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TESTS FOR THE OSCILLATION OF AUTONOMOUS DIFFERENTIAL EQUATIONS WITH BOUNDED AFTEREFFECT

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Abstract: Considering autonomous delay functional differential equations, we establish some oscillation criterion that reduces the oscillation problem to computing the only root of the real-valued function defined by the coefficients of the initial equation. Using the criterion, we obtain effectively verifiable oscillation tests for equations with various aftereffects.

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For second- and higher-order ordinary differential equations, the problem of describing the oscillating solutions has been posed and solved starting from Chaplygin's classical works [1]. Passage to functional-differential equations (FDEs) makes this problem significant already for first-order equations: As is known (see [2]), solutions of the latter can have zeros and oscillate. On the other hand, the substantial extension of the class of equations complicates inspection and requires some new ideas and methods.

Consider an FDE of the form

$$\dot{x}(t) + \mu \int_0^r x(t-s)\,dk(s) = 0, \quad t \ge 0. \tag{1}$$

The parameters of (1) obey the following constraints: μ and r are reals and $r > 0$, while k is a function of bounded variation with $k(0) = 0$, the integral is understood in the Riemann–Stieltjes sense. For nonpositive values of the arguments, a solution is extended by a continuous (a priori defined) initial function. Under these assumptions, as is known (see [3]), a solution to (1) exists and is unique in the class of locally absolutely continuous functions.

Call a continuous function on the half-axis *oscillating* if it has a sequence of zeros unbounded from the right. Call equation (1) *oscillating* if its all solutions (under any choice of the initial function) are oscillating functions.

With (1), associate the characteristic function

$$F(\lambda) \equiv -\lambda + \mu \int_0^r e^{\lambda s}\,dk(s), \quad \lambda \in \mathbb{C}.$$

The following assertion relates the oscillation of (1) to the properties of the characteristic function:

Theorem 1 [4–6]. *For* (1) *to be oscillating, it is necessary and sufficient that the equation* $F(\lambda) = 0$ *have no real roots.*

Using Theorem 1, it is easy to obtain oscillation criteria for simple equations of the form (1); alongside the criteria obtained at the beginning of the 1980s, these criteria were systemized in [4].

Over the past decades, there had appeared no substantially new results for autonomous FDEs. The focus of the researchers' attention has shifted towards *nonautonomous* FDEs to which Theorem 1 is

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inapplicable. It is natural for the latter to pose the task of obtaining exact sufficient oscillation tests and use other ideas and methods (see [7–11]).

The absence of any new tests for equations of the form (1) is explained by the fact that, with the complication of the structure of the aftereffect, the structure of the characteristic function becomes more intricate: Direct examination of the set of its real roots turns out to be a rather difficult task.

This paper presents a criterion that makes it possible to solve the oscillation problem for (1) more effectively than by Theorem 1.

We exclude from consideration the trivial case when (1) is an ordinary differential equation since the latter has no oscillating solutions.

Let k be a monotone function on $[0, r]$. Without loss of generality, consider only the case of nondecreasing k (the case of a nonincreasing function is reduced to this case by the change of the sign of the coefficient μ). If $\mu = 0$ then $F(0) = 0$, and if $\mu < 0$ then $F(0)F(-\infty) < 0$, i.e., for $\mu \le 0$ the characteristic function has real roots. Therefore, we assume henceforth that $\mu > 0$.

Put

$$G(\zeta) \equiv \zeta \int_0^r s e^{\zeta s}\, dk(s) - \int_0^r e^{\zeta s}\, dk(s),$$

where $\zeta \in \mathbb{R}$. Note that G is continuously differentiable (see [12, Lemma 3.1]). It is easy to see that $G(\zeta) < 0$ for $\zeta \le 0$, and if $\zeta > 0$ then $G'(\zeta) > 0$; i.e., G increases monotonically on the positive axis. Since $G(\zeta) \to +\infty$ as $\zeta \to +\infty$, the equation $G(\zeta) = 0$ has a unique root on the real axis (which is positive). Denote the latter by ζ_*.

