hence, we extract a sequence $\varkappa_l \in (0, r_l) \setminus \sigma_{x,r_l}$ of radii for which

$$\sup_{y \in \varphi_0(S(x,\varkappa_l)) \cap D_F} d(y, \varphi_0(x)) \geq \vartheta, \quad l \in \mathbb{N}.$$

Since x is a positive density point, for all r_l there are $\tau_l \in (0, r_l) \setminus \sigma_{x,r_l}$ with $S(x, \tau_l) \cap T_k \neq \varnothing$. Since φ_0 is continuous on $T_k \cap D_F$, see Proposition 56, for each choice of the points $u_l \in S(x, \tau_l) \cap T_k \neq \varnothing$ we have $u_l \to x$ and $\varphi_0(u_l) \to y = \varphi_0(x) \in D'_F$ as $l \to \infty$. Hence, the sequence u_l satisfies all hypotheses of Lemma 58, which yields (29). $\square$

Proposition 56 and Corollary 59 imply the following properties of φ_0.

Corollary 60. *Assume that $x \in T \cap D_F$. Given a sufficiently small $\rho > 0$, there is $\delta_{x,\rho} > 0$ such that*

(1) for the spheres $S(x,r) \subset D_F$ of radius $r \in (0, \delta_{x,\rho}) \setminus \sigma_{x,r}$ their images $\varphi_0(S(x,r))$ lie in $B(\varphi_0(x), \rho) \subset D'_F$; i.e.,

$$\varphi_0(\widehat{B}(x, \delta_{x,\rho})) \subset B(\varphi_0(x), \rho) \subset D'_F; \tag{30}$$

(2) for almost all integral lines γ of the basis vector fields, $\varphi_0(\gamma \cap B(x, \delta_{x,\rho}))$ lie in $B(\varphi_0(x), \rho) \subset D'_F$.

Corollary 61. *Assume that $x \in T \cap D_F$. The balls satisfying* (30) *enjoy the following property: For every point $y \in B(x, \delta_{x,\rho})$ the images $\varphi_0(\widehat{B}(y, \tau))$ tend to a unique point $z \in \overline{B(\varphi_0(x), \rho)} \subset D'_F$ as $\tau \to 0$.*

Proof. Fix a sequence $\tau_k \to 0$ as $k \to \infty$. For $k \in \mathbb{N}$ we can find $u_k \in B(x, \delta_{x,\rho}) \cap \widehat{B}(y, \tau)$. We have $u_k \to x$ as $k \to \infty$, while $\varphi_0(u_k) \in \overline{B(\varphi_0(x), \rho)} \subset D'_F$. Extracting a subsequence, we may assume that $\varphi_0(u_k) \to y \in D'_F$. Hence, u_k satisfies all hypotheses of Lemma 58, which implies the claim. $\square$

Definition 62. Assume that $x \in T \cap D_F$. For $\rho > 0$ sufficiently small use Corollary 60 to find $\delta_{x,\rho} > 0$ such that

$$\varphi_0(\widehat{B}(x, \delta_{x,\rho})) \subset B(\varphi_0(x), \rho) \subset D'_F.$$

If $y \in B(x, \delta_{x,\rho})$ lies in $T \cap D_F$ then

$$\lim_{z \to y, z \in \widehat{B}(y, \delta_1)} \varphi_0(z) = \varphi_0(y),$$

where δ_1 is a sufficiently small positive real; otherwise the value of φ_0 at y is unspecified, but by Corollary 61 the limit

$$\lim_{z \to y, z \in \widehat{B}(y, \delta_2)} \varphi_0(z) \in D'_F$$

exists, which we take as $\varphi_0(y)$. Here δ_2 is a sufficiently small positive real.

Consequently, at all points of $B(x, \delta_{x,\rho})$ we have defined some mapping that we denote by the same symbol φ_0. It has the following property:

$$\varphi_0(y) \in \overline{B(\varphi_0(x), \rho)} \subset D'_F \quad \text{for all } y \in B(x, \delta_{x,\rho}). \tag{31}$$

Moreover, in this case

$$\{\varphi_0(y)\} = \bigcap_{r \to 0} \overline{\varphi_0(\widehat{B}(y, r))} \in D'_F.$$

Proposition 63. *The mapping $\varphi_0 : B(x,\delta_{x,\rho}) \to \overline{B(\varphi_0(x),\rho)} \subset D'_F$, where $x \in T \cap D_F$, is continuous.*

Proof. Case 1: $y \in B(x,\delta_{x,\rho}) \cap T \cap D_F$. By Definition 62, for $\tau > 0$ sufficiently small there exists $\delta_{y,\tau} > 0$ such that (31) yields the inclusions $\varphi_0(\widehat{B}(y,\delta_{y,\tau})) \subset B(\varphi_0(y),\tau) \subset \overline{B(\varphi_0(x),\rho)} \subset D'_F$, showing that φ_0 is continuous at $y \in B(x,\delta_{x,\rho}) \cap T \cap D_F$.

Case 2: $y \in B(x,\delta_{x,\rho}) \setminus (T \cap D_F)$. By Definition 62, for $\tau > 0$ sufficiently small there exists $\delta_{y,\tau} > 0$ such that $\varphi_0(\widehat{B}(y,\delta_{y,\tau})) \subset B(\varphi_0(y),\tau) \subset \overline{B(\varphi_0(y),\tau)} \subset D'_F$.

As in case 1, $\varphi_0(z) \in \overline{B(\varphi_0(y),\tau)} \subset D'_F$ for every $z \in B(y,\delta_{y,\tau})$. Hence, we similarly infer that φ_0 is continuous at $y \in B(x,\delta_{x,\rho}) \setminus (T \cap D_F)$. $\square$

Proposition 64. *The mappings*

$$\varphi_0 : B(x,\delta_{x,\rho}) \to \overline{B(\varphi_0(x),\rho)}, \quad \varphi_0 : B(y,\delta_{y,\rho}) \to \overline{B(\varphi_0(y),\rho)},$$

where $x, y \in T \cap D_F$, coincide on the intersection $B(x,\delta_{x,\rho}) \cap B(y,\delta_{y,\rho})$ whenever the latter is nonempty.

