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ON SPLITTINGS, SUBGROUPS, AND THEORIES OF PARTIALLY COMMUTATIVE METABELIAN GROUPS

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Abstract: We consider two splittings of a partially commutative metabelian group G. The universal theories and splittings of G are compared. We prove that all nilpotent subgroups of G are abelian and give description of the Fitting subgroup of G.

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1. Introduction

A group G of solubility length at most 2 is called *metabelian*. Metabelian groups form a variety. Denote the free group of rank n of this variety by S_n, and the basis of S_n, by $X = \{x_1, \dots, x_n\}$.

Recall the definition of partially commutative group in the variety of metabelian groups. Let Γ be an undirected finite graph without loops and multiple edges with vertex set X and edge set E. Denote the edge joining vertices x_i and x_j by (x_i, x_j). Given Γ, define the partially commutative metabelian group S_Γ as the quotient group S_n/R_Γ, where R_Γ is generated as the normal subgroup by those commutators $[x_i, x_j] = x_i^{-1}x_j^{-1}x_ix_j$ for which the vertices x_i and x_j are adjacent in Γ, i.e., $(x_i, x_j) \in E$. The graph Γ is called the *defining graph* for S_Γ. If the edge set E of Γ is empty then $S_\Gamma = S_n$. If Γ is complete then S_Γ is the free abelian group A_n of rank n.

The *universal* or $\forall$*-theory* of a group G is the class of all $\forall$-propositions of the group signature true on G.

The universal theory of S_n is decidable [1, 2]. Starting from the decidability of the universal theory of S_n, it is easy also to give other trivial examples of partially commutative metabelian groups with decidable universal theory, for example, the direct sum of several free metabelian groups $S_n \times \dots \times S_m$.

A nontrivial example of a partially commutative metabelian group with decidable universal theory is the group defined by the linear graph L_4 on four vertices (see [3, Corollary 2]).

The proof in [1] of the decidability of the universal theory of a free metabelian group is based on the two facts: Firstly, the universal theory of S_n coincides with the universal theory of the group W_n that is the semidirect product of the free abelian group A_n of rank n and the free $\mathbb{Z}[A_n]$-module of rank n. Secondly, it is proved that the universal theory of W_n is decidable.

Note that W_n is a free splitting of S_n over its commutant $[S_n, S_n]$. In [4], the definitions and properties are given of splittings of a group over an abelian normal subgroup; it was also observed in [4] that a free splitting is isomorphic to a Magnus splitting.

In the general case, the question of the decidability of the universal theory of a partially commutative metabelian group S_Γ is open and included in the *Kourovka Notebook* [5, Question 17.104].

We, however, have

Theorem A [6]. *Let a group W be the semidirect product of a free abelian group A of finite rank and a finitely generated $\mathbb{Z}[A]$-module T, $W = AT$. Then the universal theory of W is decidable.*

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Therefore, for proving the decidability of the universal theory of a partially commutative metabelian group S_Γ, it seems natural to proceed analogously with proving the decidability of the universal theory of S_n; i.e., to find a free splitting W_Γ of S_Γ over the commutant $[S_\Gamma, S_\Gamma]$ and compare the universal theories of S_Γ and W_Γ.

In Section 2, alongside a free splitting W_Γ of S_Γ, we construct one more splitting $\overline{W}_\Gamma$ of S_Γ for the case when Γ is a connected graph. In Section 3, we carry out the comparison of the universal theories of the groups. There we prove that the universal theories of S_Γ and W_Γ and also of S_Γ and $\overline{M}_\Gamma$ are in general distinct for a wide class of graphs Γ.

But if Γ is the linear graph on three or four vertices then the semidirect products mentioned in Theorem A are constructed whose universal theories coincide with the universal theory of S_Γ.

The *Fitting subgroup* $\mathrm{Fit}(G)$ of a group G is the product of all nilpotent normal subgroups in G.

In [7], some theorem was proved describing $\mathrm{Fit}(G)$ for metabelian groups G with a "small" number of relations. More exactly, we have

Theorem B [7]. *Suppose that a group G is defined in the variety of metabelian groups by n generators and m defining relations, where $n - m \geq 2$. Then $\mathrm{Fit}(G) = [G, G]$.*

Section 4 is devoted to the study of nilpotent subgroups in partially commutative metabelian groups. We prove that their nilpotent subgroups are always abelian. We also give full description of the Fitting subgroup of a partially commutative metabelian group.

