Siberian Mathematical Journal, vol. 60, *no.* 1, *pp.* 27–40, 2019

FUNCTIONAL LIMIT THEOREMS FOR COMPOUND RENEWAL PROCESSES

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UDC 519.21

Abstract: We generalize Anscombe's Theorem to the case of stochastic processes converging to a continuous random process. As applications, we find a simple proof of an invariance principle for compound renewal processes (CRPs) in the case of finite variance of the elements of the control sequence. We find conditions, close to minimal ones, of the weak convergence of CRPs in the metric space $\mathbb{D}$ with metrics of two types to stable processes in the case of infinite variance. They turn out narrower than the conditions for convergence of a distribution in this space.

DOI: 10.1134/S003744661901004X

Keywords: Anscombe's theorem, functional limit theorems, compound renewal processes, invariance principle, convergence to a stable process

1. An Analog of Anscombe's Theorem in the Case of Convergence to a Continuous Process

First of all recall the original theorem of Anscombe.

Theorem 1.1. (1) *Consider a sequence $s(n)$ of random variables weakly converging in distribution to a random variable s; i.e., $s(n) \Rightarrow s$ as $n \to \infty$ such that*

$$(2) \quad \max_{|k|<\delta n} |s(n+k) - s(n)| \underset{p}{\longrightarrow} 0 \text{ as } n \to \infty \quad \text{for } \delta = \delta(n) \to 0.$$

(3) *Furthermore, consider a sequence θ_T of integer-valued random variables defined on the same probability space as $s(n)$ and depending on a parameter T such that there exists a sequence $h_T \to \infty$ as $T \to \infty$ with*

$$\frac{\theta_T}{h_T} \underset{p}{\longrightarrow} 1 \quad \text{as } T \to \infty.$$

Then $s(\theta_T) \Rightarrow s$ as $T \to \infty$.

Theorem 1.1 differs from the original Anscombe's Theorem only in that here the "continuity condition" (2) is slightly weaker and simpler than the continuity condition in the original version; see [1; 2, § 1.3]. See [3] for a proof of Theorem 1.1.

Now proceed to random processes. Consider a sequence $s_T(u)$ of processes on the segment, $u \in [0, u_0]$, defined in the measure space $(\mathbb{D}(0, u_0), \mathfrak{B}_{\mathbb{D}})$, where $\mathbb{D}(0, u_0)$ is the space of functions on $[0, u_0]$ without discontinuities of the second kind and right-continuous (or left-continuous) at each point, and $\mathfrak{B}_{\mathbb{D}}$ is the σ-algebra of subsets of $\mathbb{D}$ generated by the cylindrical sets. Also consider the space $\mathbb{C}(0, u_0)$ of continuous functions on $[0, u_0]$, the σ-algebra $\mathfrak{B}_{\mathbb{C}}$ of sets in $\mathbb{C}$ generated by the cylindrical sets, and the uniform metric $\rho_{\mathbb{C},u_0}$: given $g_i \in \mathbb{D}(0, u_0)$ for $i = 1, 2$, we have

$$\rho_{\mathbb{C},u_0}(g_1, g_2) = \sup_{u \le u_0} |g_1(u) - g_2(u)|.$$

The author was partially supported by the Program of Basic Scientific Research of the Siberian Branch of the Russian Academy of Sciences (Program No. I.1.3, Project 0314–2016–0008) and the Russian Foundation for Basic Research (Grant 18–01–00101a).

Novosibirsk. Translated from *Sibirskii Matematicheskii Zhurnal*, vol. 60, no. 1, pp. 37–54, January–February, 2019; DOI: 10.17377/smzh.2019.60.104. Original article submitted May 19, 2018; revised May 19, 2018; accepted May 23, 2018.

Say that a sequence $s_T(u)$ of processes on $(\mathbb{D}(0,u_0),\mathfrak{B}_{\mathbb{D}})$ is $\mathbb{C}$-*converging* in distribution as $T\to\infty$ to a process $s(u)$, $u\in[0,u_0]$, on $(\mathbb{C}(0,u_0),\mathfrak{B}_{\mathbb{C}})$ whenever, given a functional f measurable in $(\mathbb{D}(0,u_0),\mathfrak{B}_{\mathbb{D}})$ and continuous in the uniform metric at the "points" of $\mathbb{C}(0,u_0)$, we have

$$f(s_T)\Rightarrow f(s)\quad \text{as } T\to\infty, \tag{1.1}$$

where $\Rightarrow$ stands for weak convergence of distributions. We write $\mathbb{C}$-convergence as

$$s_T \underset{\mathbb{C}}{\Rightarrow} s \quad \text{as } T\to\infty.$$

Here is an analog of Theorem 1.1 for random processes.

Theorem 1.2. *Suppose that a random process $s_T(v)$ in $(\mathbb{D}(0,v_0),\mathfrak{B}_{\mathbb{D}})$ is $\mathbb{C}$-converging as $T\to\infty$ to a process $s(v)$, $v\in[0,v_0]$, which is continuous with probability* 1.

Consider a nondecreasing random process $\theta_T(u)$ in $(\mathbb{D}(0,v_0/b),\mathfrak{B}_{\mathbb{D}})$ on the same probability space as s_T such that for prescribed $b>0$ we have

$$\max_{u\le v_0/b}|\theta_T(u)-bu| \underset{p}{\longrightarrow} 0 \quad \text{as } T\to\infty. \tag{1.2}$$

Then for all $u_0<v_0/b$ we have

$$\mathbf{P}\{\theta_T(u_0)<v_0\}\to 1 \quad \text{as } T\to\infty, \tag{1.3}$$

while we can define the processes $s_T(u)$ and $s(u)$ on the same probability space so that

$$\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)),s(bu)) \underset{p}{\longrightarrow} 0 \quad \text{as } T\to\infty. \tag{1.4}$$

The processes $s_T(\theta_T(u))$ are $\mathbb{C}$-converging on $[0,u_0]$ to $s(bu)$.

Instead of bu in (1.2) we can take an arbitrary strictly increasing continuous function $b(u)$ with $b(0)=0$, while choosing $u_0<b^{(-1)}(v_0)$. The process $s(b(u))$ serves as the limit process.

The case that $\theta_T(u)=\frac{N(Tu)}{T}$, where $\frac{N(t)}{t} \underset{\text{almost surely}}{\longrightarrow} b$ as $T\to\infty$, as applied to stopped random walks was considered in [2, Theorem 5.2.1].

As the analog of continuity condition 2 in Anscombe's Theorem, in Theorem 1.2 we have the continuity assumption of the limit process $s(v)$, requiring the continuity of the processes $s_T(v)$ in a certain sense.

Along with the metric $\rho_{\mathbb{C},u_0}$ we consider below a series of other metrics on the space $\mathbb{D}$ and use the following simple statement.