Proof. In accordance with Definition 62, we can define the value of φ_0 at $z \in B(x,\delta_{x,\rho}) \cap B(y,\delta_{y,\rho})$ starting with either $\varphi_0 : B(x,\delta_{x,\rho}) \to \overline{B(\varphi_0(x),\rho)}$ or $\varphi_0 : B(y,\delta_{y,\rho}) \to \overline{B(\varphi_0(y),\rho)}$. There is a ball $B(z,r_z) \subset B(x,\delta_{x,\rho}) \cap B(y,\delta_{y,\rho})$ on which both ways yield the same. $\square$

Definition 65. For the points $x \in T \cap D_F$, consider the family of balls $B(x,\delta_{x,\rho}) \subset D_F$ from Definition 62. By Proposition 64, on the open set

$$U = \bigcup_{x \in T \cap D_F} B(x,\delta_{x,\rho})$$

a continuous mapping is well-defined; we denote it by $\widetilde{\varphi}_0$. Furthermore, $U \subset D_F$ and $|D_F \setminus U| = 0$.

The mapping $\widetilde{\varphi}_0 : U \to D'_F$ obviously extends $\varphi_0 : T \cap D_F \to D'_F$ to a continuous mapping of the open set U. Since $T \cap D_F$ is dense in U, this continuation is unique.

Proposition 66. *The mapping $\widetilde{\varphi}_0 : U \to \widetilde{\varphi}_0(U)$ is a homeomorphism.*

Proof. Proposition 13 implies that
(1) $\varphi : T \cap D_F \to D'_F$ is injective;
(2) $\varphi(T \cap D_F)$ is dense in D'_F and $|D'_F \setminus \varphi(T \cap D_F)| = 0$;
(3) $\varphi : T \cap D_F \to D'_F$ enjoys the Luzin $\mathscr{N}$- and $\mathscr{N}^{-1}$-properties.

Consequently, by Lemma 14 the inverse mapping $\varphi^{-1} : T' \to D_F$, where $T' = \varphi_0(T)$, induces the composition operator $\varphi^{*-1} : L^1_p(D) \cap C^\infty(D) \to L^1_p(D')$.

Applying the results established above to $\varphi^{-1} : T' \to D_F$, we obtain the continuous mapping $\widetilde{\varphi^{-1}}_0 : V \to D_F$ defined on some open set $V \subset D'_F$ with values in D_F; furthermore, $|D'_F \setminus V| = 0$. We can do it so that $\widetilde{\varphi}_0(U) = V$.

Since $\varphi_0(T \cap D_F)$ is dense in V, the above implies that $\widetilde{\varphi}_0 : U \to V \subset D'_F$ is injective and a homeomorphism. $\square$

4.4. The mapping $\widetilde{\varphi}_0 : U \to V$ is quasiconformal. In this subsection $U \subset D_F$ is an open set of Definition 65, while $V = \widetilde{\varphi}_0(U)$. Here is the main result of the subsection.

Proposition 67. *$\widetilde{\varphi}_0 : U \to V$ is quasiconformal.*

Proof. See [35] for instance. However, we will present other arguments here, which are more widely applicable.

A proof of Proposition 67 reduces essentially to establishing the absolute continuity of $\widetilde{\varphi}_0 : U \to V$ on almost all integral lines of the basis vector fields (briefly, $\widetilde{\varphi}_0 \in \mathrm{ACL}(U)$) and the pointwise inequality

$$|D\widetilde{\varphi}_0(x)| \le K|J(x,\widetilde{\varphi}_0)|^{\frac{1}{n}} \quad \text{a.e. on } U. \tag{32}$$

Since $\widetilde{\varphi}_0 : U \to V$ is an approximatively differentiable homeomorphism [10, Lemma 21], (2) implies that the Jacobian $J(x,\widetilde{\varphi}_0)$ is locally integrable on U. Moreover, by Hölder's inequality, so is the power $J(x,\widetilde{\varphi}_0)^{\frac{1}{n}}$ of the Jacobian.

Lemma 68 [10, Lemma 20]. *Consider two Riemannian manifolds* $\mathbb{M}$ *and* $\mathbb{M}'$ *of the same dimension* $n \geq 2$ *with two domains* $D \subset \mathbb{M}$ *and* $D' \subset \mathbb{M}'$ *and a measurable mapping* $\varphi : D \to D'$ *of class* IL^1_n. *If* $u \in \operatorname{Lip}(D') \cap L^1_n(D')$ *and* $\|u \mid L^1_n(D')\| \leq 1$ *then*

$$|\nabla(u \circ \varphi)|(x) \leq K \cdot J(x, \varphi)^{\frac{1}{n}} \quad \text{a.e. on } D,$$

where K *is some constant.*

Lemma 69. *Given two Riemannian manifolds* $\mathbb{M}$ *and* $\mathbb{M}'$ *of the same dimension* $n \geq 2$ *with two domains* $D \subset \mathbb{M}$ *and* $D' \subset \mathbb{M}'$ *and a measurable mapping* $\varphi : D \to D'$ *of class* IL^1_n, *we have* $\widetilde{\varphi}_0 \in W^1_{n,\mathrm{loc}}(U)$.

Proof. Verify that $\widetilde{\varphi}_0 \in \operatorname{ACL}(U)$. Take a countable dense set $\{z_j\}$ of points in V. Define the countable family of functions $d^r_{z_j} : V \to \mathbb{R}^+$ by $d^r_{z_j}(y) = (r - d_{z_j}(y))^+$, where $r \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ and $d_{z_j}(y) = d(z_j, y)$. Each of these functions satisfies the pointwise equality $\varphi^* d^r_{z_j}(x) = d^r_{z_j} \circ \widetilde{\varphi}_0(x)$ for $r \in \mathbb{Q}^+$, $j \in \mathbb{N}$, and all $x \in U$. Moreover, each of these functions satisfies the hypotheses of Lemma 68. Therefore, $\left|\nabla(d^r_{z_j} \circ \widetilde{\varphi}_0)\right|(x) \leq CJ(x, \widetilde{\varphi}_0)^{\frac{1}{n}}$ for almost all $x \in U$.

Consider the foliation Γ_s of the open set U determined by the vector field X_s and its integral line γ. Almost all lines γ in Γ_s satisfy the conditions:

(1) $\widetilde{\varphi}_0$ is continuous on γ; see Proposition 56.

