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THE FOURIER–FABER–SCHAUDER SERIES UNCONDITIONALLY DIVERGENT IN MEASURE

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Abstract: We prove that, for every $\varepsilon \in (0,1)$, there is a measurable set $E \subset [0,1]$ whose measure $|E|$ satisfies the estimate $|E| > 1 - \varepsilon$ and, for every function $f \in C_{[0,1]}$, there is $\tilde{f} \in C_{[0,1]}$ coinciding with f on E whose expansion in the Faber–Schauder system diverges in measure after a rearrangement.

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§ 1. Introduction

In the article we describe the structure of a continuous function whose expansion in the Faber–Schauder system diverges in measure after a rearrangement. The Faber–Schauder system $\{\varphi_n(x)\}_{n=0}^{\infty}$ [1] is one of the most popular systems of functions and many articles are devoted to its study (the main properties are presented, for instance, in [2, Chapter 6, Section 1]).

Recall that the Faber–Schauder system is a basis for $C[0,1]$ (see [3]), i.e., every $f \in C[0,1]$ is uniquely representable by the series

$$f(x) := \sum_{n=0}^{\infty} A_n(f)\varphi_n(x)$$

in the Faber–Schauder system uniformly convergent to f on $[0,1]$ (we call it the *Fourier–Faber–Schauder series* of f). The Fourier coefficients $A_n(f)$ of this series are defined as

$$A_0(f) = f(0), \quad A_1(f) = f(1) - f(0),$$
$$A_n(f) = A_{k,i}(f) = f\left(\frac{2i-1}{2^{k+1}}\right) - \frac{1}{2}\left[f\left(\frac{i-1}{2^k}\right) + f\left(\frac{i}{2^k}\right)\right]. \tag{1}$$

There are many interesting results on properties of the Faber–Schauder system (see the survey [4] by Ulyanov and also [5–8]). We describe the results that concern those of this article.

As is known, there are no unconditional bases for $C_{[0,1]}$ (see [9]); i.e., there is a continuous function whose Fourier–Faber–Schauder series after some rearrangement does not converge to it in the norm of $C_{[0,1]}$.

The results of [10, 11] demonstrate that the Faber–Schauder system is even more unreliable in this respect.

Theorem A. *There is a function in* $C_{[0,1]}$ *whose Fourier–Faber–Schauder series diverges in measure after some rearrangement.*

However, it is proven in [12] that changing the values of a continuous function on a set of arbitrarily small measure we can correct this apparent “hopeless” situation (see also [13]) and the Fourier–Faber–Schauder series of the new function converges unconditionally.

The aim of this article is to prove that by changing the values of a continuous function on a set of arbitrarily small measure we can construct a poor function whose Fourier–Faber–Schauder series diverges in measure after some rearrangement.

Moreover, we prove

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Theorem 1. *For every $\varepsilon \in (0,1)$, there is a measurable set $E \subset [0,1]$ such that $|E| > 1-\varepsilon$ and, for every $f \in C_{[0,1]}$, there is $\tilde{f} \in C_{[0,1]}$ coinciding with f on E whose Fourier–Faber–Schauder series diverges in measure after some rearrangement.*

Moreover, the following is established in [14]:

Theorem B. *For every $\varepsilon \in (0,1)$, there is a measurable set $E \subset [0,1]$ such that $|E| > 1-\varepsilon$ and, for every $f \in C_{[0,1]}$, there is $\tilde{f} \in C_{[0,1]}$ coinciding with f on E whose Fourier–Faber–Schauder series converges unconditionally in $C_{[0,1]}$.*

Thus, Theorems 1 and B justify the following statement: For every $\varepsilon \in (0,1)$, there is a measurable set $E \subset [0,1]$ such that $|E| > 1-\varepsilon$ and, for every $f \in C_{[0,1]}$, we can construct two functions $f_1, f_2 \in C_{[0,1]}$ coinciding with f on E such that the Fourier–Faber–Schauder series of f_1 converges unconditionally in $C_{[0,1]}$ and this series of f_2 diverges in measure after some rearrangement.