Lemma 1.1. *Suppose that the space $\mathbb{D}(0,u_0)$ is equipped with a metric ρ and take a measurable functional f on $(\mathbb{D}(0,u_0),\mathfrak{B}_{\mathbb{D}})$ continuous in ρ. Suppose furthermore that s_T and s are random processes in $(\mathbb{D}(0,u_0),\mathfrak{B}_{\mathbb{D}})$ defined on the same probability space such that $\rho(s_T,s)\underset{p}{\longrightarrow} 0$ as $T\to\infty$. Then $f(s_T)\Rightarrow f(s)$ as $T\to\infty$.*

Proof. Put $B_\delta=\{\rho(s_T,s)<\delta\}$. Then $\mathbf{P}(\overline{B}_\delta)\to 0$ as $T\to\infty$ for arbitrary $\delta>0$. This means that there is a sequence $\delta=\delta_T$ vanishing sufficiently slowly as $T\to\infty$ such that $\mathbf{P}(\overline{B}_\delta)\to 0$ still holds. Moreover, for $\delta=\delta_T$ we have

$$\mathbf{P}(f(s_T)<v)=\mathbf{P}(f(s_T)<v,B_\delta)+\mathbf{P}(f(s_T)<v,\overline{B}_\delta),$$

where the second term on the right-hand side is $o(1)$ as $T\to\infty$. Moreover, the continuity of f with $\delta=\delta_T$ yields

$$\begin{aligned}\mathbf{P}(f(s_T)<v,B_\delta)&=\mathbf{P}(f(s)+o_p(1)<v,B_\delta)\\&=\mathbf{P}(f(s)<v+o_p(1))-\mathbf{P}(f(s)<v+o_p(1),\overline{B}_\delta)=\mathbf{P}(f(s)<v+o_p(1))+o(1).\end{aligned}$$

This implies that

$$\mathbf{P}(f(s_T) < v) \to \mathbf{P}(f(s) < v) \quad \text{as } T \to \infty$$

at an arbitrary continuity point v of the distribution of the random variable $f(s)$. $\square$

Proof of Theorem 1.2. It is known that we can define the processes $s_T(v)$ and $s(v)$ on the same probability space so that

$$\rho_{\mathbb{C},v_0}(s_T(v), s(v)) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty; \tag{1.5}$$

for a general theorem about that, see [4, Theorem 1.6.7] for instance. Therefore, by Lemma 1.1 verifying (1.3) and (1.4) suffices to prove the theorem. Put

$$A_{T,\varepsilon} = \{\max_{u \le u_0} |\theta_T(u) - bu| \le \varepsilon\}.$$

Then condition (1.2) implies that there exists a sequence $\varepsilon = \varepsilon_T$ vanishing sufficiently slowly as $T \to \infty$ such that

$$\mathbf{P}(\overline{A}_{T,\varepsilon}) \to 0 \quad \text{as } T \to 1. \tag{1.6}$$

It is clear that $A_{T,\varepsilon} \subset \{\theta_T(u_0) \le v_0\}$ for all $u_0 < v_0/b$ and so (1.3) holds.

Let us now verify (1.4). Given $h > 0$, we have

$$\begin{aligned} &\mathbf{P}(\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)), s(bu) > 2h) \\ &\le \mathbf{P}(\overline{A}_{T,\varepsilon}) + \mathbf{P}(\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)), s(\theta_T(u))) > h; A_{T,\varepsilon}) \\ &\quad + \mathbf{P}(\rho_{\mathbb{C},u_0}(s(\theta_T(u)), s(bu)) > h; A_{T,\varepsilon}). \end{aligned} \tag{1.7}$$

The value of $\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)), s(\theta_T(u)))$ on the right-hand side of (1.7) on $A_{T,\varepsilon}$ is at most $\rho_{\mathbb{C},v_0}(s_T(v), s(v))$; therefore, by (1.5) the second term on the right-hand side vanishes as $T \to \infty$. The vanishing of the third term on the right-hand side follows from the definition of $A_{T,\varepsilon}$ and the continuity of $s(v)$ with probability 1. Since $\mathbf{P}(\overline{A}_{T,\varepsilon}) \to 0$ as $T \to \infty$, this justifies (1.4). $\square$

2. The Invariance Principle for Compound Renewal Processes

Given a sequence (τ, ζ), (τ_1, ζ_1), $(\tau_2, \zeta_2), \dots$ of independent identically distributed random vectors, where $\tau > 0$, let

$$T_n := \sum_{j=1}^{n} \tau_j, \quad Z_n := \sum_{j=1}^{n} \zeta_j \text{ for } n \ge 1, \quad T_0 = Z_0 = 0.$$

Given $t \ge 0$, put

$$\eta(t) := \min\{k \ge 0 : T_k > t\}, \quad \nu(t) := \eta(t) - 1.$$

It is clear that $\nu(t) = \max\{k \ge 0 : T_k \le t\}$ for all $t \ge 0$.

Refer as a *compound renewal process* (CRP) to the process

$$Z(t) := Z_{\nu(t)}, \quad t \ge 0. \tag{2.1}$$

Along with Z we consider the random process

$$Y(t) := Z_{\eta(t)} = Z_{\nu(t)} + \zeta_{\eta(t)}, \quad t \ge 0, \tag{2.2}$$

which we also call an CRP. The limit laws which we are study will be the same for $Z(t)$ and $Y(t)$ under appropriate conditions. Since $\eta(t)$ is a Markov moment, the processes $Y(t) = Z_{\eta(t)}$ are somewhat simpler, and in some cases it is more convenient to study them.

Henceforth we assume that $\mathbf{E}\tau =: a_\tau$ and $\mathbf{E}\zeta =: a_\zeta$ exist, so that the "average drift" of the CRP

$$a := \frac{a_\zeta}{a_\tau}$$

is defined ($\mathbf{E}Z(t)/t \to a$ and $Z(t)/t \underset{\text{almost surely}}{\longrightarrow} a$ as $t \to \infty$).

Define the random processes

$$\chi(t) = T_{\eta(t)} - t, \quad \gamma(t) = t - T_{\nu(t)}, \quad \zeta(t) = \zeta_{\eta(t)}$$

and consider the asymptotically centered processes

$$Y(t) - at = Z_{\eta(t)} - aT_{\eta(t)} + a\chi(t), \quad Z(t) - at = Z_{\nu(t)} - aT_{\nu(t)} - a\gamma(t).$$

Introduce the random variables

$$\xi = \zeta - a\tau, \quad \xi_i = \zeta_i - a\tau_i, \quad S_n = \sum_{i=1}^{n} \xi_i = Z_n - aT_n.$$

Then

$$Y(t) - at = S_{\eta(t)} + a\chi(t), \quad Z(t) - at = S_{\nu(t)} - a\gamma(t), \tag{2.3}$$

where $\chi(t)$ and $\gamma(t)$ have proper limit distributions as $t \to \infty$; see [5] for instance.

Let us find the limit laws for the normalized processes $Y(t) - at$ and $Z(t) - at$ in the case that $\mathbf{E}\tau^2 < \infty$ and $\mathbf{E}\zeta^2 < \infty$.

Put $\sigma_\xi^2 = \mathbf{E}\xi^2$ and $\sigma^2 = \frac{\sigma_\xi^2}{a_\tau}$ and verify the invariance principle for the processes

$$y_T(u) = \frac{Y(uT) - auT}{\sigma\sqrt{T}}, \quad z_T(u) = \frac{Z(uT) - auT}{\sigma\sqrt{T}} \quad \text{as } T \to \infty \tag{2.4}$$

on an arbitrary finite interval $[0, u_0]$.

