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Σ-DEFINABILITY IN HEREDITARILY FINITE SUPERSTRUCTURES AND COMPUTABLE ANALYSIS

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Abstract: We construct a computable real function not Σ-definable in hereditarily finite superstructures over the extensions with decidable theory of the reals.

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1. Introduction

There are several approaches to computability over uncountable sets, and currently it seems impossible to choose the one that generalizes the concepts of classical computability theory most naturally. It is therefore important to study the innate restrictions of various models of computation and their mutual properties.

The real is one of the most natural and significant objects for studying generalized computability. A widely known approach to computability over the reals is the computable analysis proposed in [1]. The latter is based on approximating the reals by computable rational sequences.

On the other hand, Ershov introduced in [2], as an analog of computability, the concept of Σ-definability in a hereditarily finite superstructure over an arbitrary structure. Later Goncharov and Sviridenko studied in [3] a similar theory of Σ-definability for list superstructures. Defining computability over the reals in this approach, we can pick as the basic model one of the formalizations of the reals, for instance, the ordered field $\langle R, +, \cdot, 0, 1, < \rangle$.

Since the expressive capability of this structure is rather limited (see [4, 5]), researchers turn often to the models extended by additional functions not Σ-definable in the original structure.

This article studies the expressive capability in computable analysis and Σ-definability. In particular, we show that the classes of computable functions in these approaches are different.

2. Preliminaries

Let us recall the definitions and statements of computable analysis and Σ-definability needed below.

Definition. A function $f : \mathrm{dom}(f) \subseteq R^n \to R$ is called *computable* whenever there exists an oracle Turing machine that, given $k \in \mathbb{N}$, can request an arbitrary good approximation for the input $x \in \mathrm{dom}(f)$; i.e., it may ask for finitely many tuples $p \in \mathbb{Q}^n$ of rationals with $d(x, p) < 2^{-i}$, where i may depend on the answer to the previous questions, and in finitely many steps the machine writes a rational q to the output tape with $|f(x) - q| < 2^{-k}$.

Here we assume that the precision i of an input is written in binary form on the special oracle tape, the machine enters a special oracle questioning state, and then the answer is provided in one step on the oracle tape. Thus, the input $x \in \mathbb{R}^n$ is given to the machine only through rational approximations that the machine has to request. As an encoding for rational approximation to a real, we use the concept of the name of a real in some alphabet.

Definition. A *representation* of a set X is a surjective function $\delta : \mathrm{dom}(\delta) \subseteq \Sigma^\omega \to X$, where Σ is some alphabet. If $x \in X$ and $p \in \Sigma^\omega$ with $\delta(p) = x$ then the sequence p is called a *δ-name* of x.

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In order to work with this model of computation, it is convenient to fix a particular representation of reals to be used below.

Definition. The *Cauchy representation* $\rho_C : \mathrm{dom}(\rho_C) \subseteq \Sigma^\omega \to \mathbb{R}$ of the reals is the representation in which each real x is given by an infinite sequence of symbols encoding a sequence of rationals converging rapidly to x:

$$\rho_C(w_1\#w_2\#\cdots) = x \Leftrightarrow |x - \nu_\mathbb{Q}(w_i)| < 2^{-i} \quad \text{for all } i \in \mathbb{N},$$

where $\nu_\mathbb{Q}(w_i) \in \mathbb{Q}$ is the number with code w_i.

We assume here that the alphabet Σ contains the symbols 0, 1, and #. Observe however that the Cauchy representation is not the unique admissible representation for reals, and the choice of representation in each particular case can be made out of convenience.

Moreover, [1] presents a slightly modified model of computation: We assume that the input to the machine via the distinguished input tape is a rapidly converging rational sequence with limit x specified in some alphabet, while the machine, instead of a rational approximation to $f(x)$ of accuracy 2^{-k} with prescribed $k \in \mathbb{N}$, outputs a rapidly converging rational sequence with limit $f(x)$.

Definition. Given some alphabet Σ, a function $F : \mathrm{dom}(F) \subseteq \Sigma^\omega \to \Sigma^\omega$ is called *computable* whenever there exists a Turing machine which, given $p \in \mathrm{dom}(F)$ on the input tape, writes to the output tape the sequence $F(p)$ symbol by symbol without stopping, while for every input $p \notin \mathrm{dom}(F)$ the machine writes at most finitely many symbols to the output tape.