(2) For measurable functions we have the pointwise inequality

$$\left|\nabla\left(\widetilde{\varphi}_0^* d^r_{z_j}\right)\right|(t) \leq KJ(t, \widetilde{\varphi}_0)^{\frac{1}{n}}, \quad r \in \mathbb{Q}^+,\ j \in \mathbb{N},$$

almost everywhere on γ, and $J(t, \widetilde{\varphi}_0)^{\frac{1}{n}}$ is integrable on each compact subset of γ.

(3) For almost all $x_0 \in \gamma$ there exists a finite limit of

$$\frac{1}{d(x_0, x)} \int\limits_{[x_0, x]} J(t, \widetilde{\varphi}_0)^{\frac{1}{n}}\, d\sigma$$

as $x \to x_0$ along γ, equal to $J(x_0, \widetilde{\varphi}_0)^{\frac{1}{n}}$; here $[x_0, x] \subset \gamma$ is a segment of an integral line.

(4) $\varphi^* d^r_{z_j}$ is absolutely continuous on γ for all $j \in \mathbb{N}$ and $r \in \mathbb{Q}^+$.

Fix a curve $\gamma \in \Gamma_s$ on which all four properties hold.

Assume that $x_0 \in U \cap \gamma$ is a point of positive linear density on γ at which condition (3) holds. Put $z = \widetilde{\varphi}_0(x_0)$. Fix a subsequence $\{z_{j_l}\}$ of $\{z_j\}$ converging to $z = \widetilde{\varphi}_0(x_0)$ and keep for the entries the notation z_l. Since $\widetilde{\varphi}_0$ is continuous on γ at x_0, we can choose δ, r, and L so that $\widetilde{\varphi}_0(B(x_0, \delta) \cap \gamma) \subset V$ (see Corollary 60) and $d^r_{z_l} \circ \widetilde{\varphi}_0(x) \neq 0$ for all $l \geq L$ and all $x \in B(x_0, \delta) \cap \gamma$.

Integrating $KJ(x, \widetilde{\varphi}_0)^{\frac{1}{n}}$, where K is independent of r and z, over the part of γ from x_0 to x, where $x \in B(x_0, \delta) \cap \gamma$, we infer that

$$\begin{gathered} K \int\limits_{[x_0, x]} J(t, \widetilde{\varphi}_0)^{\frac{1}{n}}\, dt \geq \int\limits_{[x_0, x]} \left|\nabla\left(\widetilde{\varphi}_0^* d^r_{z_j}\right)\right|(t)\, dt \\ \geq \left|d^r_{z_l} \circ \widetilde{\varphi}_0(x_0) - d^r_{z_l} \circ \widetilde{\varphi}_0(x)\right| = |r - d_{z_l}(\widetilde{\varphi}_0(x_0)) - r + d_{z_l}(\widetilde{\varphi}_0(x))| \\ = | - d_{z_l}(\widetilde{\varphi}_0(x_0)) + d_{z_l}(\widetilde{\varphi}_0(x))| \to d_z(\widetilde{\varphi}_0(x)) = d(\widetilde{\varphi}_0(x_0), \widetilde{\varphi}_0(x)) \quad \text{as } l \to \infty. \end{gathered}$$

Therefore,

$$d(\widetilde{\varphi}_0(x_0), \widetilde{\varphi}_0(x)) \leq K \int\limits_{[x_0, x]} J(t, \widetilde{\varphi}_0)^{\frac{1}{n}}\, d\sigma \tag{33}$$

for all $x \in B(x_0, \delta) \cap \gamma$. With (33) the absolute continuity of the integral implies that $\widetilde{\varphi}_0$ is absolutely continuous on $B(x_0, \delta) \cap \gamma$.

Since the choices of basis fields X_j, the integral lines $\gamma \in \Gamma_j$, and $z_0 \in \gamma$ are arbitrary, $\widetilde{\varphi}_0$ is absolutely continuous along almost all integral lines.

From (33) we have

$$\frac{d(\widetilde{\varphi}_0(x_0), \widetilde{\varphi}_0(x))}{d(x_0, x)} \leq \frac{K}{d(x_0, x)} \int\limits_{[x_0,x]} J(t, \widetilde{\varphi}_0)^{\frac{1}{n}}\, d\sigma.$$

Passing to the limit as $x \to x_0$, we obtain

$$|X_s \widetilde{\varphi}_0(x_0)| \leq K J(x_0, \widetilde{\varphi}_0)^{\frac{1}{n}}. \tag{34}$$

Consequently, $|X_s \widetilde{\varphi}_0| \in L_{n,\mathrm{loc}}(D_F)$ for all s and $\widetilde{\varphi}_0 \in W^1_{n,\mathrm{loc}}(U)$. $\square$

For other properties of mappings of Sobolev classes on a Riemannian manifold, in particular the change-of-variables formula, see [36] for instance.

Proof of Proposition 67. Lemma 69 shows that $\widetilde{\varphi}_0 : U \to V$ lies in the Sobolev class $W^1_{n,\mathrm{loc}}(U)$. Inequality (32) follows from (34).

4.5. The local connectedness of U and V. Put $S = D_F \setminus U$. Assume that $x \in S$. The two cases are possible:

(1) there is $r_0 > 0$ such that $\overline{\widetilde{\varphi}_0(\widehat{B}(x,r))} \subset D'_F$ for all $r < r_0$;

(2) $\widetilde{\varphi}_0(S(x, r_k)) \cap \partial D'_F \neq \varnothing$ for some sequence $r_k \to 0$.

In case 1 we can assign the value of $\widetilde{\varphi}_0$ at x by putting

$$\widetilde{\varphi}_0(x) = \bigcap_{r\to 0} \overline{\widetilde{\varphi}_0(\widehat{B}(x,r))} \in D'_F.$$

This value of $\widetilde{\varphi}_0(x)$ leads to a situation like in Definition 62. Consequently, we can prescribe $\widetilde{\varphi}_0(x)$ not only at x, but also at the points in some ball $B(x, \delta_{x,\rho})$ by the method of Definition 62. As in Proposition 63, we prove that $\widetilde{\varphi}_0 : B(x, \delta_{x,\rho}) \to D'_F$ is continuous at all points in $B(x, \delta_{x,\rho})$. Therefore, we can increase U and V while decreasing S.

Assume henceforth that (2) holds for all $x \in S$.

