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TO THE SPECTRAL THEORY OF PARTIALLY ORDERED SETS

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Abstract: We suggest an approach to advance the spectral theory of posets. The validity of the Hofmann–Mislove Theorem is established for posets and a characterization is obtained of the sober topological spaces as spectra of posets with topology. Also we describe the essential completions of topological spaces in terms of spectra of posets with topology. Apart from that, some sufficient conditions are found for two extensions of a topological space to be homeomorphic.

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§ 1. Introduction

The foundations of the spectral theory of lattices were laid in the papers by M. H. Stone [1, 2] and those by H. A. Priestley [3, 4]. Some results on the spectral theory of join semilattices are presented in the monographs of G. Gierz et al. [5] and G. Grätzer [6, 7]. In the paper [8] by the first author, the characterization of sober topological spaces was obtained as spectra of semitopological semilattices, together with a description of essential completions of semitopological semilattices. We refer here also to the two other papers of the first author, [9] and [10], which are related to these topics.

This paper is devoted to advance the spectral theory of posets, i.e. partially ordered sets. The authors suggest two approaches to defining an ideal of a poset. Within the frames of the first approach which is presented in Sections 3 and 4, the topology on a set plays a key role, i.e., it defines some partial order on this set (the specialization order). We note that there are at least two ways to embed an arbitrary topological T_0-space into a space that is a join semilattice (and even a lattice) with respect to its specialization order—embedding into an injective space and embedding into its own essential completion. Further, the ideals are defined as restrictions of the ideals of join semilattices on the original space. Theorem 2 presents an inner characterization of ideals. Also, Theorem 3 contains sufficient conditions for every two extensions of a topological space to be homeomorphic.

The second, more general, approach taken from Section 5 on does not establish a similar connection between partial order and topology, but allows us to generalize some results of [8], which were established for join semilattices. For example, Theorem 4 is an abstraction of the Hofmann–Mislove Theorem for posets; also see Corollary 5. Corollaries 7 and 8 characterize the (almost) sober spaces as spectra of posets with topology (or, equivalently, semitopological posets), while Theorem 7 describes the essential completions of posets with topology. All main ideas of our proofs stem from [8]. Adapting those ideas to arbitrary posets involves the definition of ideal of a poset which is given in [11]; see Definition 4.

§ 2. Definitions and Auxiliary Results

Example 1. Let $\langle X;\leq\rangle$ be a poset. We list three topologies on X for which the specialization order coincides with $\leq$.

(i) The topology $\mathscr{T}_m(\leq)$ defined by the subbase of open sets: $\{X\setminus{\downarrow}x \mid x\in X\}$.

(ii) The topology $\mathscr{T}_M(\leq)=\{Y\subseteq X \mid {\uparrow}Y=Y\}$ consisting of all upper cones.

(iii) The *Scott topology* $\mathscr{T}_S(\leq)$, where $U\in\mathscr{T}_S(\leq)$ if and only if ${\uparrow}U=U$ and $D\cap U\neq\varnothing$ whenever $\sup D$ exists and $\sup D\in U$ for each set $D\subseteq X$ up-directed with respect to $\leq$.

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Lemma 1. *Let $\langle X; \leq \rangle$ be a poset and let $\mathscr{T}$ be a topology on X such that the specialization order $\leq_{\mathscr{T}}$ coincides with $\leq$. Then $\mathscr{T}_m(\leq) \subseteq \mathscr{T} \subseteq \mathscr{T}_M(\leq)$.*

PROOF. The inclusion $\downarrow x \subseteq \mathrm{cl}_{\mathscr{T}}\{x\}$ is obvious for every $x \in X$. Conversely, take $y \in \mathrm{cl}_{\mathscr{T}}\{x\}$. Suppose that $y \in U \in \mathscr{T}$. Since y is a limit point for $\{x\}$, we conclude that $x \in U$. Therefore, $y \leq_{\mathscr{T}} x$, whence $y \leq x$. Thus $\downarrow x = \mathrm{cl}_{\mathscr{T}}\{x\}$ and $X \backslash \downarrow x \in \mathscr{T}$. This means that $\mathscr{T}_m(\leq) \subseteq \mathscr{T}$. The inclusion $\mathscr{T} \subseteq \mathscr{T}_M(\leq)$ follows from the fact that each open set of $\mathscr{T}$ is an upper cone with respect to $\leq_{\mathscr{T}}$.

In the sequel, we drop the subscript in the denotation of the specialization order each time when this leads to no confusion. Given a topological space $\mathbb{X}$ and $x \in X$, put $\mathscr{T}(x) = \{U \in \mathscr{T}(\mathbb{X}) \mid x \in U\}$.

A subset $S \subseteq X$ of a topological space $\mathbb{X}$ is *irreducible*, if for every two closed sets $F_0, F_1 \subseteq X$ the inclusion $S \subseteq F_0 \cup F_1$ implies that $S \subseteq F_i$ for some $i < 2$. Equivalently, $S \subseteq X$ is irreducible, if for all $U_0, U_1 \in \mathscr{T}(\mathbb{X})$ from $S \cap U_i \neq \varnothing$ for all $i < 2$ it follows that $S \cap U_0 \cap U_1 \neq \varnothing$. A topological T_0-space $\mathbb{X}$ is *sober*, if for every nonempty irreducible closed subset S in $\mathbb{X}$ there is $x \in S$ such that $S = \downarrow_{\mathbb{X}} x$. A space $\mathbb{X}$ is *almost sober*, if for every proper irreducible closed subset S in $\mathbb{X}$ there is $x \in S$ such that $S = \downarrow_{\mathbb{X}} x$.

If $\mathbb{X} \leq \mathbb{Y}$ is an extension of topological spaces, then for each set $U \in \mathscr{T}(\mathbb{X})$ there is a biggest set $U^* \in \mathscr{T}(\mathbb{Y})$ such that $U^* \cap X = U$. The following is an immediate corollary of the definition:

Lemma 2 [12, Lemma 2.2]. *If $U, V \in \mathscr{T}(\mathbb{X})$ then $(U \cap V)^* = U^* \cap V^*$. In particular, $U \subseteq V$ if and only if $U^* \subseteq V^*$.*

Let $\mathscr{T}_*$ denote the topology defined by the subbase $\{U^* \mid U \in \mathscr{T}(\mathbb{X})\}$ of open sets. An extension $\mathbb{X} \leq \mathbb{Y}$ is *strict*, if $\mathscr{T}_* = \mathscr{T}(\mathbb{Y})$. An extension $\mathbb{X} \leq \mathbb{Y}$ is *semistrict* if for all $V \in \mathscr{T}(\mathbb{Y})$ and $x \in V \cap X$ there is $U \in \mathscr{T}(\mathbb{X})$ such that $x \in U$ and $U^* \subseteq V$. It is trivial that every strict extension is semistrict.

DEFINITION 1 [12]. Let $\mathbb{X} \leq \mathbb{Y}$ be an extension of topological T_0-spaces. An element $y \in Y$ is *essential for $\mathbb{X}$ in $\mathbb{Y}$* if one of the two conditions holds:

(1) $y \notin X$ is a bottom of Y with respect to the specialization order and $\mathbb{X}$ has no bottom;

(2) for each $U \in \mathscr{T}(\mathbb{Y})$ such that $y \in U$, there are $x_0, \ldots, x_n \in \downarrow y \cap X$ and their neighborhoods $U_0, \ldots, U_n \in \mathscr{T}(\mathbb{Y})$ such that $\bigcap_{i \leqslant n} U_i \subseteq U$.

Lemma 3. *If an extension $\mathbb{X} \leq \mathbb{Y}$ is semistrict and each element $y \in Y$ is essential for $\mathbb{X}$ in $\mathbb{Y}$, then $\mathbb{X} \leq \mathbb{Y}$ is a strict extension.*

PROOF. Indeed, let $y \in V \in \mathscr{T}(\mathbb{Y})$. As $y \in Y$ is an essential element for $\mathbb{X}$ in $\mathbb{Y}$, the two cases are possible:

In the first case, y is a bottom of $\mathbb{Y}$. Then $V = Y = X^*$.

In the second case, there are $x_0, \ldots, x_n \in \downarrow y \cap X$ and $V_0, \ldots, V_n \in \mathscr{T}(\mathbb{Y})$ such that $x_i \in V_i$ for all $i \leqslant n$ and $V_0 \cap \cdots \cap V_n \subseteq V$. Since the extension $\mathbb{X} \leq \mathbb{Y}$ is semistrict, there are $U_0, \ldots, U_n \in \mathscr{T}(\mathbb{X})$ such that $x_i \in U_i$ and $U_i^* \subseteq V_i$ for all $i \leqslant n$. Putting $U = U_0 \cap \cdots \cap U_n$ we get $U \in \mathscr{T}(\mathbb{X})$ and $y \in U^* = U_0^* \cap \cdots \cap U_n^* \subseteq V_0 \cap \cdots \cap V_n \subseteq V$. Thus, $\{U^* \mid U \in \mathscr{T}(\mathbb{X})\}$ is a base of the open sets of $\mathscr{T}(\mathbb{Y})$.

An extension $\mathbb{X} \leq \mathbb{Y}$ of topological T_0-spaces is *essential*, if for every T_0-space $\mathbb{Z}$ a continuous mapping $f : \mathbb{Y} \to \mathbb{Z}$ is a homeomorphic embedding whenever $f|_{\mathbb{X}}$ is a homeomorphic embedding.

The following is the content of Corollary 1 and Proposition 4 of [13]:

Theorem 1. *For an arbitrary topological T_0-space $\mathbb{X}$, there is a biggest essential extension $\mathbb{H}_e(\mathbb{X})$ (called the essential completion of $\mathbb{X}$). If an extension $\mathbb{X} \leq \mathbb{Y}$ is essential, then $\mathbb{Y}$ is homeomorphic to some subspace of $\mathbb{H}_e(\mathbb{X})$ that includes $\mathbb{X}$.*

Together with Theorem 1.3 from the paper [12], Lemma 3 yields the following

Corollary 1. *An extension $\mathbb{X} \leq \mathbb{Y}$ is essential if and only if $\mathbb{X} \leq \mathbb{Y}$ is a semistrict extension and each element $y \in Y$ is essential for $\mathbb{X}$ in $\mathbb{Y}$.*

§ 3. Ideals in Topological Spaces

We recall that an *ideal of a join semilattice* $\langle S; \vee\rangle$ is every set $I \subseteq S$ with the following property: for all $a, b \in I$ and $s \in S$, if $s \leq a \vee b$ then $s \in I$.

Hence, the ideals of join semilattices are exactly the ψ-ideals; see Definition 4 and Example 4.