It is of interest to find out the analogous aspects of behavior of other bases for the space of continuous functions.

There is proven in [14] that Theorem B fails for the Franklin system (and so it is of interest to answer the question whether there is an orthonormal basis for the space $C[0,1]$ such that Theorem B holds).

§ 2. Proof of the Main Lemma

The functions of the Faber–Schauder system $\Phi = \{\varphi_n(x)\}_{n=0}^{\infty}$, $x \in [0,1]$, are defined as follows: $\varphi_0(x) = 1$, $\varphi_1(x) = x$, and

$$\varphi_n(x) = \varphi_k^{(i)}(x) = \begin{cases} 0 & \text{for } x \notin \Delta_n = \Delta_k^{(i)} \equiv (\frac{i-1}{2^k}, \frac{i}{2^k}), \\ 1 & \text{for } x = x_n = x_k^{(i)} = \frac{2i-1}{2^{k+1}}, \\ \text{is linear and continuous on } [\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}] \text{ and } [\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}] \end{cases} \tag{2}$$

for $n = 2^k + i$, $k = 0, 1, \dots$, $i = 1, 2, \dots, 2^k$. Using the definition of the Faber–Schauder system, we can see easily that, for every $f \in C_{[0,1]}$ and every nonnegative integer m, the graph of $\sum_{n=0}^{m} A_n(f)\varphi_n(x)$ on $[0,1]$ is a broken line with vertices on the graph of f so that

$$\left\| \sum_{n=0}^{m} A_n(f)\varphi_n \right\|_C \le \|f\|_C, \quad m = 0, 1, \dots. \tag{3}$$

Lemma. *For all $\varepsilon \in (0,1)$, $\delta > 0$, $B > 0$, $\gamma > 0$, and a positive integer $N_0 > 1$, there are a measurable set $E_\varepsilon \subset [0,1]$, a polynomial in the Faber–Schauder system of the form*

$$P(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x),$$

a permutation $\{\sigma(n)\}_{n=N_0}^{N}$ (not unique) of positive integers $N_0, \dots, N$, and a positive integer $M \in [N_0, N]$ such that

(1) $|E_\varepsilon| > 1-\varepsilon$,

(2) $\delta > |A_{\sigma(n)}| \ge |A_{\sigma(n+1)}| \ge 0$, $n = N_0, \dots, N-1$,

(3) $\sum_{n=N_0}^{M} A_{\sigma(n)}\varphi_{\sigma(n)}(x) > B$ *for all* $x \in E_\varepsilon$,

(4) $\|P\|_C < \delta + \gamma$,

(5) $\frac{\gamma}{2} < P(x) = \text{const} < \gamma$ *for all* $x \in E_\varepsilon$.

Proof. Let $N_0 = 2^{k_0} + i_0$ ($i_0 \in [1, 2^{k_0}]$). Choose a positive integer $k > \max\{k_0, \log_2 \frac{4}{\varepsilon}\}$ and for every positive integer j define numbers $q_j = \sum_{m=k}^{k+j-1} m$ and sequences of indices I_j^1 and I_j^2 such that

$$\begin{aligned} I_j^1 &= \bigcup_{n_1=1}^{2^k-2} \bigcup_{n_2=1}^{2^{k+1}-2} \cdots \bigcup_{n_{j-1}=1}^{2^{k+j-2}-2} \bigcup_{n=2}^{2^{k+j-1}-1} \left\{ \sum_{l=0}^{j-1} n_l \cdot 2^{q_j-q_l} + n \right\}, \\ I_j^2 &= \bigcup_{n_1=1}^{2^k-2} \cdots \bigcup_{n_{j-1}=1}^{2^{k+j-2}-2} \left\{ \sum_{l=0}^{j-1} n_l \cdot 2^{q_j-q_l} + 1; \sum_{l=0}^{j-1} n_l \cdot 2^{q_j-q_l} + 2^{k+j-1} \right\}, \end{aligned} \tag{4}$$

where $n_0 = 0$ and $q_0 = 1$.