Take $x \in S$. There is a sequence $\{x_k \in U\}$ converging to x such that $\widetilde{\varphi}_0(x_k) \to \partial D'$ as $k \to \infty$. In this case the following lemma applies:

Lemma 70. *$B(x,r) \cap U$ is connected for every ball $B(x,r) \subset D_F$ centered at some $x \in S$.*

Proof. Suppose that the claim is false and $B(x,r) \cap U$ consists of several connected components. Then $\partial D'_F$ divides $\widetilde{\varphi}_0(B(x,r))$ into several connected components: $\widetilde{\varphi}_0(B(x,r)) = V_1 \cup V_2 \cup \cdots$ or $D'_F \setminus \widetilde{\varphi}_0(S(x,r)) = V_0 \cup V_1 \cup V_2 \cup \cdots$.

In D_F consider the smooth cutoff

$$\eta = \begin{cases} 1 & \text{on } B(x, r/2), \\ 0 & \text{outside } B(x,r). \end{cases}$$

Assume, for instance, that $\left|\widetilde{\varphi}_0^{-1}(V_1) \cap B(x, r/2)\right| > 0$. Construct $g : D'_F \to \mathbb{R}$ satisfying

$$g(y) = \begin{cases} \eta \circ \widetilde{\varphi}_0^{-1}(y) & \text{on } V_1, \\ 0 & \text{on } V_0 \cup V_2 \cup V_3 \cup \cdots. \end{cases}$$

Clearly, g is a continuous function on V. Verify that $g \in L^1_{n,\varphi(F)}(D')$. Since

$$\widetilde{\varphi}_0 : B(x,r) \cap U \to \widetilde{\varphi}_0(B(x,r)) \cap V$$

is quasiconformal, g belongs to $L^1_n(\widetilde{\varphi}_0(B(x,r)) \cap V)$. Consequently, g is locally integrable and has generalized derivatives in $\widetilde{\varphi}_0(B(x,r)) \cap V$ summable to power n. In particular, in some coordinate neighborhood W' of $\mathbb{M}'$ the function g is absolutely continuous on almost all integral lines of the basis vector fields $\frac{\partial}{\partial x_j}$, and the derivatives $v_j = \frac{\partial g}{\partial x_j}$ exist a.e. in $\widetilde{\varphi}_0(B(x,r)) \cap V$ for $j = 1, 2, \dots, n$. It remains to show that v_j is a generalized derivative of g in W'; namely,

$$\int\limits_{W'} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy = -\int\limits_{W'} v_j(y) \cdot \eta(y)\, dy \tag{35}$$

for every test function $\eta \in C_0^\infty(W')$. Fubini's Theorem yields

$$\int\limits_{W'} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy = \int\limits_{\operatorname{Pr}_j W'} dy_1 \dots \widehat{dy_j} \dots dy_N \int\limits_{\operatorname{Pr}_j^{-1}(y) \cap W'} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j,$$

where $\operatorname{Pr}_j W'$ is the projection of W' to the hypersurface transversal to the vector field $\frac{\partial}{\partial x_j}$ and $\operatorname{Pr}_j^{-1}(y)$ is the integral line of $\frac{\partial \eta}{\partial x_j}$ passing through $y \in \operatorname{Pr}_j W'$. Since $g = 0$ on $V_0 \cup V_2 \cup V_3 \cup \cdots$, we obtain

$$\int\limits_{\operatorname{Pr}_j^{-1}(y) \cap W'} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j = \int\limits_{\operatorname{Pr}_j^{-1}(y) \cap V_0 \cup V_1 \cap V_2 \cup \dots} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j = \int\limits_{\operatorname{Pr}_j^{-1}(y) \cap V_1} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j.$$

We can express $\operatorname{Pr}_j^{-1}(y) \cap V_1$ as the countable union of intervals: $\operatorname{Pr}_j^{-1}(y) \cap V_1 = \bigcup_l \gamma_l$. Then

$$\int\limits_{\operatorname{Pr}_j^{-1}(y) \cap V_1} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j = \sum_l \int\limits_{\gamma_l} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j.$$

Integrating by parts, we see that

$$\int\limits_{\gamma_l} g(y) \cdot \frac{\partial \eta}{\partial x_j}(y)\, dy_j = g(y)\psi(y)\Big|_{\gamma_l(t_0^l)}^{\gamma_l(t_1^l)} - \int\limits_{\gamma_l} \frac{\partial g}{\partial x_j}(y) \cdot \psi(y)\, dy_j.$$

Observe that $\gamma_l\big(t_0^l\big) \in \partial V_1$ and the two cases are possible:

(1) $\gamma_l\big(t_0^l\big) \in D'_F$ and then $g\big(\gamma_l\big(t_0^l\big)\big) = 0$;

(2) $\gamma_l\big(t_0^l\big) \in \partial D'_F$ and then $\psi\big(\gamma_l\big(t_0^l\big)\big) = 0$.

The situation is similar for $\gamma_l\big(t_1^l\big)$. Therefore, the term outside the integral vanishes:

$$g(y)\psi(y)\Big|_{\gamma_l(t_0^l)}^{\gamma_l(t_1^l)} = 0,$$

and we have justified (35). Consequently, $g \in L^1_{n,\varphi(F)}(W')$. Since $W' \subset D'_F$ is an arbitrary coordinate neighborhood, $g \in L^1_{n,\varphi(F)}(D'_F)$.

Moreover, $\varphi^* g$ belongs to $L^1_{n,F}(D)$ and a.e. on $B(x, r/2)$ takes only two values 0 and 1. Consequently, $\nabla \varphi^* g = 0$ a.e. on $B(x, r/2)$, and so $\varphi^* g = g \circ \widetilde{\varphi}_0$ is a constant function on $B(x, r/2)$. The resulting contradiction leads to the conclusion that $B(x, r) \cap U$ is connected. $\square$

The image has a similar property.