Lemma 4. *Let $\mathbb{X}$ be a topological T_0-space that is a join semilattice with respect to the specialization order. Then*

(i) *every ideal in $\langle X; \vee\rangle$ is an irreducible subset in $\mathbb{X}$;*

(ii) *if the operation $\vee$ is continuous, then each irreducible closed subset in $\mathbb{X}$ is an ideal in $\langle X; \vee\rangle$.*

Proof. (i) Let $I \subseteq X$ be an ideal in $\langle X; \vee\rangle$ and let $a_i \in I \cap U_i$ for some $U_0, U_1 \in \mathscr{T}(\mathbb{X})$. Then $a_0 \vee a_1 \in I \cap U_0 \cap U_1$.

(ii) Assume that the operation $\vee$ is continuous and consider an arbitrary irreducible closed subset I in $\mathbb{X}$. Further, let $a, b \in I$ and $s \in S$ be such that $s \leq a \vee b$ but $s \notin I$. This means that $a \vee b \in X \backslash I \in \mathscr{T}(\mathbb{X})$. Since $\vee$ is continuous, there are $U \in \mathscr{T}(a)$ and $V \in \mathscr{T}(b)$ such that $U \times V \subseteq \vee^{-1}(X \backslash I)$. Therefore, $a \in I \cap U$ and $b \in I \cap V$. As I is an irreducible set, there exists $c \in I \cap U \cap V$. But then $c = c \vee c \in I \cap (X \backslash I) = \varnothing$, which is impossible. This contradiction proves that $s \in I$.

The following simple examples demonstrate that none of the two statements of Lemma 4 can be reversed.

Example 2. There is an irreducible set that is not an ideal. Indeed, consider the Sierpiński space $\mathbb{S}$, where $S = \{\bot, \top\}$ and $\mathscr{T}(\mathbb{S}) = \big\{\varnothing, \{\top\}, S\big\}$. Then $\{\top\}$ is obviously irreducible but not an ideal in $\mathbb{S}$ as $\bot < \top$.

Example 3. There is an ideal that is not closed. Indeed, consider the space $\mathbb{X}$, where $X = \omega + 1$, $\mathscr{T}(\mathbb{X}) = \mathscr{T}_m(\leq)$, and $\leq$ defines the natural order on $\omega + 1$. It is not hard to see that $\vee$ is continuous. However, $I = \omega$ is an ideal but not closed in $\mathbb{X}$, since ω is a limit point of I not belonging to I.

Definition 2. Let $\mathbb{X}$ be a topological T_0-space and let $x, x_0, \dots, x_n \in X$. We write $x \leq \{x_0, \dots, x_n\}$, if for every $U \in \mathscr{T}(x)$ there are $U_0 \in \mathscr{T}(x_0), \dots, U_n \in \mathscr{T}(x_n)$ such that $U_0 \cap \dots \cap U_n \subseteq U$.

Furthermore, take $y_0, \dots, y_m \in X$. We write that $\{y_0, \dots, y_m\} \leq \{x_0, \dots, x_n\}$, if $y_i \leq \{x_0, \dots, x_n\}$ for all $i \leqslant m$.

A set $I \subseteq X$ is an *ideal of a topological space* $\mathbb{X}$, if $x \leq \{x_0, \dots, x_n\}$ implies that $x \in I$ for all $x_0, \dots, x_n \in I$.

It is not hard to see that $x \leq y$ in $\mathbb{X}$ if and only if $x \leq \{y\}$.

Lemma 5. *Let an extension $\mathbb{X} \leq \mathbb{Y}$ of topological T_0-spaces be semistrict, let $\mathbb{Y}$ be a join semilattice with respect to the specialization order, and let the corresponding operation $\vee$ be continuous. Then $\{x'_0, \dots, x'_m\} \leq \{x_0, \dots, x_n\}$ in $\mathbb{X}$ if and only if $x'_0 \vee \dots \vee x'_m \leq x_0 \vee \dots \vee x_n$ in $\mathbb{Y}$ for all $x'_0, \dots, x'_m$, $x_0, \dots, x_n \in X$.*

Proof. Suppose first that $\{x'_0, \dots, x'_m\} \leq \{x_0, \dots, x_n\}$ in $\mathbb{X}$. By hypothesis, $x'_i \leq \{x_0, \dots, x_n\}$ for all $i \leqslant m$. Assume that $i \leqslant m$ and $x'_i \in V \in \mathscr{T}(\mathbb{Y})$. As $\mathbb{X} \leq \mathbb{Y}$ is a semistrict extension, there is $U \in \mathscr{T}(\mathbb{X})$ such that $x'_i \in U$ and $U^* \subseteq V$. Since $x'_i \leq \{x_0, \dots, x_n\}$, there are $U_0, \dots, U_n \in \mathscr{T}(\mathbb{X})$ such that $x_0 \in U_0, \dots, x_n \in U_n$ and $U_0 \cap \dots \cap U_n \subseteq U$. But then $x_0 \vee \dots \vee x_n \in U_0^* \cap \dots \cap U_n^* = (U_0 \cap \dots \cap U_n)^* \subseteq U^* \subseteq V$ by Lemma 2, which proves the inequality $x'_i \leq x_0 \vee \dots \vee x_n$ for all $i \leqslant m$. Therefore, $x'_0 \vee \dots \vee x'_m \leq x_0 \vee \dots \vee x_n$.

Conversely, assume that $x'_0 \vee \dots \vee x'_m \leq x_0 \vee \dots \vee x_n$ in $\langle Y; \vee\rangle$. This means that $x'_i \leq x_0 \vee \dots \vee x_n$ for all $i \leqslant m$. Consider $U \in \mathscr{T}(\mathbb{X})$ such that $x'_i \in U$; then $U = W \cap X$ for some $W \in \mathscr{T}(\mathbb{Y})$. Thus, $x_0 \vee \dots \vee x_n \in W$. As $\vee$ is continuous, there are $W_0, \dots, W_n \in \mathscr{T}(\mathbb{Y})$ such that $x_0 \in W_0, \dots, x_n \in W_n$ and $W_0 \times \dots \times W_n \subseteq \vee^{-1}(W)$. The last inclusion yields that $W_0 \cap \dots \cap W_n \subseteq W$, whence $U_0 \cap \dots \cap U_n \subseteq U$ where $x_0 \in U_0 = W_0 \cap X \in \mathscr{T}(\mathbb{X}), \dots, x_n \in U_n = W_n \cap X \in \mathscr{T}(\mathbb{X})$. Hence, $x'_i \leq \{x_0, \dots, x_n\}$ in $\mathbb{X}$ for all $i \leqslant m$; i.e., $\{x'_0, \dots, x'_m\} \leq \{x_0, \dots, x_n\}$.

Theorem 2. *Let an extension $\mathbb{X} \leq \mathbb{Y}$ of topological T_0-spaces be semistrict, let $\mathbb{Y}$ be a join semilattice with respect to the specialization order, and let the corresponding operation $\vee$ be continuous. The following are equivalent for $I \subseteq X$:*

(i) *$I = I' \cap X$ for some ideal I' of $\langle Y; \vee \rangle$.*

(ii) *I is an ideal of $\mathbb{X}$.*

Proof. Show first that (i) implies (ii). Indeed, let $I = I' \cap X$ for some ideal I' of $\langle Y; \vee \rangle$ and let $x_0, \ldots, x_n \in I$ and $x \in X$ be such that $x \leq \{x_0, \ldots, x_n\}$. By Lemma 5 $x \leq x_0 \vee \cdots \vee x_n \in I'$, whence $x \in I' \cap X = I$.

Show now that (ii) implies (i). Indeed, consider an arbitrary ideal I of $\mathbb{X}$. Let I' denote the ideal of the join semilattice $\langle Y; \vee \rangle$ which is generated by I. It is not hard to see that

$$I' = \{y \in Y \mid y \leq x_0 \vee \cdots \vee x_n \text{ for some } x_0, \ldots, x_n \in I\}.$$

Obviously, $I \subseteq I' \cap X$. To prove the reverse inclusion, take an arbitrary $x \in I' \cap X$. Then there are $x_0, \ldots, x_n \in I$ such that $x \leq x_0 \vee \cdots \vee x_n$ in $\langle Y; \vee \rangle$. By Lemma 5, $x \leq \{x_0, \ldots, x_n\}$ in $\mathbb{X}$, whence $x \in I$ by assumption. Therefore, $I = I' \cap X$.

Remark. It follows from the results of [12] and [13] that the extension $\mathbb{X} \leq \mathbb{H}_e(\mathbb{X})$ is strict and $H_e(\mathbb{X})$ is a complete lattice with respect to the specialization order $\leq$. In particular, $H_e(\mathbb{X})$ is a join semilattice. We will denote the corresponding join operation by $\vee$. By Corollary 2 of [13] $\vee$ is a continuous operation. Thus, Theorem 2 applies to the extension $\mathbb{X} \leq \mathbb{H}_e(\mathbb{X})$.

§ 4. Homeomorphic Extensions

We recall one construction from [13]. Let $\mathbb{X}$ be an arbitrary topological T_0-space and let $\mathscr{P}_\omega(X)$ denote the set of all finite subsets of X. Given $U \in \mathscr{T}(\mathbb{X})$, put

$$\overline{U} = \Big\{F \in \mathscr{P}_\omega(X) \mid \text{ there are } U_x \in \mathscr{T}(x),\ x \in F, \text{ such that } \bigcap_{x \in F} U_x \subseteq U\Big\}.$$

Observe that $\varnothing \in \overline{U}$ if and only if $U = X$. Let us define the equivalence relation $\sim$ on $\mathscr{P}_\omega(X)$ assuming that $F_0 \sim F_1$ if and only if the conditions $F_0 \in \overline{U}$ and $F_1 \in \overline{U}$ are equivalent for all $U \in \mathscr{T}(\mathbb{X})$. Put $[F] = \{G \in \mathscr{P}_\omega(X) \mid F \sim G\}$ for all $F \in \mathscr{P}_\omega(X)$. Also,

$$H_0^\vee(\mathbb{X}) = \{[F] \mid F \in \mathscr{P}_\omega(X)\}, \quad U^\sharp = \{[F] \mid F \in \overline{U}\} \quad \text{for all } U \in \mathscr{T}(\mathbb{X}),$$

$\mathscr{T}_0$ denotes the topology defined by the subbase $\{U^\sharp \mid U \in \mathscr{T}(\mathbb{X})\}$ of open sets, and

$$H^\vee(\mathbb{X}) = \{[F] \mid F \in \mathscr{P}_\omega(X),\ F \neq \varnothing\}, \quad \mathscr{T} = \{W \cap H^\vee(\mathbb{X}) \mid W \in \mathscr{T}_0\},$$
$$\mathbb{H}_0^\vee(\mathbb{X}) = \langle H_0^\vee(\mathbb{X}), \mathscr{T}_0 \rangle, \quad \mathbb{H}^\vee(\mathbb{X}) = \langle H^\vee(\mathbb{X}), \mathscr{T} \rangle,$$
$$\lambda : \mathbb{X} \to \mathbb{H}^\vee(\mathbb{X}), \quad \lambda : x \mapsto [\{x\}].$$

It is clear that $\mathbb{H}^\vee(\mathbb{X}) \leq \mathbb{H}_0^\vee(\mathbb{X})$. Moreover, if $\mathbb{X}$ contains a bottom $\perp$ with respect to the specialization order, then $\{\perp\} \in \overline{U}$ if and only if $U = X$ for every $U \in \mathscr{T}(\mathbb{X})$. Therefore, $\varnothing \sim \{\perp\}$, whence $\mathbb{H}^\vee(\mathbb{X}) = \mathbb{H}_0^\vee(\mathbb{X})$ in this case.