2. Embeddings

In the theory of soluble groups, an important role is played by the *Magnus embedding* [8]. Recall the latter for the free metabelian group S_n, $n \geq 2$.

Let $\{x_1, \dots, x_n\}$ be the basis of S_n, let A_n be the free abelian group with basis $\{a_1, \dots, a_n\}$, let $B = \mathbb{Z}[A_n]$ be the group ring of A_n, let F be a free right B-module with basis $\{f_1, \dots, f_n\}$, and let $W_n = \begin{pmatrix} A_n & 0 \\ F & 1 \end{pmatrix}$ be a matrix group. The mapping

$$\mu : x_i \mapsto \begin{pmatrix} a_i & 0 \\ f_i & 1 \end{pmatrix}, \quad i = 1, \dots, n,$$

extends to the Magnus embedding of the group S_n into W_n, for which we preserve the notation μ. The group W_n is a Magnus splitting for S_n over $[S_n, S_n]$.

Define the epimorphism (differential) d of the module F onto the fundamental ideal Δ of the ring B generated by the elements $a_1 - 1, \dots, a_n - 1$ as follows:

$$d(f_i) = a_i - 1, \quad i = 1, \dots, n.$$

It is known (see [9, 10]) that $\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \in W_n$ belongs to the image of $[S_n, S_n]$ under μ if and only if

$$d(f) = 0. \tag{1}$$

It is not hard to calculate that, under the Magnus embedding, the commutators $[x_i, x_j]$ are taken to the matrices $\begin{pmatrix} 1 & 0 \\ \tau_{ij} & 1 \end{pmatrix}$, where $\tau_{ij} = f_i(a_j - 1) + f_j(1 - a_i)$. The normal subgroup R_Γ is mapped into the submodule L of F generated by those τ_{ij} for which $(x_i, x_j) \in E$.

Let $T = F/L$. The Magnus embedding μ of S_n into W_n induces the embedding μ_Γ of S_Γ into the matrix group $W_\Gamma = \begin{pmatrix} A_n & 0 \\ T & 1 \end{pmatrix}$. Since $d(\tau_{ij}) = 0$, we have $d(L) = 0$. Therefore, d induces an epimorphism d_Γ of the module T onto Δ. Obviously, $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in \mu_\Gamma(S_\Gamma)$ if and only if

$$d_\Gamma(t) = 0. \tag{2}$$

In the terminology of [4], the group W_Γ is a free extension of S_Γ over the abelian normal subgroup $[S_\Gamma, S_\Gamma]$ and d_Γ is the differential of this extension.

Consider another embedding of S_Γ into a matrix group.

Proposition 1. *Suppose that Γ is a connected graph, $\{x_1, \dots, x_n\}$ is the vertex set of Γ and simultaneously the basis of the free metabelian group S_n, and a_i is the image of x_i under the natural homomorphism $S_n \to A_n = S_n/[S_n, S_n]$, while $\delta = (a_1 - 1)\dots(a_n - 1)$. Then S_Γ embeds in the matrix group*

$$\overline{W}_\Gamma = \begin{pmatrix} A_n & 0 \\ T/T\delta & 1 \end{pmatrix}, \tag{3}$$

when the module $T = F/L$ was defined above.

Proof. Consider the natural homomorphism

$$\nu : \begin{pmatrix} A_n & 0 \\ T & 1 \end{pmatrix} \to \begin{pmatrix} A_n & 0 \\ T/T\delta & 1 \end{pmatrix}.$$

Show that $\mu_\Gamma(S_\Gamma) \cap \ker(\nu) = 1$. Indeed, suppose that the matrix

$$C = \begin{pmatrix} b & 0 \\ t & 1 \end{pmatrix}, \quad b \in A_n,\ t \in T,$$

lies in the intersection $\mu_\Gamma(S_\Gamma) \cap \ker(\nu)$. Since b is unchanged under ν, we have $b = 1$. Further, t lies in $T\delta$. Hence, there exists $\tau \in T$ such that $t = \tau\delta$. So, $C = \begin{pmatrix} 1 & 0 \\ \tau\delta & 1 \end{pmatrix}$ belongs to $\mu_\Gamma(S_\Gamma)$. From (2) we obtain $d_\Gamma(\tau)\delta = 0$. Hence, $d_\Gamma(\tau) = 0$. Thus, $C = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}$ belongs to $\mu_\Gamma([S_\Gamma, S_\Gamma])$. The theorem on the structure of annihilators [6, Theorem 1] implies that the commutant of S_Γ is annihilated by δ. Therefore, $c^\delta = 1$; i.e., $\tau\delta = 0$. Hence, C is the identity matrix. The proposition is proved.