Lemma 71. *$B(y, r) \cap V$ is connected for every ball $B(y, r) \subset D'_F$ centered at some $y \in \varphi_0(S)$.*

4.6. Continuation of $\widetilde{\varphi}_0$ to S and the properties of the continuation. In this subsection we need the following lemma:

Lemma 72. *Given two curves $\gamma_1, \gamma_2 : [0,1) \to V$ with positive distance between them, no point of D_F can be a limit point for both preimages $\beta_1 = \widetilde{\varphi}_0{}^{-1}(\gamma_1)$ and $\beta_2 = \widetilde{\varphi}_0{}^{-1}(\gamma_2)$.*

Proof. Assume on the contrary that some $y \in D_F$ is a limit point for both $\widetilde{\varphi}_0{}^{-1}(\gamma_1)$ and $\widetilde{\varphi}_0{}^{-1}(\gamma_2)$: there exist two sequences $t_k \in [0,1)$ and $\tau_k \in [0,1)$ with $t_k \to 1$ and $\tau_k \to 1$ as $k \to \infty$ such that $\beta_1(t_k) \to y$ and $\beta_2(\tau_k) \to y$ as $k \to \infty$. Consider a continuous function $g \in L^1_n(D')$ with $g = 0$ on γ_1 and $g = 1$ on γ_2. Then $f = g \circ \widetilde{\varphi}_0 : U \to \mathbb{R}$ is a continuous function taking the values 0 on β_1 and 1 on β_2. Moreover, $\varphi^*(g) \in L^1_n(D)$. The existence of this function contradicts Proposition 45. □

Verify that the mapping $\widetilde{\varphi}_0$ extends to the subset of S excluding the points that are possibly mapped to the point at infinity. Take $x \in S$. The two cases are possible:

(1) For some sequence $\{x_n \in U\}$ converging to x the sequence of images $\widetilde{\varphi}_0(x_n)$ converges to some $z \in \partial D'_F$.

(2) For every sequence $\{x_n \in U\}$ converging to x we have the convergence $d(\widetilde{\varphi}_0(x_n), x_0) \to \infty$, where x_0 is a fixed point of the compact set $\varphi(F)$. This case is treated below.

Let us show that in the first case $\widetilde{\varphi}_0$ extends by continuity at $x \in S$.

Proposition 73. *The mapping $\widetilde{\varphi}_0 : U \to V$ extends by continuity at all points $x \in S$, for each of which there exists a sequence $\{x_n \in U\}$ converging to x such that the sequence of images $\widetilde{\varphi}_0(x_n)$ converges to some $z \in \partial D'_F$. The extended mapping is injective.*

Proof. Verify that the limit z is independent of the choice of $\{x_n\}$. Take another sequence $U \ni x'_n \to x$ with $\widetilde{\varphi}_0(x'_n) \to z' \in \partial D'_F$ and suppose that $z \neq z'$. Since V is locally connected, we can construct two curves $\gamma, \gamma' \subset V$ lying at some positive distance $\operatorname{dist}(\gamma, \gamma') \geq \delta > 0$ from each other and passing through the images $\widetilde{\varphi}_0(x_n)$ and $\widetilde{\varphi}_0(x'_n)$ respectively starting with some $n > n_0$. Then $\widetilde{\varphi}_0{}^{-1}(\gamma)$ and $\widetilde{\varphi}_0{}^{-1}(\gamma')$ have the limit point $x \in D_F$. Lemma 72 yields a contradiction.

Extend $\widetilde{\varphi}_0$ by putting $\widetilde{\varphi}_0(x) = z$. This yields a continuous extension of $\widetilde{\varphi}_0$ to S with the exception of the points mapped to the point at infinity. Denote the extension by the same symbol.

Verify that $\widetilde{\varphi}_0$ is injective. Suppose that there is $z \in Z$ with $z = \widetilde{\varphi}_0(x_1) = \widetilde{\varphi}_0(x_2)$, where $x_1, x_2 \in S$ and $x_1 \neq x_2$. Consider two curves γ_1 and γ_2 passing through x_1 and x_2 respectively and lying at some positive distance $\delta = \operatorname{dist}(\gamma_1, \gamma_2)$ from each other. Consider arbitrary sequences $\{x^1_n \in U\}$ and $\{x^2_n \in U\}$ such that $x^i_n \to x_i$ as $n \to \infty$ and $x^i_n \in \gamma_i$. Construct a sequence of curves σ_n connecting the points $\widetilde{\varphi}_0(x^1_n)$ and $\widetilde{\varphi}_0(x^2_n)$ so that $\operatorname{diam} \sigma_n \to 0$. Then $\operatorname{Cap}(\widetilde{\varphi}_0^{-1}(\sigma_n); L^1_n(U)) \to 0$ and $\operatorname{diam} \widetilde{\varphi}_0^{-1}(\sigma_n) \to 0$. We arrive at a contradiction because $\operatorname{diam} \widetilde{\varphi}_0^{-1}(\sigma_n) \geq \delta$. □

Thus, of S only those points remain that match the second case preceding Proposition 73. The following statement shows that if S is nonempty then S amounts to a single point.

Lemma 74. *At most one point $x_{\rm inv} \in S$ may exist such that for every sequence $\{x_n\} \subset U$ converging to $x_{\rm inv}$ we have $d(\widetilde{\varphi}_0(x_n), x_0) \to \infty$ as $n \to \infty$ (the case of inversion).*

Proof. Firstly, verify that S has capacity zero. Choose a ball $B(0, r_0)$ with $\varphi(F) \subset B(0, r_0)$, and a sequence of balls $B(0, R_k)$, for $k \in \mathbb{N}$, such that $R_k > r$ and $\lim_{k\to\infty} R_k = \infty$. Observe that $S \subset \bigcap_k \overline{\widetilde{\varphi}_0^{-1}(\mathbb{M}' \setminus B(0, R_k))}$.

We have

$$\begin{aligned}\overline{\operatorname{Cap}}\big(S; L^1_{n,F}(D)\big) &\leq \overline{\operatorname{Cap}}\big(\widetilde{\varphi}_0^{-1}(\mathbb{M}' \setminus B(0, R_k)) \cap D_F; L^1_{n,F}(D)\big) \\ &\leq K^n \overline{\operatorname{Cap}}\big((\mathbb{M}' \setminus B(0, R_k)) \cap D'_F; L^1_{n,\varphi(F)}(D')\big) \leq K^n \operatorname{Cap}\big(\mathbb{M}' \setminus B(0, R_k); L^1_{n,\varphi(F))}(\mathbb{M}')\big).\end{aligned}$$

By Definition 3, $\operatorname{Cap}\big(\mathbb{M}' \setminus B(0, R_k); L^1_{n,\varphi(F))}(\mathbb{M}')\big)$ vanishes as $k \to \infty$. Thus, $\overline{\operatorname{Cap}}\big(S; L^1_{n,F}(D)\big) = 0$.