Definition. A function $f : \mathrm{dom}(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ is called *computable* whenever there exist a computable function $F : \mathrm{dom}(F) \subseteq \Sigma^\omega \to \Sigma^\omega$ and a representation $\rho_C : \mathrm{dom}(\rho_C) \subseteq \Sigma^\omega \to \mathbb{R}$ such that for every ρ_C-name p of some $x \in \mathrm{dom}(f)$ the value of $F(p)$ is a ρ_C-name of the value $f(x)$ of f.

Observe that in this approach the concept of computability stays the same, while the input and output have identical forms (rapidly converging rational sequences); see [1].

For a more detailed study of the theory of computable analysis, the reader can address [1]. Let us also recall some statements of the theory of Σ-functions.

Consider a structure $\mathfrak{M}$ of signature σ and the set $\mathscr{P}_\omega(M)$ of all finite subsets of M. Start with defining the underlying set of the hereditarily finite superstructure $\mathbb{HF}(\mathfrak{M})$ as

$$HF_0(M) = \{\varnothing\}; \quad HF_{n+1}(M) = \mathscr{P}_\omega(HF_n(M) \cup M); \quad HF(M) = \bigcup_{n<\omega} HF_n(M).$$

Then the *hereditarily finite superstructure* over M is the structure $\mathbb{HF}(\mathfrak{M}) = \langle HF(M) \cup M, \sigma' \rangle$, where $\sigma' = \sigma \cup \{\varnothing, \in^2, U^1\}$, while $\varnothing$ is interpreted as the empty set, $\in^2$ is the containment relation, and U^1 is the predicate selecting the elements of M.

Definition. Each predicate $P(x_1, x_2, \ldots, x_n)$ representable by a Σ-formula is called a Σ-*predicate*. A function f is called Σ-*definable* whenever the graph of f is a Σ-predicate.

Denote by $\mathbb{R}$ the structure $\mathbb{R} = \langle R, +, \cdot, 0, 1, < \rangle$ of the ordered reals. Therefore, a function $f : \mathbb{R} \to \mathbb{R}$ is Σ-definable in $\mathbb{HF}(\mathbb{R})$ whenever there exists a Σ-formula $\Phi(x, y)$ such that $\mathbb{HF}(\mathbb{R}) \models \Phi(x, y) \Leftrightarrow f(x) = y$.

The following statement is well known in the theory of Σ-definability; for a proof see, for instance, [6].

Proposition 1. *Each Σ-formula $\varphi(\bar{x})$ of signature σ_{HF} can be effectively represented by a computable disjunction $\bigvee_{i\in\omega} \varphi_i(\bar{x})$ of $\exists$-formulas of signature σ such that $\mathfrak{M} \models \bigvee_{i\in\omega} \varphi_i(\bar{a}) \Leftrightarrow \mathbb{HF}(\mathfrak{M}) \models \varphi(\bar{a})$ for all $\bar{a} \in \mathfrak{M}$.*

For a more detailed exposition of the available results on Σ-definability; see, for instance, [6].

3. Construction of a Computable Function

Now we explain the main result of this article.

Theorem 1. *For every extension $\mathbb{R}^*$ of the ordered field of reals with decidable theory there exists a total computable function $F : \mathbb{R} \to \mathbb{R}$ not definable in $\mathbb{HF}(\mathbb{R}^*)$ by any Σ-formula.*

PROOF. Let us describe an algorithm for constructing a total computable function $F : \mathbb{R} \to \mathbb{R}$ distinct from all functions which are Σ-definable in $\mathbb{HF}(\mathbb{R}^*)$.

To start with, fix a computable enumeration ν of pairs of rationals. Assume also that we are given a computable enumeration $\{\Phi_n(x,y)\}_{n\in\omega}$ of all Σ-formulas of signature σ_{HF} in two variables, as well as decompositions $\Phi_n(x,y) = \bigvee_{i\in\omega} \varphi_i(\bar{x})$ of Proposition 1 for each formula. We now proceed to the algorithm.