Lemma 6 [13]. *$(U_0 \cap U_1)^\sharp = U_0^\sharp \cap U_1^\sharp$ for arbitrary $U_0, U_1 \in \mathscr{T}(\mathbb{X})$, whence $\{U^\sharp \mid U \in \mathscr{T}(\mathbb{X})\}$ is a base of $\mathscr{T}_0$.*

Lemma 7. *The mapping λ is a homeomorphic embedding and $\lambda(\mathbb{X})$ is a smooth subspace of $\mathbb{H}^\vee(\mathbb{X})$.*

Proof. The fact that λ is one-to-one follows from the T_0-separation axiom for $\mathscr{T}(\mathbb{X})$. Moreover, $x \in U$ if and only if $\{x\} \in \overline{U}$ for all $x \in X$ and $U \in \mathscr{T}(\mathbb{X})$, which is equivalent to the containment $[\{x\}] \in U^\sharp$. This proves the first claim.

We prove now the second. Assume that $x_0 \prec_{\mathbb{X}} x_1$. This means that $x_1 \in U \subseteq \uparrow_{\mathbb{X}} x_0$ for some $U \in \mathscr{T}(\mathbb{X})$. Thus, $\lambda(x_1) \in U^\sharp$. In order to prove that $\lambda(x_0) \prec \lambda(x_1)$, it suffices to verify that $U^\sharp \subseteq \uparrow\lambda(x_0)$. Let $[F] \in U^\sharp$ for some finite $F \subseteq X$. By definition, for each $x \in F$, there is $U_x \in \mathscr{T}(\mathbb{X})$ such that $x \in U_x$ and $\bigcap_{x\in F} U_x \subseteq U$. Suppose that $\lambda(x_0) \in V^\sharp$ for some $V \in \mathscr{T}(\mathbb{X})$. This implies that $x_0 \in V$, whence $\bigcap_{x\in F} U_x \subseteq U \subseteq \uparrow_{\mathbb{X}} x_0 \subseteq V$. Consequently, $[F] \in V^\sharp$; i.e., $\lambda(x_0) \leq [F]$ which was to be proved. $\square$

Theorem 3. *Let $\mathbb{X} \leq \mathbb{Y}$ be an extension of topological T_0-spaces. Then $\mathbb{Y} \cong \mathbb{H}^\vee(\mathbb{X})$ if and only if*

(i) *$\mathbb{X} \leq \mathbb{Y}$ is a semistrict extension;*

(ii) *$\mathbb{Y}$ is a join semilattice with respect to the specialization order and the corresponding join operation $\vee$ is continuous on $\mathbb{Y}$;*

(iii) *the join semilattice $\langle Y; \vee\rangle$ is generated by X.*

Proof. Suppose that an extension $\mathbb{X} \leq \mathbb{Y}$ satisfies (i)–(iii) and define the mapping $f : \mathbb{H}^\vee(\mathbb{X}) \to \mathbb{Y}$ on assuming that $f\big([x_0,\dots,x_n]\big) = x_0 \vee \dots \vee x_n$ for all $n < \omega$ and $x_0,\dots,x_n \in X$.

Claim 1. *The equality $x_0 \vee \dots \vee x_n = x'_0 \vee \dots \vee x'_m$ is equivalent to the comparison $\{x_0,\dots,x_n\} \sim \{x'_0,\dots,x'_m\}$ for all $x_0,\dots,x_n,\ x'_0,\dots,x'_m \in X$.*

Proof of Claim 1. Indeed, let $x'_0,\dots,x'_m \in X$ be such that $\{x_0,\dots,x_n\} \sim \{x'_0,\dots,x'_m\}$. Prove that $\{x_0,\dots,x_n\} \leq \{x'_0,\dots,x'_m\}$ in $\mathbb{X}$. Take $i \leqslant n$ and $x_i \in U \in \mathscr{T}(\mathbb{X})$. Putting $U_{x_i} = U$ and $U_{x_j} = X$ for each $j \leqslant n$ such that $j \neq i$ implies that $\{x_0,\dots,x_n\} \in \overline{U}$. Since $\{x_0,\dots,x_n\} \sim \{x'_0,\dots,x'_m\}$; therefore, $\{x'_0,\dots,x'_m\} \in \overline{U}$. This means that for every $j \leqslant m$ there is $U_j \in \mathscr{T}(\mathbb{X})$ such that $x'_j \in U_j$ and $\bigcap_{j\leqslant m} U_j \subseteq U$. By Definition 2 $x_i \leq \{x'_0,\dots,x'_m\}$ for each $i \leqslant n$, whence $\{x_0,\dots,x_n\} \leq \{x'_0,\dots,x'_m\}$. By Lemma 5, $x_0 \vee \dots \vee x_n \leq x'_0 \vee \dots \vee x'_m$. By symmetry, $x'_0 \vee \dots \vee x'_m \leq x_0 \vee \dots \vee x_n$.

Conversely, assume that $x_0 \vee \dots \vee x_n = x'_0 \vee \dots \vee x'_m$ for some $x_0,\dots,x_n,\ x'_0,\dots,x'_m \in X$ and show that $\{x_0,\dots,x_n\} \sim \{x'_0,\dots,x'_m\}$. Indeed, let $U \in \mathscr{T}(\mathbb{X})$ be such that $\{x_0,\dots,x_n\} \in \overline{U}$. This means that there are $U_0,\dots,U_n \in \mathscr{T}(\mathbb{X})$ such that $x_i \in U_i$ for each $i \leqslant n$ and $\bigcap_{i\leqslant n} U_i \subseteq U$. By Lemma 5, $x_i \leq \{x'_0,\dots,x'_m\}$ for all $i \leqslant n$. Therefore, for all $i \leqslant n$ and $j \leqslant m$, there is $U_{ij} \in \mathscr{T}(\mathbb{X})$ satisfying $x'_j \in U_{ij}$ and $\bigcap_{j\leqslant m} U_{ij} \subseteq U_i$. Hence, $x'_j \in \bigcap_{i\leqslant n} U_{ij} = V_j \in \mathscr{T}(\mathbb{X})$ for all $j \leqslant m$. Moreover, $\bigcap_{j\leqslant m} V_j = \bigcap\{U_{ij} \mid i \leqslant n,\ j \leqslant m\} \subseteq \bigcap_{i\leqslant n} U_i \subseteq U$ whence $\{x'_0,\dots,x'_m\} \in \overline{U}$. By symmetry, $\{x'_0,\dots,x'_m\} \in \overline{U}$ implies that $\{x_0,\dots,x_n\} \in \overline{U}$.

It follows from Claim 1 that f is well-defined and one-to-one. Since each element from Y is a finite join of elements from X, f is also onto.

Claim 2. *The mapping f is continuous and open.*

Proof of Claim 2. In order to prove that f is open, it suffices to show that $f(U^\sharp) \in \mathscr{T}(\mathbb{Y})$ for arbitrary $U \in \mathscr{T}(\mathbb{X})$. Indeed, let $x_0,\dots,x_n \in X$ be such that $\{x_0,\dots,x_n\} \in \overline{U}$; i.e., $x_0 \vee \dots \vee x_n \in f(U^\sharp)$. Then there exist $U_0,\dots,U_n \in \mathscr{T}(\mathbb{X})$ such that $x_0 \in U_0,\dots,x_n \in U_n$ and $U_0 \cap \dots \cap U_n \subseteq U$. It follows that $x_0 \vee \dots \vee x_n \in U_0^* \cap \dots \cap U_n^* = (U_0 \cap \dots \cap U_n)^* \subseteq U^* \in \mathscr{T}(\mathbb{Y})$. We show that $U^* \subseteq f(U^\sharp)$. Indeed, let $y \in U^*$ and let $x'_0,\dots,x'_m \in X$ be such that $y = x'_0 \vee \dots \vee x'_m$. Since $\vee$ is continuous, there are $V_0,\dots,V_m \in \mathscr{T}(\mathbb{Y})$ such that $x'_j \in V_j$ for all $j \leqslant m$ and $V_0 \times \dots \times V_m \subseteq \vee^{-1}(U^*)$; in particular, $V_0 \cap \dots \cap V_m \subseteq U^*$. So $x'_i \in V_i \cap X = W_i \in \mathscr{T}(\mathbb{X})$ for all $i \leqslant m$ and $W_0 \cap \dots \cap W_m = V_0 \cap \dots \cap V_m \cap X \subseteq U^* \cap X = U$. By definition, it means that $[x'_0,\dots,x'_m] \in U^\sharp$; i.e., $y = f\big([x'_0,\dots,x'_m]\big) \in f(U^\sharp)$ and $x_0 \vee \dots \vee x_n \in U^* \subseteq f(U^\sharp)$, which proves that $f(U^\sharp)$ is an open set.

In order to prove that f is continuous, take $V \in \mathscr{T}(\mathbb{Y})$ and $[x_0,\dots,x_n] \in f^{-1}(V)$. Then $x_0 \vee \dots \vee x_n = f\big([x_0,\dots,x_n]\big) \in V$. Since $\vee$ is continuous, there are $V_0,\dots,V_n \in \mathscr{T}(\mathbb{Y})$ such that $x_i \in V_i$ for all $i \leqslant n$ and $V_0 \times \dots \times V_n \subseteq \vee^{-1}(V)$; in particular, $V_0 \cap \dots \cap V_n \subseteq V$. As $\mathbb{X} \leq \mathbb{Y}$ is a semistrict extension, there are $U_0,\dots,U_n \in \mathscr{T}(\mathbb{X})$ such that $x_i \in U_i$ and $U_i^* \subseteq V_i$ for all $i \leqslant n$. We put $U = U_0 \cap \dots \cap U_n$; then

$U \in \mathscr{T}(\mathbb{X})$. It is clear that $[x_0, \dots, x_n] \in U^\sharp$. We show that $U^\sharp \subseteq f^{-1}(V)$. Indeed, let $[x'_0, \dots, x'_m] \in U^\sharp$ for some elements $x'_0, \dots, x'_m \in X$. This means that there are $W_0, \dots, W_m \in \mathscr{T}(\mathbb{X})$ such that $x'_j \in W_j$ for all $j \leqslant m$ and $W_0 \cap \dots \cap W_m \subseteq U$. By Lemma 2

$$\begin{aligned} f([x'_0, \dots, x'_m]) = x'_0 \vee \dots \vee x'_m \in W_0^* \cap \dots \cap W_m^* = (W_0 \cap \dots \cap W_m)^* \subseteq U^* \\ = (U_0 \cap \dots \cap U_n)^* = U_0^* \cap \dots \cap U_n^* \subseteq V_0 \cap \dots \cap V_n \subseteq V. \end{aligned}$$

Therefore, $x_0 \vee \dots \vee x_n \in U^\sharp \subseteq f^{-1}(V)$; in other words, $f^{-1}(V)$ contains an open neighborhood of each of its elements, which proves that $f^{-1}(V)$ is open.