Verify that S cannot contain more than one point. Assume on the contrary that there are two distinct points $x_1, x_2 \in S$ satisfying the properties mentioned. Consider two sequences $\{x^1_n\}, \{x^2_n\} \subset U$ with $\lim_{n\to\infty} x^1_n = x_1$ and $\lim_{n\to\infty} x^2_n = x_2$. Choose two spheres $S(x_1, r_1), S(x_2, r_2) \subset U$ on which $\widetilde{\varphi}_0$ is continuous (see Proposition 56) and such that $\overline{B}(x_1, r_1) \cap \overline{B}(x_2, r_2) = \varnothing$. Since $\widetilde{\varphi}_0$ is continuous and injective,

$\widetilde{\varphi}_0(S(x_1, r_1))$ subdivides D'_F into two connected components, one bounded and one unbounded; furthermore, $\widetilde{\varphi}_0(B(x_1, r_1) \setminus S)$ lies in the unbounded component, whereas $\widetilde{\varphi}_0(U \setminus B(x_1, r_1))$ lies in the bounded component. On the other hand, $B(x_2, r_2) \setminus S \subset U \setminus B(x_1, r_1)$, and consequently $\widetilde{\varphi}_0(B(x_2, r_2) \setminus S)$ lies in the bounded component $D'_F \setminus \widetilde{\varphi}_0(S(x_1, r_1))$, which contradicts the assumption that $d\big(\widetilde{\varphi}_0\big(x_n^2\big), x_0\big) \to \infty$ as $n \to \infty$. $\square$

As a result, we obtain a continuous injective mapping $\widetilde{\varphi}_0 : D_F \setminus \{x_{\rm inv}\} \to \overline{D'_F}$.

Proposition 75. $\widetilde{\varphi}_0 : D_F \setminus \{x_{\rm inv}\} \to \widetilde{\varphi}_0(D_F \setminus \{x_{\rm inv}\})$ *is a homeomorphism.*

Proof. It suffices to verify that

$$\widetilde{\varphi}_0 : D_F \setminus \{x_{\rm inv}\} \to \widetilde{\varphi}_0(D_F \setminus \{x_{\rm inv}\})$$

is open. Indeed, for every ball $B(x, r) \subset D_F \setminus \{x_{\rm inv}\}$ we have $\mu(\widetilde{\varphi}_0, B(x, r), \varphi_0(x)) \neq 0$. From this we see that $\widetilde{\varphi}_0(x)$ is an interior point of the image.

Now we can prove that $\widetilde{\varphi}_0$ is of Sobolev class $W^1_{n,{\rm loc}}(D_F \setminus \{x_{\rm inv}\})$, extending Lemma 69.

Lemma 76. *Given two Riemannian manifolds $\mathbb{M}$ and $\mathbb{M}'$ of the same dimension $n \geq 2$ with two domains $D \subset \mathbb{M}$ and $D' \subset \mathbb{M}'$ and a measurable mapping $\varphi : D \to D'$ of class IL^1_n, we have $\widetilde{\varphi}_0 \in W^1_{n,{\rm loc}}(D_F \setminus \{x_{\rm inv}\})$.*

Proof. This lemma is straightforward from Lemma 69. In the hypotheses of the latter, we should take $D_F \setminus \{x_{\rm inv}\}$ as U. $\square$

The argument above implies

Proposition 77. $\widetilde{\varphi}_0 : D_F \setminus \{x_{\rm inv}\} \to \mathbb{M}'$ *is quasiconformal.*

Proof. The previous statements show that the homeomorphism $\widetilde{\varphi}_0$ is of class $W^1_n(D_F \setminus \{x_{\rm inv}\})$, and the pointwise inequality $|D(x, \widetilde{\varphi}_0)| \leq K|J(x, \varphi)|^{\frac{1}{n}}$ holds almost everywhere in $D_F \setminus \{x_{\rm inv}\}$ because $|S| = 0$; observe that $J(x, \varphi) = J(x, \widetilde{\varphi}_0)$ almost everywhere. Consequently, $\widetilde{\varphi}_0 : D_F \setminus \{x_{\rm inv}\} \to \mathbb{M}'$ is quasiconformal. $\square$

Proposition 78. $\widetilde{\varphi}_0 : D \setminus \{x_{\rm inv}\} \to \mathbb{M}'$ *is quasiconformal.*

Proof. Choose another closed set $F_1 \subset T_{k_0}$ of positive measure without isolated points lying at positive distance from F. Repeating the procedure described above, we can show that $\widetilde{\varphi}_0$ is quasiconformal on the open set $D \setminus \{x_{\rm inv}\}$. $\square$

4.7. Proof of Theorem 2. Let us now prove the main result of this article.

Sufficiency. We may assume that $\varphi : D \to D'$ is quasiconformal. By Definition 4, the quasiconformal mapping φ is locally of Sobolev class, $\varphi \in W^1_{n,{\rm loc}}$. Moreover, φ is differentiable and has the Luzin $\mathscr{N}$- and $\mathscr{N}^{-1}$-properties [36].

For every $f \in L^1_n(D') \cap C^\infty(D')$ the composition $f \circ \varphi$ is absolutely continuous on almost all integral lines of the basis vector fields because so is f. Moreover, [17, p. 263] shows that $\nabla_{\mathscr{L}}(f \circ \varphi) = D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x))$, where $D\varphi(x) = \{X_i\varphi_j(x)\}$ for $i, j = 1, \dots, n_1$ is the differential. Hence,

$$\begin{aligned}\int\limits_D |\nabla_{\mathscr{L}}(f \circ \varphi)|^n \, dx &= \int\limits_D |D_h\varphi^T(x)\nabla_{\mathscr{L}} f(\varphi(x))|^n \, dx \\ \leq \int\limits_D |D_h\varphi^T(x)|^n \cdot |\nabla_{\mathscr{L}} f(\varphi(x))|^n \, dx &= \int\limits_D |\nabla_{\mathscr{L}} f|^n(\varphi(x)) \cdot |D_h\varphi(x)|^n \, dx \\ \leq K \int\limits_D |\nabla_{\mathscr{L}} f|^n(\varphi(x)) \cdot |J(x, \varphi)| \, dx &= \int\limits_{D'} |\nabla_{\mathscr{L}} f|^n(y) \, dy.\end{aligned}$$

Here we use the pointwise inequality $|D_h\varphi(x)|^n \leq K|J(x, \varphi)|$ for almost all $x \in D$ and (2).