Consider

$$E_j = \bigcup_{i \in I_j^1} \Delta_{q_j}^{(i)}, \quad j \in \mathbb{N}, \tag{5}$$

where E_1 is obtained by dividing $[0,1]$ into 2^k equal parts $\big(\Delta_{q_1}^{(i)} = \Delta_k^{(i)}\big)$ and discarding the first and last of them, in turn $E_2 \subset E_1$ is obtained from E_1 by dividing each of the intervals $\Delta_{q_1}^{(i)}$ in E_1 into 2^{k+1} equal parts $\big(\Delta_{q_2}^{(i)} = \Delta_{2k+1}^{(i)}\big)$ and discarding the first and last of them; for every $j > 1$, the set $E_j \subset E_{j-1}$ is obtained from E_{j-1} by dividing each of the intervals $\Delta_{q_{j-1}}^{(i)}$ in E_{j-1} into 2^{k+j-1} equal parts $\big(\Delta_{q_j}^{(i)} = \Delta_{\frac{2k+j-1}{2}j}^{(i)}\big)$ and discarding the first and last of them.

Put

$$E_\varepsilon = \bigcap_{j=1}^{\infty} E_j. \tag{6}$$

Using the definitions of sequences of indices (4), it is not difficult to see that

$$\begin{aligned} |E_\varepsilon| &= 1 - \sum_{j=1}^{\infty} \sum_{i \in I_j^2} \big|\Delta_{q_j}^{(i)}\big| \\ &= 1 - 2 \cdot \frac{1}{2^k} - 2 \cdot \sum_{j=2}^{\infty} \frac{(2^k-2)(2^{k+1}-2)\dots(2^{k+j-2}-2)}{2^k 2^{k+1} \dots 2^{k+j-1}} > 1 - \varepsilon. \end{aligned}$$

Next, we consider the system of continuous functions $\{\psi_j(x)\}_{j=1}^{\infty}$ on $[0,1]$ such that (see (5))

$$\psi_j(x) = \begin{cases} 0 & \text{for } x \notin E_j, \\ 1 & \text{for } \ x \in E_{j+1}, \\ \text{is linear and continuous for } x \in E_j \setminus E_{j+1}. \end{cases} \tag{7}$$

From (6), (7), and (1), it is easy to see that, for every positive integer j, we have

$$E_\varepsilon \subset E_j = \mathscr{S}_{\psi_j}, \quad \|\psi_j\|_C = 1, \quad \psi_j(x) = 1 \text{ for all } x \in E_\varepsilon, \tag{8}$$

$$\psi_j(x) = \sum_{i \in I_j^1} \left(\varphi_{q_j}^{(i)}(x) + \frac{1}{2} \left\{ \sum_{p=1}^{k+j-1} \varphi_{q_j+p}^{(2t_p-1)}(x) + \sum_{p=1}^{k+j-1} \varphi_{q_j+p}^{(2h_p)}(x) \right\} \right), \tag{9}$$

where $\mathscr{S}_{\psi_j}$ is the support of $\psi_j(x)$, $t_1 = h_1 = i = \sum_{l=0}^{j} n_l \cdot 2^{q_j-q_l} + 1$, $t_{p+1} = 2t_p - 1$, and $h_{p+1} = 2h_p$.

It follows from (5)–(9) that, for all positive integer j and $x_0 \in E_\varepsilon$, we have

$$\varphi_{q_j}^{(i_{x_0})}(x_0) \neq 0 \tag{10}$$

only for the sole index $i_{x_0} \in I_j^1$.