3. Universal Theories

Let us give the two examples when the universal theory of a partially commutative group defined by a tree coincides with the universal theory of a group of the form mentioned in Theorem A.

Example 1. Suppose that the graph Γ_1 defining a partially commutative group is a star with vertices $x_0, x_1, \dots, x_n$; i.e., $S_{\Gamma_1} \cong S_n(x_1, \dots, x_n) \times \langle x_0 \rangle$. The universal theory of S_{Γ_1} coincides with the universal theory of $W_1 = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, where A is the free abelian group with basis $\{a_0, a_1, \dots, a_n\}$, T is a free $\mathbb{Z}[a_1^{\pm 1}, \dots, a_n^{\pm 1}]$-module of rank n, and the element a_0 annihilates T. The group S_{Γ_1} embeds in W_1, and the universal theories of the groups S_{Γ_1} and W_1 coincide.

Example 2. Let $\Gamma_2 = L_4$ be the linear graph on four vertices and let S_2 be the free metabelian group of rank 2. As was shown in [10, Theorem 2], the universal theory of S_{Γ_2} coincides with the universal theory of the direct product $S_2 \times S_2$ of two copies of a free metabelian group of rank 2. It is known [1, Theorem 2.5, item 2] that the universal theory of S_2 coincides with the universal theory of the group $M_1 = \begin{pmatrix} A_1 & 0 \\ T_1 & 1 \end{pmatrix}$, where A_1 is a free abelian group of rank 2 and T_1 is a free $\mathbb{Z}[A_1]$-module of rank 2. Suppose that $A_2 \cong A_1$, $T_2 \cong T_1$, and $M_2 = \begin{pmatrix} A_2 & 0 \\ T_2 & 1 \end{pmatrix}$. Consider the abelian group $T = T_1 \oplus T_2$, on which $A = A_1 \times A_2$ acts as follows: $(t_1, t_2)(a_1, a_2) = (t_1 a_1, t_2 a_2)$. With respect to this action, T is endowed with the structure of an A-module. Since the universal theories of S_{Γ_2} and $M_1 \times M_2$ coincide and $W_2 = M_1 \times M_2$ is representable as $W_2 = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, the universal theories of S_{Γ_2} and W_2 coincide. By Theorem A, the universal theory of S_{Γ_2} is decidable.

However, in the case that the defining graph is a tree containing the linear graph on five vertices L_5 as a subgraph, the universal theories of S_Γ and W_Γ are distinct as follows from Theorem 2.

Recall that a vertex x_i in a graph Γ is called *hanging* if the degree $\deg(x_i)$ of x_i equals 1.

Theorem 2. *Let Γ be a tree whose vertex set $\{x_1, \dots, x_n\}$ contains at least two nonadjacent nonhanging vertices. Then the universal theories of S_Γ and W_Γ are distinct.*

PROOF. Suppose that $G = S_\Gamma$ or $G = W_\Gamma$. Let $\{x_1, \dots, x_m\}$ be nonhanging vertices and $[x_1, x_2] \neq 1$. Consider the proposition

$$\Phi \leftrightharpoons \exists b_1 \dots b_m u_1 \dots u_m v_1 \dots v_m \Bigg(\bigwedge_{i=1}^{m} c_i = [u_i, v_i] \wedge \bigwedge_{1 \le i,j \le m} [b_i, b_j] = 1$$
$$\wedge \bigwedge_{i=1}^{m} [b_i, c_i] = 1 \wedge \bigwedge_{1 \le i \neq j \le m} [b_i, c_j] \neq 1 \Bigg). \tag{4}$$

Show that (4) is false on G if $G = S_\Gamma$.

Since $c_i \in [G, G]$ and $[b_i, c_j] \neq 1$, we have $b_i \notin [G, G]$ for all $i = 1, \dots, m$.

Each element $b \in G$ is uniquely representable as $b = \prod_{i=1}^{n} x_i^{l_i} c$, where $c \in [G, G]$ and $l_i \in \mathbb{Z}$. Denote by $\operatorname{supp}(b)$ the set of those x_i that occur in the representation of b with nonzero exponents l_i.