Theorem 2. *The function F is positive for all $\zeta \ge 0$ if and only if $\mu \int_0^r e^{\zeta_* s}\, dk(s) > \zeta_*$.*

Proof. The hypotheses of the theorem and the properties of ζ_* imply that

$$\int_0^r e^{\zeta_* s}\, dk(s) > 0.$$

Put

$$\mu_* = \zeta_* \left(\int_0^r e^{\zeta_* s}\, dk(s) \right)^{-1}$$

and consider

$$F_*(\zeta) = -\zeta + \mu_* \int_0^r e^{\zeta s}\, dk(s)$$

on $\mathbb{R}$. By [12, Lemma 3.1], F_* is infinitely differentiable and

$$F_*'(\zeta) = -1 + \mu_* \int_0^r s\, e^{\zeta s}\, dk(s), \quad F_*''(\zeta) = \mu_* \int_0^r s^2\, e^{\zeta s}\, dk(s).$$

Since $F_*''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$; therefore, F_*' increases monotonically on $\mathbb{R}$ from -1 to $+\infty$. Hence, there exists a unique $\zeta_0 \in \mathbb{R}$ satisfying $F_*'(\zeta_0) = 0$. On the other hand,

$$\int_0^r s e^{\zeta_* s}\, dk(s) = \frac{1}{\zeta_*} \int_0^r e^{\zeta_* s}\, dk(s) = \frac{1}{\mu_*};$$

consequently, $F_*'(\zeta_*) = -1 + \mu_*/\mu_* = 0$, and so $\zeta_0 = \zeta_*$. Thus, $F_*(\zeta_0) = F_*'(\zeta_0) = 0$ and ζ_0 is a minimum point for F_*; therefore, $F_*(\zeta) \ge 0$ for all $\zeta \ge 0$.

Turn to proving Theorem 2. Let the assumption of Theorem 2 be fulfilled which can be rewritten as $\mu > \mu_*$ in view of the above notations. Then, for every $\zeta \in [0, \infty)$, we have

$$F(\zeta) = -\zeta + \mu \int_0^r e^{\zeta s}\, dk(s) = F_*(\zeta) + (\mu - \mu_*) \int_0^r e^{\zeta s}\, dk(s) > 0,$$

which was required. Conversely, if $\mu \le \mu_*$ then

$$F(\zeta_*) = F_*(\zeta_*) + (\mu - \mu_*) \int_0^r e^{\zeta s}\, dk(s) = (\mu - \mu_*) \int_0^r e^{\zeta s}\, dk(s) \le 0.$$

Consequently, F is not positive on $[0, \infty)$. The theorem is proved.

Combining Theorems 1 and 2, we obtain

Theorem 3. *The following are equivalent:*
(i) *Equation* (1) *is oscillating.*
(ii) *The characteristic function of* (1) *is positive on* $[0, \infty)$.
(iii) $\mu \int_0^r e^{\zeta_* s}\, dk(s) > \zeta_*$.
(iv) $\mu \int_0^r s e^{\zeta_* s}\, dk(s) > 1$.

PROOF. Obviously, $F(\zeta) > 0$ for $\zeta < 0$; therefore, by Theorem 1, (i) and (ii) are equivalent. Theorem 2 guarantees the equivalence of items (ii) and (iii), and the definition of ζ_* ensures the equivalence of (iii) and (iv).

Corollary 1. *If* $\mu \int_0^r s\, dk(s) > 1/e$ *then* (1) *is oscillating.*

As any function of bounded variation, k is representable as the sum of "jump functions," an absolutely continuous component, and a singular component (see [13]). Each of these functions corresponds to its own type of FDEs. Let us demonstrate the application of Theorem 3 by the examples of equations with various aftereffects.