Choose positive integers $K > \delta^{-1}$ and L so that

$$\frac{1}{K}\left(\sum_{l=1}^{L}\frac{1}{2l-1}-\sum_{l=L}^{2L-1}\frac{1}{2l}\right) > B, \tag{11}$$

and consider the series

$$\frac{1}{K}+\frac{1}{2K}-\frac{1}{2K}+\dots+\frac{1}{(2m-1)K}+\frac{1}{(2m)K}-\frac{1}{(2m)K}+\dots=+\infty. \tag{12}$$

Rearrange the summands in (12) by the Riemann scheme (see [15, Chapter 11, Section 4, Subsection 388, the Riemann Theorem]) in order to ensure convergence to γ. The so-obtained summands are denoted by C_j, $j = 1, 2, \dots$; i.e.,

$$\sum_{j=1}^{\infty} C_j = \gamma. \tag{13}$$

Let J be a positive integer such that $C_{J+1} = \frac{1}{(4L-1)K}$. Based on (12) and the Riemann scheme, we can assume that L is so large that

$$-\frac{\gamma}{2} < -\frac{1}{(4L-2)K} < C_J < \sum_{j=1}^{J} C_j - \gamma < 0 \tag{14}$$

and $\left\{\frac{1}{mK}\right\}_{m=1}^{4L-2}$ and $\left\{-\frac{1}{2mK}\right\}_{m=1}^{2L-1}$ belong to $\{C_j\}_{j=1}^{J}$.

Examine the function

$$Q_J(x) = \sum_{j=1}^{J} C_j \psi_j(x). \tag{15}$$

Using (9) and (12), we conclude that $Q_J(x)$ is a polynomial $P(x)$ in the Faber–Schauder system of the form

$$Q_J(x) = P(x) = \sum_{n=N_0}^{N} A_n \varphi_n(x), \quad \text{where } 0 \le |A_n| < \delta,\ n = N_0, \dots, N. \tag{16}$$

Next, (7), (8), the scheme of construction of (13) and (14) yield

$$\|P\|_C < C_1 + \gamma < \delta + \gamma, \quad \frac{\gamma}{2} < P(x) = \sum_{j=1}^{J} C_j < \gamma \quad \text{for all } x \in E_\varepsilon.$$

Let $\{\sigma(n)\}_{n=N_0}^{N}$ be a permutation of positive integers $N_0, \dots, N$ such that $|A_{\sigma(n)}| \ge |A_{\sigma(n+1)}|$ and let M be a positive integer satisfying

$$|A_{\sigma(M)}| = \frac{1}{(4L-2)K}, \quad |A_{\sigma(M+1)}| < \frac{1}{(4L-2)K}. \tag{17}$$

Denote by $\{\tilde{\sigma}(j)\}_{j=1}^{J}$ some permutation of positive integers $1, \dots, J$ such that $|C_{\tilde{\sigma}(j)}| \ge |C_{\tilde{\sigma}(j+1)}|$ (obviously, $\{\tilde{\sigma}(j)\}_{j=1}^{J}$ is not a unique permutation). In view of (9), (10), (12)–(17), it is easy to see that, for every $x \in E_\varepsilon$, we have

$$\sum_{n=N_0}^{M} A_{\sigma(n)}\varphi_{\sigma(n)}(x) = \frac{1}{K}\sum_{l=1}^{L-1}\left\{\frac{\psi_{\tilde{\sigma}(3l-2)}(x)}{2l-1} + \lambda_{\tilde{\sigma}}^{2l}\frac{\psi_{\tilde{\sigma}(3l-1)}(x)}{2l} - \lambda_{\tilde{\sigma}}^{2l}\frac{\psi_{\tilde{\sigma}(3l)}(x)}{2l}\right\}$$

$$+\frac{\psi_{\tilde{\sigma}(3L-2)}(x)}{(2L-1)K} + \frac{1}{K}\sum_{l=L}^{2L-2}\left(\lambda_{\tilde{\sigma}}^{2l}\frac{\varphi^{(i_x)}_{q_{\tilde{\sigma}(3l-1)}}(x)}{2l} - \lambda_{\tilde{\sigma}}^{2l}\frac{\varphi^{(i_x)}_{q_{\tilde{\sigma}(3l)}}(x)}{2l} + \frac{\varphi^{(i_x)}_{q_{\tilde{\sigma}(3l+1)}}(x)}{2l+1}\right)$$