Thus f is a homeomorphism. The fact that the extension $\mathbb{X} \leq \mathbb{H}^\vee(\mathbb{X})$ satisfies (i)–(iii) follows from results of [13].

Corollary 2. *If extensions $\mathbb{X} \leq \mathbb{Y}_0$ and $\mathbb{X} \leq \mathbb{Y}_1$ of topological T_0-spaces satisfy conditions* (i)–(iii) *of Theorem* 3 *then* $\mathbb{Y}_0 \cong \mathbb{Y}_1$.

Corollary 3. *If an extension $\mathbb{X} \leq \mathbb{Y}$ of topological T_0-spaces satisfies conditions* (i)–(iii) *of Theorem* 3 *then the extension is essential.*

Proof. By Corollary 1, it suffices to prove that each $y \in Y$ is essential for $\mathbb{X}$ in $\mathbb{Y}$. Indeed, if $y \notin X$ then since X generates the join semilattice $\langle Y; \vee \rangle$, y cannot be a bottom of $\mathbb{Y}$. Furthermore, if $y \in U \in \mathscr{T}(\mathbb{Y})$ then $y = x_0 \vee \dots \vee x_n$ for some $x_0, \dots, x_n \in X$. By continuity of $\vee$, there exist $U_0, \dots, U_n \in \mathscr{T}(\mathbb{Y})$ such that $x_i \in U_i$ for all $i \leqslant n$ and $U_0 \times \dots \times U_n \subseteq \vee^{-1}(U)$; in particular, $U_0 \cap \dots \cap U_n \subseteq U$. It remains to refer to Definition 1.

§ 5. Ideals and Filters of Posets

Given a poset $\langle S; \leq \rangle$ and a subset $X \subseteq S$, denote by $U(X)$ the set of all *upper bounds* and by $L(X)$, the set of all *lower bounds* of X in $\langle S; \leq \rangle$. Herein, we assume that $L(\varnothing) = U(\varnothing) = S$. We also write $L(s_0, \dots, s_n)$ instead of $L(\{s_0, \dots, s_n\})$ and $U(s_0, \dots, s_n)$ instead of $U(\{s_0, \dots, s_n\})$.

The following definition of a φ-ideal corresponds to that of [11].

Definition 3. For a poset $\langle S; \leq \rangle$, an algebraic closure operator φ on S defines a *completion* of $\langle S; \leq \rangle$, if the mapping $s \mapsto \varphi(s)$ is an order embedding of $\langle S; \leq \rangle$ into the poset $\mathrm{Cl}(S, \varphi)$ of φ-closed subsets of S with respect to the set-theoretic inclusion.

It is clear that for all $X, Y \in \mathrm{Cl}(S, \varphi)$ the set $X \cap Y$ is closed with respect to φ, whence $X \cap Y$ is the meet of X and Y in the lattice $\mathrm{Cl}(S, \varphi)$. Let $+$ denote the join in $\mathrm{Cl}(S, \varphi)$.

Lemma 8. *Let an algebraic closure operator φ on S define a completion of $\langle S; \leq \rangle$. Then $\varphi(s) = L(s)$ for each $s \in S$.*

Proof. Let $s \in S$. By Definition 3, $s' \in \varphi(s') \subseteq \varphi(s)$ for each $s' \leq s$; i.e., $L(s) \subseteq \varphi(s)$. On the other hand, if $s' \in \varphi(s)$ then $\varphi(s') \subseteq \varphi(s)$. Since $s \mapsto \varphi(s)$ embeds $\langle S; \leq \rangle$ into $\langle \mathrm{Cl}(S, \varphi); \subseteq \rangle$, conclude that $s' \leq s$; whence $\varphi(s) \subseteq L(s)$.

Definition 4. Let an algebraic closure operator φ define a completion of $\langle S; \leq \rangle$. In this case, each $I \in \mathrm{Cl}(S, \varphi)$ is called a *φ-ideal* of $\langle S; \leq \rangle$.

A *filter* of $\langle S; \leq \rangle$ is a subset $F \subseteq S$ possessing the following property:

$$s_0, s_1 \in F \text{ if and only if } L(s_0, s_1) \cap F \neq \varnothing \quad \text{for all } s_0, s_1 \in S.$$

Moreover, a φ-ideal I of $\langle S; \leq \rangle$ is *prime*, if I is proper (i.e., $I \neq \varnothing$ and $I \neq S$) and $S \backslash I$ is a filter. A filter F of $\langle S; \leq \rangle$ is *φ-prime*, if F is proper and $S \backslash F$ is a φ-ideal.

Let $\mathrm{Spec}_\varphi S$ denote the set of all prime φ-ideals of $\langle S; \leq \rangle$.

The terminology of Definition 4 can be justified by the next

Proposition 1 [11, Proposition 2.4]. *Let an algebraic closure operator φ define a completion of a poset $\langle S;\leq\rangle$. Then $I\subseteq S$ is a φ-ideal if and only if $I=P\cap I'$ for some ideal I' of the lattice* $\mathrm{Cl}(S,\varphi)$.

In the sequel, we denote the lattice of all φ-ideals of a poset $\langle S;\leq\rangle$ by $\mathrm{Id}(S,\varphi)$ instead of $\mathrm{Cl}(S,\varphi)$.

It follows from Definition 4 that $F\subseteq S$ is a filter of $\langle S;\leq\rangle$ if and only if the two conditions are satisfied:

(i) if $s_0\leq s_1$ and $s_0\in F$ then $s_1\in F$;

(ii) if $s_0,s_1\in F$ then there is $s\leq s_0,s_1$ such that $s\in F$.

The following examples illustrate Definition 4.

EXAMPLE 4. (i) Let $\langle S;\leq\rangle$ be a poset. We consider the closure operator φ_0 on S, defined by $\varphi_0(X)={\downarrow}X$ for an arbitrary set $X\subseteq S$. We also consider the closure operator φ_1 on S such that $X\subseteq S$ is closed if and only if $LU(a,b)\subseteq X$ for all $a,b\in X$. It is obvious that both closure operators, φ_0 and φ_1, are algebraic and define a completion of $\langle S;\leq\rangle$.

(ii) Let $\langle S;\vee\rangle$ be a join semilattice. Given $X\subseteq S$, put

$$\psi(X)=\{s\in S\mid s\leq \bigvee F,\ \text{where } F\subseteq X,\ 0<|F|<\omega\}.$$

It is straightforward to verify that ψ is an algebraic closure operator on S, which defines a completion of $\langle S;\leq\rangle$, where $\leq$ denotes the natural order defined by $\vee$. Then the φ_0-ideals are exactly the order ideals of $\langle S;\leq\rangle$ and the ψ-ideals are exactly the ideals of the join semilattice $\langle S;\vee\rangle$.

(iii) We consider an arbitrary topological T_0-space $\mathbb{X}$ and define a closure operator ξ on X by assuming $\xi(I)=I$, with $I\subseteq X$, if and only if $x\leq\{x_0,\dots,x_n\}$ implies that $x\in I$ for all $x_0,\dots,x_n\in I$; cf. Definition 2. It is clear that the closure operator ξ is algebraic and defines a completion of $\langle X;\leq\rangle$. In this case, the ξ-ideals are exactly ideals of the topological space $\mathbb{X}$; cf. Theorem 2.

Definition 4 yields

Lemma 9. *Let an algebraic closure operator φ define a completion of a poset $\langle S;\leq\rangle$. Then*

(i) *intersection of any family of φ-ideals of $\langle S;\leq\rangle$ is a φ-ideal of $\langle S;\leq\rangle$. Thus* $\mathrm{Id}(S,\varphi)$ *is a complete lattice;*

(ii) *a set $I\subseteq S$ is a prime φ-ideal if and only if $S\backslash I$ is a φ-prime filter.*

The proof of the next statement is also easy; see [11, Proposition 3.1].

Lemma 10. *Let an algebraic closure operator φ define a completion of a poset $\langle S;\leq\rangle$. For an arbitrary proper φ-ideal $I\in\mathrm{Id}(S,\varphi)$, the following are equivalent:*

(i) *I is a prime φ-ideal;*

(ii) *I is a $\cap$-prime element in the lattice* $\mathrm{Id}(S,\varphi)$*;*

(iii) *$L(s_0,s_1)\subseteq I$ implies that either $s_0\in I$ or $s_1\in I$.*

DEFINITION 5 [11]. Let an algebraic closure operator φ define a completion of a poset $\langle S;\leq\rangle$. The poset $\langle S;\leq\rangle$ is *φ-distributive*, if the lattice $\mathrm{Id}(S,\varphi)$ is distributive.

We recall that a join semilattice $\langle S;\vee\rangle$ is *distributive*, if for all $s,s_0,s_1\in S$ the inequality $s\leq s_0\vee s_1$ implies the existence of $s_0'\leq s_0$ and $s_1'\leq s_1$ in S such that $s=s_0'\vee s_1'$. We note that (distributive) semilattices are an instance of (φ-distributive) posets. In this case, the closure operator ψ assigns to each subset $X\subseteq S$ the smallest subset of S, containing X, is a lower cone closed with respect to $\vee$; cf. Example 4 and Lemma 12.

§ 6. Semitopological Posets and the Hofmann–Mislove Theorem

DEFINITION 6. Let an algebraic closure operator φ define a completion of a poset $\langle S;\leq\rangle$ and let $\mathscr{T}$ be a topology on S. Given each finite $F\subseteq S$ and each $X\subseteq S$, put $\sigma_F^{-1}(X)=\bigl\{a\in S\mid \varphi\bigl(F\cup\{a\}\bigr)\cap X\neq\varnothing\bigr\}$. The triple $\mathbb{S}=\langle S,\leq,\mathscr{T}\rangle$ is a *φ-semitopological poset* whenever for each finite $F\subseteq S$ and each $X\subseteq S$, we have that if $X\in\mathscr{T}$ then $\sigma_F^{-1}(X)\in\mathscr{T}$.