Step 0: Put $F_0(x) \equiv 0$. For each index n of a formula, it is convenient to keep a label $r_n = 0$ indicating that no witness has been found for the formula $\Phi_n(x,y)$.

Step $k+1$: Consider all formulas $\Phi_n(x,y)$ with $n \leq k+1$ and $r_n = 0$.

For these $\Phi_n(x,y)$, find the first $k+1$ formulas in the decomposition $\Phi_n(x,y) = \bigvee_{i\in\omega} \varphi_i(\bar{x})$. Then among these find, for each $\Phi_n(x,y)$, those satisfying the condition

$$\exists a_n \exists b_n (n < a_n < b_n < n+1) \& (\forall x((a_n \leq x \leq b_n)) \to \exists! y \varphi_i(x,y)).$$

To check the above condition is possible because $\mathbb{R}^*$ is a decidable theory.

If the condition fails for $\Phi_n(x,y)$ then proceed immediately to the next formula. If the condition holds for some formula $\varphi_i(x,y)$ in the decomposition of $\Phi_n(x,y)$ then say that the segment $[n, n+1]$ is *stabilized*. In this case fix $\varphi_i(x,y)$ with the smallest index i and find $a_n, b_n \in \mathbb{Q}$ with the smallest index ν satisfying the formula.

Check the condition $\varphi_i\big(\frac{b_n+a_n}{2}, 0\big)$. If it fails then mark the index n as inspected, $r_n := 1$, and proceed to the next formula $\Phi_n(x,y)$. If the condition holds then put

$$F_{k+1}(x) = \begin{cases} F_k(x) & \text{for } x \notin [n, n+1], \\ 2^{-2k} \frac{\frac{b+a}{2} - x}{\frac{b+a}{2} - n} & \text{for } x \in [n, \frac{b+a}{2}], \\ 2^{-2k} \frac{x - \frac{b+a}{2}}{n+1-\frac{b+a}{2}} + 2^{-2k} & \text{for } x \in [\frac{b+a}{2}, n+1]. \end{cases}$$

After that put $r_n := 1$ and proceed to the next formula. Having read all formulas with $n \leq k$, proceed to step $k+2$.

Verify that the function constructed cannot be defined by any Σ-formula in $\mathbb{HF}(\mathbb{R}^*)$.

Suppose that Φ_n determines a total function from $\mathbb{R}$ to $\mathbb{R}$. Then its decomposition includes a formula $\varphi_i(x,y)$ defining this function on some segment $[a,b] \subseteq [n, n+1]$; i.e., the condition

$$\exists a \exists b (n < a < b < n+1) \& (\forall x((a \leq x \leq b)) \to \exists! y \varphi_i(x,y))$$

is met. Since $Th(\mathbb{R}^*)$ is decidable, we can effectively test this condition.

By construction, for each formula of this kind we have defined the point $x_n = \frac{b_n+a_n}{2}$ at which the value of F is distinct from the one given by $\varphi_i(x,y)$ and consequently by $\Phi_n(x,y)$ as well.

Verify that the function constructed is computable in the sense of Weihrauch; i.e., there is a Turing machine that for a given precision k and the input name ρ outputs the value of the function with this accuracy in finitely many steps.

Suppose that some name of a Cauchy number $x \in \mathbb{R}$ is given, namely, a sequence $\{q_s\}_{s\in\mathbb{N}}$ rapidly converging to x, as well the accuracy 2^{-k} with which it is necessary to calculate $F(x)$. Find all Φ_n with $[q_s - 2^{-s}, q_s + 2^{-s}] \cap [n, n+1] \neq \varnothing$ and $n > q_s + 2^{-s}$. Make k steps of the above algorithm for each required Φ_n.

As the output of F we take the values that are calculated at each step:

$$F_k(q_s) = \begin{cases} 2^{-2m} & \text{if } q_s = \frac{b_n+a_n}{2}, \\ \frac{2^{-2k}}{\frac{b_n+a_n}{2}-n}(q_s - n) & \text{if the function on the segment } [n, n+1] \\ & \text{has stabilized at some step } m < k \\ & \text{and } n < q_s < \frac{b_n+a_n}{2}, \\ \frac{2^{-2m}}{n+1-\frac{b_n+a_n}{2}}(n+1-q_s) & \text{if the function on the segment } [n, n+1] \\ & \text{has stabilized at some step } m < k \\ & \text{and } \frac{b_n+a_n}{2} < q_s < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $F_k(q_s) \in \mathbb{Q}$ anyway. Estimate the difference $|F_k(q_s) - F(x)|$.