By Lemma 14, the resulting inequality holds for all $f \in L^1_n(D')$; i.e.,

$$\|\varphi^*(f) \mid L^1_n(D)\| \le K^{\frac{1}{n}} \|f \mid L^1_n(D')\|.$$

The mapping φ^{-1} is also quasiconformal. Then for $g \in L^1_n(D)$ we have

$$\|\varphi^{-1*}(g) \mid L^1_n(D')\| \le K_1^{\frac{1}{n}} \|g \mid L^1_n(D)\|, \tag{36}$$

where K_1 is the quasiconformality coefficient of the inverse mapping. Observe that $f \in L^1_n(D') \cap C^\infty(D')$ satisfies $\varphi^{-1*}(f \circ \varphi) = f$. Consequently, (36) becomes $K_1^{-\frac{1}{n}} \|f \mid L^1_n(D')\| \le \|\varphi^*(f) \mid L^1_n(D)\|$. Thus,

$$K_1^{-\frac{1}{n}} \|f \mid L^1_n(D')\| \le \|\varphi^*(f) \mid L^1_n(D)\| \le K^{\frac{1}{n}} \|f \mid L^1_n(D')\|, \tag{37}$$

where the constants K and K_1 depend only on the properties of φ.

In order to verify that $\varphi^*\big(L^1_n(D') \cap C^\infty(D')\big)$ is dense in $L^1_n(D)$, take $g \in L^1_n(D)$. There is a sequence $g_n \in L^1_n(D) \cap C^\infty(D)$ with $\|g - g_n \mid L^1_n(D)\| \to 0$. On the other hand, (37) yields $g_n \circ \varphi^{-1} \in L^1_n(D')$. Hence, there is a sequence $f_{nk} \in L^1_n(D') \cap C^\infty(D')$ with $\|g_n \circ \varphi^{-1} - f_{nk} \mid L^1_n(D')\| \to 0$ as $k \to \infty$. Then for some sequence l_n of positive integers we have $\varphi^* f_{nl_n} \in \varphi^*\big(L^1_n(D') \cap C^\infty(D')\big)$ and $\|g - \varphi^* f_{nl_n} \mid L^1_n(D)\| \to 0$ as $n \to \infty$.

Necessity. The existence of a quasiconformal mapping Φ is established in Lemma 76; moreover, $\Phi = \widetilde{\varphi}_0 : D \setminus \{x_{\mathrm{inv}}\} \to \mathbb{M}'$. Basing on the argument above, we see that the composition operator

$$\Phi^* : L^1_n(\Phi(D \setminus \{x_{\mathrm{inv}}\})) \to L^1_n(D \setminus \{x_{\mathrm{inv}}\})$$

is an isomorphism. Since evidently $L^1_n(D \setminus \{x_{\mathrm{inv}}\}) = L^1_p(D)$, this yields the isomorphism

$$\varphi^{*-1} \circ \Phi^* : L^1_n(\Phi(D \setminus \{x_{\mathrm{inv}}\})) \to L^1_n(D')$$

satisfying $\varphi^{*-1} \circ \Phi^*(f)(x) = f(x)$ for all points $x \in \Phi(D \setminus \{x_{\mathrm{inv}}\}) \cap D'$, where $f \in L^1_n(\Phi(D \setminus \{x_{\mathrm{inv}}\})$ is an arbitrary function.

Consequently, the restriction operator makes the space $L^1_n(\Phi(D \setminus \{x_{\mathrm{inv}}\}) \cup D')$ isomorphic to both $L^1_n(\Phi(D \setminus \{x_{\mathrm{inv}}\}))$ and $L^1_n(D')$. Thus, $\Phi(D \setminus \{x_{\mathrm{inv}}\})$ and D' are $(1, n)$-equivalent domains.

By analogy with Theorem 3.1 of [12] and Proposition 6.10 of [13], we can obtain the properties:

(1) $|\Phi(D) \Delta D'| = 0$;

(2) $B \setminus \Phi(D) \Delta D'$ is connected for every ball $B \subset D'$. $\square$

4.8. Corollary: removable sets for quasiconformal mappings. Recall that a closed set $E \subset D$ is called *removable* for quasiconformal mappings whenever each quasiconformal mapping $\varphi : D \setminus E \to \mathbb{M}'$ extends to a quasiconformal mapping of D.

Corollary 79. *Consider $U \subset D$ such that U and D are $(1, n)$-equivalent. Then $D \setminus U$ is removable for quasiconformal mappings.*

Proof. Take a quasiconformal mapping $\varphi_1 : U \to \mathbb{M}'$. To prove the corollary, we need to construct a quasiconformal continuation of φ_1 to D.

By Theorem 2, the composition operator $\varphi_1^* : L^1_n(\varphi_1(U)) \to L^1_n(U)$ is an isomorphism. Since U and D are $(1, n)$-equivalent sets, the restriction operator $r^* : L^1_n(D) \to L^1_n(U)$ is also an isomorphism.

Consider a measurable mapping $\varphi : D \to \varphi_1(U)$ with $\varphi(x) = \varphi_1(x)$ for $x \in U$. The composition operator $\varphi^* : L^1_n(\varphi_1(U)) \cap C^\infty(\varphi_1(U)) \to L^1_n(D)$ defined as $\varphi^* f = f \circ \varphi$ extends to an isomorphism between $L^1_n(\varphi_1(U))$ and $L^1_n(D)$ because $\varphi^* f = r^{*-1} \circ \varphi_1^* f$ for $f \in L^1_n(\varphi_1(U)) \cap C^\infty(\varphi_1(U))$. By Theorem 2, there is a quasiconformal mapping $\Phi : D \to \mathbb{M}'$ coinciding with φ almost everywhere. Furthermore, $\Phi(x) = \varphi(x)$ if $x \in U$. Thus, Φ is a required continuation. $\square$