Since $[b_i, c_i] = 1$ but $[b_j, c_i] \neq 1$, $1 \le i \neq j \le m$, it is easy to understand that all sets $\operatorname{supp}(b_i)$, $i = 1, \dots, m$, are nonempty and different.

Note (see [11, Corollary 2]) that if Γ is a tree, b is an element in S_Γ for which $\operatorname{supp}(b)$ consists of more than one element, and an element $d \in [G, G]$ commutes with b then $d = 1$.

For all i, we have $c_i \neq 1$, $b_i \notin [G, G]$, and $[b_i, c_i] = 1$; hence, $\operatorname{supp}(b_i)$ consists of a single element x_{t_i}.

The hanging vertices $x_{m+1}, \dots, x_n$ commute only with the identity element of the commutant (see [6, Theorem 4]). Therefore, $\{x_1, \dots, x_m\}$ and $\{x_{t_1}, \dots, x_{t_m}\}$ coincide.

By hypothesis, $[x_1, x_2] \neq 1$. Hence, the commutators $[b_i, b_j]$ contain a nonidentity commutator (see [11, Lemma 4]). Therefore, Φ is false on S_Γ.

Show that Φ is true on G if $G = W_\Gamma$. Put $b_i = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}$ and $c_i = [x_{u_i}, x_{v_i}]$, where $i = 1, \dots, m$, (x_i, x_{u_i}) and (x_i, x_{v_i}) are different edges of Γ.

The theorem on the structure of annihilators (see [6, Theorem 1]) yields

$$[x_{u_i}, x_{v_i}, x_i] = 1, \quad [x_{u_i}, x_{v_i}, x_j] \neq 1, \quad i \neq j.$$

The theorem is proved.

By complete analogy to Theorem 2, we prove

Theorem 3. *Let Γ be a tree whose vertex set $\{x_1, \dots, x_n\}$ contains at least two nonadjacent nonhanging vertices. Then the universal theories of the groups S_Γ and $\overline{W}_\Gamma$ are different.*

Observe some more properties connected with the embeddings and the universal theories of partially commutative metabelian groups.

Lemma 4. *The property of a finitely generated metabelian group G to be approximable by torsion-free nilpotent groups can be written down with the use of a countable system of universal axioms.*

PROOF. Consider a free metabelian group S of countable rank with basis $\{x, y_1, y_2, \dots\}$. Write down all elements $v(x, y_1, \dots, y_n)$, $n \ge 0$, of S, for which the sum of the exponents $\log_x(v)$ with respect to x is nonzero. Let those be the elements $v_1, v_2, \dots$. Given $v_i(x, y_{j_1}, \dots, y_{j_m})$, write down the proposition

$$\varphi_i \rightleftharpoons \forall a b g_1, \dots, g_m (v_i(a, g_1, \dots, g_m) = 1 \wedge v_i(b, g_1, \dots, g_m) = 1 \Rightarrow a = b). \tag{5}$$

It is known that a finitely generated metabelian group G is approximable by torsion-free nilpotent groups if and only if each equation

$$v(x, g_1, \dots, g_n) = 1,\ g_i \in G, \quad \log_x(v) \neq 0,$$

has at most one solution [12, p. 2]. Consequently, if all propositions φ_i are true on G then G is approximable by torsion-free nilpotent groups, which was required to prove.

Proposition 5. *Suppose that S_Γ is a partially commutative metabelian group, A is an abelian group, T is a right $\mathbb{Z}[A]$-module, $W = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$, and the universal theories of W and S_Γ coincide. Then $[W, W] \lneq \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$.*

Proof. Suppose that $[W, W] = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix}$. Then each $t \in T$ is representable as

$$t = \sum_i t_i(b_i - 1), \quad t_i \in T,\ b_i \in A. \tag{6}$$

Each t_i is again representable in the form (6)

$$t_i = \sum_j t_{ij}(b_{ij} - 1), \quad t_{ij} \in T,\ b_{ij} \in A. \tag{7}$$

From (6) and (7) we deduce that $C = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ lies in $\gamma_3(W)$ for every $t \in T$. Obviously, C lies in the intersection of all terms of the lower central series of W. But the universal theories of W and S_Γ coincide. Hence, by Lemma 4, the group W is approximable by nilpotent groups. Consequently, $\bigcap \gamma_i(W) = E$. We get a contradiction if we choose a nonidentity matrix as C.

4. Subgroups

The groups whose all nilpotent subgroups are abelian constitute the quasivariety which we will denote by $\mathscr{Q}$. Corollary 4 in [13] states that all partially commutative metabelian nilpotent group are abelian. We will prove more, namely, that all partially commutative metabelian groups belong to $\mathscr{Q}$.