1. Equations with concentrated delay. Let $\mu = 1$ and $k(t) = \sum_{k=1}^n a_k \chi(t - r_k)$, where $a_k > 0$, $0 < r_1 < r_2 < \dots < r_n = r$, and χ is the characteristic function of $[0, \infty)$. Equation (1) takes the form

$$\dot{x}(t) + \sum_{k=1}^n a_k x(t - r_k) = 0, \quad t \ge 0. \tag{2}$$

Applying Corollary 1 to (2), we obtain the familiar oscillation test:

Test 1 [4]. *If* $a_k > 0$ *for each* $k = \overline{1, n}$ *and* $\sum_{k=1}^n a_k r_k > 1/e$ *then* (2) *is oscillating.*

Putting $n = 1$ in (2), we obtain the equation

$$\dot{x}(t) + a x(t - r) = 0, \quad t \ge 0, \tag{3}$$

for which $G(\zeta) = e^{\zeta r}(\zeta r - 1)$ and $\zeta_* = 1/r$. Applying (iii) or (iv) of Theorem 3, we obtain an oscillation criterion for (3).

Test 2 [2]. *Equation* (3) *is oscillating if and only if* $ar > 1/e$.

Thus, in the case of a single summand, Test 1 becomes a criterion. This implies in particular that the constant $1/e$ in Test 1 is unimprovable; moreover, the strict inequality cannot be replaced by the nonstrict inequality.

2. Equations with distributed delay. Let

$$k(t) = \int_0^t p(s)\, ds,$$

where p is a nonnegative integrable function on $[0, r]$. Equation (1) takes the form

$$\dot{x}(t) + \mu \int_0^r p(s)x(t-s)\,ds = 0, \quad t \geq 0. \tag{4}$$

Specify Corollary 1 for equation (4): in this case the strict inequality can be replaced by the nonstrict inequality.

Test 3. *If* $\mu \int_0^r p(s)s\,ds \geq 1/e$ *then* (4) *is oscillating.*

PROOF. Since $e^{\zeta_* s} > es\zeta_*$ for all $s \neq 1/\zeta_*$ and the hypotheses of Test 3 imply that $\mu > 0$ and $\int_0^r p(s)s\,ds > 0$, we have

$$\mu \int_0^r p(s)\,e^{\zeta_* s}\,ds > \mu e\zeta_* \int_0^r p(s)s\,ds \geq \frac{e\zeta_*}{e} = \zeta_*.$$

It remains to use Theorem 3(3). The test is proved.

Apply Theorem 3 to an equation of the form (4) for some particular families of functions p.

FAMILY 1. Let $p(t) = t^\alpha$, where $\alpha > -1$. Equation (4) takes the form

$$\dot{x}(t) + \mu \int_0^r s^\alpha x(t-s)\,ds = 0, \quad t \geq 0. \tag{5}$$

Put

$$I(\xi) = \int_0^1 s^\alpha e^{\xi s}\,ds.$$

Then

$$\int_0^r p(s)e^{\zeta s}\,ds = r^{\alpha+1}I(r\zeta), \quad \zeta \int_0^r p(s)se^{\zeta s}\,ds = r^{\alpha+1}(e^{\zeta r} - (\alpha+1)I(r\zeta)).$$

Put $\xi = r\zeta$. It is easy to see that for family (4) the equation $G(\zeta) = 0$ is equivalent to the equation

$$(\alpha + 2)I(\xi) = e^\xi,$$

which, by the above-established properties of G, has a unique positive root. Denote the latter by ξ_α.

Rewriting the inequality of item (3) of Theorem 3 as $\mu r^{\alpha+2} I(\xi) > \xi$ or $\mu r^{\alpha+2} > (\alpha+2)\xi\, e^{-\xi}$, we obtain the following oscillation criterion for equation (5):

Test 4. *Equation* (5) *is oscillating if and only if* $\mu r^{\alpha+2} > (\alpha+2)\xi_\alpha e^{-\xi_\alpha}$.

In particular, for $\alpha = 0$ we have

Test 5. *The equation* $\dot{x}(t) + \mu \int_0^r x(t-s)\,ds = 0$ *is oscillating if and only if* $\mu r^2 > 2\xi_0 e^{-\xi_0}$, *where* ξ_0 *is the positive root of the equation* $1 - \xi/2 = e^{-\xi}$.

The numerical methods give $\xi_0 \approx 1.594$ and $2\xi_0 e^{-\xi_0} \approx 0.648$.