$$+\frac{1}{K}\left(\lambda_{\tilde{\sigma}}^{4L-2}\frac{\varphi^{(i_x)}_{q_{\tilde{\sigma}(6L-4)}}(x)}{4L-2} - \lambda_{\tilde{\sigma}}^{4L-2}\frac{\varphi^{(i_x)}_{q_{\tilde{\sigma}(6L-3)}}(x)}{4L-2}\right),$$

where $\lambda_{\tilde{\sigma}}^{2l}$, $l = 1, \dots, 2L-1$, is equal to 1 or -1 in dependence on $\{\tilde{\sigma}(j)\}_{j=1}^{J}$. Hence, by (8), (11), and the definition of (2), we establish that

$$\sum_{n=N_0}^{M} A_{\sigma(n)}\varphi_{\sigma(n)}(x) > \frac{1}{K}\left(\sum_{l=1}^{L} \frac{1}{2l-1} - \sum_{l=L}^{2L-1} \frac{1}{2l}\right) > B \quad \text{for all } x \in E_\varepsilon.$$

The lemma is proven.

§ 3. Proof of Theorem 1

To prove Theorem 1, we need the following

Theorem 2. *For every $\varepsilon \in (0,1)$, there are a measurable set $E \subset [0,1]$ with $|E| > 1-\varepsilon$, $f_0 \in C_{[0,1]}$, a permutation $\sigma(n)$ of positive integers, and some sequences of positive integers $\{M_k\}_{k=1}^{\infty}$ and $\{N_k\}_{k=1}^{\infty}$ such that*

(1) $f_0(x) = 0$ *for all* $x \in E$,

(2) $\lim_{k\to\infty} \left\| \sum_{n=0}^{N_k} A_{\sigma(n)}\varphi_{\sigma(n)} - f_0 \right\|_C = 0$,

(3) $\lim_{k\to\infty} \sum_{n=0}^{M_k} A_{\sigma(n)}\varphi_{\sigma(n)}(x) = +\infty$ *a.e. on* $[0,1]$.

Proof. Let ε be an arbitrary real number in $(0,1)$. Apply the lemma on assuming that $\varepsilon = \frac{\varepsilon}{2}$, $\delta = 1$, $B = 1$, $\gamma = 1$, and $N_0 := N_1 + 1 = 2$ in the formulation. We can find a measurable set $E_1 \subset [0,1]$, a polynomial of the form

$$P_1(x) = \sum_{n=N_1+1}^{N_2} A_n^{(1)}\varphi_n(x),$$

a permutation $\{\sigma_1(n)\}_{n=N_1+1}^{N_2}$ of positive integers $N_1+1, \dots, N_2$, and a positive integer $M_1 \in (N_1, N_2]$ such that

$$|E_1| > 1 - \frac{\varepsilon}{2},$$

$$1 > \left|A_{\sigma_1(n)}^{(1)}\right| \geq \left|A_{\sigma_1(n+1)}^{(1)}\right| \geq 0, \quad n = N_1+1, \dots, N_2-1,$$

$$\|P_1\|_C < 2, \tag{18}$$

$$\frac{1}{2} < P_1(x) := p_1 < 1 \quad \text{for all } x \in E_1, \tag{19}$$

$$\sum_{n=2}^{M_1} A_{\sigma_1(n)}^{(1)}\varphi_{\sigma_1(n)}(x) > 1 \quad \text{for all } x \in E_1.$$

It follows from (3), (18), and (19) that

$$\max_{N_1+1\leq m\leq N_2} \left\| \sum_{n=2}^{m} A_n^{(1)}\varphi_n \right\|_C = \|P_1\|_C < 2$$

and

$$0 < 1 - p_1 < \frac{1}{2}. \tag{20}$$

Denote by a_1 the least number among the modules of the coefficients $A_n^{(1)}$ occurring in $P_1(x)$; i.e., $a_1 := \min\{\left|A_n^{(1)}\right| \neq 0,\ n \in (N_1, N_2]\}$. Apply the lemma again on assuming that $\varepsilon = \frac{\varepsilon}{4}$, $\delta = \frac{a_1}{2}$, $B = 2$, $\gamma = 1 - p_1$, and $N_0 = N_2 + 1$ in its formulation. We can define a measurable set $E_2 \subset [0,1]$, a polynomial of the form