Theorem 6. *Every nilpotent subgroup of a metabelian partially commutative group is abelian.*

Proof. Let S_Γ be a partially commutative metabelian nonabelian group. In [14, Corollary 1], it was proved that all partially commutative metabelian nonabelian groups generate the same quasivariety, which we will denote by $\mathscr{Q}_0$. In particular, the free metabelian nonabelian group S_n generates the quasivariety $\mathscr{Q}_0$.

The quasi-identity

$$\Phi = \forall xy([x, y, x] = 1 \land [x, y, y] = 1 \to [x, y] = 1)$$

is true on some group H if and only if all nilpotent subgroups of H are abelian. Indeed, suppose that H has a nilpotent nonabelian subgroup N. Then in N we can find a nilpotent subgroup M of class 2. For this it suffices to consider the upper central series of N. Choose two elements $g, h \in M$ so that $[g, h] \neq 1$. These elements do not satisfy Φ. Conversely, suppose that a group H does not satisfy Φ. Then there are two elements g and h in H such that

$$[g, h] \neq 1, \quad [g, h, g] = [g, h, h] = 1.$$

This means that the nilpotency class of the group $\langle g, h \rangle$ generated by the elements g and h, is equal to 2.

The group S_n satisfies the quasi-identity Φ. Indeed, its commutant $[S_n, S_n]$ is torsion-free as a $\mathbb{Z}[S_n/[S_n, S_n]]$-module. Therefore, if some elements $x, y \in S_n$ satisfy $[x, y]^{1-x} = 1$ and $[x, y] \neq 1$ then $x \in [S_n, S_n]$. Similarly, from $[x, y]^{1-y} = 1$ and $[x, y] \neq 1$ we obtain $y \in [S_n, S_n]$. But then $[x, y] = 1$; a contradiction.

Every nonabelian partially commutative group S_Γ generates the same quasivariety as S_n. Hence, S_Γ satisfies Φ. Consequently, all nilpotent subgroups S_Γ are abelian. The theorem is proved.

We obtain the following description of the Fitting subgroup for a partially commutative metabelian group:

Proposition 7. *The Fitting subgroup* $\mathrm{Fit}(S_\Gamma)$ *of a partially commutative metabelian group* S_Γ *is the direct product of the center* Z *of* S_Γ *and the commutant* $[S_\Gamma, S_\Gamma]$.

Proof. As we have just proved, all nilpotent subgroups in S_Γ are abelian. Hence, $\mathrm{Fit}(S_\Gamma)$ is an abelian group. It contains the commutant $[S_\Gamma, S_\Gamma]$. Suppose that $g \in \mathrm{Fit}(S_\Gamma) \setminus [S_\Gamma, S_\Gamma]$. Let $\{x_1, \dots, x_n\}$ be all vertices of the defining graph. Without loss of generality, we may assume that the element g has the form

$$g = x_1^{l_1} \dots x_m^{l_m} c,$$

where $c \in [S_\Gamma, S_\Gamma]$ and $l_1, \dots, l_m$ are integers and $m \geq 1$. Since $[S_\Gamma, S_\Gamma] \subseteq \mathrm{Fit}(S_\Gamma)$ and $\mathrm{Fit}(S_\Gamma)$ is an abelian group, $[g, [S_\Gamma, S_\Gamma]] = 1$. Hence,

$$\left[x_1^{l_1} \dots x_m^{l_m}, [S_\Gamma, S_\Gamma]\right] = 1.$$

From [11, Theorem 2] we conclude that $[x_i, [S_\Gamma, S_\Gamma]] = 1$ for $i = 1, \dots, m$. Suppose that one of the vertices x_i, $i = 1, \dots, m$, and some vertex of the graph x_j are nonadjacent. Then the commutator $[x_i, x_j]$ is not equal to 1. By Theorem 1 in [6] on the structure of annihilators, we have $[x_i, [x_i, x_j]] \neq 1$. We get a contradiction to $[x_i, [S_\Gamma, S_\Gamma]] = 1$. Thus, for all $i = 1, \dots, m$, the vertex x_i is adjacent to any other vertex of the graph. In other words, the vertices x_i lie in the center of the group. Therefore, $g \in \langle S'_\Gamma, Z\rangle$. By [13, Corollary 1], the center and the commutant are disjoint. The proposition is proved.

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