References

1. Vodopyanov S. K. and Goldshtein V. M., “Lattice isomorphisms of the spaces W_n^1 and quasiconformal mappings,” Sib. Math. J., vol. 16, no. 2, 174–189 (1975).
2. Vodopyanov S. K. and Goldshtein V. M., “Functional characteristics of quasi-isometric mappings,” Sib. Math. J., vol. 17, no. 4, 580–584 (1976).
3. Romanov A. S., “A change of variable in the Bessel and Riesz potential spaces,” in: *Functional Analysis and Mathematical Physics* [Russian], Inst. Mat., Novosibirsk, 1985, 117–133.
4. Vodop’yanov S. K., “Composition operators on Sobolev spaces,” in: *Complex Analysis and Dynamical Systems II. Contemporary Mathematics*, Amer. Math. Soc., Providence, 2005, 247–301 (Contemp. Math.; V. 382).
5. Vodopyanov S. K., “L_p-Potential theory and quasiconformal mappings on homogeneous groups,” in: *Modern Problems of Geometry and Analysis* [Russian], Nauka, Novosibirsk, 1989, 45–89.
6. Vodopyanov S. K. and Evseev N. A., “Isomorphisms of Sobolev spaces on Carnot groups and quasi-isometric mappings,” Sib. Math. J., vol. 55, no. 5, 817–848 (2014).
7. Vodopyanov S. K. and Evseev N. A., “Isomorphisms of Sobolev spaces on Carnot groups and metric mapping properties,” Dokl. Math., vol. 82, no. 2, 232–236 (2015).
8. Vodopyanov S. K. and Evseev N. A., “Isomorphisms of Sobolev spaces on Carnot groups and quasiconformal mappings,” Sib. Math. J., vol. 56, no. 5, 789–821 (2015).
9. Vodopyanov S. K., “On admissible changes of variables for Sobolev functions on (sub)Riemannian manifolds,” Dokl. Math., vol. 93, no. 3, 318–321 (2016).
10. Vodopyanov S. K., “Admissible changes of variables for Sobolev functions on (sub-)Riemannian manifolds,” Sb. Math., vol. 210, no. 1, 59–104 (2019).
11. Federer H., *Geometric Measure Theory*, Springer-Verlag, New York (1969).
12. Vodopyanov S. K. and Goldshtein V. M., “Criteria for the removability of sets in spaces of L_p^1 quasiconformal and quasi-isometric mappings,” Sib. Math. J., vol. 18, no. 1, 35–50 (1977).
13. Chernikov V. M. and Vodopyanov S. K., “Sobolev spaces and hypoelliptic equations. I,” Siberian Adv. Math., vol. 6, no. 3, 27–67 (1996).
14. Chernikov V. M. and Vodopyanov S. K., “Sobolev spaces and hypoelliptic equations. II,” Siberian Adv. Math., vol. 6, no. 4, 64–96 (1996).
15. Reshetnyak Yu. G., “Sobolev-type classes of functions with values in a metric space,” Sib. Math. J., vol. 38, no. 3, 567–582 (1997).
16. Vodopyanov S. K., “P-Differentiability on Carnot groups in various topologies and related topics,” in: *Studies on Analysis and Geometry* [Russian], Izdat. Inst. Mat., Novosibirsk, 2000, 603–670.
17. Vodop’yanov S. K., “Geometry of Carnot–Carathéodory spaces and differentiability of mappings,” in: *The Interaction of Analysis and Geometry. Contemporary Mathematics*, Amer. Math. Soc., Providence, 2007, 247–301 (Contemp. Math.; V. 424).
18. Hajłasz P., “Change of variable formula under the minimal assumptions,” Colloq. Math., vol. 64, no. 1, 93–101 (1993).
19. Isangulova D. V. and Vodopyanov S. K., “Coercive estimates and integral representation formulas on Carnot groups,” Eurasian Math. J., vol. 1, no. 3, 58–96 (2010).
20. Jerison D., “The Poincaré inequality for vector fields satisfying Hörmander’s condition,” Duke Math. J., vol. 53, 503–523 (1986).
21. Lu G., “The sharp Poincaré inequality for free vector fields: an endpoint result,” Rev. Mat. Iberoam., vol. 10, no. 2, 453–466 (1994).
22. Hajłasz P. and Koskela P., *Sobolev Met Poincaré*, Amer Math. Soc., Ann Arbor (2000) (Mem. Amer. Math. Soc.; No. 688).
23. Franchi B., Lu G., and Wheeden R. L., “Representation formulas and weighted Poincaré inequalities for Hörmander vector fields,” Ann. Inst. Fourier (Grenoble), vol. 45, no. 2, 577–604 (1992).
24. John F., “Rotation and strain,” Comm. Pure Appl. Math., vol. 14, 391–413 (1961).
25. Stein E. M., *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ (1993).
26. Reshetnyak Yu. G., “The concept of capacity in the theory of functions with generalized derivatives,” Sib. Math. J., vol. 10, no. 5, 818–842 (1969).
27. Mazya V. G. and Havin V. P., “Non-linear potential theory,” Russian Math. Surveys, vol. 27, no. 6, 71–148 (1972).

28. Heinonen J. and Kilpeläinen T., and Martio O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, Oxford (1993).
29. Choquet G., "Theory of capacities," Ann. Inst. Fourier (Grenoble), vol. 9, 83–89 (1959).
30. Bojarski B., "Remarks on some geometric properties of Sobolev mappings," in: *Functional Analysis and Related Topics* (S. Koshi (ed.)), World Scientific, Singapore, 1991, 65–76.
31. Kinnunen J. and Latvala V., "Lebesgue points for Sobolev functions on metric spaces," Rev. Mat. Iberoam., vol. 18, 685–700 (2002).
32. De Rham G., *Variétés Différentiables*, Hermann, Paris (1955).
33. Vodopyanov S. K. and Kudryavtseva N. A., "Nonlinear potential theory for Sobolev spaces on Carnot groups," Sib. Math. J., vol. 50, no. 5, 803–819 (2009).
34. Troyanov M. and Vodopyanov S. K., "Liouville type theorems for mappings with bounded (co)-distortion," Ann. Inst. Fourier, vol. 52, no. 6, 1753–1784 (2002).
35. Vodopyanov S. K., "Monotone functions and quasiconformal mappings on Carnot groups," Sib. Math. J., vol. 37, no. 6, 1113–1136 (1996).
36. Vodopyanov S. K., "Differentiability of maps of Carnot groups of Sobolev classes," Sb. Math., vol. 194, no. 6, 857–877 (2003).

S. K. Vodopyanov
Sobolev Institute of Mathematics
Novosibirsk State University, Novosibirsk, Russia
E-mail address: vodopis@math.nsc.ru