Theorem 2.1. *If $\mathbf{E}\tau^2 < \infty$ and $\mathbf{E}\zeta^2 < \infty$ then the sequences $y_T(u)$ and $z_T(u)$ of processes defined in* (2.1)–(2.4) *are $\mathbb{C}$-converging for arbitrary $u_0 > 0$ as $T \to \infty$ to the standard Wiener process $w(u)$; $u \in [0, u_0]$, i.e.,* (1.1) *holds with the replacement of s_T by y_T and both z_T and s by w.*

The invariance principle for $y_T(u)$ was established in [6] by rather complicated constructions on the same probability space as a corollary of strong approximation of the processes y_T by the Wiener process w; see also [7, 8]. A series of articles deals with convergence rate in the invariance principle; see [9, 10] and the references therein for instance.

It will be clear from the proof of Theorem 2.1 that the main contribution in the limit distribution for y_T and z_T comes from the distribution of stopped normalized sums $S_{\eta(uT)}$ and $S_{\nu(uT)}$. Functional limit theorems for the stopped sums $S_{N(uT)}$, where $\frac{N(t)}{t} \underset{\text{almost surely}}{\longrightarrow} b \in (0, \infty)$ as $t \to \infty$, are established in [2, Chapter 5]. They imply the theorems for the processes $y_T(u)$ and $z_T(u)$ in the case $a = 0$.

In order to establish Theorem 2.1, we can use the "classical" approach, according to which

for the $\mathbb{C}$-convergence $y_T \underset{\mathbb{C}}{\Rightarrow} w$ it suffices that

(1) *we have the weak convergence of the finite-dimensional distributions of y_T to the corresponding distributions of w;*

(2) *for every $\varepsilon > 0$ we have*

$$\lim_{\Delta \to 0} \overline{\lim_{T \to \infty}} \mathbf{P}(\omega(y_T, \Delta) > \varepsilon) = 0, \tag{2.5}$$

where for $g \in \mathbb{D}(0, 1)$ the functional

$$\omega(g, \Delta) = \sup_{(u,v) \in B_\Delta} |g(u + v) - g(u)|$$

is the continuity modulus of g, and

$$B_\Delta := \{(u, v) : v \in (0, \Delta), u \in [0, u_0], u + v \le u_0\}.$$

For instance, see Chapter 2 in [11] and Theorem 1.2.1 in [12].

To this end we should have to repeat in a more complicated setup the available proof of (2.5) for the processes $s_T(u)$ generated by the trajectories of random walks and prove the convergence of finite-dimensional distributions.

In our opinion, the simplest and most lucid approach to proving Theorem 2.1 consists in using the already available invariance principle for random walks and Theorem 1.2 (our analog of Anscombe's Theorem).

Proof of Theorem 2.1. For $\bar{\zeta}(T) := \max_{t\le T} \zeta(t)$ it is not difficult to verify that

$$\frac{\bar{\zeta}(T)}{\sqrt{T}} \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty. \tag{2.6}$$

Indeed, for all $\delta > 0$ and $h > \frac{1}{a_\tau}$ we obtain

$$\mathbf{P}(\bar{\zeta} > \delta\sqrt{T}) \le \mathbf{P}(\eta(T) \ge hT) + \mathbf{P}(\max_{i\le hT} \zeta_i > \delta T)$$
$$= o(1) + hT\mathbf{P}(\zeta > \delta T) = o(1) + hTo\left(\frac{\mathbf{E}\zeta^2}{\delta^2 T}\right) = o(1) \quad \text{as } T \to \infty.$$

By (2.4) for $u_0 = 1$ we have

$$\rho_{\mathbb{C},1}(y_T, z_T) \le \frac{1}{\sigma\sqrt{T}}\bar{\zeta}(T),$$

which implies that

$$\rho_{\mathbb{C},1}(y_T, z_T) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty,$$

and the processes y_T and z_T are asymptotically equivalent. Everything clearly remains valid for arbitrary $u_0 > 0$. Therefore, to establish Theorem 2.1, we may confine discussion to the processes y_T.

Consider the processes

$$s_T(v) = \frac{S_{vT}}{\sigma_\xi\sqrt{T}}, \quad \text{where } S_{vT} := S_{[vT]}. \tag{2.7}$$

It is known that there exists a standard Wiener process $w(v)$ on the same probability space as $s_T(v)$ such that for arbitrary $v_0 > 0$, which is assumed fixed, we have

$$\rho_{\mathbb{C},v_0}(s_T(v), w(v)) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty. \tag{2.8}$$

We can always enlarge the probability space that underlies the processes $s_T(v)$ and $w(v)$ in order to define on it the random variables τ_i and ζ_i so that ξ_i be the differences between ζ_i and $a\tau_i$ for all i. Therefore, the processes $Z(t)$ and $Y(t)$ will be defined on the whole semiaxis, and the random variables $\eta(t)$ will be defined with the property $\frac{\eta(t)}{t} \underset{\text{almost surely}}{\longrightarrow} \frac{1}{a_\tau}$ as $t \to \infty$. Thus,

$$\theta_T(u) := \frac{\eta(uT)}{T} \underset{\text{almost surely}}{\longrightarrow} \frac{u}{a_\tau} \quad \text{as } uT \to \infty.$$

This implies that for u_0 arbitrary we have

$$\sup_{u\le u_0}\left|\theta_T(u) - \frac{u}{a_\tau}\right| = \rho_{\mathbb{C},u_0}(\theta_T(u), ub) \underset{\text{almost surely}}{\longrightarrow} 0$$

as $T \to \infty$, so that in our case (1.2) and (1.3) of Theorem 1.2 are satisfied for $b = 1/a_\tau$. Theorem 1.2 implies (see (1.4)) that

$$\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)), w(bu)) \underset{p}{\longrightarrow} 0 \quad \text{for all } u_0 < \frac{v_0}{b} \text{ as } T \to \infty. \tag{2.9}$$

Consider the processes

$$y_T(u) = \frac{S_{\eta(uT)} + a\chi(uT)}{\sigma\sqrt{T}},$$

see (2.3) and (2.4), and put

$$y_T^0(u) := \frac{S_{\eta(uT)}}{\sigma\sqrt{T}} = \sqrt{a_\tau} s_T\left(\frac{\eta(uT)}{T}\right) = \sqrt{a_\tau} s_T(\theta_T(u)).$$

Introduce the standard Wiener process

$$w^*(u) = \frac{w(bu)}{\sqrt{b}}.$$

By Lemma 1.1, in order to prove the theorem, it suffices to verify that

$$\rho_{\mathbb{C},v_0}(y_T(u), w^*(u)) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty.$$

We have

$$\rho_{\mathbb{C},u_0}(y_T, w^*) \le \rho_{\mathbb{C},u_0}(y_T, y_T^0) + \rho_{\mathbb{C},u_0}(y_T^0, w^*),$$

where

$$\rho_{\mathbb{C},u_0}(y_T, y_T^0) = \frac{|a|}{\sigma} \sup_{u \le u_0} \frac{\chi(uT)}{\sqrt{T}}.$$

By complete analogy with (2.6) we verify that

$$\sup_{u \le u_0} \frac{\chi(uT)}{\sqrt{T}} \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty.$$