Definition 7 [8]. Let $\langle S;\vee\rangle$ be a join semilattice and let $\mathscr{T}$ be a topology on S. The triple $\mathbb{S}=\langle S,\vee,\mathscr{T}\rangle$ is a *semitopological (join) semilattice* if for all $s\in S$, the mapping $\sigma_s:\mathbb{S}\to\mathbb{S}$, $\sigma_s: x\mapsto x\vee s$, is continuous. Moreover, $\mathbb{S}$ is a *topological semilattice*, if $\vee$ is continuous.

A lattice $\langle S;\vee,\wedge\rangle$ endowed with a topology $\mathscr{T}$ is a *topological lattice* if $\mathbb{S}=\langle S,\vee,\mathscr{T}\rangle$ is a topological semilattice.

Corollary 4. *The following hold:*

(i) *Each topological semilattice is a semitopological semilattice.*

(ii) *A semilattice with topology $\mathbb{S}=\langle S,\vee,\mathscr{T}\rangle$ is a semitopological semilattice if and only if $\mathbb{S}$ is a ψ-semitopological poset.*

Lemma 11. *Let $\langle S;\leq\rangle$ be a φ-distributive meet semilattice, where the algebraic operator φ on S defines a completion of $\langle S;\leq\rangle$. Then for every finite set $F\subseteq S$ and every filter $P\subseteq S$ the set $\sigma_F^{-1}(P)$ is a filter containing P.*

Proof. Let the algebraic closure operator φ define a completion of $\langle S;\leq\rangle$ and let $F\subseteq S$ be a finite set, while $P\subseteq S$ is a filter. It is obvious that $P\subseteq\sigma_F^{-1}(P)$. Since $\sigma_\varnothing^{-1}(P)=P$, we may assume without loss of generality that $F\neq\varnothing$. Suppose that $L(s_0,s_1)\cap\sigma_F^{-1}(P)\neq\varnothing$ and let $a\in L(s_0,s_1)\cap\sigma_F^{-1}(P)$. This means that $a\in L(s_0)\cap L(s_1)$. By the φ-distributivity of $\langle S;\leq\rangle$, we get

$$\varnothing\neq\varphi(F\cup\{a\})\cap P\subseteq(\varphi(F)+(L(s_0)\cap L(s_1)))\cap P$$
$$=(\varphi(F)+(\varphi(s_0)\cap\varphi(s_1)))\cap P=\varphi(F\cup\{s_0\})\cap\varphi(F\cup\{s_1\})\cap P;$$

i.e., $\varphi\big(F\cup\{s_i\}\big)\cap P\neq\varnothing$ for all $i<2$. This means that $s_0,s_1\in\sigma_F^{-1}(P)$.

Conversely, assume that $s_0,s_1\in\sigma_F^{-1}(P)$. This means that there exist $a_i\in\varphi\big(F\cup\{s_i\}\big)\cap P$, $i<2$. Since P is a filter, there exists $a\in L(a_0,a_1)\cap P$. So,

$$a\in L(a_0)\cap L(a_1)\cap P\subseteq\varphi(F\cup\{s_0\})\cap\varphi(F\cup\{s_1\})\cap P$$
$$=(\varphi(F)+(L(s_0)\cap L(s_1)))\cap P=(\varphi(F)+L(s_0,s_1))\cap P=\varphi(F\cup\{s_0\wedge s_1\})\cap P,$$

i.e., $s_0\wedge s_1\in L(s_0,s_1)\cap\sigma_F^{-1}(P)$. By Definition 4, $\sigma_F^{-1}(P)$ is a filter.

We recall that the completion ψ of join semilattices was considered in Example 4 (ii).

Lemma 12. *A join semilattice $\langle S;\vee\rangle$ is distributive if and only if $\langle S;\vee\rangle$ is ψ-distributive.*

Proof. We assume first that the semilattice $\langle S;\vee\rangle$ is distributive and consider arbitrary ideals $X,Y_0,Y_1\in\operatorname{Id}(S,\psi)$. In order to establish ψ-distributivity, it suffices to show that $X\cap\psi(Y_0\cup Y_1)\subseteq\psi\big((X\cap Y_0)\cup(X\cap Y_1)\big)$. Indeed, take $s\in X\cap\psi(Y_0\cup Y_1)$. This means that $s\leq s_0\vee s_1$ for some $s_0\in Y_0$ and $s_1\in Y_1$. By the distributivity of $\langle S;\vee\rangle$, there are $y_0\leq s_0$ and $y_1\leq s_1$ such that $s=y_0\vee y_1$. Since $s\in X$, we have $y_0\in X\cap Y_0$ and $y_1\in X\cap Y_1$; i.e., $s\in\psi\big((X\cap Y_0)\cup(X\cap Y_1)\big)$.

Conversely, let $\langle S;\vee\rangle$ be ψ-distributive and let $s\leq s_0\vee s_1$ for some $s,s_0,s_1\in S$. Then $s\in L(s)\cap\psi\big(\{s_0,s_1\}\big)=\big(L(s)\cap\psi(s_0)\big)+\big(L(s)\cap\psi(s_1)\big)=L(s,s_0)+L(s,s_1)=\psi\big(L(s,s_0)\cup L(s,s_1)\big)$. By the definition of ψ, there are $s_0'\in L(s,s_0)$ and $s_1'\in L(s,s_1)$ such that $s\leq s_0'\vee s_1'$. Since $s_0',s_1'\leq s$, we have $s=s_0'\vee s_1'$.

Lemma 13 [8, Lemma 11]. *Let $\langle S;\vee\rangle$ be a distributive join semilattice. Then $\sigma_F^{-1}(P)$ (defined with respect to ψ) is a filter and contains P for every finite set $F\subseteq S$ and every filter $P\subseteq S$.*

Proof. Indeed, let $F\subseteq S$ be finite and let $P\subseteq S$ be a filter. It is clear that $P\subseteq\sigma_F^{-1}(P)=\{s\in S\mid f\vee s\in P\}$, where $f=\bigvee F$. It is not hard to see that the set $\sigma_F^{-1}(P)$ is an upper cone. Suppose that $s_0,s_1\in\sigma_F^{-1}(P)$. This means that $f\vee s_0$, $f\vee s_1\in P$. By the definition of filter, there is $c\in P$ such that $c\leq f\vee s_0$, $c\leq f\vee s_1$. By the distributivity of $\langle S;\vee\rangle$, there are $s_0'\leq s_0$ and $f'\leq f$ such that $c=f'\vee s_0'$. As $s_0'\leq c\leq f\vee s_1$, there are elements $s_1'\leq s_1$ and $f''\leq f$ such that $s_0'=s_1'\vee f''$. So $s_1'\leq s_0'\leq s_0$, $s_1'\leq s_1$. Moreover,

$$s_1'\vee f=s_1'\vee f''\vee f=s_0'\vee f=s_0'\vee f'\vee f=c\vee f\geq c\in P;$$

i.e., $s_1'\vee f\in\psi\big(F\cup\{s_1'\}\big)\cap P$, whence $s_1'\in L(s_0,s_1)\cap\sigma_F^{-1}(P)$. By Definition 4, $\sigma_F^{-1}(P)$ is a filter.

Lemma 14. *Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq\rangle$ and let $\langle S, \leq, \mathscr{T}\rangle$ be a φ-semitopological poset such that $\sigma_F^{-1}(F)$ is a filter for each filter $F \subseteq S$. Also let $\varnothing \neq I \subseteq S$ be a φ-ideal and let $\varnothing \neq P \subseteq S$ be a filter such that $P \in \mathscr{T}$ and $I \cap P = \varnothing$. Then there is a φ-prime filter $P^* \in \mathscr{T}$ such that $P \subseteq P^*$ and $I \cap P^* = \varnothing$.*

Proof. By Zorn's Lemma, the set

$$\mathscr{X} = \{X \in \mathscr{T} \mid X \text{ is a filter, } P \subseteq X,\ I \cap X = \varnothing\}$$

contains a maximal element P^*; in particular, $P^* \neq \varnothing$. Moreover, $P^* \neq S$ since otherwise $I \cap P^* = I \neq \varnothing$.

Claim 1. *$\sigma_F^{-1}(P^*) = P^*$ for every finite $F \subseteq S \backslash P^*$.*

Proof of Claim 1. Suppose that Claim 1 does not hold and choose a finite $F \subseteq S$ satisfying $\sigma_F^{-1}(P^*) \neq P^*$ and such that $|F \backslash I|$ is minimal. By Definition 6, $\sigma_F^{-1}(P^*) \in \mathscr{T}$ is a filter, obviously containing P^*. By the maximality of P^* in $\mathscr{X}$, we conclude that $I \cap \sigma_F^{-1}(P^*) \neq \varnothing$, whence there is some $b \in I \cap \sigma_F^{-1}(P^*)$.

If $F \subseteq I$ then

$$\varnothing \neq \varphi(F \cup \{b\}) \cap P^* \subseteq \varphi(I) \cap P^* = I \cap P^* = \varnothing,$$

which is impossible. Therefore, $F \nsubseteq I$, whence $|F \backslash I| > 0$ and there is some $a \in F \backslash I$. But then $a \in F \cap \sigma_G^{-1}(P^*)$, where $G = \{b\} \cup (F \backslash \{a\})$. Since $|G \backslash I| = |F \backslash I| - 1$, conclude by the choice of F that $a \in F \cap \sigma_G^{-1}(P^*) = F \cap P^* = \varnothing$, which is a contradiction. This contradiction shows the falsity of the assumption that Claim 1 does not hold.

Since P^* is a filter, we have $L(a) \cap P^* = \varnothing$ for all $a \notin P^*$. Furthermore, show that $S \backslash P^*$ is a φ-ideal. To this end, prove that $\varphi(S \backslash P^*) \subseteq S \backslash P^*$. By the algebraicity of the closure operator φ, we have to show that $\varphi(F) \subseteq S \backslash P^*$ for an arbitrary nonempty finite set $F \subseteq S \backslash P^*$. Indeed, if $\varphi(F) \cap P^* \neq \varnothing$ then $a \in (S \backslash P^*) \cap \sigma_{F \backslash \{a\}}^{-1}(P^*) = (S \backslash P^*) \cap P^* = \varnothing$ for every $a \in F$ by Claim 1, which is impossible. Therefore, $\varphi(S \backslash P^*) = S \backslash P^*$; i.e., $S \backslash P^*$ is a φ-ideal. Thus P^* is a φ-prime filter.

The following generalization of the Hofmann–Mislove Theorem holds; cf. Corollary V–5.4 in [5].

Theorem 4. *Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq\rangle$ and let $\langle S, \leq, \mathscr{T}\rangle$ be a φ-semitopological poset such that $\sigma_F^{-1}(F)$ is a filter for each filter $F \subseteq S$. Also let $F \in \mathscr{T}$ be an open filter such that $F \notin \{\varnothing, S\}$. Then F is the intersection of a nonempty family of open φ-prime filters.*

Proof. By assumption, there is some $a \in S \backslash F$. Then $L(a)$ is a nonempty φ-ideal and $L(a) \cap F = \varnothing$. By Lemma 14, $F \subseteq P$ and $a \notin P$ for some open φ-prime filter $P \in \mathscr{T}$. Thus

$$F = \bigcap \{P \in \mathscr{T} \mid P \text{ is a prime } \varphi\text{-prime filter, } F \subseteq P\},$$

which was to be proven.