$$P_2(x) = \sum_{n=N_2+1}^{N_3} A_n^{(2)}\varphi_n(x),$$

a permutation $\{\sigma_2(n)\}_{n=N_2+1}^{N_3}$ of the numbers $N_2+1,\dots,N_3$, and a positive integer $M_2 \in (N_2, N_3]$ satisfying the conditions

$$|E_2| > 1 - \frac{\varepsilon}{4},$$

$$\frac{1}{2} > \frac{a_1}{2} > \left|A^{(2)}_{\sigma_2(n)}\right| \geq \left|A^{(2)}_{\sigma_2(n+1)}\right| \geq 0, \quad n = N_2+1,\dots,N_3-1,$$

$$\|P_2\|_C < 1, \tag{21}$$

$$\frac{1-p_1}{2} < P_2(x) := p_2 < 1 - p_1 \quad \text{for all } x \in E_2, \tag{22}$$

$$\sum_{n=N_2+1}^{M_2} A^{(2)}_{\sigma_2(n)} \varphi_{\sigma_2(n)}(x) > 2 \quad \text{for all } x \in E_2.$$

Relations (3) and (20)–(22) yield

$$\max_{N_2+1 \leq m \leq N_3} \left\| \sum_{n=N_2+1}^{m} A^{(2)}_n \varphi_n \right\|_C = \|P_2\|_C < 1 \quad \text{and} \quad 0 < 1 - p_1 - p_2 < \frac{1}{4}.$$

Arguing as before, for every integer $k > 1$, by induction we can define a measurable set $E_k \subset [0,1]$, a polynomial

$$P_k(x) = \sum_{n=N_k+1}^{N_{k+1}} A^{(k)}_n \varphi_n(x), \tag{23}$$

a permutation $\{\sigma_k(n)\}_{n=N_k+1}^{N_{k+1}}$ of positive integers $N_k+1,\dots,N_{k+1}$, and a positive integer M_k such that

$$|E_k| > 1 - \varepsilon 2^{-k}, \tag{24}$$

$$\frac{1}{2^{-k+1}} > \frac{a_{k-1}}{2} > \left|A^{(k)}_{\sigma_k(n)}\right| \geq \left|A^{(k)}_{\sigma_k(n+1)}\right| \geq 0, \quad n = N_k+1,\dots,N_{k+1}-1,$$

$$\|P_k\|_C < 2^{-k+2}, \tag{25}$$

$$\frac{1 - \sum\limits_{l=1}^{k-1} p_l}{2} < P_k(x) := p_k < 1 - \sum_{l=1}^{k-1} p_l \quad \text{for all } x \in E_k, \tag{26}$$

$$\sum_{n=N_k+1}^{M_k} A^{(k)}_{\sigma_k(n)} \varphi_{\sigma_k(n)}(x) > k \quad \text{for all } x \in E_k, \tag{27}$$

where $a_{k-1} = \min\left\{\left|A^{(k-1)}_n\right| \neq 0,\, n \in (N_{k-1}, N_k]\right\}$, and

$$0 < 1 - \sum_{l=1}^{k-1} p_l < 2^{-k+1}. \tag{28}$$

From (3), (25), (26), and (28), we derive that

$$\max_{N_k+1 \leq m \leq N_{k+1}} \left\| \sum_{n=N_k+1}^{m} A^{(k)}_n \varphi_n \right\|_C < 2^{-k+2}, \tag{29}$$

$$0 < 1 - \sum_{l=1}^{k} p_l < 2^{-k}. \tag{30}$$

Inequalities (25) and (29) imply that the series $-\varphi_0(x)+\sum_{n=2}^{\infty} A_n\varphi_n(x)$, where $A_n = A_n^{(k)}$, $n \in (N_k, N_{k+1}]$, converges uniformly on $[0,1]$.