This implies that $\rho_{\mathbb{C},u_0}(y_T, y_T^0) \underset{p}{\longrightarrow} 0$ as $T \to \infty$. Therefore, it remains to verify that for $b = 1/a_\tau$ we have

$$\begin{aligned}\rho_{\mathbb{C},u_0}(y_T^0(u), w^*(u)) &= \sqrt{a_\tau}\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)), 1/\sqrt{a_\tau} w^*(u)) \\ &= \sqrt{a_\tau}\rho_{\mathbb{C},u_0}(s_T(\theta_T(u)), w(ub)) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty,\end{aligned} \tag{2.10}$$

which is established in (2.9). □

3. The S-Convergence of a Normalized CRP to Stable Processes in the Case of the Infinite Variance of ξ

Suppose that the processes $s_T(u)$ and $s(u)$ are defined on the space $(\mathbb{D}(0,u_0), \mathfrak{B}_{\mathbb{D}})$ with the Skorokhod metric

$$\rho_{S,u_0}(g_1, g_2) := \inf_h\Big[\sup_{u\in[0,u_0]} |g_1(u) - g_2(h(u))| + \sup_{u\in[0,u_0]} |h(u) - u|\Big], \tag{3.1}$$

where $\inf_h$ is taken over all increasing continuous functions $h(u)$ with $h(0) = 0$ and $h(u_0) = u_0$.

Say that the sequence $s_T(u)$ of processes *S-converges* in distribution as $T \to \infty$ to the process $s(u)$ on the segment $[0, u_0]$, and write $s_T \underset{S}{\Rightarrow} s$, if every functional f on $(\mathbb{D}(0,u_0), \mathfrak{B}_{\mathbb{D}})$ which is measurable and continuous in the Skorokhod metric satisfies

$$f(s_T) \Rightarrow f(s) \quad \text{as } T \to \infty. \tag{3.2}$$

The criteria for the S-convergence of processes can be found in [13, § 6.5; 4, 12] for instance.

Consider now an CRP in the case that $\sigma_\xi^2 = \mathbf{D}\xi = \infty$ on assuming the condition of regular decay of the distribution $\xi = \zeta - a\tau$ at infinity. Put

$$F_-(t) = \mathbf{P}(\xi \le -t), \quad F_+(t) = \mathbf{P}(\xi \ge t), \quad F(t) = F_-(t) + F_+(t) = \mathbf{P}(|\xi| \ge t).$$

Assume also that the following condition is satisfied:

$[\mathbf{R}_{\alpha,\beta}]$ *The function $F(t)$ is regularly varying at infinity, meaning that we can express it as*

$$F(t) = t^{-\alpha} l(t), \quad \alpha \in (1,2),$$

where the function $l(t)$ is slowly varying at infinity, furthermore,

$$\lim_{t\to\infty} \frac{F_+(t)}{F(t)} =: \beta_+ \in [0,1]$$

exists, and we put $\beta := 2\beta_+ - 1$.

Denote by $F^{(-1)}(u)$ the generalized inverse function to $F(t)$:

$$F^{(-1)}(u) = \inf\{t : F(t) < u\},$$

and put

$$\sigma_\xi(n) := F^{(-1)}(1/n). \tag{3.3}$$

The function $\sigma_\xi(n)$ is of the form $n^{1/\alpha} l_\sigma(n)$, where $l_\sigma(n)$ is a slowly varying sequence; see [5, § 8.8] for instance.

Under Condition $[\mathbf{R}_{\alpha,\beta}]$ the normalized sums $s(n) := \frac{S_n}{\sigma_\xi(n)}$ converge weakly in distribution to the stable law $\Phi_{\alpha,\beta}$ with parameters (α,β); see [5] for instance. Moreover, the normalized process

$$s_T(u) = \frac{S_{uT}}{\sigma_\xi(T)}, \quad u \le u_0, \tag{3.4}$$

S-converges as $T \to \infty$ to the stable process $w_{\alpha,\beta}(u)$ with parameters α and β: $s_T \underset{S}{\Rightarrow} w_{\alpha,\beta}$; see [13, § 6.5].

Put $\sigma(t) = \sigma_\xi(t) a_\tau^{-1/\alpha}$ and consider the sequence of processes

$$y_T(u) = \frac{Y(uT) - auT}{\sigma(T)} = \frac{Z_{\eta(uT)} - auT}{\sigma(T)}, \quad u \in [0, u_0].$$

Under Condition $[\mathbf{R}_{\alpha,\beta}]$, do we have the S-convergence of the distributions of y_T to distribution of the stable process $w_{\alpha,\beta}(u)$, where $w_{\alpha,\beta}(1)$ has distribution $\Phi_{\alpha,\beta}$? Not always, it turns out.

The proof of the S-convergence of an CRP to $w_{\alpha,\beta}$ faces fundamental difficulties because by (2.3) the trajectory of $y_T(u)$ is $y_T(u) = \frac{S_{\eta(ut)}}{\sigma(T)} + \frac{a\chi(uT)}{\sigma(T)}$ and involves an "irregular" sawtooth term $\frac{a\chi(uT)}{\sigma(T)}$, which can become significant in some cases. Thus, the assertion $y_T \underset{S}{\Rightarrow} w_{\alpha,\beta}$ is valid only under additional assumptions. Given a random variable ω, put $F_\omega(t) := \mathbf{P}(\omega > t)$.

Theorem 3.1. *Suppose that $\xi = \zeta - a\tau$ satisfies condition $[\mathbf{R}_{\alpha,\beta}]$ and at least one of the following additional conditions is satisfied:*

$$a = 0 \text{ or } F_\tau(t) = o(F(t)) \quad \text{as } t \to \infty. \tag{3.5}$$

Then for $u_0 > 0$ arbitrary we have

$$y_T(u) \underset{S}{\Rightarrow} w_{\alpha,\beta}(u) \quad \text{on } [0, u_0]. \tag{3.6}$$

It is not difficult to see that for τ and ζ independent the second condition in (3.5) is met as soon as

$$F_\tau(t) = o(F_{|\zeta|}(t)) \quad \text{as } t \to \infty;$$

so that, for instance, it holds for compound Poisson processes.

Convergence (3.6) in the case $a = 0$ is established in [2, Theorem 5.2.2].

If (3.5) is violated then the S-convergence (3.6) is absent in general, although we have the convergence of finite-dimensional distributions. For more detail, see Remarks 3.1 and 4.1 below.

Proof of Theorem 3.1. As before, put $\theta_T(u) = \frac{\eta(uT)}{T}$. Then (3.4) yields

$$y_T(u) = a_\tau^{1/\alpha} s_T(\theta_T(u)) + \frac{a\chi(uT)}{\sigma(T)}. \tag{3.7}$$

Verify that by (3.5) the second term on the right-hand side of (3.7) is negligibly small. It is obvious for $a = 0$. If $a \neq 0$ then we verify that

$$\rho_{\mathbb{C},u_0}\big(y_T(u), a_\tau^{1/\alpha} s_T(\theta_T(u))\big) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty \tag{3.8}$$

or, which is the same,

$$\sup_{u \le u_0} \frac{\chi(uT)}{\sigma(T)} \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty. \tag{3.9}$$

Indeed, for $b = 1/a_\tau$ on the set $A_T = \{\theta_T(u_0) < v_0\}$, where $u_0 < v_0/b$ and $P(A_T) \to 1$, we have

$$\max_{u \le u_0} \chi(uT) \le \max_{i \le v_0 T} \tau_i =: \bar{\tau}_T,$$

where

$$\begin{aligned}\mathbf{P}(\bar{\tau}_T > \varepsilon\sigma(T)) &= o(1) + \mathbf{P}(\bar{\tau}_T > \varepsilon\sigma(T), A_T) \le o(1) + v_0 T F_\tau(\varepsilon\sigma(T)) \\ &= o(1) + o(TF(\sigma_\xi(T))) = o(1)\end{aligned}$$

for arbitrary $\varepsilon > 0$ as $T \to \infty$. This justifies (3.9) and (3.8).