At each step k of the calculation the value of the function at an arbitrary point changes by at most 2^{-2k}; moreover, it changes at most once on each segment $[n, n+1]$. Suppose that the function changes on $[n, n+1]$ at some subsequent step $k + s$. Then $|F_k - F| \le 2^{-2(k+s)} < 2^{-2k}$ at each point of $[n, n+1]$.

In the case that the function has stabilized on $[n, n+1]$ at some step $m \le k$, we have

$$|F_k(q_s) - F(x)| \le \left| \frac{2^{-2m}}{\frac{b_n+a_n}{2} - n}(q_s - n) - \frac{2^{-2k}}{\frac{b_n+a_n}{2} - n}(x - n) \right|$$
$$\le \left| \frac{2^{-2m}}{\frac{b_n+a_n}{2} - n}(q_s - x) \right| < \left| \frac{2^{-2m}}{\frac{b_n+a_n}{2} - n} 2^{-s} \right| = \left| \frac{2^{-2m-s}}{\frac{b_n+a_n}{2} - n} \right|.$$

Then it suffices to choose the input precision s with $2^{-2m-s} < \frac{b_n+a_n}{2} 2^{-2k}$. We can do this because of the rapid convergence of $\{q_s\}_{s \in \mathbb{N}}$ to x. Thus, in k steps of the calculation we know the function with accuracy 2^{-2k}.

Remark. It is easy to obtain the converse because all Weihrauch computable functions are continuous. Even the simple function

$$\operatorname{sgn}(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \ge 0, \end{cases}$$

is not computable in this approach [1].

On the other hand, $\operatorname{sgn}(\cdot)$ can obviously be defined in $\mathbb{HF}(\mathbb{R})$ by the Δ_0-formula

$$((x < 0)\&(y = 0)) \vee ((x \ge 0)\&(y = 1)).$$

Denote by $\mathbb{R}_{\exp}$ the algebraic system $\langle \mathbb{R}, +, \cdot, \exp(x), <, 0, 1 \rangle$, where $\mathbb{R}$ is the real, the operations $+$ and $\cdot$, the relation $<$, and the constants 0 and 1 are interpreted as usual, while $\exp(x)$ is the exponential function.

As [7] shows, $\mathbb{R}_{\exp}$ is decidable provided that the following (Schanuel's) conjecture holds.

For a positive integer n take $a_1, \dots, a_n \in \mathbb{C}$ which are linearly independent over $\mathbb{Q}$. Then the transcendence degree of the extension $\mathbb{Q}(a_1, \dots, a_n, e^{a_1}, \dots, e^{a_n})$ is at least n.

Thus, Theorem 1 implies the following corollary.

Corollary 1. *Under Schanuel's conjecture, there exists a computable real function which is not Σ-definable in the hereditarily finite superstructure over the real exponential field.*

Remark. For the latter extension, Corollary 1 has a simpler proof. It is shown [7] that under Schanuel's conjecture the number π cannot be defined by a Σ-formula in $\mathbb{R}_{\exp}$. Using the decomposition of a Σ-formula to an infinite disjunction of $\exists$-formulas, it is easy to show that in this case it is impossible to define the sine function by any Σ-formula in $HW(\mathbb{R}_{\exp})$.

Indeed, suppose that $\Phi(x, y)$ is a Σ-formula defining the sine in $HW(\mathbb{R}_{\exp})$. It is equivalent to the disjunction of $\exists$-formulas $\bigvee_{n\in\omega} \varphi_n(x, y)$, where $\varphi_n(x, y)$ is an $\exists$-formula in $\mathbb{R}_{\exp}$. Since $\sin(\pi) = 0$, the formula $\Phi(\pi, 0)$ is true, and accordingly the disjunction $\bigvee_{n\in\omega} \varphi_n(x, y)$ is true. Then there exists at least one formula $\varphi_n(x, y)$ such that $\varphi_n(x, 0)$ defines π in $\mathbb{R}_{\exp}$, which is known to be impossible.

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