From Lemmas 11 and 12 together with Theorem 4 we obtain the following

Corollary 5. *Let an algebraic closure operator φ define a completion of $\langle S; \leq\rangle$, let $\langle S, \leq, \mathscr{T}\rangle$ be a φ-distributive semitopological meet semilattice (a distributive semitopological join semilattice, respectively), and let $F \in \mathscr{T}$ be an open filter such that $F \notin \{\varnothing, S\}$. Then F is the intersection of a nonempty family of open φ-prime filters.*

We note that the statement of Corollary 5 was established for distributive semitopological join semilattices in [8].

§ 7. Spectral Theory

In what follows, we will assume that each poset is endowed with some T_0-topology and we call it a *poset with topology.*

Definition 8. Let an algebraic closure operator φ define a completion of a poset $\langle S;\leq\rangle$. Further, let $\operatorname{Spec}_\varphi \mathbb{S}$ consist of all prime φ-ideals of $\langle S;\leq\rangle$ closed in $\mathscr{T}$, and let $\mathscr{T}_*$ denote the topology with the subbase $\{V_s \mid s\in S\}$ of open sets, where $V_s=\{I\in \operatorname{Spec}_\varphi \mathbb{S}\mid s\notin I\}$ for all $s\in S$. Then the topological space $\mathbb{S}\mathrm{pec}_\varphi \mathbb{S}=\langle \operatorname{Spec}_\varphi \mathbb{S},\mathscr{T}_*\rangle$ is called the *φ-spectrum* of the poset with topology $\mathbb{S}=\langle S,\leq,\mathscr{T}\rangle$.

It is not hard to see that $\mathscr{T}_*$ satisfies the T_0-separation axiom.

Remark 2. We should take into account that a poset with topology can include no (closed) prime ideal. Hence its spectrum can be empty. Indeed, consider the poset $\langle X;\leq\rangle$, where $X=\{a,b,c\}$ and every two elements are incomparable; i.e., $x\leq y$ if and only if $x=y$. Endow X with the discrete topology and consider the corresponding poset with topology $\mathbb{X}$. By Lemma 1, $\leq$ coincides with the specialization order. It is not hard to see that $I\subseteq X$ is an ideal of $\mathbb{X}$ if and only if $I\in\big\{\varnothing,\{a\},\{b\},\{c\},X\big\}$. Moreover, the complement of any singleton of X is not a filter. Therefore, $\mathbb{X}$ has no prime ideals.

Lemma 15. *Let $\mathbb{S}$ be a poset with topology and let an algebraic closure operator φ define a completion of $\langle S;\leq\rangle$. Then*

$$V_{s_0}\cap V_{s_1}=\bigcup_{s\in L(s_0,s_1)} V_s$$

for all $s_0,s_1\in S$. In particular $\{V_s\mid s\in S\}$ forms a base of open sets for $\mathscr{T}_*$.

Proof. Take $I\in V_{s_0}\cap V_{s_1}$. This means that $s_0,s_1\in S\backslash I$. Since I is a prime φ-ideal, by Lemma 10, there exists $s\in L(s_0,s_1)\cap(S\backslash I)$. Hence $I\in\bigcup_{s\in L(s_0,s_1)}V_s$. Suppose that $I\in V_s$ for some $s\in L(s_0,s_1)$. If $s_i\in I$ for some $i<2$ then $s\in\varphi(s_i)\subseteq\varphi(I)=I$, which contradicts our assumption. Therefore, $s_0,s_1\notin I$ and $I\in V_{s_0}\cap V_{s_1}$.

Given $X\subseteq S$, put $V_X=\{I\in\operatorname{Spec}_\varphi\mathbb{S}\mid X\nsubseteq I\}$. Lemma 15 proves that $V_{s_0}\cap V_{s_1}=V_{L(s_0,s_1)}$ for arbitrary $s_0,s_1\in S$.

Corollary 6. *For an arbitrary poset with topology $\mathbb{S}$ and an algebraic completion φ of $\mathbb{S}$, the following are true:*

(i) $V_X=\bigcup_{s\in X}V_s\in\mathscr{T}_*$ *for each subset $X\subseteq S$; in particular, $X_0\subseteq X_1$ implies that $V_{X_0}\subseteq V_{X_1}$.*

(ii) $V_{J_0}\cap V_{J_1}=V_{J_0\cap J_1}$ *for all φ-ideals $J_0,J_1\subseteq S$.*

(iii) $V_X=V_{\varphi(X)}$ *for all $X\subseteq S$.*

(iv) If $U\in\mathscr{T}_*$ then there is a φ-ideal $J\subseteq S$ such that $U=V_J$; in particular, $\varnothing=V_\varnothing$ and $\operatorname{Spec}_\varphi\mathbb{S}=V_S$.

Proof. (i) is obvious.

(ii) The inclusion $V_{J_0\cap J_1}\subseteq V_{J_0}\cap V_{J_1}$ is straightforward. Suppose that $I\in V_{J_0}\cap V_{J_1}$. This means that there are $s_i\in J_i\backslash I$, $i<2$. By Lemma 10, there is some $s\in L(s_0,s_1)\backslash I$. Hence, $s\in\varphi(s_0)\cap\varphi(s_1)\subseteq J_0\cap J_1$ and $I\in V_{J_0\cap J_1}$.

(iii) From (i) it follows that $V_X=\bigcup_{s\in X}V_s\subseteq V_{\varphi(X)}$. Conversely, if $X\subseteq I\in\operatorname{Spec}_\varphi\mathbb{S}$ then $\varphi(X)\subseteq\varphi(I)=I$; i.e., $V_{\varphi(X)}\subseteq\bigcup_{s\in X}V_s=V_X$.

(iv) Take $U\in\mathscr{T}_*$. By Lemma 15, $U=\bigcup_{s\in X}V_s$ for some $X\subseteq S$. So the claim follows from (iii).

Lemma 16. *Let $\mathbb{S}$ be a poset with topology and let φ define an algebraic completion of $\mathbb{S}$. For all $I_0,I_1\in\operatorname{Spec}_\varphi\mathbb{S}$, the inequality $I_0\leq_{\mathscr{T}_*}I_1$ holds if and only if $I_1\subseteq I_0$.*

Proof. Suppose first that $I_0\leq_{\mathscr{T}_*}I_1$ for some closed prime φ-ideals I_0 and I_1. Take $s\in I_1$. This means that $I_1\notin V_s\in\mathscr{T}_*$. By the definition of specialization order, $I_0\notin V_s$, whence $s\in I_0$ and $I_1\subseteq I_0$.

Conversely, take $I_1\subseteq I_0$ and $I_0\in U\in\mathscr{T}_*$. By Lemma 15 and Definition 8, $I_0\in V_s\subseteq U$ for some $s\in S$. In particular, $s\notin I_0$. By assumption, $s\notin I_1$, whence $I_1\in V_s\subseteq U$. Therefore, $I_0\leq_{\mathscr{T}_*}I_1$.

Theorem 5. *Let $\mathbb{S}$ be a poset with topology and let an algebraic closure operator φ define a completion of $\langle S;\leq\rangle$. Then $\mathbb{S}\mathrm{pec}_\varphi\mathbb{S}$ is an almost sober space. The space $\mathbb{S}\mathrm{pec}_\varphi\mathbb{S}$ is sober whenever S contains an element that belongs to each nonempty closed prime φ-ideal in $\mathbb{S}$ (in particular, if $\langle S;\leq\rangle$ has a bottom).*

Proof. Let $\mathscr{T} = \mathscr{T}(\mathbb{S})$ and let $F \subseteq \operatorname{Spec}_\varphi \mathbb{S}$ be a nonempty irreducible closed set in $\operatorname{Spec}_\varphi \mathbb{S}$. Put $U = (\operatorname{Spec}_\varphi \mathbb{S})\backslash F$; it is obvious that $U \in \mathscr{T}_*$. Also, put $J = \{s \in S \mid V_s \subseteq U\}$.

Claim 1. *J is a largest φ-ideal in $\langle S; \leq\rangle$ satisfying $U = V_J$.*

Proof of Claim 1. Using Lemma 15, Corollary 6(i), and the definition of $\mathscr{T}_*$, observe that $U = V_J = V_{\varphi(J)}$. Applying Corollary 6(i) once again, we see that $V_s \subseteq U$ and so $s \in J$ for every $s \in \varphi(J)$; i.e., $J = \varphi(J)$ is a φ-ideal. Let J' be a φ-ideal in $\langle S; \leq\rangle$ such that $U = V_{J'}$ and take $s \in J'$. By Corollary 6(i), $V_s \subseteq U$, whence $s \in J$ and $J' \subseteq J$.

Suppose now that $F \neq \operatorname{Spec}_\varphi \mathbb{S}$; in this case, $U \notin \{\varnothing, \operatorname{Spec}_\varphi \mathbb{S}\}$ and so $J \notin \{\varnothing, \operatorname{Spec}_\varphi \mathbb{S}\}$.

Claim 2. *$J \in \operatorname{Spec}_\varphi \mathbb{S}$.*

Proof of Claim 2. We show first that $S\backslash J$ is a filter. Indeed, take $s_0, s_1 \in S\backslash J$. By the definition of J, this means that $V_{s_0}, V_{s_1} \nsubseteq U$, whence $V_{s_0} \cap F \neq \varnothing$ and $V_{s_1} \cap F \neq \varnothing$. Since F is irreducible, we see that $V_{s_0} \cap V_{s_1} \cap F \neq \varnothing$. By Lemma 15, there is $s \in L(s_0, s_1)$ such that $V_s \cap F \neq \varnothing$. It follows that $V_s \nsubseteq U$ and $s \in S\backslash J$ and so $L(s_0, s_1) \cap (S\backslash J) \neq \varnothing$. If $L(s_0, s_1) \cap (S\backslash J) \neq \varnothing$ then $s_0, s_1 \in S\backslash J$, as otherwise, we would have by the definition of φ-ideal that $L(s_0, s_1) \subseteq \varphi(s_i) \subseteq J$ for some $i < 2$, which is impossible. Therefore, J is a prime φ-ideal.

Establish now that $V = S\backslash J \in \mathscr{T}$. Indeed, take $s \in V$. Since $s \notin J$, this means that $V_s \nsubseteq U$ and so $V_s \cap F \neq \varnothing$; let $I \in V_s \cap F$. Since I is a closed prime φ-ideal, $V' = S\backslash I \in \mathscr{T}$ and $s \in V'$. Show that $V' \subseteq V$; this will prove that V is open in $\mathscr{T}$. Take $s' \in V'$. This means that $s' \notin I$, whence $I \in V_{s'}$ and so $I \in V_{s'} \cap F$. Hence, $V_{s'} \nsubseteq U$ and $s' \notin J$. Therefore, $s' \in V$ and J is closed indeed.