Under the conditions mentioned the processes $y_T(u)$ and $a_\tau^{1/\alpha} s_T(\theta_T(u))$ are thus asymptotically equivalent. The convergence of the processes $a^{1/\alpha} s_T(\theta_T(u))$ to $w_{\alpha,\beta}(u)$, as we already noted, was established in [2, Theorem 5.2.2]. At this point we might finish proving Theorem 3.1; however, the proof of the required convergence in [2] has many steps and relies on other, sometimes more difficult results. At the same time, there is a direct and simple proof of the required convergence, which we now present.

Without loss of generality, assume that $a_\tau = 1$, $u_0 = 1$, and $\rho_{S,u_0} = \rho_S$. According to Theorem 1.6.7 of [4], we can define the CRP $s_T(u)$ and the process $w_{\alpha,\beta}(u)$ on the same probability space so that

$$\rho_S(s_T(u), w_{\alpha,\beta}(u)) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty. \tag{3.10}$$

As in the proof of Theorem 2.1, enlarge the probability space underlying the processes $s_T(u)$ and $w_{\alpha,\beta}(u)$ so that the whole sequence (τ_i, ζ_i) is defined on it, and therefore the processes $Z(t)$ and $Y(t)$ as well. Then the random variables $\eta(t)$ and $\theta_T(u) = \frac{\eta(uT)}{T}$ are defined on this space. Furthermore,

$$\rho_S(s_T(\theta_T(\cdot)), w_{\alpha,\beta}(\cdot)) \le \rho_S(s_T(\theta_T(\cdot)), s_T(\cdot)) + \rho_S(s_T(\cdot), w_{\alpha,\beta}(\cdot)), \tag{3.11}$$

where the second term on the right-hand side of (3.11) converges in probability to 0 as $T \to \infty$. Consequently, it suffices to establish the vanishing of the first term; i.e., the asymptotic equivalence of $s_T(\theta_T(u))$ and $s_T(u)$. Since

$$\rho_S(s_T(\theta_T(\cdot)), s_T(\cdot)) \le \sup_{u \le 1} |s_T(\theta_T(u)) - s_T(h(u))| + \sup_{u \le 1} |h(u) - u| \tag{3.12}$$

for every increasing continuous function $h(u)$ with $h(0) = 0$ and $h(1) = 1$, it suffices to choose a function $h(u) = h_T(u)$ for which both terms on the right-hand side of (3.12) converge in probability to 0 as $T \to \infty$.

Given $\varepsilon > 0$ small, consider $A_{T,\varepsilon} = \{|\eta(T) - T| < \varepsilon T + 1\}$. It is clear that

$$P(A_{T,\varepsilon}) \to 1 \quad \text{as } T \to \infty \text{ for all } \varepsilon > 0. \tag{3.13}$$

Thus, there is a sequence $\varepsilon = \varepsilon_T$ vanishing sufficiently slowly as $T \to \infty$ such that (3.13) still holds.

Choose as h the smoothed version $h(u) = \tilde{\theta}_T(u)$ of the function $\theta_T(u)$, defining it on $A_{T,\varepsilon}$ as a continuous broken line passing through the nodes

$$\left(t_k = \frac{T_k}{T}, u_k = \frac{k}{T}\right) \quad \text{for } k = 0, 1, \dots,\ k_T := [T(1-\varepsilon)]$$

and the point $(1, 1)$, so that

$$\theta_T(t_k) = \tilde{\theta}_T(t_k) \tag{3.14}$$

for $k \le k_T$.

Since $s_T(u)$ is piecewise constant, taking the same value on (u_{k-1}, u_k) for $k \ge 1$, it follows that

$$s_T(\theta_T(u)) = s_T(\tilde{\theta}_T(u)) \quad \text{for } u \le t_{k_T}. \tag{3.15}$$

For the first term on the right-hand side of (3.12) on $A_{T,\varepsilon}$ we have $u_{k_T} < 1$ and

$$\sup_{u \le 1} |s_T(\theta_T(u)) - s_T(\tilde{\theta}_T(u))| \le d_1 + d_2,$$

where

$$d_1 = \sup_{u \in [u_{k_T}, 1]} |s_T(\theta_T(u)) - s_T(\theta_T(u_{k_T}))| \underset{p}{=} \frac{1}{\sigma(T)} \max_{k \le \eta(T) - T(1-\varepsilon)} |S_k^*| \le \frac{1}{\sigma(T)} \max_{k \le 2\varepsilon T} |S_k^*|,$$

$$d_2 = \sup_{u \in [u_{k_T}, 1]} |s_T(\tilde{\theta}_T(u)) - s_T(\tilde{\theta}_T(u_{k_T}))|$$

by (3.15) and (3.14). Here the sequence $\{S_k^*\}$ has the same distribution as $\{S_k\}$, the variable

$$\frac{1}{\sigma(2\varepsilon T)} \max_{k \le 2\varepsilon T} |S_k^*|$$

has proper limit distribution for $\varepsilon = \varepsilon_T$ as $\varepsilon T \to \infty$; furthermore, $\frac{\sigma(2\varepsilon T)}{\sigma(T)} \to 0$. Thus, for arbitrary $\delta > 0$ we have

$$\mathbf{P}(d_1 > \delta, A_{T,\varepsilon}) \to 0 \quad \text{as } T \to \infty.$$

Similarly we can estimate d_2.

For the second term on the right-hand side of (3.12) on the set $A_{T,\varepsilon}$ we have

$$d_3 := \sup_{u \le 1} |h(u) - u| = \frac{1}{T} \sup_{k \le k_T} |T_k - k|, \quad \mathbf{P}(d_3 > \delta, A_{T,\varepsilon}) \to 0 \quad \text{as } T \to \infty.$$

Therefore, (3.12) yields

$$\mathbf{P}(\rho_S(s_T(\theta_T(\cdot)), s_T(\cdot)) > \delta) = o(1) + \mathbf{P}(\rho_S(s_T(\theta_T(\cdot)), s_T(\cdot)) > \delta, A_{T,\varepsilon}) = o(1) \quad \text{as } T \to \infty.$$