Finding the roots of the equation $(\alpha+2)I(\xi) = e^\xi$ is as easy also for other $\alpha > -1$: $\xi_{-1/2} \approx 1.914$; $\xi_{1/2} \approx 1.447$; $\xi_1 \approx 1.361$; $\xi_{3/2} \approx 1.303$; $\xi_2 \approx 1.262$; $\xi_{5/2} \approx 1.231$; and $\xi_3 \approx 1.206$. To each root there corresponds its own oscillation test.

Observe the following fact: Test 4 makes it possible to demonstrate the accuracy of the constant $1/e$ in Test 3. To this end, establish some properties of the family ξ_α.

Lemma 1. *$\xi_\alpha > 1$ for all $\alpha > -1$; moreover,* $\lim_{\alpha\to\infty} \xi_\alpha = 1$.

PROOF. The definition of ξ_α yields

$$(\alpha+2)\int_0^1 s^\alpha e^{\xi_\alpha(s-1)}\,ds = 1.$$

Integrating this equality by parts, we obtain

$$(\alpha+2)\int_0^1 s^{\alpha+1} e^{\xi_\alpha(s-1)}\,ds = 1/\xi_\alpha.$$

Subtracting the second equality from the first, we have

$$1 - 1/\xi_\alpha = (\alpha+2)\int_0^1 s^\alpha(1-s)e^{\xi_\alpha(s-1)}\,ds, \tag{6}$$

which immediately implies that $\xi_\alpha > 1$ for all $\alpha > -1$. Reckoning with this inequality, from (6) we obtain

$$0 < 1 - 1/\xi_\alpha \le (\alpha+2)\int_0^1 (s^\alpha - s^{\alpha+1})\,ds = 1/(\alpha+1),$$

whence $\lim_{\alpha\to\infty} \xi_\alpha = 1$. The lemma is proved.

Show that the constant $1/e$ in Test 3 cannot be decreased. Put $r = 1$ in (5) and apply Test 3 to it. We obtain a sufficient oscillation condition: $\frac{\mu r^{\alpha+2}}{\alpha+2} \ge 1/e$.

The application of Test 4 gives the oscillation criterion: $\frac{\mu r^{\alpha+2}}{\alpha+2} > \xi_\alpha e^{-\xi_\alpha}$. By Lemma 1, $\xi_\alpha > 1$; therefore, $\xi_\alpha e^{-\xi_\alpha} < 1/e$ but also $\xi_\alpha e^{-\xi_\alpha} \to 1/e$; hence, the constant $1/e$ in Test 3 cannot be decreased.

FAMILY 2. Put $p(t) = (r-t)^\alpha$ in (2), where $\alpha > -1$, and consider the family of equations

$$\dot{x}(t) + \mu\int_0^r (r-s)^\alpha x(t-s)\,ds = 0, \quad t \ge 0. \tag{7}$$

Put $J(\eta) = \int_0^1 (1-s)^\alpha e^{\eta s}\,ds = \int_0^1 s^\alpha e^{\eta(1-s)}\,ds$. Then

$$\int_0^r p(s)e^{\zeta s}\,ds = r^{\alpha+1}J(r\zeta),$$

$$\zeta\int_0^r p(s)se^{\zeta s}\,ds = r^{\alpha+1}\left(r\zeta J(r\zeta) + 1 - (\alpha+1)J(r\zeta)\right).$$

Put $\eta = r\zeta$. It is easy to see that, for family (7), the equation $G(\zeta) = 0$ is equivalent to the equation $(\alpha+2-\eta)J(\eta) = 1$, which, by the above-established properties of G, has a unique positive root. Denote the latter by η_α. Rewriting the inequality of (iii) of Theorem 3 as $\mu r^{\alpha+2}J(\eta) > \eta$ or $\mu r^{\alpha+2} > \eta(\alpha+2-\eta)$, we obtain the following oscillation criterion for (7):

Test 6. *Equation* (7) *is oscillating if and only if* $\mu r^{\alpha+2} > \eta_\alpha(\alpha+2-\eta_\alpha)$.

For $\alpha = 0$, Test 6 coincides obviously with Test 5.