The following is obvious:

Claim 3. *$F = {\downarrow} J = \{I \in \operatorname{Spec} \mathbb{S} \mid J \subseteq I\}$.*

It follows from Claims 2 and 3 that the sets of the form ${\downarrow} I$, where $I \in \operatorname{Spec}_\varphi \mathbb{S}$, exhaust all proper irreducible closed sets in $\operatorname{Spec}_\varphi \mathbb{S}$. If $\langle S; \leq\rangle$ contains an element s_0 belonging to each nonempty closed prime ideal and $F = \operatorname{Spec}_\varphi \mathbb{S}$ is irreducible, then $U = \varnothing$ and $V_{s_0} = \varnothing$; whence $J \notin \{\varnothing, \operatorname{Spec}_\varphi \mathbb{S}\}$ as $s_0 \in J$. Therefore, J is a closed prime φ-ideal in $\mathbb{S}$ in this case too. Moreover,

$$F = \operatorname{Spec}_\varphi \mathbb{S} = \{I \in \operatorname{Spec}_\varphi \mathbb{S} \mid J \subseteq I\} = {\downarrow} J.$$

The proof of Theorem 5 is complete.

§ 8. Characterization of Sober Spaces

Definition 9. Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq\rangle$. The topology $\mathscr{T}_\varphi^\pi \subseteq \mathscr{T}$ on a semitopological poset $\mathbb{S} = \langle S, \leq, \mathscr{T}\rangle$ is defined by the subbase of φ-prime filters of $\mathbb{S}$ open in $\mathscr{T}$. Put $\mathbb{S}_\varphi^\pi = \langle S, \vee, \mathscr{T}_\varphi^\pi\rangle$.

Lemma 17 [8, Corollary of Lemma 5]. *The structure $\mathbb{S}^\pi$ is a topological lattice for every semitopological semilattice $\mathbb{S}$.*

Lemma 18. *Let an algebraic closure operator φ define a completion of a poset $\langle S; \leq\rangle$, let $\mathbb{S} = \langle S, \leq, \mathscr{T}\rangle$ be a T_0-space, and let $\mathscr{T} = \mathscr{T}^\pi$. Then for all $s_0, s_1 \in S$, the following hold:*

(i) *If $s_0 \leq s_1$ then $s_0 \leq_{\mathscr{T}} s_1$.*

(ii) *If $\langle S; \leq\rangle$ is a join (meet) semilattice and $s_0 \leq_{\mathscr{T}} s_1$ then $s_0 \leq s_1$.*

Proof. If $\langle S; \leq\rangle$ is a join semilattice, then assume that $\varphi = \psi$; cf. Example 4(ii). Item (i) is obvious.

(ii) Suppose that $s_0 \nleq s_1$ and consider the two possible cases:

Case 1: $\langle S; \leq\rangle$ is a join semilattice. Then $s_1 < s_0 \vee s_1$, and using the T_0-separation axiom and (i), note that there is a φ-prime filter $P \in \mathscr{T}$ such that $s_0 \vee s_1 \in P$ but $s_1 \notin P$. If $s_0 \notin P$ then $s_0 \vee s_1 \in \varphi(s_0, s_1) \subseteq S\backslash P$, which is impossible. This means that $s_0 \in P$; whence $s_0 \nleq_{\mathscr{T}} s_1$.

Case 2: $\langle S; \leq\rangle$ is a meet semilattice. Then $s_0 \wedge s_1 < s_0$, and using the T_0-separation axiom and (i), note that there is a φ-prime filter $P \in \mathscr{T}$ such that $s_0 \in P$, but $s_0 \wedge s_1 \notin P$. This means that $L(s_0, s_1) = L(s_0 \wedge s_1) \subseteq S\backslash P$. Since the last set is a φ-ideal, $s_1 \notin P$ by Lemma 10, whence $s_0 \nleq_{\mathscr{T}} s_1$.

EXAMPLE 5 [8]. Fix a topological space $\mathbb{X}$. The algebraic structure $\langle \mathscr{T}(\mathbb{X}); \cup, \cap\rangle$ is obviously a distributive lattice. Note that in case that $\mathbb{X}$ is irreducible (i.e., X is a set irreducible in $\mathbb{X}$), the set $\mathscr{T}(\mathbb{X})\backslash\{\varnothing\}$ is a sublattice of $\mathscr{T}(\mathbb{X})$. We consider several topologies defined on $\mathscr{T}(\mathbb{X})$.

(i) $\mathscr{T}_\omega$ stands for the discrete topology on $\mathscr{T}(\mathbb{X})$.

(ii) $\mathscr{T}_{dk}$ denotes the Scott topology $\mathscr{T}_S(\subseteq)$ on $\mathscr{T}(\mathbb{X})$ (which is also called the *Day–Kelly topology*).

(iii) The *φ-topology* $\mathscr{T}_\varphi$ on $\mathscr{T}(\mathbb{X})$ is defined by the subbase of open sets which consists of all proper filters of the semilattice $\langle \mathscr{T}(\mathbb{X}); \cup\rangle$ open in the Day–Kelly topology $\mathscr{T}_{dk}$.

(iv) The *π-topology* $\mathscr{T}_\pi$ on $\mathscr{T}(\mathbb{X})$ is defined by the subbase of open sets which consists of all prime filters of the semilattice $\langle \mathscr{T}(\mathbb{X}); \cup\rangle$ open in the Day–Kelly topology $\mathscr{T}_{dk}$ (or, equivalently, open in $\mathscr{T}_\varphi$).

It is obvious that $\mathscr{T}_\pi \subseteq \mathscr{T}_\varphi \subseteq \mathscr{T}_{dk} \subseteq \mathscr{T}_\omega$. Moreover, it is not hard to verify that $\mathscr{T}_\pi = (\mathscr{T}_{dk})^\pi$. Put

$$\mathbb{T}_\omega(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}), \cup, \mathscr{T}_\omega\rangle, \quad \mathbb{T}_{dk}(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}), \cup, \mathscr{T}_{dk}\rangle,$$

$$\mathbb{T}_\varphi(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}), \cup, \mathscr{T}_\varphi\rangle, \quad \mathbb{T}_\pi(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}), \cup, \mathscr{T}_\pi\rangle.$$

If $\mathbb{X}$ is irreducible, then put

$$\mathbb{T}^\varnothing_\omega(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}) \setminus \{\varnothing\}, \cup, \mathscr{T}_\omega\rangle, \quad \mathbb{T}^\varnothing_{dk}(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}) \setminus \{\varnothing\}, \cup, \mathscr{T}_{dk}\rangle,$$
$$\mathbb{T}^\varnothing_\varphi(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}) \setminus \{\varnothing\}, \cup, \mathscr{T}_\varphi\rangle, \quad \mathbb{T}^\varnothing_\pi(\mathbb{X}) = \langle \mathscr{T}(\mathbb{X}) \setminus \{\varnothing\}, \cup, \mathscr{T}_\pi\rangle.$$

We use the above notation to denote also the corresponding topological spaces.

Lemma 19 [8, Lemma 3]. *For a topological space $\mathbb{X}$, the following are true:*

(i) *$\mathbb{T}_\omega(\mathbb{X})$, $\mathbb{T}_{dk}(\mathbb{X})$, $\mathbb{T}_\varphi(\mathbb{X})$, and $\mathbb{T}_\pi(\mathbb{X})$ are T_0-separable semitopological (semi)lattices;*

(ii) $\operatorname{Spec}\mathbb{T}_{dk}(\mathbb{X}) = \operatorname{Spec}\mathbb{T}_\varphi(\mathbb{X}) = \operatorname{Spec}\mathbb{T}_\pi(\mathbb{X})$;

(iii) *if $\mathbb{X}$ is irreducible, then $\mathbb{T}^\varnothing_\omega(\mathbb{X})$, $\mathbb{T}^\varnothing_{dk}(\mathbb{X})$, $\mathbb{T}^\varnothing_\varphi(\mathbb{X})$, and $\mathbb{T}^\varnothing_\pi(\mathbb{X})$ are T_0-separable semitopological semilattices and* $\operatorname{Spec}\mathbb{T}^\varnothing_{dk}(\mathbb{X}) = \operatorname{Spec}\mathbb{T}^\varnothing_\varphi(\mathbb{X}) = \operatorname{Spec}\mathbb{T}^\varnothing_\pi(\mathbb{X})$.

Proposition 2 [8, Lemma 4]. *Let $\mathbb{X}$ be a topological T_0-space, while $x \in X$, and $I_x = \{U \in \mathscr{T}(\mathbb{X}) \mid x \notin U\}$. Then*

(i) *I_x is a closed prime ideal of the semitopological semilattice $\mathbb{T}_{dk}(\mathbb{X})$;*

(ii) *the mapping $\lambda : \mathbb{X} \to \mathbb{S}\mathrm{pec}\,\mathbb{T}_{dk}(\mathbb{X})$, $\lambda : x \mapsto I_x$, is a homeomorphic embedding;*

(iii) *if X is irreducible in $\mathbb{X}$, then $\lambda^\varnothing : \mathbb{X} \to \mathbb{S}\mathrm{pec}\,\mathbb{T}^\varnothing_{dk}(\mathbb{X})$, $\lambda^\varnothing : x \mapsto I_x\backslash\{\varnothing\}$, is a homeomorphic embedding;*

(iv) $\mathbb{S}\mathrm{pec}\,\mathbb{T}_{dk}(\mathbb{X}) = \mathbb{S}\mathrm{pec}\,\mathbb{T}_\pi(\mathbb{X})$ *is a sobrification of $\mathbb{X}$.*

As a corollary of Proposition 2, the next fact is obtained in [8]:

Theorem 6 [8, Theorem 2]. *Given an arbitrary topological T_0-space $\mathbb{X}$, the following are true:*

(i) *if $\mathbb{X}$ is sober, then $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\,\mathbb{T}_{dk}(\mathbb{X})$;*

(ii) *if $\mathbb{X}$ is almost sober but not sober, then $\mathbb{X} \cong \mathbb{S}\mathrm{pec}\,\mathbb{T}^\varnothing_{dk}(\mathbb{X})$;*

Corollary 7. *Given a topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) *$\mathbb{X}$ is almost sober;*

(ii) *$\mathbb{X}$ is homeomorphic to the spectrum of some (distributive) poset with topology;*

(iii) *$\mathbb{X}$ is homeomorphic to the spectrum of some φ-semitopological poset;*

(iv) *$\mathbb{X}$ is homeomorphic to the spectrum of some (distributive) lattice with topology;*

(v) *$\mathbb{X}$ is homeomorphic to the spectrum of some (distributive) semitopological lattice.*

PROOF. It is obvious that (v) implies (iv), while (iv) implies (iii), and (iii) implies (ii). By Theorem 5, (ii) implies (i). Finally, (i) implies (v) by Lemma 19 and Theorem 6.