The proof of Theorem 3.1 is complete. □

Remark 3.1. Suppose that

$$a = 1, \quad F_\tau(t) \gg F(t) \tag{3.16}$$

as $t \to \infty$, so that (3.5) is violated. This happens, for instance, in the case $\zeta = \tau + \omega$, where $\mathbf{E}\omega = 0$ and $\omega = \xi$ satisfies condition $[\mathbf{R}_{\alpha,\beta}]$ and is independent of τ, while τ satisfies condition $[\mathbf{R}_{\alpha',1}]$ for $1 < \alpha' < \alpha$, so that the possible jumps of τ are greater than the jumps of $\xi = \omega$. Verify that in this case the convergence $y_T \underset{S}{\Rightarrow} w_{\alpha,\beta}$ is impossible. Observe firstly that

$$h_{T,\tau} := \frac{\sup \chi(uT)}{\sigma_\tau(T)},$$

where $\sigma_\tau(T) = F_\tau^{(-1)}(1/T)$, has proper limit distribution as $T \to \infty$. Indeed, it is not difficult to verify, using the set $A_{T,\varepsilon}$ once again, that $h_{T,\tau}$ has the same limit distribution as $\frac{\bar{\tau}_T}{\sigma_\tau(T)}$, where $\bar{\tau}_T = \max_{k \le T} \tau_k$. However, $\frac{\bar{\tau}_T}{\sigma_\tau(T)}$ as $T \to \infty$ has proper nondegenerate limit distribution. Indeed, assuming for simplicity that T is an integer,

$$\mathbf{P}(\bar{\tau}_T < v\sigma_\tau(T)) = (1 - F_\tau(v\sigma_\tau(T)))^T, \tag{3.17}$$

where $F_\tau(v\sigma_\tau(T)) \sim \frac{v^{-\alpha'}}{T}$ as $T \to \infty$. Thus, the left-hand side of (3.17) equals

$$\left(1 - \frac{v^{-\alpha'}(1 + o(1))}{T}\right)^T \to \exp\{-v^{-\alpha'}\} \quad \text{as } T \to \infty;$$

see also [14, 15]. Since $\sigma(T) \ll \sigma_\tau(T)$ as $T \to \infty$, the above implies that

$$h_T := \frac{\sup \chi(uT)}{\sigma(T)} \to \infty \quad \text{as } T \to \infty \tag{3.18}$$

with probability 1. At the same time, $s_T(\theta_T(\cdot)) \underset{S}{\Rightarrow} w_{\alpha,\beta}(\cdot)$ and so

$$\inf_{u \le 1} s_T(\theta_T(u)) \Rightarrow \inf_{u \le 1} w_{\alpha,\beta}(u).$$

Since the right-hand side here is finite with probability 1, it follows from (3.7) and (3.8) that

$$\sup_{u \le 1} y_T(u) \underset{p}{\longrightarrow} \infty \quad \text{as } T \to \infty. \tag{3.19}$$

But the functionals $f(s) = \inf_{u \le 1} s(u)$ and $f(s) = \sup_{u \le 1} s(u)$ are ρ_S-continuous. Thus, (3.19) means that the S-convergence $y_T \underset{S}{\Rightarrow} w_{\alpha,\beta}$ is absent.

Observe also that meanwhile for every $u \le 1$ we have

$$\frac{\chi(uT)}{\sigma(T)} \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty$$

and the finite-dimensional distributions $\frac{\chi(uT)}{\sigma(T)}$ correspond to the zero limit process.

If τ and ζ are independent, while $a \ne 0$, then the distribution F is formed as the convolution of the distributions τ and ζ, and instead of (3.16) only the inequality

$$F_\tau(t) \ge cF(t) \text{ as } t \to \infty, \quad c = \text{const},$$

is possible. In this case, by analogy with the above, the limit process for y_T in (3.7) is formed as the sum of $w_{\alpha,\beta}$ and the *sawtooth* term $\frac{\chi(uT)}{\sigma(T)}$, whose jumps are comparable with 1, so that in this case, in addition to the vertical jumps of the process $s_T(\theta_T(u))$ in (3.7) there are *slanted jumps* of the process $-\frac{\chi(uT)}{\sigma(T)}$, and the S-convergence $y_T \underset{S}{\Rightarrow} w_{\alpha,\beta}$ is also absent. This is related to a property of the metric ρ_S preventing its use to describe the proximity of processes, one of which has vertical jumps and the other has jumps of the same size, but slanted. To avoid this inconvenience, in the next section we consider the metric $\rho_{\mathbb{D}}$ close to and slightly weaker than ρ_S, but free of this drawback. But even for it we need additional conditions under which the processes y_T and z_T weakly converge to stable processes.

Remark 3.2. In the proof of Theorem 3.1, instead of proving the intermediate assertion about the vanishing of $\rho_S(s_T(\theta_T(\cdot)), w_{\alpha,\beta})$, see (3.11), we could apply Theorem 5.2.2 of [2] on the S-convergence of $s_T(\theta_T(\cdot))$ to $w_{\alpha,\beta}(\cdot)$. But, in our opinion, the above proof is simpler, more lucid, and self-contained.

4. $\mathbb{D}$-Convergence to Stable Processes

In order to widen the conditions under which the distributions of CRPs y_T and z_T weakly converge in $(D(0,u_0),\mathfrak{B}_{\mathbb{D}})$ to stable processes, introduce the new metric $\rho_{\mathbb{D},u_0}$ which is close to ρ_{S,u_0} but weaker.

Assume for simplicity that $u_0=1$. Write $\rho_{\mathbb{D},u_0}=\rho_{\mathbb{D}}$ and $\mathbb{D}=\mathbb{D}(0,1)$. The metric $\rho_{\mathbb{D}}$ was introduced and studied in [16] for the space $\mathbb{F}$ larger than $\mathbb{D}$. The topology induced by it coincides with the Skorokhod topology M_2 described in [17]. We can also regard the metric $\rho_{\mathbb{D}}$ as extending the Levi metric on the space of nondecreasing functions to the larger space $\mathbb{D}$; see [18] for instance.

The metric $\rho_{\mathbb{D}}$ is defined as follows: Given $g\in\mathbb{D}$, define the *graph* of g as the simply-connected set Γg in $[0,1]\times\mathbb{R}$ coinciding with the curve $(t,g(t))$ everywhere but the discontinuity points of g, where $(t,g(t-0))$, $(t,g(t+0))\in\mathbb{R}^2$ are joined by the straight line segment. The distance $\rho_{\mathbb{D}}(g_1,g_2)$ is defined as the Hausdorff distance between Γg_1 and Γg_2 in $\mathbb{R}^2$; i.e., we write $\rho_{\mathbb{D}}(g_1,g_2)<\varepsilon$ whenever simultaneously $\Gamma g_1\in(\Gamma g_2)_\varepsilon$ and $\Gamma g_2\in(\Gamma g)_\varepsilon$, where we take ε-neighborhoods in $\mathbb{R}^2$ in the Euclidean metric. In other words,

$$\rho_{\mathbb{D}}(g_1,g_2):=\max\{r(g_1,g_2),r(g_2,g_1)\},\quad r(g_1,g_2):=\max_{\boldsymbol{v}\in\Gamma g_1}\min_{\boldsymbol{u}\in\Gamma g_2}|\boldsymbol{u}-\boldsymbol{v}|, \tag{4.1}$$

where $\boldsymbol{u}=(u_1,u_2)$ with $u_1,v_1\in[0,1]$ and $\boldsymbol{v}=(v_1,v_2)$ with $u_2,v_2\in\mathbb{R}$; Moreover, $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^2$.