For other $\alpha > -1$, the roots of the equation $(\alpha+2-\eta)J(\eta) = 1$ are easily found by numerical methods: for example, $\eta_1 \approx 2.149$; $\eta_2 \approx 2.688$; $\eta_3 \approx 3.217$; $\eta_4 \approx 3.741$; $\eta_5 \approx 4.260$; $\eta_{10} \approx 6.822$; $\eta_{20} \approx 11.880$; and $\eta_{100} \approx 51.965$.

Examine the asymptotic behavior of the family η_α.

Lemma 2. $\eta_\alpha > \frac{\alpha+2}{2}$ *for all* $\alpha > -1$*; moreover,* $\lim_{\alpha\to\infty} \frac{\eta_\alpha}{\alpha+2} = 1/2$.

PROOF. Put

$$K(\eta) = (\alpha+2-\eta)\int_0^1 s^\alpha e^{\eta(1-s)}\,ds.$$

Obviously, the function K is defined and continuous for all $\eta \geq 0$. Establish two inequalities for K. It is easy to see that

$$\int_0^1 s^\alpha e^{\frac{\alpha+2}{2}(1-s)}\,ds = \int_0^1 s^\alpha e^{\frac{\alpha+2}{2}(1-s)}\,ds + \frac{2}{\alpha+2}\int_0^1 \left(s^{\alpha+2}e^{\frac{\alpha+2}{2}(1-s)}\right)'\,ds$$

$$-\frac{2}{\alpha+2}\int_0^1 \left((\alpha+2)s^{\alpha+1} - \frac{\alpha+2}{2}s^{\alpha+2}\right)e^{\frac{\alpha+2}{2}(1-s)}\,ds$$

$$= \frac{2}{\alpha+2} + \int_0^1 (1-s)^2 s^\alpha e^{\frac{\alpha+2}{2}(1-s)}\,ds > \frac{2}{\alpha+2},$$

whence $K\left(\frac{\alpha+2}{2}\right) > 1$. On the other hand,

$$\int_0^1 s^\alpha e^{\frac{\alpha+4}{2}(1-s)}\,ds = \int_0^1 s^\alpha e^{\frac{\alpha+4}{2}(1-s)}\,ds$$

$$+\frac{2}{\alpha}\int_0^1 \left(s^\alpha e^{\frac{\alpha}{2}(1-s)}\right)'\,ds - \frac{2}{\alpha}\int_0^1 \left(\alpha s^{\alpha-1} - \frac{\alpha}{2}s^\alpha\right)e^{\frac{\alpha}{2}(1-s)}\,ds$$

$$= \frac{2}{\alpha} + \int_0^1 \left(se^{2(1-s)} + s - 2\right)s^{\alpha-1}e^{\frac{\alpha}{2}(1-s)}\,ds < \frac{2}{\alpha};$$

therefore, $K\left(\frac{\alpha+4}{2}\right) < 1$. Since η_α is the only root of the equation $K(\eta) = 1$, we have $\frac{\alpha+2}{2} < \eta_\alpha < \frac{\alpha+4}{2}$, which implies the first claim of the lemma. Rewrite the obtained estimates as $\frac{1}{2} < \frac{\eta_\alpha}{\alpha+2} < \frac{\alpha+4}{2(\alpha+2)}$ and pass to the limit as $\alpha \to +\infty$. We infer that $\lim_{\alpha\to\infty} \frac{\eta_\alpha}{\alpha+2} = 1/2$. The lemma is proved.

Using Lemma 2, we can obtain a simple and easily verifiable oscillation test for (7).

Test 7. *If* $\mu r^{\alpha+2} \geq \frac{1}{4}(\alpha+2)^2$ *then* (7) *is oscillating.*

PROOF. Obviously, $x(1-x) < 1/4$ for any $x \neq 1/2$. It was established in Lemma 2 that $\frac{\eta_\alpha}{\alpha+2} > 1/2$; consequently,

$$\mu r^{\alpha+2} \geq \frac{(\alpha+2)^2}{4} > \frac{\eta_\alpha}{\alpha+2}\left(1 - \frac{\eta_\alpha}{\alpha+2}\right)(\alpha+2)^2 = \eta_\alpha(\alpha+2-\eta_\alpha).$$

A reference to Test 6 completes the proof.