The next corollary can be proved similarly.

Corollary 8. *Given a topological T_0-space $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ *is sober;*

(ii) $\mathbb{X}$ *is homeomorphic to the spectrum of some (distributive) poset with topology, which contains a bottom with respect to the defined order;*

(iii) $\mathbb{X}$ *is homeomorphic to the spectrum of some φ-semitopological poset that contains a bottom with respect to the defined order;*

(iv) $\mathbb{X}$ *is homeomorphic to the spectrum of some (distributive) lattice with topology, which contains a bottom with respect to the natural order;*

(v) $\mathbb{X}$ *is homeomorphic to the spectrum of some (distributive) semitopological lattice that contains a bottom with respect to the natural order.*

§ 9. Essential Completions

Proposition 3. *Let $\mathbb{S} = \langle S, \leq, \mathscr{T} \rangle$ be a poset with topology and let an algebraic closure operator φ define a completion of $\langle S; \leq \rangle$. Consider the mapping $\mu_{\mathbb{S}} : \mathbb{S}^{\pi}_{\varphi} \to \mathbb{T}_{\pi}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$, $\mu_{\mathbb{S}} : s \mapsto V_s$. Then*

(i) $\mu_{\mathbb{S}}$ *is open and continuous;*

(ii) $\mu_{\mathbb{S}}$ *is one-to-one (or, equivalently, a homomorphic embedding) if and only if $\mathbb{S}^{\pi}_{\varphi}$ satisfies the T_0-separation axiom.*

Proof. Recall that $\mathscr{T}_{\pi} = (\mathscr{T}_{dk})^{\pi}$. Suppose that $P \subseteq \mathscr{T}_*$ is a prime filter in $\langle \mathscr{T}_*; \subseteq \rangle$ which is open in $\mathscr{T}_{dk}$. By Theorem 5, $\mathbb{S}\mathrm{pec}\, \mathbb{S}$ is almost sober, but not sober (and sober in certain circumstances). Therefore, by Proposition 2(ii) and Theorem 6, $\lambda^{\varnothing}$ (the mapping λ, respectively) is a homeomorphism between $\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S}$ and $\mathrm{Spec}\, \mathbb{T}^{\varnothing}_{dk}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$ (between $\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S}$ and $\mathrm{Spec}\, \mathbb{T}_{dk}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$, respectively). Hence, by Proposition 2(i), there is $I \in \mathrm{Spec}_{\varphi}\, \mathbb{S}$ such that $P = P_I = \{U \in \mathscr{T}_* \mid I \in U\}$.

(i) To prove continuity of $\mu_{\mathbb{S}}$, it suffices to verify in view of the above considerations that $\mu^{-1}_{\mathbb{S}}(P_I) = S \backslash I$ for every $I \in \mathrm{Spec}_{\varphi}\, \mathbb{S}$. Indeed, $s \in \mu^{-1}_{\mathbb{S}}(P_I)$ if and only if $V_s \in P_I$, which is equivalent to the fact that $I \in V_s$; i.e., $s \notin I$.

Establish now that $\mu_{\mathbb{S}}$ is open. By the definition of $\mathscr{T}_{\pi}$, it suffices to verify that $\mu_{\mathbb{S}}(S \backslash I)$ is open in $\mu_{\mathbb{S}}(\mathbb{S}^{\pi}_{\varphi})$ for each $I \in \mathrm{Spec}_{\varphi}\, \mathbb{S}$. Indeed, take $I \in \mathrm{Spec}_{\varphi}\, \mathbb{S}$. By Proposition 2(i), $P_I = \{U \in \mathscr{T}_* \mid I \in U\}$ is an open prime filter in $\mathbb{T}_{dk}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$. To complete the proof of openness of $\mu_{\mathbb{S}}$, it remains to check that $\mu_{\mathbb{S}}(S \backslash I) = P_I \cap \mu_{\mathbb{S}}(S)$. Indeed, the containment $s \in S \backslash I$ is equivalent to $I \in V_s = \mu_{\mathbb{S}}(s)$ for every $s \in S$; i.e., to the containment $\mu_{\mathbb{S}}(s) \in P_I \cap \mu_{\mathbb{S}}(S)$. Item (i) is therefore proved.

(ii) Suppose that $\mathbb{S}^{\pi}_{\varphi}$ is a T_0-space and let $s_0 \nleq_{\mathbb{S}^{\pi}_{\varphi}} s_1$. Then there is a φ-prime filter $P \subseteq S$ open in $\mathscr{T}$ such that $s_0 \in P$, but $s_1 \notin P$. By Lemma 9, $I = S \backslash P \in \mathrm{Spec}_{\varphi}\, \mathbb{S}$, $I \in V_{s_0} = \mu_{\mathbb{S}}(s_0)$, and $I \notin V_{s_1} = \mu_{\mathbb{S}}(s_1)$. This means that $\mu_{\mathbb{S}}(s_0) \nsubseteq \mu_{\mathbb{S}}(s_1)$; whence the mapping $\mu_{\mathbb{S}}$ is one-to-one. Conversely, suppose that $\mu_{\mathbb{S}}$ is one-to-one and let $s_0 \neq s_1$ in $\mathbb{S}$. This means that $\mu_{\mathbb{S}}(s_0) \neq \mu_{\mathbb{S}}(s_1)$. Without loss of generality, we may assume that $\mu_{\mathbb{S}}(s_0) \nsubseteq \mu_{\mathbb{S}}(s_1)$. Since $\mathbb{T}_{\pi}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$ is a T_0-space by Lemma 19(i), there is a prime filter $P \subseteq \mathscr{T}_*$ open in the Day–Kelly topology and such that $V_{s_0} = \mu_{\mathbb{S}}(s_0) \in P$, $V_{s_1} = \mu_{\mathbb{S}}(s_1) \notin P$. By the argument in the beginning of this proof, $P = P_I$ for some $I \in \mathrm{Spec}_{\varphi}\, \mathbb{S}$. This means that $I \in V_{s_0}$ but $I \notin V_{s_1}$, whence $s_0 \in S \backslash I = P$ and $s_1 \notin S \backslash I = P$. Since $P \in \mathscr{T}^{\pi}_{\varphi}$, we have that $\mathscr{T}^{\pi}_{\varphi}$ is a T_0-topology. The proof is complete.

Theorem 7. *Let $\mathbb{S} = \langle S, \leq, \mathscr{T} \rangle$ be a poset with topology, let an algebraic closure operator φ define a completion of $\langle S; \leq \rangle$, and let $\mathscr{T} = \mathscr{T}^{\pi}_{\varphi}$ be a T_0-topology. Then $\mathbb{T}_{\pi}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$ is the biggest essential extension of $\mathbb{S}$.*

Proof. By Proposition 3(ii), $\mu_{\mathbb{S}}$ embeds $\mathbb{S}$ into $\mathbb{T}_{\pi}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$ homeomorphically. Assume without loss of generality that $\mathbb{S}$ is a subspace of $\mathbb{T}_{\pi}(\mathbb{S}\mathrm{pec}\, \mathbb{S})$.

Claim 1. *The extension $\mathbb{S} \leq \mathbb{T}_{\pi}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$ is essential.*

Proof of Claim 1. By Theorem 5 and Lemma 19(i), the two cases are possible:

Case 1: $V_s \neq \varnothing$ for all $s \in S$. We show first that $\mathbb{T}^{\varnothing}_{\pi}(\mathbb{S}\mathrm{pec}_{\varphi}\, \mathbb{S})$ has no bottom with respect to the specialization order which coincides with the set-theoretic inclusion by Lemma 18. We assume on the contrary that a nonempty $U \in (\mathscr{T}_*)_\pi$ is such that $U \subseteq V$ for each nonempty set $V \in (\mathscr{T}_*)_\pi$. Since $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a T_0-space; cf. Lemma 19(iii); we have $U = \{I\}$ for some $I \in \mathrm{Spec}_\varphi \mathbb{S}$. By the definition of specialization order, $J \leq_{(\mathscr{T}_*)_\pi} I$ for all $J \in \mathrm{Spec}_\varphi \mathbb{S}$; whence $I \subseteq J$ by Lemma 16. Therefore, there is $s \in I$ belonging to each prime φ-ideal. This implies in particular that $V_s = \varnothing$, which contradicts our assumption. Hence, $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ does not contain a bottom with respect to the specialization order. This means that $\varnothing$ is essential for $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ in $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$. It is clear that the extension $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S}) \leq \mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is strict. By Corollary 1, $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S}) \leq \mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is essential.

Note that $\mu_\mathbb{S}(s) = V_s \neq \varnothing$ for all $s \in S$; whence $\mu_\mathbb{S}(S) \subseteq \mathscr{T}_* \backslash \{\varnothing\}$. Therefore, $\mu_\mathbb{S}$ embed $\mathbb{S}$ into $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ homeomorphically. Moreover, $U = \bigcup\{V_s \mid s \in S,\ V_s \subseteq U\}$ for every nonempty set $U \in \mathscr{T}(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ by Corollary 6(iv). It follows that $\mu_\mathbb{S}(S)$ is a $\vee$-dense subset and so a d-basis of $\mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$. By [14, Lemma 2.7], $\mathbb{S} \leq \mathbb{T}_\pi^\varnothing(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a u-extension. By [14, Corollary 3.4], this extension is essential. Finally, $\mathbb{S} \leq \mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is an essential extension by [13, Lemma].

Case 2: $V_s = \varnothing$ for some $s \in S$. By Corollary 6(iv), $U = \bigcup\{V_s \mid s \in S,\ V_s \subseteq U\}$ for every nonempty set $U \in \mathscr{T}(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$. It follows that $\mu_\mathbb{S}(S)$ is a $\vee$-dense subset and so a d-basis of $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$. By [14, Lemma 2.7], $\mathbb{S} \leq \mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a u-extension. By [14, Corollary 3.4], this extension is essential.

Claim 2. $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ *is an essentially complete space.*

Proof of Claim 2. By Lemma 18, the specialization order coincides with the set-theoretic inclusion, whence $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a join semilattice with a bottom. By Lemma 19(i), $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a semitopological semilattice. In view of the equality $\mathscr{T} = (\mathscr{T}_{dk})^\pi$ and Lemma 17, $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a topological semilattice; i.e., $\cup$ is a continuous operator. Furthermore, it is not hard to see that $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is a d-space. By [13, Proposition 2], $\mathbb{T}_\pi(\mathbb{S}\mathrm{pec}_\varphi \mathbb{S})$ is essentially complete.

The proof of Theorem 7 follows from Claims 1 and 2.

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