It is obvious that

$$\rho_{\mathbb{D}}(g_1,g_2)\le\rho_{\mathbb{C}}(g_1,g_2):=\sup_{0\le t\le 1}|g_1(t)-g_2(t)|. \tag{4.2}$$

Compare the distances ρ_S and $\rho_{\mathbb{D}}$. Suppose for simplicity that for these functions g_1 and g_2 from $\mathbb{D}$ there exists a function $h_0(t)$ on which $\inf_h$ in (3.1) is attained:

$$\rho_S(g_1,g_2)=\rho_1+\rho_2,\ \text{where } \rho_1=\sup_u|g_1(u)-g_2(h_0(u))|,\quad \rho_2=\sup|h_0(u)-u|.$$

Then for each point u the distance between $(u,g_1(u))$ and $(h_0(u),g_2(h_0(u)))$ in $\mathbb{R}^2$ is at most

$$\sqrt{\rho_1^2+\rho_2^2}\le\rho_1+\rho_2=\rho_S(g_1,g_2).$$

The pairs of points $(u,g_2(u))$ and $(h_1(u),g_1(h_1(u)))$ satisfy the same inequality for a suitable function $h_1(u)$. This implies that $\rho_{\mathbb{D}}(g_1,g_2)\le\rho_S(g_1,g_2)$. On the other hand, for the functions

$$g(u)=\begin{cases}0, & u\in[0,p),\quad p\in(0,1),\\ 1, & u\in[p,1];\end{cases}\qquad g_n(u)=\begin{cases}0, & u\in[0,p),\\ n(u-p), & u\in[p,p+1/n),\\ 1, & u\in[p+1/n,1],\end{cases} \tag{4.3}$$

we have

$$\rho_{\mathbb{D}}(g,g_n)\le\frac{1}{n}\to 0 \text{ as } n\to\infty,\quad \rho_S(g,g_n)=1 \text{ for all } n.$$

The above means that the metric $\rho_{\mathbb{D}}$ is weaker than ρ_S, and so the class of functionals f continuous with respect to $\rho_{\mathbb{D}}$, is slightly narrower than the class of functionals continuous with respect to ρ_S. Therefore, the convergence $s_T\underset{S}{\Rightarrow}s$ implies the convergence $s_T\underset{\mathbb{D}}{\Rightarrow}s$, but not conversely. The difference between $\rho_{\mathbb{D}}$ and ρ_S is small; we are unaware of examples of functionals with interesting applications which are continuous with respect to ρ_S, but discontinuous with respect to $\rho_{\mathbb{D}}$. The widely applicable functionals like $\max_{u\in[0,1]}(g(u)-g_0(u))$, where $g_0(u)$ is continuous, integral functionals, and others, are $\rho_{\mathbb{D}}$-continuous. The metric $\rho_{\mathbb{D}}$ is used successfully in [19, 20] to establish the extended large deviation principle for random walks.

Return to an CRP on $[0,u_0]$ for arbitrary $u_0>0$. Define the distance $\rho_{\mathbb{D},u_0}$ by (4.1), where g_1 and g_2 are functions on $[0,u_0]$.

By analogy with the above, we write $s_T\underset{\mathbb{D}}{\Rightarrow}s$ as $T\to\infty$ provided that (1.1) holds for every functional f on $(\mathbb{D}(0,u_0),\mathfrak{B}_{\mathbb{D}})$ continuous with respect to $\rho_{\mathbb{D},u_0}$.

Let us now give the main statement of this section.

Theorem 4.1. *Suppose that $\xi = \zeta - a\tau$ satisfies condition $[\mathbf{R}_{\alpha,\beta}]$ and at least one of the conditions*

$$a = 0 \text{ or } \mathbf{P}(\tau > t, a\zeta > t) = o(F(t)) \quad \text{as } t \to \infty \tag{4.4}$$

is satisfied. Then for all $u_0 > 0$ we have

$$y_T(u) \underset{\mathbb{D}}{\Rightarrow} w_{\alpha,\beta}(u) \quad \text{on } [0, u_0]. \tag{4.5}$$

If τ and ζ are independent then we always have the convergence (4.5).

The same assertions apply to z_T.

PROOF. If $a = 0$ then (4.5) follows from Theorem 3.1.

Suppose now that $a \neq 0$ and for simplicity, but without loss of generality, put

$$u_0 = 1, \quad a_\tau = 1, \quad \rho_{\mathbb{D},u_0} = \rho_{\mathbb{D}}, \quad \rho_{S,u_0} = \rho_S.$$

The proof of Theorem 3.1 shows, see (3.11), that we can define the processes $s_T(u)$, $\theta_T(u) = \frac{\eta(uT)}{T}$, $y_T(u)$, and $w_{\alpha,\beta}(u)$ on the same probability space so that

$$\rho_S(s_T(\theta_T(\cdot)), w_{\alpha,\beta}(\cdot)) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty. \tag{4.6}$$

We have

$$\rho_{\mathbb{D}}(y_T, w_{\alpha,\beta}) \leq \rho_{\mathbb{D}}(y_T(\cdot), s_T(\theta_T(\cdot))) + \rho_{\mathbb{D}}(s_T(\theta_T(\cdot)), w_{\alpha,\beta}(\cdot)). \tag{4.7}$$

Since $\rho_{\mathbb{D}} \leq \rho_S$, by (4.6) the second term on the right-hand side of (4.7) converges in probability to 0 as $T \to \infty$. Hence, by Lemma 1.1 it suffices to verify that

$$\rho_{\mathbb{D}}(y_T(\cdot), s_T(\theta(\cdot))) \underset{p}{\longrightarrow} 0 \quad \text{as } T \to \infty. \tag{4.8}$$

Split the half-open interval $[0, 1 + \chi(T)/T)$ into the half-open intervals

$$[t_{i-1}, t_i) = [T_{i-1}/T, T_i/T), \quad i = 1, 2, \dots, \eta(T),$$

and consider the restrictions of the trajectories of $y_T(u)$ and $s_T(\theta_T(u))$ to these.

The trajectory of $y_T(u)$ on $[t_{i-1}, t_i)$ consists of the vertical jump $\frac{\zeta_i}{\sigma(T)}$ at t_{i-1} and the inclined line $y_T(t_{i-1} + 0) - \frac{auT}{\sigma(T)}$ on (t_{i-1}, t_i), so that

$$y_T(t_i - 0) = y_T(t_{i-1} - 0) + \frac{\zeta_i - a\tau_i}{\sigma(T)}, \quad \zeta_i - a\tau_i = \xi_i.$$

The trajectory of $s_T(\theta(u))$ on $[t_{i-1}, t_i)$ consists of the vertical jump ξ_i at t_{i-1} and the horizontal fragment on (t_{i-1}, t_i), so that at t_i the values of $y_T(\cdot)$ and $s_T(\theta_T(\cdot))$ coincide.