Put

$$f_0(x) := -\varphi_0(x) + \sum_{n=2}^{\infty} A_n\varphi_n(x) = -\varphi_0(x) + \sum_{k=1}^{\infty} P_k(x)$$

$$= -\varphi_0(x) + \sum_{k=1}^{\infty}\sum_{n=N_k+1}^{N_{k+1}} A_n^{(k)}\varphi_n(x) = -\varphi_0(x) + \sum_{k=1}^{\infty}\sum_{n=N_k+1}^{N_{k+1}} A_{\sigma_k(n)}^{(k)}\varphi_{\sigma_k(n)}(x), \tag{31}$$

$$\sigma(n) = \sigma_k(n), \quad n \in (N_k, N_{k+1}], \tag{32}$$

$$E' := \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k, \quad E'' := \bigcap_{k=1}^{\infty} E_k \subset E_k. \tag{33}$$

Relying on (24), (26), (30), (31), and (33), we can easily see that

$$|E'| = 1, \quad |E''| > 1-\varepsilon, \quad f_0(x) = -1 + \sum_{k=1}^{+\infty} p_k = 0 \quad \text{for all } x \in E''$$

and, for every $x \in E'$, there is a positive integer k_x such that $x \in E_k$ for all $k \geq k_x$. Hence, from (27) and (32) we conclude that

$$\sum_{n=N_k+1}^{M_k} A_{\sigma(n)}\varphi_{\sigma(n)}(x) > k$$

for all $k \geq k_x$ and so, in view of (25), (31), and (32), for every $x \in E'$ as $k \to +\infty$, we have

$$-\varphi_0(x) + \sum_{n=2}^{M_k} A_{\sigma(n)}\varphi_{\sigma(n)}(x) = -\varphi_0(x) + \sum_{l=1}^{k-1}\sum_{n=N_l+1}^{N_{l+1}} A_{\sigma(n)}\varphi_{\sigma(n)}(x)$$

$$+ \sum_{n=N_k+1}^{M_k} A_{\sigma(n)}\varphi_{\sigma(n)}(x) \geq \sum_{n=N_k+1}^{M_k} A_{\sigma(n)}\varphi_{\sigma(n)}(x) - \left\|\sum_{l=1}^{k-1} P_l\right\|_C - 1 > k-5 \to +\infty;$$

on the other hand, the series

$$-\varphi_0(x) + \sum_{n=2}^{N_k} A_{\sigma(n)}\varphi_{\sigma(n)}(x) = -\varphi_0(x) + \sum_{l=1}^{k-1} P_l(x)$$

converges to $f_0(x)$ uniformly on $[0,1]$.

Theorem 2 is proved.

Note that Theorem 2 is stronger than Theorem A, since in the former we constructed a continuous function that differs from zero only on a set of arbitrarily small measure whose Fourier–Faber–Schauder series diverges in measure after some rearrangement. Note that also the absolute values of the nonzero coefficients in this rearrangement are enumerated in decreasing order.

Let $\varepsilon \in (0,1)$ be arbitrary. Find the sets $E_1 \subset [0,1]$ and $E_2 \in [0,1]$, with $|E_1| > 1-\frac{\varepsilon}{2}$ and $|E_2| > 1-\frac{\varepsilon}{2}$, from the formulations of Theorems B and 2, respectively, and consider the set $E = E_1 \cap E_2$, which is a subset of E_1 and E_2 simultaneously, such that $|E| > 1-\varepsilon$. Given $f \in C_{[0,1]}$, by Theorem B we can find $f_1 \in C_{[0,1]}$ that coincides with f on E whose Fourier–Faber–Schauder series converges unconditionally in $C_{[0,1]}$; on the other hand, by Theorem 2 we can construct $f_0 \in C_{[0,1]}$ vanishing on E whose series diverges in measure after some rearrangement. It is easy to see that the series of $f_2 = f_1 + f_0$ (which coincides with f on E) diverges in measure again after some rearrangement.

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