Observe that the constant 1/4 in Test 7 is sharp. To verify that, rewrite the oscillation criterion for equation (7) in the form $\frac{\mu r^{\alpha+2}}{(\alpha+2)^2} > \frac{\eta_\alpha}{\alpha+2}\big(1-\frac{\eta_\alpha}{\alpha+2}\big)$. By Lemma 2, $\frac{\eta_\alpha}{\alpha+2} \to 1/2$; therefore, $\frac{\eta_\alpha}{\alpha+2}\big(1-\frac{\eta_\alpha}{\alpha+2}\big) \to 1/4$, and so the constant 1/4 in Test 7 cannot be decreased by any however small value.

3. Equations with singular component. There are quite a few available FDEs with singular component. As an example, consider the equation of the form (1), where $r = 1$ and $k(t) = c(t)$ is the "Cantor ladder" (see [13]):

$$\dot{x}(t) + \mu \int_0^1 x(t-s)\,dc(s) = 0, \quad t \ge 0. \tag{8}$$

Put $\phi(\zeta) = \int_0^1 e^{\zeta s}\,dc(s)$. Obviously, $\phi(0) = 1$.

Observe the following elementary properties of the function c: $c(s/3) = (1/2)c(s)$ and $c(2/3 + s/3) = 1/2 + (1/2)c(s)$ for all $s \in [0,1]$. This implies that ϕ satisfies the functional equation

$$\phi(\zeta) = \frac{1 + e^{2\zeta/3}}{2}\phi(\zeta/3)$$

whose solution is representable as the infinite product $\phi(\zeta) = \prod_{k=1}^{\infty} \frac{1+e^{2\zeta/3^k}}{2}$. The equation $G(\zeta) = 0$ for (8) takes the form $\frac{1}{\zeta} = w(\zeta)$, where

$$w(\zeta) = \frac{\phi'(\zeta)}{\phi(\zeta)} = 2\sum_{k=1}^{\infty} \frac{1}{3^k(1 + e^{-2\zeta/3^k})}.$$

It is easy to see that the function w is defined and continuously differentiable on the whole axis. Since $w'(\zeta) > 0$, w increases monotonically from $0 = \lim_{\zeta\to-\infty} w(\zeta)$ to $1 = \lim_{\zeta\to+\infty} w(\zeta)$. Moreover,

$$w(\zeta) + w(-\zeta) = 1$$

for all $\zeta \in \mathbb{R}$, which means that (0,1/2) is a symmetry point of the graph of w.

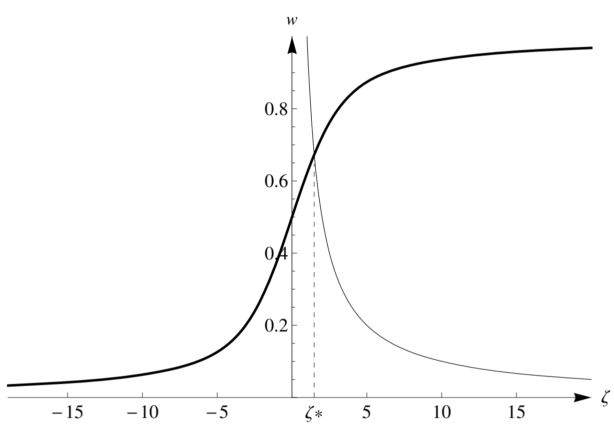

Fig. 1. The thick line is the graph of $w(\zeta)$;
the thin line is the graph of $1/\zeta$

Denote the only root of the equation $\frac{1}{\zeta} = w(\zeta)$ by ζ_* (Fig. 1) and put $\mu_* = \frac{\zeta_*}{\phi(\zeta_*)}$. Theorem 3 implies an oscillation test for (8):

Test 8. *Equation* (8) *is oscillating if and only if* $\mu > \mu_*$.

Numerical methods yield $\zeta_* \approx 1.48$ and $\mu_* \approx 0.618$.

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