The distance $\rho_{\mathbb{D}}^{(i)}$ between these two *restrictions* of $y_T(\cdot)$ and $s_T(\theta(\cdot))$ to $[t_{i-1}, t_i)$ in the metric $\rho_{\mathbb{D}}$ is obviously at most $\frac{\tau_i}{T}$ if ζ_i and ξ_i are of the same sign, and at most $\max\left(\frac{\tau_i}{T}, \frac{|\zeta_i|}{\sigma(T)}\right)$ if ζ_i and ξ_i are of opposite signs. More precisely, suppose that $a > 0$. Then

$$\rho_{\mathbb{D}}^{(i)} = \frac{\tau_i}{T} I(\{\zeta_i \geq 0, \xi_i \geq 0\} \cup \{\zeta_i \leq 0, \xi_i \leq 0\}) + \max\left(\frac{\tau_i}{T}, \frac{\zeta_i}{\sigma(T)}\right) I(\zeta_i > 0, \xi_i < 0);$$

for $a > 0$ the event $\{\zeta_i < 0, \xi_i > 0\}$ is impossible because $\xi_i < \zeta_i$. Thus,

$$\rho_{\mathbb{D}}^{(i)} \leq \max\left(\frac{\tau_i}{T}, \frac{X_i^+}{\sigma(T)}\right),$$

where $X_i^+ := \zeta_i I(\zeta_i > 0, \xi_i < 0)$, and for $v > 0$ we have

$$\mathbf{P}(X_i^+ > v) = \mathbf{P}(\zeta_i > v, \zeta_i - a\tau_i < 0) = \mathbf{P}\left(\tau > \frac{\zeta_i}{a}, \zeta > v\right) \le \mathbf{P}\left(\tau > \frac{v}{a}, \zeta > v\right). \tag{4.9}$$

If $a < 0$ then we find similarly that

$$\rho_{\mathbb{D}}^{(i)} \le \max\left(\frac{\tau_i}{T}, -\frac{X_i^-}{\sigma(T)}\right),$$

where

$$X_i^- := \zeta_i I(\zeta_i < 0, \xi_i > 0),$$

and for $v > 0$ we see that

$$\mathbf{P}(X_i^- < -v) = \mathbf{P}(\zeta_i < -v, \zeta_i - a\tau_i > 0) = \mathbf{P}\left(\tau > \frac{\zeta_i}{a}, \zeta < -v\right) \le \mathbf{P}\left(\tau > \frac{v}{|a|}, \mathrm{e}_a\zeta > v\right), \tag{4.10}$$

where $\mathrm{e}_a = \frac{a}{|a|}$ and the right-hand side in the case $a > 0$ coincides with the right-hand side of (4.9). Finally,

$$\mathbf{P}\big(\rho_{\mathbb{D}}^{(i)} > \delta\big) \le \mathbf{P}(\tau > \delta T) + \mathbf{P}\left(\tau > \frac{\delta\sigma(T)}{|a|}, \mathrm{e}_a\zeta > \delta\sigma(T)\right) \tag{4.11}$$

for all $\delta > 0$, and so

$$\rho_{\mathbb{D}}(y_T(\cdot), s_T(\theta(\cdot))) \le \max_{i \le \eta(T)} \rho_{\mathbb{D}}^{(i)}.$$

Introduce

$$A_{T,\varepsilon} = \{\theta_T(1) < 1 + \varepsilon\} = \{\eta(T) < T(1+\varepsilon)\}.$$

It is clear that $\mathbf{P}(A_{T,\varepsilon}) \to 1$ as $T \to \infty$ for arbitrary $\varepsilon > 0$. On this set

$$\rho_{\mathbb{D}}(y_T(\cdot), s_T(\theta(\cdot))) \le \max_{i \le T(1+\varepsilon)} \rho_{\mathbb{D}}^{(i)}$$

and

$$\begin{aligned}\mathbf{P}(\rho_{\mathbb{D}}(y_T(\cdot), s_T(\theta(\cdot))) > \delta) &\le \mathbf{P}(\overline{A}_{T,\varepsilon}) + \mathbf{P}(\rho_{\mathbb{D}}(y_T(\cdot), s_T(\theta(\cdot)))) > \delta, A_{T,\varepsilon}) \\ &\le \mathbf{P}(\overline{A}_{T,\varepsilon}) + T(1+\varepsilon)\mathbf{P}\big(\rho_{\mathbb{D}}^{(i)} > \delta\big).\end{aligned} \tag{4.12}$$

Since $\mathbf{P}(\overline{A}_{T,\varepsilon}) \to 0$ as $T \to \infty$, in order to justify (4.8), by (4.11) it suffices to show that

$$T\mathbf{P}(\tau > \delta T) \to 0, \quad T\mathbf{P}\left(\tau > \frac{\delta\sigma(T)}{|a|}, \mathrm{e}_a\zeta > \delta\sigma(T)\right) \to 0 \tag{4.13}$$

as $T \to \infty$. The first relation is obvious because $\mathbf{E}\tau$ exists. To verify the second relation, use condition (4.4), which yields

$$\mathbf{P}\left(\tau > \frac{\delta\sigma(T)}{|a|}, \mathrm{e}_a\zeta > \delta\sigma(T)\right) = o\left(F\left(\delta \min\left(|a|, \frac{1}{|a|}\right)\sigma(T)\right) = o(F(\sigma(T)))\right) = o\left(\frac{1}{T}\right)$$

as $T \to \infty$. Therefore, the second estimated value in (4.13) is $o(1)$ as $T \to \infty$. This proves (4.8) and (4.5).

If τ and ζ are independent then

$$\mathbf{P}(\tau > t, a\zeta > t) = \mathbf{P}(\tau > t)\mathbf{P}(a\zeta > t) = o(t^{-2}) \quad \text{as } t \to \infty$$

and condition (4.4) obviously holds, which implies (4.5).

The argument for the processes z_T is similar. The proof of Theorem 4.1 is complete. □

Remark 4.1. The assumptions in Theorem 4.1 are close to the minimal ones. If condition (4.4), which is wider than (3.5), is violated then the S- and $\mathbb{D}$-convergences of the distributions of y_T and z_T to the process $w_{\alpha,\beta}$ are absent in general. The example in Remark 3.1 demonstrates that when $\zeta = \tau + \omega$ with independent τ and ω, and $a = 1$. In that example not only (3.5) is violated, but also (4.4). Indeed, we can always choose the random variable ω so that $p := \mathbf{P}(\omega > 0) > 0$; if $\beta > -1$ then this always holds. In this case

$$\mathbf{P}(\tau > t, \zeta > t) \geq \mathbf{P}(\tau > t, \tau + \omega > t, \omega > 0)$$
$$= \mathbf{P}(\tau > t, \omega > 0) = p\mathbf{P}(\tau > t) \gg F(t) \quad \text{as } t \to \infty.$$

Since in the example of Remark 3.1 $\sup_{u \leq 1} y_T(u) \underset{p}{\longrightarrow} \infty$ as $T \to \infty$, it follows that the $\mathbb{D}$-convergence $y_T \underset{D}{\Rightarrow} w_{\alpha,\beta}$ is absent for the same reason as the S-convergence. (The functionals $f(s) = \inf_{u \leq 1} s(u)$ and $f(s) = \sup_{u \leq 1} s(u)$ are $\rho_{\mathbb{D}}$-continuous; cf. Remark 3.1.) The convergence is absent in the metric space $\mathbb{D}$ with each metric ρ in which the functional $f(s) = \sup_{u \leq 1} s(u)$ is continuous.

The author is grateful to K. A. Borovkov for useful remarks.

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